Refinement Algebra on Jacobians: Polynomial Stratification and σ-Algebra Exhaustion

Zachary Mullaghy

Abstract

We introduce the **Refinement Algebra**, a measure-theoretic framework that reformulates geometric mesh refinement as a $\mathrm{GL}(n)$ scaling action on a hierarchy of positive Jacobian density fields. Departing from the traditional combinatorial view of refinement as simplicial subdivision, we define refinement fundamentally as an operator-theoretic action on the density bundle, where a branching factor m induces the scaling $J \mapsto m^{-1}J$. We prove that the resulting filtration of Jacobian fields forms a martingale sequence with respect to the refinement σ-algebra.

By stratifying the space of Jacobian fields via polynomial degree, we utilize a Stone-Weierstrass density argument to establish that any continuous positive density on a compact domain can be uniformly approximated by refinable polynomial Jacobians. We further demonstrate that the moduli space of these Jacobians is governed by the quotient $\mathrm{GL}(n)/\mathrm{SL}(n) \cong \mathbb{R}_+$. Consequently, we prove that under iterated refinement, the "shape" of the mesh flows to a constant density (Stratum 0). This framework provides a rigorous analytical foundation for Geometric Refinement Theory (GRT), replacing heuristic geometric axioms with algebraic derivations validated by formal proof.

Keywords: Refinement Algebra, Jacobian Fields, $\mathrm{GL}(n)$ Action, Martingales, Stone-Weierstrass.

1 Canonical Refinement on Jacobian Fields

We work on a fixed measurable space $(\mathbb{R}^n, \mathcal{B})$ equipped with Lebesgue measure. A *Jacobian field* is a measurable map

$$J : \mathbb{R}^n \to (0, \infty)$$

representing a positive density. In geometric applications, $J(x)$ encodes the infinitesimal volume distortion of a reference metric or the local density of a mass distribution.

1.1 Multi-dimensional p-adic refinement

For a space of dimension n, the refinement structure is governed by a vector of natural numbers $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, where each $p_i \in \mathbb{N}_{\geq 2}$ represents the number of partitions along the i-th dimension. These correspond to independent p-adic refinements per axis.

The total **refinement multiplicity** m is the product of these dimensional factors:

$$m := \prod_{i=1}^{n} p_i.$$

This integer m dictates the total number of subcells generated per refinement step.

1.2 The unit cell condition

We ground the refinement logic on the normalization of the density field. We define the fundamental cell condition such that the total mass of the Jacobian over a domain Ω is unitary:

$$\int_{\Omega} J(x)\, d\mu = 1.$$

This condition establishes the baseline for mass redistribution. Refinement is then viewed as the operator that preserves this unit condition locally across the m new subpartitions.

1.3 Refinement action via $\mathrm{GL}(n)$

Rather than defining the partition boundaries combinatorially, we define the refinement action algebraically. Consider the **flat Jacobian** $J_{\text{flat}}(x) = 1$.

The refinement action is induced by the diagonal scaling matrix $S_{\mathbf{p}} \in \mathrm{GL}(n)$ defined by:

$$S_{\mathbf{p}} := \mathrm{diag}(p_1^{-1}, p_2^{-1}, \ldots, p_n^{-1}).$$

Note that the determinant of this scaling matrix is exactly the inverse of the multiplicity:

$$\det(S_{\mathbf{p}}) = \prod_{i=1}^{n} p_i^{-1} = \frac{1}{m}.$$

The **canonical refinement operator** is defined by the action of this matrix on the density. If we start with the flat Jacobian $J = 1$, the refinement action yields:

$$\mathrm{Refine}(1) := |\det(S_{\mathbf{p}})| \cdot 1 = \frac{1}{m}. \tag{1}$$

This operator-theoretic definition recovers the intuitive notion that refining a unit mass into m pieces results in a density of $1/m$ per piece, but derives it strictly from the $\mathrm{GL}(n)$ action on the density bundle.

