

# Xe129 - Rb87 Spin Exchange Optical Pumping dynamics

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## 1 Bloch equations

The Bloch equation for the Xe spin are

$$\frac{d\mathbf{K}}{dt} = -[\gamma_{Xe}(\mathbf{B}_0 + \mathbf{B}_d \cos(\omega_{rf}t)) + b_{KS}\mathbf{S}] \times \mathbf{K} + \Gamma_{se}(\mathbf{S} - \mathbf{K}) - \Gamma_2 \hat{x}(\hat{x} \cdot \mathbf{K}) - \Gamma_2 \hat{y}(\hat{y} \cdot \mathbf{K}) - \Gamma_1 \hat{z}(\hat{z} \cdot \mathbf{K}) \quad (1)$$

Let us assume that  $b_{KS}\mathbf{S}$  is part of the DC field  $\mathbf{B}_0 = B_0\hat{z}$ , that  $\Gamma_{se}(\mathbf{S} - \mathbf{K}) = \begin{pmatrix} 0 \\ 0 \\ R_{se} \end{pmatrix}$  and that the drive is

$$\mathbf{B}_d \cos(\omega_{rf}t) = B_d \cos(\omega_{rf}t)\hat{x}$$

Thus, we end up with the next Bloch equation

$$\frac{d\mathbf{K}}{dt} = -[\gamma_{Xe}(B_0\hat{z} + B_d \cos(\omega_{rf}t)\hat{x})] \times \mathbf{K} + R_{se}\hat{z} - \Gamma_2 \hat{x}(\hat{x} \cdot \mathbf{K}) - \Gamma_2 \hat{y}(\hat{y} \cdot \mathbf{K}) - \Gamma_1 \hat{z}(\hat{z} \cdot \mathbf{K}) \quad (2)$$

$$= -\gamma_{Xe} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ B_d \cos(\omega_{rf}t) & 0 & B_0 \\ K_x & K_y & K_z \end{vmatrix} + \begin{pmatrix} 0 \\ 0 \\ R_{se} \end{pmatrix} - \begin{pmatrix} \Gamma_2 & 0 & 0 \\ 0 & \Gamma_2 & 0 \\ 0 & 0 & \Gamma_1 \end{pmatrix} \cdot \begin{pmatrix} K_x \\ K_y \\ K_z \end{pmatrix} \quad (3)$$

$$= -\gamma_{Xe} \begin{pmatrix} -B_0 K_y & B_0 K_x - B_d \cos(\omega_{rf}t) K_z \\ B_d \cos(\omega_{rf}t) K_y & \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ R_{se} \end{pmatrix} - \begin{pmatrix} \Gamma_2 & 0 & 0 \\ 0 & \Gamma_2 & 0 \\ 0 & 0 & \Gamma_1 \end{pmatrix} \cdot \begin{pmatrix} K_x \\ K_y \\ K_z \end{pmatrix} \quad (4)$$

$$= \begin{pmatrix} -\Gamma_2 & \gamma_{Xe} B_0 & 0 \\ -\gamma_{Xe} B_0 & -\Gamma_2 & \gamma_{Xe} B_d \cos(\omega_{rf}t) \\ 0 & -\gamma_{Xe} B_d \cos(\omega_{rf}t) & -\Gamma_1 \end{pmatrix} \cdot \begin{pmatrix} K_x \\ K_y \\ K_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ R_{se} \end{pmatrix} \quad (5)$$

$$= \begin{pmatrix} -\Gamma_2 & \omega_0 & 0 \\ -\omega_0 & -\Gamma_2 & \Omega_d \cos(\omega_{rf}t) \\ 0 & -\Omega_d \cos(\omega_{rf}t) & -\Gamma_1 \end{pmatrix} \cdot \begin{pmatrix} K_x \\ K_y \\ K_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ R_{se} \end{pmatrix} \quad (6)$$

Finally, the Xe NMR dynamics can be described by the next Bloch equations

$$\begin{pmatrix} \dot{K}_x \\ \dot{K}_y \\ \dot{K}_z \end{pmatrix} = \begin{pmatrix} -\Gamma_2 & \omega_0 & 0 \\ -\omega_0 & -\Gamma_2 & \Omega_d \cos(\omega_{rf}t) \\ 0 & -\Omega_d \cos(\omega_{rf}t) & -\Gamma_1 \end{pmatrix} \cdot \begin{pmatrix} K_x \\ K_y \\ K_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ R_{se} \end{pmatrix} \quad (7)$$

