Supplemental Materials for

Lee, Se Yoon. "The Use of a Log-Normal Prior for the Student *t*-Distribution." *Axioms* (2022): 11(9):462.

Explaining an R Package **bayesestdft** to estimate degree of freedoms of the student t-distribution

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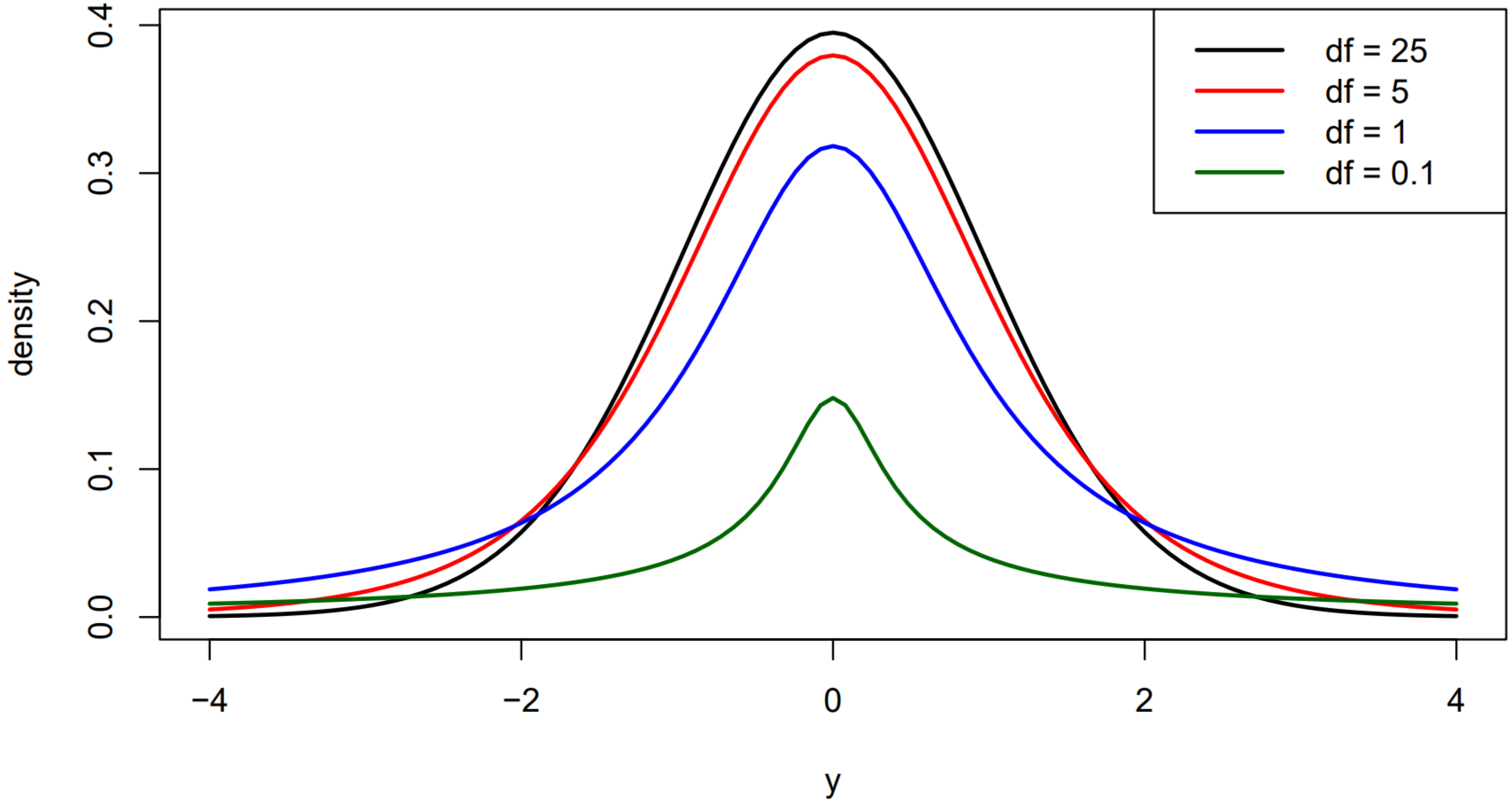
Student t-distribution

- **Probability density function**

$$f(y|\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \cdot \left(1 + \frac{y^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \qquad -\infty < y < \infty$$



Student t-distribution

- **Probability density function of Student t-distribution:** $Y \sim t_\nu(y),\ (\nu > 0)$:

$$f(y|\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} \cdot \left(1 + \frac{y^2}{\nu}\right)^{-\frac{\nu+1}{2}} \quad , \qquad -\infty < y < \infty$$

  - **Special case 1:** when $\nu = 1, Y \sim t_{\nu=1}(y) = Cauchy(y)$

$$f(y|\nu = 1) = \frac{\Gamma\left(\frac{1+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{2}\right)} \cdot (1 + y^2)^{-1} \ = \frac{1}{\pi(1+y^2)}, \qquad -\infty < y < \infty$$

  - **Special case 2:** when $\nu = \infty,\ Y \sim t_{\nu=+\infty}(y) \sim N(y|0,1)$

$$f(y|\nu = \infty) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \qquad -\infty < y < \infty$$

- **Likelihood for N i.i.d data points $(\boldsymbol{y_1}, \boldsymbol{y_2}, \cdots, \boldsymbol{y_N})$ from the t-distribution:**

$$L(\nu) = \prod_{i=1}^{N} t(y_i|\nu) = \prod_{i=1}^{N} \left(\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} \cdot \left(1 + \frac{y_i^2}{\nu}\right)^{-\frac{\nu+1}{2}} \right) = \left(\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} \right)^N \prod_{i=1}^{N} \left(1 + \frac{y_i^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

