

# Geodesic Filtrations on Flag Manifolds: A Variational Foundation for Multiscale Stability

Z. Mullaghy, M.S.

November 2025

## Abstract

This paper establishes a variational principle for the construction of optimal multiresolution transforms. We show that the Geometric Refinement Transform (GRT), which generates a nested sequence of resolution spaces, is the discrete sampling of a *\*\*Riemannian geodesic\*\** on the Flag Manifold of subspaces. The minimal energy required for this refinement is shown to be a geometric measure of distance between the coarse and fine scales. Crucially, we demonstrate that this geodesic path yields the maximal possible  $L^2$  stability bound for multiscale time-stepping schemes. This framework provides a geometric foundation for stability analysis in parabolic and multiscale PDE solvers.

## 1 Introduction

The Geometric Refinement Transform (GRT) is a framework proposed for unifying continuous geometric analysis with discrete computational structures. Central to this approach is the definition of an optimal filtration  $\mathcal{F}$  of finite-dimensional nested subspaces  $\{V_k\}_{k=0}^K$  of the continuous Hilbert space  $\mathcal{H}$ . In this work, we demonstrate that this optimal filtration is a geometric entity—the shortest path between subspaces—and that its geometric properties control the stability of any numerical scheme constructed upon it.

## 2 The Variational Principle

We consider the space of all admissible filtrations, the generalized *\*\*Flag Manifold\*\**  $\mathcal{F}(d_0, \dots, d_K)$ , where the dimensions  $d_k = \dim V_k$  are fixed. The orthogonal projection onto  $V_k$  is denoted  $P_k : \mathcal{H} \rightarrow V_k$ . The distance between successive subspaces is measured using the Hilbert–Schmidt metric on the Grassmannian.

**Lemma 1** (Geodesic Path on the Grassmannian). *Let  $V_0, V_K \subset \mathcal{H}$  be subspaces with nonzero principal angles  $\theta_i \in (0, \pi/2)$ . Then the path of orthogonal projectors  $P(t) = U(t)P_0U(t)^*$ , where  $U(t) = \exp(tA)$  with  $A^* = -A$  and  $U(1)V_0 = V_K$ , is the unique curve that minimizes the energy functional  $\mathcal{E}[\gamma] = \int_0^1 \|\dot{P}(t)\|_{\text{HS}}^2 dt$ . The minimal energy is exactly  $\mathcal{E}_{\min} = \sum_i \theta_i^2$ .*

*Proof.* Let  $P_0$  and  $P_1$  be the orthogonal projectors onto  $V_0$  and  $V_K$ , respectively. Any smooth path  $P(t)$  of orthogonal projectors satisfies

$$P(t)^2 = P(t), \quad P(t)^* = P(t),$$

and differentiating gives

$$\dot{P}(t) = [A(t), P(t)] := A(t)P(t) - P(t)A(t),$$

for some skew-adjoint operator  $A(t)^* = -A(t)$ . This  $A(t)$  represents the infinitesimal generator of the unitary motion on  $\mathcal{H}$ .

The *energy functional* on the Grassmannian is

$$\mathcal{E}[P] = \int_0^1 \|\dot{P}(t)\|_{\text{HS}}^2 dt = \int_0^1 \| [A(t), P(t)] \|_{\text{HS}}^2 dt.$$

Since  $\|\cdot\|_{\text{HS}}$  is unitarily invariant, we can work in a basis in which  $P_0$  and  $P_1$  are simultaneously block-diagonalized with respect to their principal angles  $\theta_i$ . In each two-dimensional principal plane  $\Pi_i = \text{span}\{e_i, f_i\}$ , the projectors take the canonical form

$$P_0|_{\Pi_i} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_1|_{\Pi_i} = \begin{pmatrix} \cos^2 \theta_i & \cos \theta_i \sin \theta_i \\ \cos \theta_i \sin \theta_i & \sin^2 \theta_i \end{pmatrix}.$$

The geodesic path that rotates  $V_0$  into  $V_K$  within each  $\Pi_i$  is generated by the constant skew-adjoint block

$$A_i = \begin{pmatrix} 0 & -\theta_i \\ \theta_i & 0 \end{pmatrix},$$

so that  $U(t)|_{\Pi_i} = e^{tA_i}$  and  $P(t) = U(t)P_0U(t)^*$ .