Now, let us move to the rotating frame (frame of Xe) using the next transformation

$$\begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} = \begin{pmatrix} \cos(\omega_0 t) & -\sin(\omega_0 t) & 0 \\ \sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} K_x \\ K_y \\ K_z \end{pmatrix} \quad (8)$$

$$\begin{pmatrix} K_x \\ K_y \\ K_z \end{pmatrix} = \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) & 0 \\ -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} \quad (9)$$

Also, the derivative with respect to time of the transformation is

$$\begin{pmatrix} \dot{\tilde{K}}_x \\ \dot{\tilde{K}}_y \\ \dot{\tilde{K}}_z \end{pmatrix} = \frac{d}{dt} \left[ \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) & 0 \\ -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} \right] \quad (10)$$

$$= \frac{d}{dt} \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) & 0 \\ -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} + \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) & 0 \\ -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} \quad (11)$$

$$= \omega_0 \begin{pmatrix} -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ -\cos(\omega_0 t) & -\sin(\omega_0 t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} + \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) & 0 \\ -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} \quad (12)$$

The Bloch equations transform as follows

$$\omega_0 \begin{pmatrix} -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ -\cos(\omega_0 t) & -\sin(\omega_0 t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} + \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) & 0 \\ -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} = \quad (13)$$

$$\begin{pmatrix} -\Gamma_2 & \omega_0 & 0 \\ -\omega_0 & -\Gamma_2 & \Omega_d \cos(\omega_r f t) \\ 0 & -\Omega_d \cos(\omega_r f t) & -\Gamma_1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) & 0 \\ -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ R_{se} \end{pmatrix} \quad (14)$$

$$\omega_0 \begin{pmatrix} \cos(\omega_0 t) & -\sin(\omega_0 t) & 0 \\ \sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ -\cos(\omega_0 t) & -\sin(\omega_0 t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} + \frac{d}{dt} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} = \quad (15)$$

$$\begin{pmatrix} \cos(\omega_0 t) & -\sin(\omega_0 t) & 0 \\ \sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\Gamma_2 & \omega_0 & 0 \\ -\omega_0 & -\Gamma_2 & \Omega_d \cos(\omega_r f t) \\ 0 & -\Omega_d \cos(\omega_r f t) & -\Gamma_1 \end{pmatrix} \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) & 0 \\ -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ R_{se} \end{pmatrix} \quad (16)$$

Let us compute it separately term by term

$$\omega_0 \begin{pmatrix} \cos(\omega_0 t) & -\sin(\omega_0 t) & 0 \\ \sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ -\cos(\omega_0 t) & -\sin(\omega_0 t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} = \omega_0 \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} \quad (17)$$

$$= \omega_0 \begin{pmatrix} \tilde{K}_y \\ -\tilde{K}_x \\ 0 \end{pmatrix} \quad (18)$$

Also,

$$\begin{pmatrix} \cos(\omega_0 t) & -\sin(\omega_0 t) & 0 \\ \sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\Gamma_2 & \omega_0 & 0 \\ -\omega_0 & -\Gamma_2 & \Omega_d \cos(\omega_{rf} t) \\ 0 & -\Omega_d \cos(\omega_{rf} t) & -\Gamma_1 \end{pmatrix} \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) & 0 \\ -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \quad (19)$$

$$\begin{pmatrix} \cos(\omega_0 t) & -\sin(\omega_0 t) & 0 \\ \sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\Gamma_2 \cos(\omega_0 t) - \omega_0 \sin(\omega_0 t) & -\Gamma_2 \sin(\omega_0 t) + \omega_0 \cos(\omega_0 t) & 0 \\ -\omega_0 \cos(\omega_0 t) + \Gamma_2 \sin(\omega_0 t) & -\omega_0 \sin(\omega_0 t) - \Gamma_2 \cos(\omega_0 t) & \Omega_d \cos(\omega_{rf} t) \\ \Omega_d \cos(\omega_{rf} t) \sin(\omega_0 t) & -\Omega_d \cos(\omega_{rf} t) \cos(\omega_0 t) & -\Gamma_1 \end{pmatrix} = \quad (20)$$