Student t-distribution

By plugging-in $\eta = \log\nu \ \ (\nu = e^{\eta})$, we have

- **Likelihood ratio:**

$$\frac{L(e^{\eta})}{L(e^{\eta^{(s)}})} = \frac{\left(\frac{\Gamma\left(\frac{e^{\eta}+1}{2}\right)}{\sqrt{e^{\eta}\pi}\,\Gamma\left(\frac{e^{\eta}}{2}\right)}\right)^{N} \cdot \prod_{i=1}^{N}\left(1+\frac{y_i^2}{e^{\eta}}\right)^{-\frac{e^{\eta}+1}{2}}}{\left(\frac{\Gamma\left(\frac{e^{\eta^{(s)}}+1}{2}\right)}{\sqrt{e^{\eta^{(s)}}\pi}\,\Gamma\left(\frac{e^{\eta^{(s)}}}{2}\right)}\right)^{N} \cdot \prod_{i=1}^{N}\left(1+\frac{y_i^2}{e^{\eta^{(s)}}}\right)^{-\frac{e^{\eta^{(s)}}+1}{2}}}$$

- **Likelihood ratio (log-scaled):**

$$\log\left(\frac{L(e^{\eta})}{L(e^{\eta^{(s)}})}\right) = \log\left(\left(\frac{\Gamma\left(\frac{e^{\eta}+1}{2}\right)}{\sqrt{e^{\eta}\pi}\,\Gamma\left(\frac{e^{\eta}}{2}\right)}\right)^{N} \cdot \prod_{i=1}^{N}\left(1+\frac{y_i^2}{e^{\eta}}\right)^{-\frac{e^{\eta}+1}{2}}\right) - \log\left(\left(\frac{\Gamma\left(\frac{e^{\eta^{(s)}}+1}{2}\right)}{\sqrt{e^{\eta^{(s)}}\pi}\,\Gamma\left(\frac{e^{\eta^{(s)}}}{2}\right)}\right)^{N} \cdot \prod_{i=1}^{N}\left(1+\frac{y_i^2}{e^{\eta^{(s)}}}\right)^{-\frac{e^{\eta^{(s)}}+1}{2}}\right)$$

$$= N\cdot\log\left(\frac{\Gamma\left(\frac{e^{\eta}+1}{2}\right)}{\sqrt{e^{\eta}\pi}\,\Gamma\left(\frac{e^{\eta}}{2}\right)}\right) + \sum_{i=1}^{N}\log\left(\left(1+\frac{y_i^2}{e^{\eta}}\right)^{-\frac{e^{\eta}+1}{2}}\right) - \left\{N\cdot\log\left(\frac{\Gamma\left(\frac{e^{\eta^{(s)}}+1}{2}\right)}{\sqrt{e^{\eta^{(s)}}\pi}\,\Gamma\left(\frac{e^{\eta^{(s)}}}{2}\right)}\right) + \sum_{i=1}^{N}\log\left(\left(1+\frac{y_i^2}{e^{\eta^{(s)}}}\right)^{-\frac{e^{\eta^{(s)}}+1}{2}}\right)\right\}$$

$$= N\cdot\log\left(\frac{\Gamma\left(\frac{e^{\eta}+1}{2}\right)}{\sqrt{e^{\eta}\pi}\,\Gamma\left(\frac{e^{\eta}}{2}\right)}\right) - \frac{e^{\eta}+1}{2}\sum_{i=1}^{N}\log\left(1+\frac{y_i^2}{e^{\eta}}\right) - \left\{N\cdot\log\left(\frac{\Gamma\left(\frac{e^{\eta^{(s)}}+1}{2}\right)}{\sqrt{e^{\eta^{(s)}}\pi}\,\Gamma\left(\frac{e^{\eta^{(s)}}}{2}\right)}\right) - \frac{e^{\eta^{(s)}}+1}{2}\sum_{i=1}^{N}\log\left(1+\frac{y_i^2}{e^{\eta^{(s)}}}\right)\right\}$$

$$= N\cdot\underbrace{\boxed{\left[\log\left(\frac{\Gamma\left(\frac{e^{\eta}+1}{2}\right)}{\sqrt{e^{\eta}\pi}\,\Gamma\left(\frac{e^{\eta}}{2}\right)}\right) - \log\left(\frac{\Gamma\left(\frac{e^{\eta^{(s)}}+1}{2}\right)}{\sqrt{e^{\eta^{(s)}}\pi}\,\Gamma\left(\frac{e^{\eta^{(s)}}}{2}\right)}\right)\right]}}_{\text{Independent of data } (y_1, y_2, \cdots, y_N)} - \underbrace{\boxed{\left[\frac{e^{\eta}+1}{2}\sum_{i=1}^{N}\log\left(1+\frac{y_i^2}{e^{\eta}}\right) - \frac{e^{\eta^{(s)}}+1}{2}\sum_{i=1}^{N}\log\left(1+\frac{y_i^2}{e^{\eta^{(s)}}}\right)\right]}}_{\text{Dependent of data } (y_1, y_2, \cdots, y_N)}$$

Student t-distribution

- **Numerical Stability:**

The first part of the likelihood ratio (log-scaled) can be made numerically stable by using the final analytic form