1.4 Refinement action as repeated $\mathrm{GL}(n)$ scaling

To make the refinement hierarchy explicit, consider the unit cube

$$Q := [0,1]^n$$

equipped with Lebesgue measure. Under the p-adic refinement structure determined by the vector $\mathbf{p} = (p_1, \ldots, p_n)$, the k-fold refinement level corresponds to subdividing Q into

$$m^k = \Big(\prod_{i=1}^{n} p_i\Big)^k$$

rectangular subcells, each of side lengths

$$p_1^{-k},\ p_2^{-k},\ \ldots,\ p_n^{-k}.$$

Rather than explicitly indexing these subcells or describing their boundaries, we characterize refinement purely through the linear scaling map

$$S_{\mathbf{p}} : \mathbb{R}^n \to \mathbb{R}^n, \qquad S_{\mathbf{p}}(x_1, \ldots, x_n) = (p_1^{-1}x_1, \ldots, p_n^{-1}x_n),$$

and its iterates

$$S_{\mathbf{p}}^k(x) = (p_1^{-k}x_1, \ldots, p_n^{-k}x_n).$$

Each application of $S_{\mathbf{p}}$ contracts the cube Q onto one of its $p_1 \times \cdots \times p_n$ subcells. Thus the k-th refinement level corresponds to the image $S_{\mathbf{p}}^k(Q)$, which is a uniformly scaled copy of Q.

Density transformation under refinement

Given a Jacobian field J, we consider its pullback by $S_{\mathbf{p}}^k$:

$$(S_{\mathbf{p}}^k)^{\#} J(x) := |\det(S_{\mathbf{p}}^k)|\, J\big(S_{\mathbf{p}}^k x\big) = m^{-k}\, J\big(S_{\mathbf{p}}^k x\big).$$

This expression contains all the refinement information:

- The factor m^{-k} expresses the total mass scaling: each refinement breaks mass into m equal parts.
- The composition $J \circ S_{\mathbf{p}}^k$ reflects the geometric zoom-in onto smaller and smaller regions.

The refinement tower

The *refinement tower* is the sequence of Jacobian fields

$$J_0 := J, \qquad J_{k+1} := (S_{\mathbf{p}})^{\#} J_k = m^{-1}\, J_k \circ S_{\mathbf{p}}.$$

By direct computation,

$$J_k = (S_{\mathbf{p}}^k)^{\#} J = m^{-k}\, J\big(S_{\mathbf{p}}^k(\cdot)\big),$$

so the tower is obtained entirely through repeated application of the same $\mathrm{GL}(n)$ action. No combinatorial structure is introduced at any stage.

This operator-theoretic formulation is the core philosophical shift of the Refinement Algebra: the refinement hierarchy is generated not by subdividing domains, but by *scaling the density*. The geometry one would normally attribute to mesh structure is instead encoded in the repeated pullback of J under the linear contraction $S_{\mathbf{p}}$.

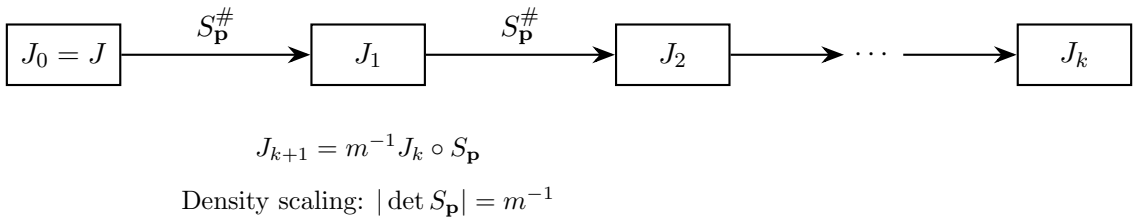

Figure 1: The refinement tower: Repeated application of the $\mathrm{GL}(n)$ pullback action $S_{\mathbf{p}}^{\#}$ generates the hierarchy of refined Jacobian fields.

Roadmap

The refinement tower raises a fundamental question: can this algebraic action recover *arbitrary* Jacobian fields, not merely those with simple algebraic structure? We address this through a three-stage program:

1. **Section 2**: We introduce polynomial Jacobians and prove via Stone–Weierstrass that they form a dense class capable of uniformly approximating any continuous positive density on compact domains. The challenge is not approximation per se, but ensuring *global positivity* of the polynomial representatives.
2. **Section 3**: We embed the refinement tower in the Hilbert space $L^2(\Omega)$ and show that the refinement operators are conditional expectations forming an L^2-bounded martingale with respect to a nested filtration of σ-algebras.

3. **Section 4**: We prove that the refinement filtration exhausts the Borel σ-algebra. By Doob's martingale convergence theorem, the refinement tower then converges in L^2 to any measurable Jacobian field.

This progression extends the theory from its algebraic origin (constant and polynomial densities) to the full analytic setting (arbitrary L^2 fields), establishing that the $\mathrm{GL}(n)$ action on densities provides a complete foundation for refinement.