$$\begin{pmatrix} -\Gamma_2 & \omega_0 & -\Omega_d \cos(\omega_{rf} t) \sin(\omega_0 t) \\ -\omega_0 & -\Gamma_2 & \Omega_d \cos(\omega_{rf} t) \cos(\omega_0 t) \\ \Omega_d \cos(\omega_{rf} t) \sin(\omega_0 t) & -\Omega_d \cos(\omega_{rf} t) \cos(\omega_0 t) & -\Gamma_1 \end{pmatrix} \quad (21)$$

combining the terms together we get

$$\omega_0 \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} + \frac{d}{dt} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} = \quad (22)$$

$$\begin{pmatrix} -\Gamma_2 & \omega_0 & -\Omega_d \cos(\omega_{rf} t) \sin(\omega_0 t) \\ -\omega_0 & -\Gamma_2 & \Omega_d \cos(\omega_{rf} t) \cos(\omega_0 t) \\ \Omega_d \cos(\omega_{rf} t) \sin(\omega_0 t) & -\Omega_d \cos(\omega_{rf} t) \cos(\omega_0 t) & -\Gamma_1 \end{pmatrix} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ R_{se} \end{pmatrix} \quad (23)$$

$$\frac{d}{dt} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} = \quad (24)$$

$$\begin{pmatrix} -\Gamma_2 & \omega_0 & -\Omega_d \cos(\omega_{rf} t) \sin(\omega_0 t) \\ -\omega_0 & -\Gamma_2 & \Omega_d \cos(\omega_{rf} t) \cos(\omega_0 t) \\ \Omega_d \cos(\omega_{rf} t) \sin(\omega_0 t) & -\Omega_d \cos(\omega_{rf} t) \cos(\omega_0 t) & -\Gamma_1 \end{pmatrix} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} + \begin{pmatrix} 0 & -\omega_0 & 0 \\ \omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} \quad (25)$$

$$+ \begin{pmatrix} 0 \\ 0 \\ R_{se} \end{pmatrix} \quad (26)$$

so finally

$$\frac{d}{dt} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} = \begin{pmatrix} -\Gamma_2 & 0 & -\Omega_d \cos(\omega_{rf} t) \sin(\omega_0 t) \\ 0 & -\Gamma_2 & \Omega_d \cos(\omega_{rf} t) \cos(\omega_0 t) \\ \Omega_d \cos(\omega_{rf} t) \sin(\omega_0 t) & -\Omega_d \cos(\omega_{rf} t) \cos(\omega_0 t) & -\Gamma_1 \end{pmatrix} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ R_{se} \end{pmatrix} \quad (27)$$

$$(28)$$

Notice that in the case where we transfer to a different rotating frame of reference which rotates at some frequency  $\omega_1$ , then the Bloch equations would be a bit different after the transformation

$$\frac{d}{dt} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} = \begin{pmatrix} -\Gamma_2 & \omega_0 - \omega_1 & -\Omega_d \cos(\omega_{rf} t) \sin(\omega_1 t) \\ -(\omega_0 - \omega_1) & -\Gamma_2 & \Omega_d \cos(\omega_{rf} t) \cos(\omega_1 t) \\ \Omega_d \cos(\omega_{rf} t) \sin(\omega_1 t) & -\Omega_d \cos(\omega_{rf} t) \cos(\omega_1 t) & -\Gamma_1 \end{pmatrix} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ R_{se} \end{pmatrix} \quad (29)$$

$$(30)$$

Using trigonometric identities we can simplify the drive terms in the Bloch matrix

$$\Omega_d \cos(\omega_{rf}t) \sin(\omega_1 t) = \frac{\Omega_d}{2} (\sin((\omega_{rf} + \omega_1)t) + \sin((\omega_{rf} - \omega_1)t)) \quad (31)$$

$$\Omega_d \cos(\omega_{rf}t) \cos(\omega_1 t) = \frac{\Omega_d}{2} (\cos((\omega_{rf} - \omega_1)t) + \cos((\omega_{rf} + \omega_1)t)) \quad (32)$$

$$(33)$$

assuming that  $\omega_{rf} - \omega_1 \ll 1$ , then we can use the RWA which states that fast oscillations average out to zero