For this constant-speed motion we have

$$\dot{P}(t) = [A, P(t)], \quad \|\dot{P}(t)\|_{\text{HS}}^2 = 2 \text{tr}(A^2 P(t) - AP(t)AP(t)) = \sum_i 2\theta_i^2 \sin^2(\theta_i t),$$

whose integral over  $t \in [0, 1]$  yields the total energy

$$\mathcal{E}_{\min} = \int_0^1 \|\dot{P}(t)\|_{\text{HS}}^2 dt = \sum_i \theta_i^2.$$

Any other admissible path  $P(t)$  with the same endpoints has strictly larger energy, because the Grassmannian equipped with the Hilbert–Schmidt metric is a symmetric Riemannian manifold of nonnegative sectional curvature, and its geodesics are exactly the curves  $P(t) = U(t)P_0U(t)^*$  with  $U(t) = \exp(tA)$  for constant skew-adjoint  $A$ . Uniqueness follows from the uniqueness of the principal angles  $\{\theta_i\}$  and the corresponding two-plane decompositions.

Hence  $P(t) = U(t)P_0U(t)^*$  is the unique minimizing curve, and  $\mathcal{E}_{\min} = \sum_i \theta_i^2$ .  $\square$

## 2.1 Inter-Scale Coupling Energy

We define the “Inter-Scale Coupling Energy”  $\mathcal{E}$  for a discrete filtration as the sum of squared Hilbert–Schmidt distances between successive projection operators. We impose the normalization condition  $\sum_{k=0}^{K-1} w_k = 1$  for the weights  $w_k > 0$ :

$$\mathcal{E}(\{V_k\}) = \sum_{k=0}^{K-1} w_k \cdot d_{\text{HS}}(P_{k+1}, P_k)^2$$

where the Hilbert–Schmidt metric is defined explicitly with the adjoint as:

$$d_{\text{HS}}(P, Q) = \|P - Q\|_{\text{HS}} = (\text{tr}((P - Q)^*(P - Q)))^{1/2}.$$

This energy quantifies the cumulative “bending” or change in orientation of the subspace path.

**Theorem 1** (Geodesic Optimality of the GRT Filtration). *For fixed endpoints  $V_0$  (Coarse Space) and  $V_K$  (Fine Space), the admissible filtration  $\mathcal{F}_{\text{GRT}} = \{V_k^*\}$  generated by the Geometric Refinement Transform (GRT) is the unique filtration that minimizes the Inter-Scale Coupling Energy  $\mathcal{E}$ , provided the set of principal angles between  $V_0$  and  $V_K$  are non-zero.*

*The minimizing filtration  $\{V_k^*\}$  is the equally-spaced discrete sampling of the “Grassmann Geodesic” connecting  $V_0$  to  $V_K$ , parameterized explicitly by:*

$$P_k^* = U_k P_0 U_k^*, \quad U_k = \exp(kA/K), \quad A^* = -A, \quad U_K V_0 = V_K.$$

*The minimal energy is directly related to the total geometric distance between the endpoints:*

$$\mathcal{E}_{\min} = \sum_{i=1}^{\min(d_0, d_K)} \theta_i^2.$$

*Proof.* Let  $P(t) = U(t)P_0U(t)^*$ , with  $U(t) = \exp(tA)$  and  $A^* = -A$ , denote the continuous Grassmannian geodesic connecting  $V_0$  and  $V_K$ . By Lemma 1, this curve minimizes the continuous energy functional

$$\mathcal{E}[P] = \int_0^1 \|\dot{P}(t)\|_{\text{HS}}^2 dt,$$

and the minimal value is  $\mathcal{E}_{\min} = \sum_i \theta_i^2$ , where  $\theta_i$  are the principal angles between  $V_0$  and  $V_K$ .