These parts take Infinity when η and $\eta^{(s)}$ take large values due to the nature of the gamma function

$$\log\left(\frac{\Gamma\left(\frac{e^{\eta}+1}{2}\right)}{\sqrt{e^{\eta}}\pi\Gamma\left(\frac{e^{\eta}}{2}\right)}\right) - \log\left(\frac{\Gamma\left(\frac{e^{\eta^{(s)}}+1}{2}\right)}{\sqrt{e^{\eta^{(s)}}}\pi\Gamma\left(\frac{e^{\eta^{(s)}}}{2}\right)}\right) =$$

$$\log\left(\frac{\Gamma\left(\frac{e^{\eta}+1}{2}\right)}{\sqrt{e^{\eta}}\pi\Gamma\left(\frac{e^{\eta}}{2}\right)}\right) - \log\left(\frac{\Gamma\left(\frac{e^{\eta^{(s)}}+1}{2}\right)}{\sqrt{e^{\eta^{(s)}}}\pi\Gamma\left(\frac{e^{\eta^{(s)}}}{2}\right)}\right) =$$

$$\log\left(\Gamma\left(\frac{e^{\eta}+1}{2}\right)\right) - \log\left(\sqrt{e^{\eta}}\pi\Gamma\left(\frac{e^{\eta}}{2}\right)\right) - \left\{\log\left(\Gamma\left(\frac{e^{\eta^{(s)}}+1}{2}\right)\right) - \log\left(\sqrt{e^{\eta^{(s)}}}\pi\Gamma\left(\frac{e^{\eta^{(s)}}}{2}\right)\right)\right\}$$

$$= \log\left(\Gamma\left(\frac{e^{\eta}+1}{2}\right)\right) - \log\left(e^{\frac{\eta}{2}}\cdot\pi^{\frac{1}{2}}\cdot\Gamma\left(\frac{e^{\eta}}{2}\right)\right) - \left\{\log\left(\Gamma\left(\frac{e^{\eta^{(s)}}+1}{2}\right)\right) - \log\left(e^{\frac{\eta^{(s)}}{2}}\cdot\pi^{\frac{1}{2}}\cdot\Gamma\left(\frac{e^{\eta^{(s)}}}{2}\right)\right)\right\}$$

$$= \log\left(\Gamma\left(\frac{e^{\eta}+1}{2}\right)\right) - \left[\frac{\eta}{2} + \log\pi^{\frac{1}{2}} + \log\left(\Gamma\left(\frac{e^{\eta}}{2}\right)\right)\right] - \left\{\log\left(\Gamma\left(\frac{e^{\eta^{(s)}}+1}{2}\right)\right) - \left[\frac{\eta^{(s)}}{2} + \log\pi^{\frac{1}{2}} + \log\left(\Gamma\left(\frac{e^{\eta^{(s)}}}{2}\right)\right)\right]\right\}$$

$$= \log\left(\Gamma\left(\frac{e^{\eta}+1}{2}\right)\right) - \left[\frac{\eta}{2} + \log\pi^{\frac{1}{2}} + \log\left(\Gamma\left(\frac{e^{\eta}}{2}\right)\right)\right] - \left\{\log\left(\Gamma\left(\frac{e^{\eta^{(s)}}+1}{2}\right)\right) - \left[\frac{\eta^{(s)}}{2} + \log\pi^{\frac{1}{2}} + \log\left(\Gamma\left(\frac{e^{\eta^{(s)}}}{2}\right)\right)\right]\right\}$$

Use R function lgamma functions for the stability of the numerical calculation

Prior Options in R Package bayesestdft

Priors	R functions in Package bayesestdft	References
$$\pi_J(\nu) \propto \left(\frac{\nu}{\nu+3}\right)^{\frac{1}{2}} \cdot \left\{\psi'\left(\frac{\nu}{2}\right) - \psi'\left(\frac{\nu+1}{2}\right) - \frac{2(\nu+3)}{\nu(\nu+1)^2}\right\}^{\frac{1}{2}}$$ $\psi(a) = \frac{d \log \Gamma(a)}{da}$ and $\psi'(a) = \frac{d\psi(a)}{da}$ represent digamma and trigamma functions	BayesJeffreys	• Fonseca, Thaís CO, Marco AR Ferreira, and Helio S. Migon. "Objective Bayesian analysis for the Student-t regression model." *Biometrika* 95.2 (2008): 325-333.
$$\nu \sim \pi_E(\nu) = Ga(\alpha = shape = 1, \beta = rate = 0.1) = Exp(rate = 0.1)$$ $$\pi_E(\nu) = \frac{\beta^1}{\Gamma(1)} \nu^{1-1} \cdot e^{-\beta\nu} = \beta \cdot e^{-\beta\nu} = \frac{1}{10} e^{-\frac{\nu}{10}}$$	BayesGA	• Fernández, Carmen, and Mark FJ Steel. "On Bayesian modeling of fat tails and skewness." *Journal of the american statistical association* 93.441 (1998): 359-371.
$$\nu \sim \pi_G(\nu) = Ga(\alpha = shape = 2, \beta = rate = 0.1)$$ $$\pi_G(\nu) = \frac{\beta^\alpha}{\Gamma(\alpha)} \nu^{\alpha-1} \cdot e^{-\beta\nu} = \frac{1}{100} \nu \cdot e^{-\nu/10}$$	BayesGA	• Juárez, Miguel A., and Mark FJ Steel. "Model-based clustering of non-Gaussian panel data based on skew-t distributions." *Journal of Business & Economic Statistics* 28.1 (2010): 52-66. • Geweke, John. "Bayesian treatment of the independent Student-t linear model." *Journal of applied econometrics* 8.S1 (1993): S19-S40.
$$\nu \sim \pi_L(\nu) = logN(\mu, \sigma^2)$$ $$\pi_L(\nu) = \frac{1}{\nu\sigma\sqrt{2\pi}} \exp\left(-\frac{(\log\nu - \mu)^2}{2\sigma^2}\right) = \frac{1}{\nu\sqrt{2\pi}} \exp\left(-\frac{(\log\nu - 1)^2}{2}\right)$$	BayesLNP	• Lee, Se Yoon. "The Use of a Log-Normal Prior for the Student *t*-Distribution." *Axioms* (2022): 11(9):462.

bayesestdft::BayesJeffreys

- **Likelihood:**

$$L(\nu) = p(y_{1:N}|\nu\ ) = \prod_{i=1}^{N} t(y_i|\nu) = \left(\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} \right)^{N} \prod_{i=1}^{N} \left(1 + \frac{y_i^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

- **Prior:**

$$\nu \sim \pi_J(\nu) \propto \left(\frac{\nu}{\nu+3}\right)^{\frac{1}{2}} \cdot \left\{\psi'\left(\frac{\nu}{2}\right) - \psi'\left(\frac{\nu+1}{2}\right) - \frac{2(\nu+3)}{\nu(\nu+1)^2}\right\}^{\frac{1}{2}},$$

$\psi(a) = \frac{d \log \Gamma\,(a)}{da}$ and $\psi'(a) = \frac{d\psi(a)}{da}$ represent digamma and trigamma functions.