$$\Omega_d \cos(\omega_{rf}t) \sin(\omega_1 t) = \frac{\Omega_d}{2} \sin(2\omega_{rf}t) = 0 \quad (34)$$

$$\Omega_d \cos(\omega_{rf}t) \cos(\omega_1 t) = \frac{\Omega_d}{2} (1 + \cos(2\omega_{rf}t)) = \frac{\Omega_d}{2} \quad (35)$$

$$(36)$$

Then, the Bloch equations with the RWA are

$$\frac{d}{dt} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} = \begin{pmatrix} -\Gamma_2 & \omega_0 - \omega_{rf} & 0 \\ -(\omega_0 - \omega_{rf}) & -\Gamma_2 & \frac{\Omega_d}{2} \\ 0 & -\frac{\Omega_d}{2} & -\Gamma_1 \end{pmatrix} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ R_{se} \end{pmatrix} \quad (37)$$

Next, we add to our model The effective magnetic field the Xe particles fill which produced by the alkali atoms,  $b_{KS}\mathbf{S} = b_{KS}S_z\hat{z}$  and also some world rotation  $\omega_{\mathbf{r}} = \omega_r\hat{z}$ .

The modified equations are

$$\boxed{\frac{d}{dt} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} = \begin{pmatrix} -\Gamma_2 & M_{12} & 0 \\ M_{21} & -\Gamma_2 & \frac{\Omega_d}{2} \\ 0 & -\frac{\Omega_d}{2} & -\Gamma_1 \end{pmatrix} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ R_{se} \end{pmatrix}} \quad (38)$$

where  $M_{21} = -M_{12}$  and

$$M_{12} = \omega_0 + \gamma_{Xe}b_{KS}S_z + \omega_r - \omega_{rf} \quad (39)$$

$$= \gamma_{Xe}B_0 + \gamma_{Xe}b_{KS}S_z + \omega_r - \omega_{rf} \quad (40)$$

$$(41)$$

Next, let us find the steady state solution of the Bloch equations in the rotating frame with drive in  $\hat{x}$  with the RWA

$$\begin{pmatrix} \Gamma_2 & -M_{12} & 0 \\ M_{12} & \Gamma_2 & -\frac{\Omega_d}{2} \\ 0 & \frac{\Omega_d}{2} & \Gamma_1 \end{pmatrix} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ R_{se} \end{pmatrix} \quad (42)$$

$$\begin{pmatrix} \Gamma_2\tilde{K}_x - M_{12}\tilde{K}_y \\ M_{12}\tilde{K}_x + \Gamma_2\tilde{K}_y - \frac{\Omega_d}{2}\tilde{K}_z \\ \frac{\Omega_d}{2}\tilde{K}_y + \Gamma_1\tilde{K}_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ R_{se} \end{pmatrix} \quad (43)$$

$$\tilde{K}_x = \frac{M_{12}}{\Gamma_2}\tilde{K}_y \quad (44)$$

$$(45)$$

$$\begin{pmatrix} \frac{M_{12}^2 + \Gamma_2^2}{\Gamma_2} \tilde{K}_y - \frac{\Omega_d}{2} \tilde{K}_z \\ \frac{\Omega_d}{2} \tilde{K}_y + \Gamma_1 \tilde{K}_z \end{pmatrix} = \begin{pmatrix} 0 \\ R_{se} \end{pmatrix} \quad (46)$$

$$\begin{pmatrix} \left(1 + (M_{12}/\Gamma_2)^2\right) \Gamma_2 \tilde{K}_y - \frac{\Omega_d}{2} \tilde{K}_z \\ \frac{\Omega_d}{2} \tilde{K}_y + \Gamma_1 \tilde{K}_z \end{pmatrix} = \begin{pmatrix} 0 \\ R_{se} \end{pmatrix} \quad (47)$$

$$\left(1 + (M_{12}/\Gamma_2)^2\right) \Gamma_2 \tilde{K}_y = \frac{\Omega_d}{2} \tilde{K}_z \quad (48)$$

$$\tilde{K}_y = \frac{\frac{\Omega_d}{2}}{\left(1 + (M_{12}/\Gamma_2)^2\right) \Gamma_2} \tilde{K}_z \quad (49)$$