Main contributions

The principal contribution of this work is *conceptual* rather than technical: we reformulate geometric mesh refinement in operator-theoretic terms, replacing the traditional combinatorial view of refinement as simplicial subdivision with a measure-theoretic framework centered on density transformations. The key innovations are:

1. **Operator-theoretic definition of refinement**: We define refinement fundamentally as a $\mathrm{GL}(n)$ scaling action on Jacobian density fields, rather than as a combinatorial subdivision of domains. This reformulation reveals refinement as a sequence of conditional expectations forming an L^2-bounded martingale.

2. **Equivalence of $\mathrm{GL}(n)$ and conditional expectation perspectives**: We establish that the $\mathrm{GL}(n)$ pullback action and the conditional expectation interpretation of refinement are two manifestations of the same underlying operator. This bridge connects geometric scaling (the traditional view) with measure-theoretic averaging (the probabilistic view).

3. **Polynomial stratification**: We prove that polynomial Jacobians, stratified by degree, form a uniformly dense class in the space of continuous positive densities (Proposition 3). This provides algebraically manageable representatives for general Jacobian fields.

4. **σ-algebra exhaustion and martingale convergence**: We prove that the refinement filtration exhausts the Borel σ-algebra (Proposition 5), which combined with Doob's martingale convergence theorem establishes that refinement recovers arbitrary L^2 Jacobian fields (Theorem 2).

The theorems themselves are applications of standard results from measure theory and martingale theory; the novelty lies in recognizing that mesh refinement can be axiomatized entirely within this functional-analytic framework, without reference to combinatorial or topological mesh structures.

1.5 Constant Jacobian fields and the $\mathrm{GL}(n)$ orbit

Before introducing polynomial Jacobians, it is essential to understand the structure of the simplest Jacobian fields: the constant densities. These fields already exhibit the central group-theoretic phenomenon underlying the refinement algebra.

Constant Jacobians

For any $\lambda > 0$, define the constant Jacobian field

$$J_\lambda(x) := \lambda.$$

Such a field represents a uniform rescaling of volume by the factor λ. These are the simplest elements of the space of positive density fields, and we now describe their algebraic structure under the $\mathrm{GL}(n)$ action.

Jacobian pullback under linear maps

Recall that $\mathrm{GL}(n)$ acts on Jacobian fields by the pullback formula

$$A^{\#}J(x) := |\det A|\, J(Ax), \qquad A \in \mathrm{GL}(n).$$

For a constant Jacobian J_λ, the pullback simplifies drastically:

$$A^{\#}J_\lambda(x) = |\det A|\, J_\lambda(Ax) = |\det A|\, \lambda.$$

Thus constant fields transform only by scalar multiplication, and the geometric distortion of A does not introduce any spatial dependence.

The GL(n) orbit of constant fields

The above identity shows that for any two positive constants λ_1, λ_2, there exists a linear transformation $A \in \mathrm{GL}(n)$ such that

$$A^{\#}J_{\lambda_1} = J_{\lambda_2}.$$

Indeed, choosing any A with

$$|\det A| = \frac{\lambda_2}{\lambda_1}$$

gives the required equality. Since $\mathrm{GL}(n)$ contains matrices with arbitrary positive determinant, such an A always exists.

Proposition 1 (GL-connectivity of constant Jacobians)**.** *The set of constant Jacobian fields*

$$\{J_\lambda : \lambda > 0\}$$

forms a single $\mathrm{GL}(n)$ *orbit. Equivalently, all constant densities are equivalent under the Jacobian pullback action.*

Proof. For any $\lambda_1, \lambda_2 > 0$, choose a matrix $A \in \mathrm{GL}(n)$ with $|\det A| = \lambda_2/\lambda_1$. Then the pullback satisfies

$$A^{\#}J_{\lambda_1} = J_{\lambda_2}.$$

Thus J_{λ_1} and J_{λ_2} lie in the same orbit, and the orbit is connected by one-step transformations. □

Moduli interpretation

The proposition shows that constant densities do not possess internal structure: their only distinguishing property is their total mass, encoded algebraically by the determinant. Taking the quotient by $\mathrm{SL}(n)$,

$$\mathrm{GL}(n)/\mathrm{SL}(n) \cong \mathbb{R}_{>0},$$

the determinant gives the moduli parameter for this orbit. This observation will reappear in Section 2, where polynomial Jacobians of fixed degree are shown to form equivalence classes parameterized by the same quotient.

2 Polynomial Jacobians and Degree Stratification

Why polynomial Jacobians?