- **Posterior distribution:**

$$\pi(\nu|y_{1:N}) \propto L(\nu) \cdot \pi_J(\nu)$$

- **Target density obtained by change of variable ($\boldsymbol{\eta} = \log \boldsymbol{\nu}\ , \boldsymbol{\nu} = \boldsymbol{e^{\eta}}$):**

$$\pi(\eta|y_{1:N}) = \pi(\nu|y_{1:N})\,\Big|_{\nu=e^{\eta}} \cdot \left|\frac{d\nu}{d\eta}\right| = \pi(\nu = e^{\eta}|y_{1:N}) \cdot e^{\eta} \propto L(e^{\eta}) \cdot \pi_J(e^{\eta}) \cdot e^{\eta}$$

$$\propto L(e^{\eta}) \cdot \left(\frac{e^{\eta}}{e^{\eta}+3}\right)^{\frac{1}{2}} \cdot \left\{\psi'\left(\frac{e^{\eta}}{2}\right) - \psi'\left(\frac{e^{\eta}+1}{2}\right) - \frac{2(e^{\eta}+3)}{e^{\eta}(e^{\eta}+1)^2}\right\}^{\frac{1}{2}} \cdot e^{\eta}$$

bayesestdft::BayesJeffreys [Metropolis-Hastings Algorithm]

[Metropolis-Hastings Algorithm]

Goal: Sampling from $\eta \sim \pi(\eta | y_{1:N})$

$$\propto L(e^{\eta}) \cdot \pi_J(e^{\eta}) \cdot e^{\eta} = \exp(-\gamma(\eta)), \qquad \eta \in \boldsymbol{R}$$

$$\gamma(\eta) = -\log(L(e^{\eta})) - \log\left(\pi_J(e^{\eta})\right) - \eta$$

$$= -\left(\log(L(e^{\eta})) + \log\left(\pi_J(e^{\eta})\right) + \eta\right)$$

Specify the step size α

Input : current state $\eta^{(s)}$.

Output : current state $\eta^{(s+1)}$.

a. Define a criterion function

$$\alpha(\eta, \eta^{(s)}) = \min\left[\frac{e^{-\gamma(\eta)}}{e^{-\gamma(\eta^{(s)})}}, 1\right] : R \to [0,1]$$

b. Choose a threshold $u \sim U[0,1]$

c. Draw a proposal

$$\eta^* \sim N_1(\eta^{(s)}, 2\delta).$$

d. If ($u < \alpha(\eta^*, \eta^{(s)})$) $\{\eta^{(s+1)} = \eta^*\}$ else $\{\eta^{(s+1)} = \eta^{(s)}\}$

- **Posterior ratio:**

$$\frac{e^{-\gamma(\eta)}}{e^{-\gamma(\eta^{(s)})}} = \frac{\exp\left(\log(L(e^{\eta})) + \log\left(\pi_J(e^{\eta})\right) + \eta\right)}{\exp\left(\log\left(L(e^{\eta^{(s)}})\right) + \log\left(\pi_J(e^{\eta^{(s)}})\right) + \eta^{(s)}\right)} = \boxed{\frac{L(e^{\eta})}{L(e^{\eta^{(s)}})}} \cdot \boxed{\frac{\pi_J(e^{\eta})}{\pi_J(e^{\eta^{(s)}})}} \cdot \frac{e^{\eta}}{e^{\eta^{(s)}}}$$

- **Posterior ratio (log-scaled):**

$$\log\left(\frac{e^{-\gamma(\eta)}}{e^{-\gamma(\eta^{(s)})}}\right) = \log\left(\boxed{\frac{L(e^{\eta})}{L(e^{\eta^{(s)}})}}\right) + \log\left(\boxed{\frac{\pi_J(e^{\eta})}{\pi_J(e^{\eta^{(s)}})}}\right) + (\eta - \eta^{(s)})$$

- **Likelihood ratio (log-scaled):**

$$\log\left(\boxed{\frac{L(e^{\eta})}{L(e^{\eta^{(s)}})}}\right) = N \cdot \left[\log\left(\frac{\Gamma\left(\frac{e^{\eta}+1}{2}\right)}{\sqrt{e^{\eta}\pi}\Gamma\left(\frac{e^{\eta}}{2}\right)}\right) - \log\left(\frac{\Gamma\left(\frac{e^{\eta^{(s)}}+1}{2}\right)}{\sqrt{e^{\eta^{(s)}}\pi}\Gamma\left(\frac{e^{\eta^{(s)}}}{2}\right)}\right)\right]$$

$$-\left[\frac{e^{\eta}+1}{2}\sum_{i=1}^{N}\log\left(1+\frac{y_i^2}{e^{\eta}}\right) - \frac{e^{\eta^{(s)}}+1}{2}\sum_{i=1}^{N}\log\left(1+\frac{y_i^2}{e^{\eta^{(s)}}}\right)\right]$$

- **Prior ratio (log-scaled):**

$$\log\left(\boxed{\frac{\pi_J(e^{\eta})}{\pi_J(e^{\eta^{(s)}})}}\right)$$

$$= \frac{1}{2}\left\{\log\left(\frac{e^{\eta}}{e^{\eta}+3}\right) - \log\left(\frac{e^{\eta^{(s)}}}{e^{\eta^{(s)}}+3}\right)\right.$$

$$+ \log\left(\psi'\left(\frac{e^{\eta}}{2}\right) - \psi'\left(\frac{e^{\eta}+1}{2}\right) - \frac{2(e^{\eta}+3)}{e^{\eta}(e^{\eta}+1)^2}\right)$$

$$\left. - \log\left(\psi'\left(\frac{e^{\eta^{(s)}}}{2}\right) - \psi'\left(\frac{e^{\eta^{(s)}}+1}{2}\right) - \frac{2\left(e^{\eta^{(s)}}+3\right)}{e^{\eta^{(s)}}\left(e^{\eta^{(s)}}+1\right)^2}\right)\right\}$$

bayesestdft:: BayesJeffreys [Metropolis Adjusted Langevin Algorithm]