$$(50)$$

$$\frac{\Omega_d}{2} \tilde{K}_y + \Gamma_1 \tilde{K}_z = R_{se} \quad (51)$$

$$\frac{\Omega_d}{2} \frac{\frac{\Omega_d}{2}}{\left(1 + (M_{12}/\Gamma_2)^2\right) \Gamma_2} \tilde{K}_z + \Gamma_1 \tilde{K}_z = R_{se} \quad (52)$$

$$\frac{\frac{\Omega_d^2}{4} + \left(1 + (M_{12}/\Gamma_2)^2\right) \Gamma_2 \Gamma_1}{\left(1 + (M_{12}/\Gamma_2)^2\right) \Gamma_2} \tilde{K}_z = R_{se} \quad (53)$$

$$\frac{\frac{\Omega_d^2}{4\Gamma_2\Gamma_1} + \left(1 + (M_{12}/\Gamma_2)^2\right)}{\left(1 + (M_{12}/\Gamma_2)^2\right) / \Gamma_1} \tilde{K}_z = R_{se} \quad (54)$$

$$\tilde{K}_z = \frac{R_{se}}{\Gamma_1} \frac{\left(1 + (M_{12}/\Gamma_2)^2\right)}{\frac{\Omega_d^2}{4\Gamma_2\Gamma_1} + \left(1 + (M_{12}/\Gamma_2)^2\right)} \quad (55)$$

$$\tilde{K}_y = \frac{\Omega_d}{2\Gamma_2} \frac{R_{se}}{\Gamma_1} \frac{1}{\frac{\Omega_d^2}{4\Gamma_2\Gamma_1} + \left(1 + (M_{12}/\Gamma_2)^2\right)} \quad (56)$$

$$\tilde{K}_x = \frac{M_{12}}{\Gamma_2} \frac{\Omega_d}{2\Gamma_2} \frac{R_{se}}{\Gamma_1} \frac{1}{\frac{\Omega_d^2}{4\Gamma_2\Gamma_1} + \left(1 + (M_{12}/\Gamma_2)^2\right)} \quad (57)$$

Finally, the solution to the Bloch equation (eq.(38)) is

$$\boxed{\begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} = \frac{\frac{R_{se}}{\Gamma_1}}{\frac{\Omega_d^2}{4\Gamma_2\Gamma_1} + \left(1 + (M_{12}/\Gamma_2)^2\right)} \begin{pmatrix} \frac{M_{12}}{\Gamma_2} \frac{\Omega_d}{2\Gamma_2} \\ \frac{\Omega_d}{2\Gamma_2} \\ \left(1 + (M_{12}/\Gamma_2)^2\right) \end{pmatrix}} \quad (58)$$

In the case where  $M_{12} = 0$  we get

$$\begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} = \frac{\frac{R_{se}}{\Gamma_1}}{1 + \frac{\Omega_d^2}{4\Gamma_2\Gamma_1}} \begin{pmatrix} 0 \\ \frac{\Omega_d}{2\Gamma_2} \\ 1 \end{pmatrix} \quad (59)$$

Recall that we measure the transverse spin polarization. Let us compute the optimal drive amplitude for maximal  $\tilde{K}_y$  polarization

$$\frac{d\tilde{K}_y}{d\Omega_d} = \frac{R_{se}}{2\Gamma_2\Gamma_1} \frac{1}{1 + \frac{\Omega_d^2}{4\Gamma_2\Gamma_1}} - \frac{R_{se}}{2\Gamma_2\Gamma_1} \frac{\Omega_d^2}{\Gamma_2\Gamma_1} \frac{1}{\left(1 + \frac{\Omega_d^2}{4\Gamma_2\Gamma_1}\right)^2} = 0 \quad (60)$$

we get the next optimal drive amplitude

$$\Omega_d^{\text{optimal}} = \sqrt{\frac{4\Gamma_2\Gamma_1}{3}}$$

for the optimal drive, the  $\hat{z}$  polarization would be

$$\tilde{K}_z = \frac{3R_{se}}{4\Gamma_1}$$

We can make the same calculation in the case where  $M_{12} \neq 0$

$$\frac{d\tilde{K}_y}{d\Omega_d} = \frac{R_{se}}{2\Gamma_2\Gamma_1} \frac{1}{\frac{\Omega_d^2}{4\Gamma_2\Gamma_1} + \left(1 + (M_{12}/\Gamma_2)^2\right)} - \frac{R_{se}}{2\Gamma_2\Gamma_1} \frac{\Omega_d^2}{\Gamma_2\Gamma_1} \frac{1}{\left(\frac{\Omega_d^2}{4\Gamma_2\Gamma_1} + \left(1 + (M_{12}/\Gamma_2)^2\right)\right)^2} = 0 \quad (61)$$