The constant Jacobian fields studied in the previous section form a single $\mathrm{GL}(n)$ orbit, providing a complete classification for the simplest case. However, the refinement tower for a general Jacobian field $J : \mathbb{R}^n \to (0, \infty)$ may involve arbitrarily complex spatial structure. To understand convergence in this broader setting, we need a tractable class of fields that is simultaneously:

1. *Algebraically manageable*: amenable to explicit calculation and degree-based stratification;

2. *Analytically dense*: capable of uniformly approximating arbitrary continuous positive densities on compact domains.

Polynomial Jacobians satisfy both requirements. By the Stone–Weierstrass theorem, they form a uniformly dense subset of continuous functions, and their finite-dimensional structure makes them accessible to classical geometric tools. This dual nature—algebraic simplicity and analytic completeness—makes polynomials the natural bridge between the elementary theory of constant fields and the general convergence theorem for arbitrary L^2 densities.

The next level of structure in the refinement algebra thus comes from Jacobian fields that are polynomial functions. These form natural finite-dimensional strata, and their behavior under refinement can be analyzed using classical results from differential geometry—most notably, Moser's theorem on the equivalence of smooth positive volume forms.

2.1 Polynomial Jacobians

For a fixed degree $d \in \mathbb{N}$, let

$$R_d := \Big\{ P : \mathbb{R}^n \to (0,\infty) \;\Big|\; P(x) = \sum_{|\alpha| \le d} c_\alpha x^\alpha \Big\}$$

denote the space of positive polynomial Jacobian fields of degree at most d. Each R_d is a finite-dimensional manifold whose coordinates are the coefficients $\{c_\alpha\}$.

Unlike constant Jacobians, polynomial fields carry nontrivial geometric information, and refinement acts on them through a mixture of scaling and composition. To understand their structure, we first show that every such polynomial density can be locally "flattened" to a constant density.

2.2 Flattening polynomial Jacobians via Moser's theorem

We recall the following classical result due to Moser [1]:

Theorem 1 (Moser, 1965)**.** *Let μ_0 and μ_1 be two smooth positive volume forms on a compact manifold K with equal total mass. Then there exists a diffeomorphism $\Phi : K \to K$ such that*

$$\Phi^* \mu_1 = \mu_0.$$

We apply this result to polynomial Jacobians restricted to compact domains. Let $K \subset \mathbb{R}^n$ be compact, and let $P \in R_d$ be a positive polynomial. The volume form $P(x)\,dx$ is smooth and positive on K. Letting $\mu_0 = J_{\text{flat}}(x)\,dx$ and $\mu_1 = P(x)\,dx$, Moser's theorem yields a diffeomorphism

$$\Phi_P : K \to K$$

such that

$$(\Phi_P)^{\#} P = 1 \quad \text{on } K.$$

This shows that—even though polynomials may exhibit complicated spatial variation—they are, in the measure-theoretic sense on compact domains, connected to the flat Jacobian.

Proposition 2 (Local flattening of polynomial Jacobians)**.** *For any $P \in R_d$ and any compact $K \subset \mathbb{R}^n$, there exists a diffeomorphism $\Phi_P : K \to K$ such that*

$$(\Phi_P)^{\#} P = 1.$$

The key consequence is that refinement of polynomial Jacobians can always be carried out in *flat coordinates*. All geometric information is encoded in the pullback map Φ_P, and refinement itself acts only by the canonical scaling described in the previous section.

This flattening result is critical for the convergence theory developed in Sections 3–4. Moser's theorem allows us to *reduce* the analysis of polynomial refinement towers to the constant case, where the algebra is explicit. Combined with the density of polynomials (Proposition 3), this reduction principle extends to arbitrary L^2 fields via approximation. The global refinement program thus rests on a local-to-global bootstrap: flatten polynomials on compact sets, approximate general fields by polynomials, and pass to the limit.

2.3 Refinement in flat coordinates

Given a polynomial Jacobian P and its flattening diffeomorphism Φ_P, we define its refinement by:

$$\mathrm{Refine}(P) := (\Phi_P^{-1})^{\#}\Big(\mathrm{Refine}(1)\Big).$$

Expanding this expression using the pullback identity gives:

$$\mathrm{Refine}(P) = (\Phi_P^{-1})^{\#}(m^{-1}) = m^{-1}\left|\det D\Phi_P^{-1}\right|.$$

Thus refinement affects P in two stages:

1. flattening the density to the constant field 1 via Φ_P,
2. applying the canonical scaling m^{-1} to the flat field,
3. pulling back the result to $\mathbb{R}^n$ via Φ_P^{-1}.

This separates refinement into a *pure density action* and a *pure coordinate action*. The interplay of these two components is the foundation of the polynomial strata introduced next.