[Metropolis Adjusted Langevin Algorithm]

Goal: Sampling from $\eta \sim \pi(\eta|y_{1:N})$

$$\propto L(e^{\eta}) \cdot \pi_J(e^{\eta}) \cdot e^{\eta} = \exp(-\gamma(\eta)), \qquad \eta \in \boldsymbol{R}$$

$$\gamma(\eta) = -\log(L(e^{\eta})) - \log\left(\pi_J(e^{\eta})\right) - \eta$$

$$= -\left(\log(L(e^{\eta})) + \log\left(\pi_J(e^{\eta})\right) + \eta\right)$$

Specify the step size δ

Input : current state $\eta^{(s)}$.

Output : current state $\eta^{(s+1)}$.

a. Define a criterion function

$$\alpha(\eta, \eta^{(s)}) = \min\left[\frac{e^{-\gamma(\eta)}}{e^{-\gamma(\eta^{(s)})}} \cdot \frac{J(\eta^{(s)}|\eta)}{J(\eta|\eta^{(s)})}, 1\right] : R \to [0,1]$$

b. Choose a threshold $u \sim U[0,1]$

c. Draw a proposal

$$\eta^* \sim N_1(\eta^{(s)} - \delta \cdot \boxed{\nabla\gamma(\eta^{(s)})}, 2\delta).$$

d. If ($u < \alpha(\eta^*, \eta^{(s)})$) $\{\eta^{(s+1)} = \eta^*\}$ else $\{\eta^{(s+1)} = \eta^{(s)}\}$

- **Posterior ratio $\times$ MH Correction:**

$$\frac{e^{-\gamma(\eta)}}{e^{-\gamma(\eta^{(s)})}} \cdot \frac{J(\eta^{(s)}|\eta)}{J(\eta|\eta^{(s)})} = \frac{\exp\left(\log(L(e^{\eta})) + \log\left(\pi_J(e^{\eta})\right) + \eta\right)}{\exp\left(\log\left(L(e^{\eta^{(s)}})\right) + \log\left(\pi_J(e^{\eta^{(s)}})\right) + \eta^{(s)}\right)} \cdot \frac{N_1(\eta^{(s)}|\eta - \delta \cdot \nabla_\theta\gamma(\eta), 2\delta)}{N_1(\eta|\eta^{(s)} - \delta \cdot \nabla_\theta\gamma(\eta^{(s)}), 2\delta)}$$

where the proposal density is $J(\eta'|\eta) = N(\eta'|\eta - \delta \cdot \nabla\gamma(\eta), 2\delta)$

- **Posterior ratio $\times$ MH Correction (log-scaled):**

$$\log\left(\frac{e^{-\gamma(\eta)}}{e^{-\gamma(\eta^{(s)})}} \cdot \frac{J(\eta^{(s)}|\eta)}{J(\eta|\eta^{(s)})}\right) = \left[\left(\log(L(e^{\eta})) + \log\left(\pi_J(e^{\eta})\right) + \eta\right) - \left(\log\left(L\left(e^{\eta^{(s)}}\right)\right) + \log\left(\pi_J\left(e^{\eta^{(s)}}\right)\right) + \eta^{(s)}\right)\right]$$
$$+ \log\left(N_1(\eta^{(s)}|\eta - \delta \cdot \nabla_\theta\gamma(\eta), 2\delta)\right) - \log\left(N_1(\eta|\eta^{(s)} - \delta \cdot \nabla_\theta\gamma(\eta^{(s)}), 2\delta)\right)$$

- **Gradient of negative of log-posterior posterior:**

$$\nabla\gamma(\eta) = \frac{\partial\gamma(\eta)}{\partial\eta} = \frac{\partial}{\partial\eta}\left\{-\left(\log(L(e^{\eta})) + \log\left(\pi_J(e^{\eta})\right) + \eta\right)\right\}$$

Use R package "numDeriv“ to obtain the gradient function $\nabla\gamma(\eta)$

bayesestdft::BayesGA

- **Likelihood:**

$$L(\nu) = p(y_{1:N}|\nu) = \prod_{i=1}^{N} t(y_i|\nu) = \left(\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\right)^N \prod_{i=1}^{N}\left(1+\frac{y_i^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

- **Prior:**

$$\nu \sim \pi_{GA}(\nu) = \frac{\beta^\alpha}{\Gamma(\alpha)}\nu^{\alpha-1}\cdot e^{-\beta\nu}.$$

- **Posterior distribution:**

$$\pi(\nu|y_{1:N}) \propto L(\nu)\cdot \pi_{GA}(\nu)$$

- **Target density obtained by change of variable ($\boldsymbol{\eta = \log\nu}$, $\boldsymbol{\nu = e^{\eta}}$):**

$$\pi(\eta|y_{1:N}) = \pi(\nu|y_{1:N})\Big|_{\nu=e^\eta}\cdot\left|\frac{d\nu}{d\eta}\right| = \pi(\nu = e^\eta|y_{1:N})\cdot e^\eta \propto L(e^\eta)\cdot\pi_{GA}(e^\eta)\cdot e^\eta$$

$$= L(e^\eta)\cdot\frac{\beta^\alpha}{\Gamma(\alpha)}e^{\eta(\alpha-1)}\cdot e^{-\beta e^\eta}\cdot e^\eta \propto L(e^\eta)\cdot e^{\eta\alpha-\beta e^\eta}$$

bayesestdft::BayesGA [Metropolis-Hastings Algorithm]

[Metropolis-Hastings Algorithm]

Goal: Sampling from $\eta \sim \pi(\eta|y_{1:N})$

$$\propto L(e^{\eta}) \cdot \pi_{GA}(e^{\eta}) \cdot e^{\eta} = L(e^{\eta}) \cdot e^{\eta\alpha - \beta e^{\eta}} = \exp(-\gamma(\eta)), \eta \in \boldsymbol{R}$$

$$\gamma(\eta) = -\log(L(e^{\eta})) - \eta\alpha + \beta e^{\eta}$$

Specify the step size α

Input : current state $\eta^{(s)}$.