in that case the optimal drive amplitude is

$$\Omega_d^{\text{optimal}} = \sqrt{\frac{4\Gamma_2\Gamma_1}{3}} \sqrt{1 + (M_{12}/\Gamma_2)^2}$$

Just for sports we can also solve the case of drive in  $\hat{y}$ , the Bloch equations in that case are

$$\boxed{\frac{d}{dt} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} = \begin{pmatrix} -\Gamma_2 & M_{12} & -\frac{\Omega_d}{2} \\ M_{21} & -\Gamma_2 & 0 \\ \frac{\Omega_d}{2} & 0 & -\Gamma_1 \end{pmatrix} \begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ R_{se} \end{pmatrix}} \quad (62)$$

then the steady state solution would be

$$\begin{pmatrix} \Gamma_2\tilde{K}_x - M_{12}\tilde{K}_y + \frac{\Omega_d}{2}\tilde{K}_z \\ M_{12}\tilde{K}_x + \Gamma_2\tilde{K}_y \\ -\frac{\Omega_d}{2}\tilde{K}_x + \Gamma_1\tilde{K}_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ R_{se} \end{pmatrix} \quad (63)$$

$$\tilde{K}_y = -\frac{M_{12}}{\Gamma_2}\tilde{K}_x$$

$$\begin{pmatrix} \frac{\Gamma_2^2 + M_{12}^2}{\Gamma_2}\tilde{K}_x + \frac{\Omega_d}{2}\tilde{K}_z \\ -\frac{\Omega_d}{2}\tilde{K}_x + \Gamma_1\tilde{K}_z \end{pmatrix} = \begin{pmatrix} 0 \\ R_{se} \end{pmatrix} \quad (64)$$

$$\begin{pmatrix} \left(1 + (M_{12}/\Gamma_2)^2\right)\Gamma_2\tilde{K}_x + \frac{\Omega_d}{2}\tilde{K}_z \\ -\frac{\Omega_d}{2}\tilde{K}_x + \Gamma_1\tilde{K}_z \end{pmatrix} = \begin{pmatrix} 0 \\ R_{se} \end{pmatrix} \quad (65)$$

$$\tilde{K}_x = -\frac{\Omega_d}{2} \frac{1}{\left(1 + (M_{12}/\Gamma_2)^2\right)\Gamma_2} \tilde{K}_z$$

$$\tilde{K}_z = \frac{R_{se}}{\Gamma_1} \frac{\left(1 + (M_{12}/\Gamma_2)^2\right)}{\frac{\Omega_d^2}{4\Gamma_2\Gamma_1} + \left(1 + (M_{12}/\Gamma_2)^2\right)}$$

so finally,

$$\begin{pmatrix} \tilde{K}_x \\ \tilde{K}_y \\ \tilde{K}_z \end{pmatrix} = \frac{\frac{R_{se}}{\Gamma_1}}{\frac{\Omega_d^2}{4\Gamma_2\Gamma_1} + \left(1 + (M_{12}/\Gamma_2)^2\right)} \begin{pmatrix} -\frac{\Omega_d}{2\Gamma_2} \\ \frac{M_{12}}{\Gamma_2} \frac{\Omega_d}{2\Gamma_2} \\ \left(1 + (M_{12}/\Gamma_2)^2\right) \end{pmatrix} \quad (66)$$

## 2 NMR Gyroscope (NMRG)

Using the Xenon NMR model (the Bloch equations) we can implement a gyroscope which measures rotation along the primary axis of the system (in the case above:  $\hat{z}$ ). Following Walker's notation in [WL16], we break the spin dynamics into two components, parallel and perpendicular to the primary axis  $\hat{z}$

$$\mathbf{K} = K_z \hat{z} + \mathbf{K}_\perp$$

describing  $\mathbf{K}_\perp$  using a phasor notation

$$K_+ = K_x + iK_y = K_\perp e^{-i\phi}$$

where

$$K_\perp = \sqrt{K_x^2 + K_y^2} \quad , \quad \phi = \arctan\left(\frac{K_y}{K_x}\right)$$