Consider now any discrete filtration  $\{V_k\}_{k=0}^K$  with orthogonal projectors  $P_k$ . The discrete inter-scale coupling energy is

$$\mathcal{E}(\{V_k\}) = \sum_{k=0}^{K-1} w_k \|P_{k+1} - P_k\|_{\text{HS}}^2, \quad w_k > 0, \quad \sum w_k = 1.$$

For sufficiently fine sampling ( $K \gg 1$ ),  $P_{k+1} - P_k \approx \dot{P}(t_k) \Delta t$ , and hence

$$\mathcal{E}(\{V_k\}) = \sum_{k=0}^{K-1} \|\dot{P}(t_k)\|_{\text{HS}}^2 \Delta t + \mathcal{O}(\Delta t^2) \rightarrow \mathcal{E}[P],$$

so that the discrete energy converges to the continuous one as  $\Delta t = 1/K \rightarrow 0$ .

Among all discrete filtrations with the same endpoints, the discrete curve that samples the continuous geodesic at equal arclength increments,

$$P_k^* = U(k/K)P_0U(k/K)^*, \quad U(t) = \exp(tA),$$

minimizes  $\mathcal{E}(\{V_k\})$  to second order in  $\Delta t$ . Any other admissible sequence  $\{P_k\}$  that departs from this uniform sampling increases the discrete action due to the strict convexity of the squared Hilbert–Schmidt distance on the Grassmannian manifold. Hence  $\{P_k^*\}$  is the unique minimizer.

Finally, by the orthogonal decomposition of  $A$  into the two-plane generators associated with the principal angles  $\theta_i$ , the discrete minimal energy approaches

$$\mathcal{E}_{\min} = \sum_{i=1}^{\min(d_0, d_K)} \theta_i^2,$$

which equals the squared Riemannian distance between  $V_0$  and  $V_K$  on the Flag Manifold.  $\square$

### 3 Numerical Stability Consequence

The geometric optimality of the GRT filtration has direct consequences for the stability of numerical schemes that use these subspaces for operator splitting or error projection.

#### 3.1 Geometric Control of Stability

The stability of an inter-scale transfer operation is bounded by the operator norm of the projector difference,  $\|P_{k+1} - P_k\|_{\text{op}}$ . This norm is equal to the sine of the largest principal angle between the successive subspaces,  $\theta_k = \sin(\theta_{\max}(V_k, V_{k+1}))$ .

**Corollary 1** ( $\theta$ -Controlled  $L^2$  Stability for Parabolic PDEs under Geodesic Filtration). *Let  $\mathcal{M}$  be a compact Riemannian manifold and consider the parabolic evolution equation:*

$$\partial_t u = \mathcal{L}u, \quad u(0) = u_0 \in L^2(\mathcal{M}),$$

where  $\mathcal{L}$  is a self-adjoint negative-semidefinite operator (e.g., the diffusion operator).

If one refinement step is defined by the composition  $u_{k+1} = P_{k+1}e^{\Delta t \mathcal{L}}P_k u_k$ , the  $L^2$  energy of the solution satisfies the  $L^2$  stability inequality:

$$\|u_{k+1}\|_{L^2}^2 \leq (1 + \theta_k^2)e^{-2\lambda_{\min}\Delta t} \|u_k\|_{L^2}^2,$$

where  $\lambda_{\min} \geq 0$  is the smallest eigenvalue magnitude of  $(-\mathcal{L})$ .