Output : current state $\eta^{(s+1)}$.

a. Define a criterion function

$$\alpha(\eta, \eta^{(s)}) = \min\left[\frac{e^{-\gamma(\eta)}}{e^{-\gamma(\eta^{(s)})}}, 1\right] : R \to [0,1]$$

b. Choose a threshold $u \sim U[0,1]$

c. Draw a proposal

$$\eta^{*} \sim N_1(\eta^{(s)}, 2\delta).$$

d. If ($u < \alpha(\eta^{*}, \eta^{(s)})$) $\{\eta^{(s+1)} = \eta^{*}\}$ else $\{\eta^{(s+1)} = \eta^{(s)}\}$

- **Posterior ratio:**

$$\frac{e^{-\gamma(\eta)}}{e^{-\gamma(\eta^{(s)})}} = \frac{\exp\left(\log(L(e^{\eta})) + \eta\alpha - \beta e^{\eta}\right)}{\exp\left(\log\left(L\left(e^{\eta^{(s)}}\right)\right) + \eta^{(s)}\alpha - \beta e^{\eta^{(s)}}\right)} = \exp\left(\log\left(\frac{L(e^{\eta})}{L\left(e^{\eta^{(s)}}\right)}\right) + \alpha(\eta - \eta^{(s)}) - \beta\left(e^{\eta} - e^{\eta^{(s)}}\right)\right)$$

- **Posterior ratio (log-scaled):**

$$\log\left(\frac{e^{-\gamma(\eta)}}{e^{-\gamma(\eta^{(s)})}}\right) = \log\left(\boxed{\frac{L(e^{\eta})}{L\left(e^{\eta^{(s)}}\right)}}\right) + \alpha(\eta - \eta^{(s)}) - \beta\left(e^{\eta} - e^{\eta^{(s)}}\right)$$

- **Likelihood ratio (log-scaled):**

$$\log\left(\boxed{\frac{L(e^{\eta})}{L\left(e^{\eta^{(s)}}\right)}}\right) = N \cdot \left[\log\left(\frac{\Gamma\left(\frac{e^{\eta}+1}{2}\right)}{\sqrt{e^{\eta}\pi}\,\Gamma\left(\frac{e^{\eta}}{2}\right)}\right) - \log\left(\frac{\Gamma\left(\frac{e^{\eta^{(s)}}+1}{2}\right)}{\sqrt{e^{\eta^{(s)}}\pi}\,\Gamma\left(\frac{e^{\eta^{(s)}}}{2}\right)}\right)\right]$$

$$-\left[\frac{e^{\eta}+1}{2}\sum_{i=1}^{N}\log\left(1 + \frac{y_i^2}{e^{\eta}}\right) - \frac{e^{\eta^{(s)}}+1}{2}\sum_{i=1}^{N}\log\left(1 + \frac{y_i^2}{e^{\eta^{(s)}}}\right)\right]$$

- **Prior ratio:**

$$\alpha(\eta - \eta^{(s)}) - \beta\left(e^{\eta} - e^{\eta^{(s)}}\right)$$

bayesestdft::BayesLNP

- **Likelihood:**

$$L(\nu) = p(y_{1:N}|\nu\ ) = \prod_{i=1}^{N} t(y_i|\nu) = \left(\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)}\right)^N \prod_{i=1}^{N}\left(1+\frac{y_i^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

- **Prior:**

$$\nu \sim \pi_L(\nu) = logN(\nu|\mu,\sigma^2).$$

- **Posterior distribution:**

$$\pi(\nu|y_{1:N}) \propto L(\nu)\cdot\pi_L(\nu)$$

- **Target density obtained by change of variable ($\boldsymbol{\eta = \log\nu\ , \nu = e^{\eta}}$):**

$$\pi(\eta|y_{1:N}) = \pi(\nu|y_{1:N})\Big|_{\nu=e^{\eta}}\cdot\left|\frac{d\nu}{d\eta}\right| = \pi(\nu = e^{\eta}|y_{1:N})\cdot e^{\eta} \propto L(e^{\eta})\cdot\pi_L(e^{\eta})\cdot e^{\eta}$$

$$\propto L(e^{\eta})\cdot logN(e^{\eta}|\mu,\sigma^2)\cdot e^{\eta} = L(e^{\eta})\cdot\frac{1}{e^{\eta}\cdot\sigma\sqrt{2\pi}}\exp\left(-\frac{(\log e^{\eta}-\mu)^2}{2\sigma^2}\right)\cdot e^{\eta}$$

$$= L(e^{\eta})\cdot\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(\eta-\mu)^2}{2\sigma^2}\right) = L(e^{\eta})\cdot N(\eta|\mu,\sigma^2)$$

bayesestdft::BayesLNP [Elliptical Slice Sampler]

[Elliptical Slice Sampler]

Goal: Sample from the posterior density

$$\pi(\eta|y_{1:N}) \propto L(e^{\eta}) \cdot N(\eta|\mu,\sigma^2),$$

where $L(\nu) = \left(\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)}\right)^N \cdot \prod_{i=1}^N \left(1+\frac{y_i^2}{\nu}\right)^{-\frac{\nu+1}{2}}$.