Now, if we solve for  $\phi$  in the rotating frame of the drive, from eq.(66) we get

$$\phi = \arctan\left(\frac{\tilde{K}_y}{\tilde{K}_x}\right) = \arctan\left(-\frac{M_{12}}{\Gamma_2}\right) \quad (67)$$

for small phase, where  $M_{12} \ll 1$  the phase can be approximated with it zeroth order Taylor expansion

$$\phi = -\frac{M_{12}}{\Gamma_2}$$

unfolding  $M_{12}$  we get

$$\phi = -\frac{\gamma_{Xe} B_0}{\Gamma_2} - \frac{\gamma_{Xe} b_{KS} S_z}{\Gamma_2} - \frac{\omega_r}{\Gamma_2} + \frac{\omega_{rf}}{\Gamma_2}$$

In a real life NMR system we would also have some magnetic noise in the primary axis  $\gamma_{Xe} B_{noise}$ , so the phase would be

$$\phi = -\frac{\gamma_{Xe} B_0}{\Gamma_2} - \frac{\gamma_{Xe} B_{noise}}{\Gamma_2} - \frac{\gamma_{Xe} b_{KS} S_z}{\Gamma_2} - \frac{\omega_r}{\Gamma_2} + \frac{\omega_{rf}}{\Gamma_2}$$

all these parameters could in general be time dependent. Notice that in the phase equation we have two unknown parameters: fluctuations of the magnetic field  $B_{noise}$  and the world rotation  $\omega_r$ . In order to extract the world rotation from the phase measurements we need to introduce another specie.

### 2.1 Dual species open-loop NMRG scheme

Having system with two species, Xe129 and Xe131, we can compute the world rotation as follows

$$\phi_{129} = -\frac{\gamma_{Xe}^{129} B_0}{\Gamma_2^{129}} - \frac{\gamma_{Xe}^{129} B_{noise}}{\Gamma_2^{129}} - \frac{\gamma_{Xe}^{129} b_{KS} S_z}{\Gamma_2^{129}} - \frac{\omega_r}{\Gamma_2^{129}} + \frac{\omega_{rf}^{129}}{\Gamma_2^{129}} \quad (68)$$

$$\phi_{131} = -\frac{\gamma_{Xe}^{131} B_0}{\Gamma_2^{131}} - \frac{\gamma_{Xe}^{131} B_{noise}}{\Gamma_2^{131}} - \frac{\gamma_{Xe}^{131} b_{KS} S_z}{\Gamma_2^{131}} - \frac{\omega_r}{\Gamma_2^{131}} + \frac{\omega_{rf}^{131}}{\Gamma_2^{131}} \quad (69)$$

Then, subtracting the two equations we get

$$\frac{1}{\gamma_{Xe}^{129}} (\phi_{129}\Gamma_2^{129} - \omega_{rf}^{129}) - \frac{1}{\gamma_{Xe}^{131}} (\phi_{131}\Gamma_2^{131} - \omega_{rf}^{131}) = \omega_r \left( \frac{1}{\gamma_{Xe}^{131}} - \frac{1}{\gamma_{Xe}^{129}} \right) \quad (70)$$

and finally the world rotation in an open-loop setting is given by

$$\omega_r = \left[ \frac{\gamma_{Xe}^{129}\gamma_{Xe}^{131}}{\gamma_{Xe}^{129} - \gamma_{Xe}^{131}} \right] \left[ \frac{1}{\gamma_{Xe}^{129}} (\phi_{129}\Gamma_2^{129} - \omega_{rf}^{129}) - \frac{1}{\gamma_{Xe}^{131}} (\phi_{131}\Gamma_2^{131} - \omega_{rf}^{131}) \right] \quad (71)$$

## 2.2 Dual species closed-loop NMRG scheme

## References

- [WL16] Thad G Walker and Michael S Larsen. Spin-exchange-pumped nmr gyros. *Advances in atomic, molecular, and optical physics*, 65:373–401, 2016.