The geodesic filtration (from Theorem 1) achieves the smallest possible  $\theta_k$ , yielding the strongest  $L^2$  stability bound among all admissible multiscale transforms. The total amplification factor converges to:

$$\exp(-2\lambda_{\min}t - \theta_{\text{tot}}^2 t),$$

where  $\theta_{\text{tot}} = \|\log(U)\|_{\text{HS}}$  is the total geometric distance between the coarse and fine spaces.

*Proof.* Because  $\mathcal{L}$  is self-adjoint and negative semidefinite, the heat semigroup  $e^{t\mathcal{L}}$  is a contraction on  $L^2(\mathcal{M})$ :

$$\|e^{t\mathcal{L}}v\|_{L^2} \leq e^{-\lambda_{\min}t} \|v\|_{L^2}, \quad \lambda_{\min} \geq 0.$$

For one refinement step,

$$u_{k+1} = P_{k+1}e^{\Delta t \mathcal{L}}P_k u_k,$$

we estimate

$$\|u_{k+1}\|_{L^2} \leq \|P_{k+1}\|_{\text{op}} \|e^{\Delta t \mathcal{L}}\|_{\text{op}} \|P_k u_k\|_{L^2} \leq e^{-\lambda_{\min}\Delta t} \|P_{k+1}P_k u_k\|_{L^2},$$

since  $\|P_k\|_{\text{op}} = \|P_{k+1}\|_{\text{op}} = 1$  for orthogonal projectors. We now bound  $\|P_{k+1}P_k\|_{\text{op}}$  in terms of the largest principal angle  $\theta_k$  between  $V_k$  and  $V_{k+1}$ . Because  $P_{k+1} - P_k$  has operator norm  $\|P_{k+1} - P_k\|_{\text{op}} = \sin(\theta_k)$ , we have

$$\|P_{k+1}P_k\|_{\text{op}} = \|P_k + (P_{k+1} - P_k)P_k\|_{\text{op}} \leq \|P_k\|_{\text{op}} + \|P_{k+1} - P_k\|_{\text{op}} \leq 1 + \sin(\theta_k) \leq \sqrt{1 + \theta_k^2},$$

using  $\sin x \leq x$  for small angles and squaring both sides to obtain a clean bound. Hence

$$\|u_{k+1}\|_{L^2}^2 \leq (1 + \theta_k^2) e^{-2\lambda_{\min}\Delta t} \|u_k\|_{L^2}^2.$$

Iterating over  $K$  steps yields

$$\|u_K\|_{L^2}^2 \leq \prod_{k=0}^{K-1} (1 + \theta_k^2) e^{-2\lambda_{\min}K\Delta t} \|u_0\|_{L^2}^2.$$

If the filtration is geodesic, Theorem 1 implies that all inter-scale angles  $\theta_k$  are equal and minimal,  $\theta_k = \theta_{\text{tot}}/K$ . In the limit  $K \rightarrow \infty$  with  $t = K\Delta t$  fixed,

$$\prod_{k=0}^{K-1} (1 + \theta_k^2) \rightarrow \exp(\theta_{\text{tot}}^2 t),$$

so that the amplification factor converges to

$$\exp(-2\lambda_{\min}t - \theta_{\text{tot}}^2 t).$$

Thus, the geodesic filtration achieves the strongest  $L^2$  stability bound among all admissible multiscale transforms, with geometric deviation quantified by  $\theta_{\text{tot}}$ .  $\square$

### 3.2 Conclusion

The term  $e^{-2\lambda_{\min}\Delta t}$  represents the physical damping due to diffusion, while the factor  $\sqrt{1 + \theta_k^2}$  introduces a \*\*geometric penalty\*\* for non-optimal subspace selection. The GRT, by following the geodesic path on the Flag Manifold, minimizes this penalty, ensuring that the refinement trajectory is maximally stable and geometrically efficient. This establishes a profound link between the Riemannian geometry of subspace manifolds and the stability of multiscale numerical analysis. Future work will extend this variational principle to nonlinear PDEs and adaptive refinement strategies via dynamic geodesic tracking on flag manifolds.