Input: Current state $\eta^{(s)}$

Output: A new state $\eta^{(s+1)}$

a. Choose an ellipse centered at μ: $\rho \sim N(\mu,\sigma^2)$.

b. Define a criterion function:

$$\alpha\left(\eta,\eta^{(s)}\right) = \min\left\{\frac{L(e^{\eta})}{L\left(e^{\eta^{(s)}}\right)}, 1\right\} : R \to [0,1].$$

c. Choose a threshold and fix:

$$u \sim Unif[0,1].$$

d. Draw an initial proposal η^* :

$$\phi \sim Unif(-\pi,\pi];$$

$$\eta^* = \left(\eta^{(s)} - \mu\right) \cdot \cos\phi + (\rho-\mu)\cdot\sin\phi + \mu.$$

e. **if** ($u < \alpha\left(\eta^*,\eta^{(s)}\right)$) $\{\eta^{(s+1)} = \eta^*\}$ **else** {

Define a bracket : $(\phi_{min},\phi_{max}] = (-\pi,\pi]$

while ($u \geq \alpha\left(\eta^*,\eta^{(s)}\right)$) {

Shrink the bracket and try a new point:

if ($\phi > 0$) $\phi_{max} = \phi$ **else** $\phi_{min} = \phi$

$\phi \sim Unif(\phi_{min},\phi_{max}]$;

$\eta^* = \left(\eta^{(s)} - \mu\right) \cdot \cos\phi + (\rho-\mu)\cdot\sin\phi + \mu.$

}

$\eta^{(s+1)} = \eta^*$

}

- **Likelihood ratio:**

$$\frac{L(e^{\eta})}{L\left(e^{\eta^{(s)}}\right)} = \frac{\left(\frac{\Gamma\left(\frac{e^{\eta}+1}{2}\right)}{\sqrt{e^{\eta}\pi}\Gamma\left(\frac{e^{\eta}}{2}\right)}\right)^N \cdot \prod_{i=1}^N \left(1+\frac{y_i^2}{e^{\eta}}\right)^{-\frac{e^{\eta}+1}{2}}}{\left(\frac{\Gamma\left(\frac{e^{\eta^{(s)}}+1}{2}\right)}{\sqrt{e^{\eta^{(s)}}\pi}\Gamma\left(\frac{e^{\eta^{(s)}}}{2}\right)}\right)^N \cdot \prod_{i=1}^N \left(1+\frac{y_i^2}{e^{\eta^{(s)}}}\right)^{-\frac{e^{\eta^{(s)}}+1}{2}}}$$

- **Likelihood ratio (log-scaled):**

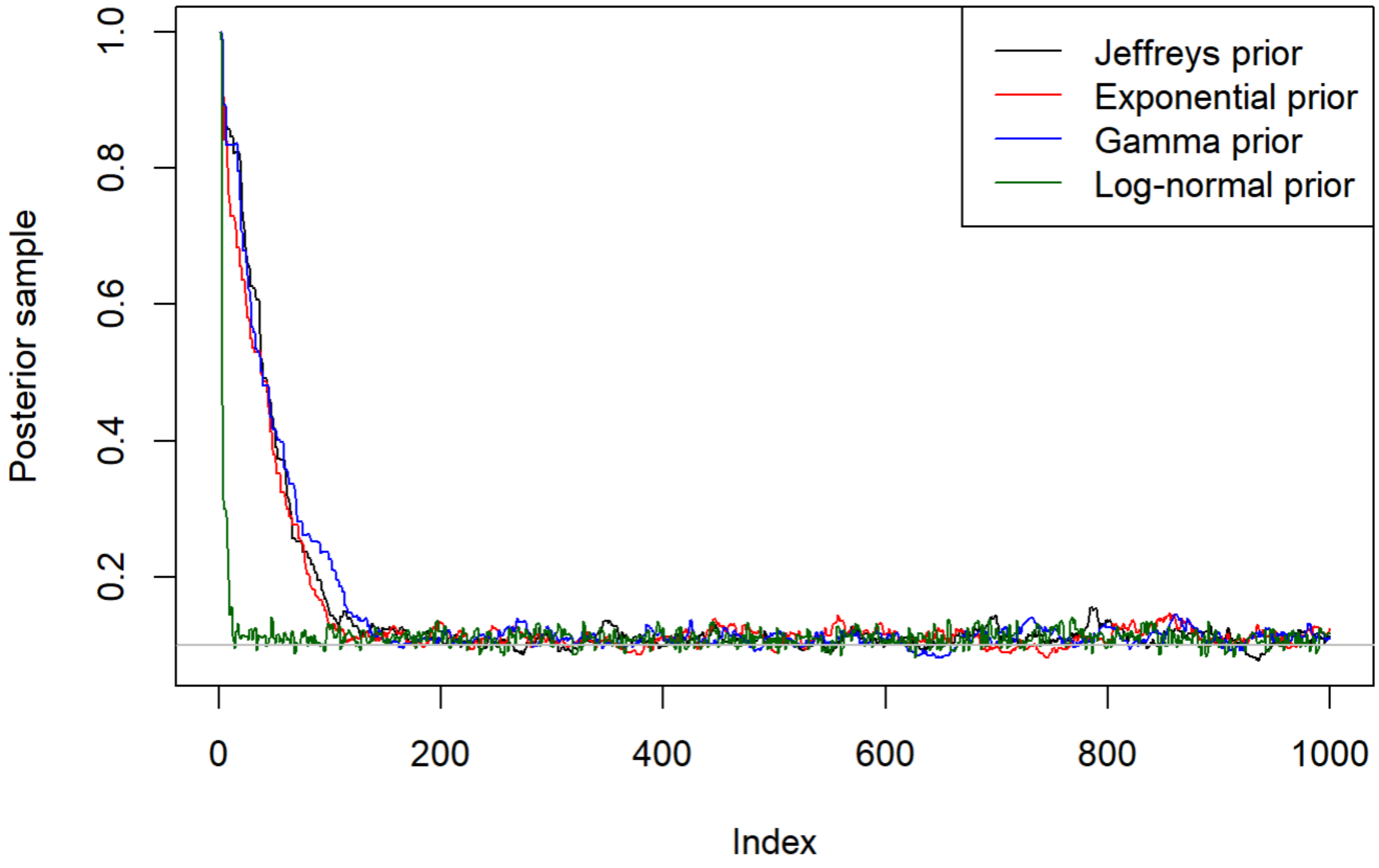
$$\log\left(\frac{L(e^{\eta})}{L\left(e^{\eta^{(s)}}\right)}\right) =$$

$$= N \cdot \left[\log\left(\frac{\Gamma\left(\frac{e^{\eta}+1}{2}\right)}{\sqrt{e^{\eta}\pi}\Gamma\left(\frac{e^{\eta}}{2}\right)}\right) - \log\left(\frac{\Gamma\left(\frac{e^{\eta^{(s)}}+1}{2}\right)}{\sqrt{e^{\eta^{(s)}}\pi}\Gamma\left(\frac{e^{\eta^{(s)}}}{2}\right)}\right)\right]$$

$$-\left[\frac{e^{\eta}+1}{2}\sum_{i=1}^N \log\left(1+\frac{y_i^2}{e^{\eta}}\right) - \frac{e^{\eta^{(s)}}+1}{2}\sum_{i=1}^N \log\left(1+\frac{y_i^2}{e^{\eta^{(s)}}}\right)\right]$$

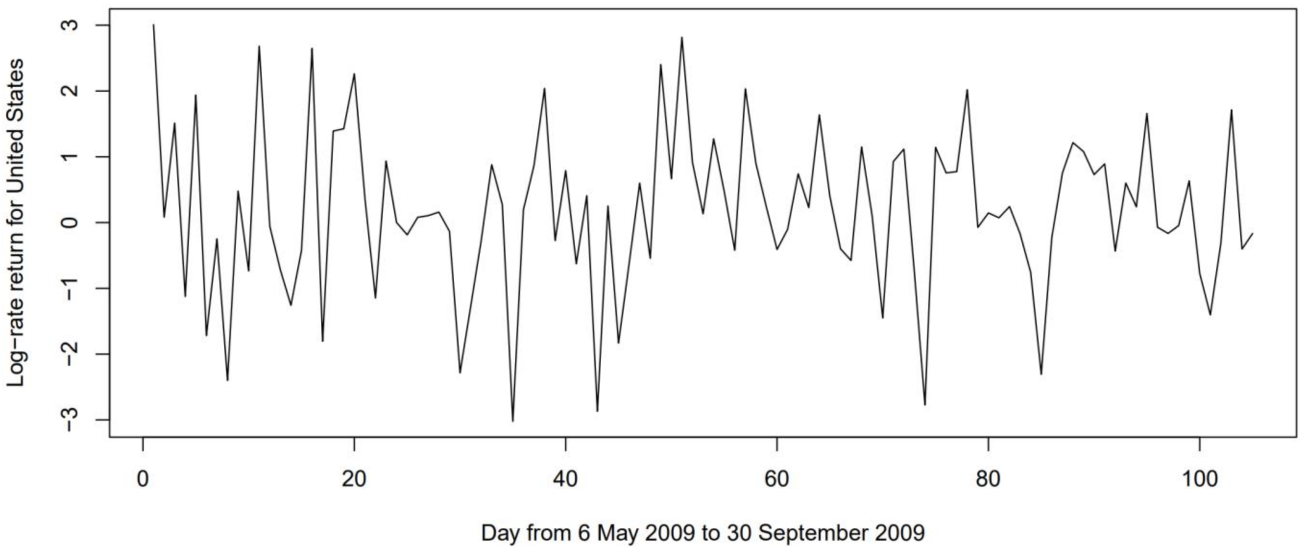
DEMO 1: comparison of the trace plots

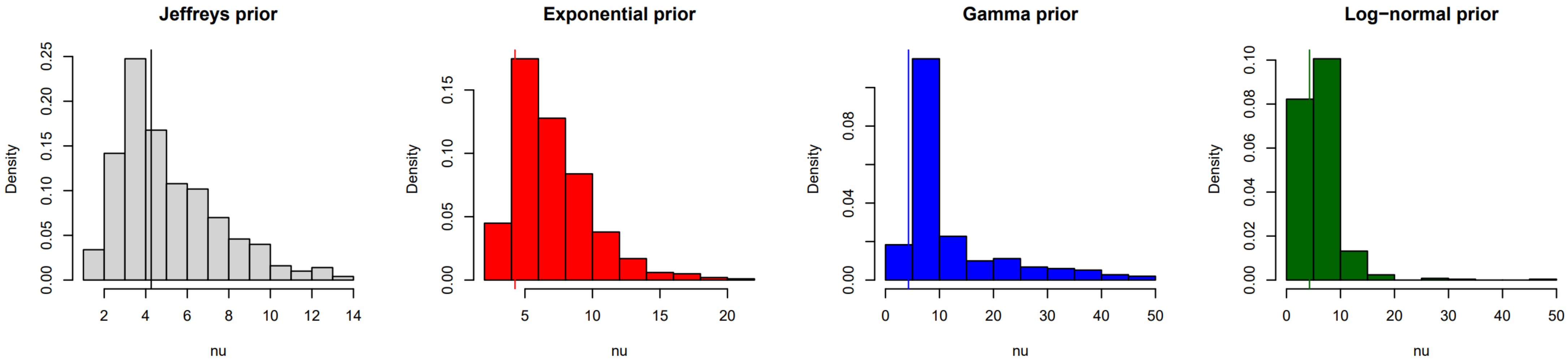
Observations are 100 number of samples drawn from Student t-distribution with df = 0.1



DEMO 2: financial data analysis

Observations are 105 number of samples of the daily log-rate return for US rom 6 May 2009 to 30 September 2009





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