Why study the algebraic structure?

Before proceeding to the convergence theory, we pause to develop the algebraic structure of polynomial Jacobians. The composite monomial algebra introduced below serves two purposes. First, it provides a systematic classification of polynomial densities by degree and $\mathrm{GL}(n)$-orbit type, organizing them into natural strata indexed by homogeneous components. Second, and more crucially, it reveals which algebraic invariants are *preserved* under refinement—namely, the determinant scaling factor $|\det A|$, which governs total mass redistribution—and which invariants are *collapsed*, such as finer orbit structure within each degree stratum.

This algebraic analysis is essential for understanding how refinement acts on the polynomial representatives constructed via Stone–Weierstrass approximation, and it foreshadows the measure-theoretic structure developed in Section 3, where the $\mathrm{GL}(n)$ action is reinterpreted as conditional expectation with respect to the refinement filtration.

2.4 Polynomial strata and the composite monomial algebra

Let R_d denote the space of positive polynomial Jacobian fields of degree at most d. Every $P \in R_d$ admits the homogeneous decomposition

$$P(x) = \sum_{k=0}^{d} P_k(x), \qquad P_k \in \mathcal{H}_k,$$

where $\mathcal{H}_k$ is the finite–dimensional space of homogeneous polynomials of degree k. This endows R_d with the structure of a graded commutative algebra

$$R_d = \bigoplus_{k=0}^{d} \mathcal{H}_k, \qquad \mathcal{H}_a \cdot \mathcal{H}_b \subset \mathcal{H}_{a+b}.$$

The refinement theory naturally interacts with this grading. The $\mathrm{GL}(n)$ pullback action

$$(A^{\#}P)(x) := |\det A|\, P(Ax) \qquad (A \in \mathrm{GL}(n))$$

preserves polynomial degree and acts linearly on each homogeneous component. Consequently, each $\mathcal{H}_k$ decomposes into a finite or countable collection of $\mathrm{GL}(n)$–orbits, distinguished by algebraic invariants such as discriminant, signature, and factorization type. Transitivity fails for $d \geq 2$, so R_d contains many distinct orbits.

What *is* preserved across all these orbits is the determinant scaling factor. The quotient

$$\mathrm{GL}(n)/\mathrm{SL}(n) \cong \mathbb{R}_{>0}$$

provides a universal "mass coordinate" for every polynomial Jacobian: the factor $|\det A|$ appearing in the pullback action governs the global rescaling of density. This recovers the constant–Jacobian case as the degree–zero stratum.

To capture the algebraic structure relevant for refinement, we introduce the *composite monomial algebra*

$$\mathcal{M} := \bigoplus_{k \geq 0} \mathcal{H}_k^{\mathrm{pos}},$$

the graded algebra of homogeneous polynomial densities with strictly positive evaluation. Two polynomial Jacobians $P, Q \in R_d$ are said to be *equivalent* if there exist $A \in \mathrm{GL}(n)$ and $\lambda > 0$ such that

$$Q = \lambda\,(A^{\#}P).$$

This relation respects the grading, the $\mathrm{GL}(n)$–action, and the determinant scaling, and it isolates precisely the invariants preserved under refinement. Each equivalence class is parametrized by

$$[(P)] \;\cong\; \Big(\mathrm{GL}(n)\text{–orbit of homogeneous components of } P\Big) \times \mathbb{R}_{>0},$$

with the $\mathbb{R}_{>0}$ factor representing the universal mass scaling.

This composite monomial structure forms the algebraic backbone for the refinement tower: while $\mathrm{GL}(n)$ distinguishes many polynomial orbit types inside R_d, the refinement operator collapses these finer invariants, retaining only the determinant scaling and the graded structure needed for the subsequent exhaustion arguments. In Section 3, we will see that this algebraic collapsing corresponds precisely to the averaging operation performed by conditional expectation operators: refinement forgets fine-scale geometric detail (captured by orbit structure) while preserving global mass (captured by the determinant).

2.5 Density of positive polynomial Jacobians

The refinement algebra on Jacobian fields is built on the flat density $J_{\mathrm{flat}} = 1$, together with the $\mathrm{GL}(n)$ pullback action. To extend the theory from constant densities to arbitrary measurable Jacobians, we require a density result for polynomial representatives.

Let $K \subset \mathbb{R}^n$ be a compact set. A positive Jacobian field on K is simply a continuous function

$$J : K \to (0, \infty).$$

By the Stone–Weierstrass theorem [2], the restriction of the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$ to K is uniformly dense in $C(K)$. Thus for any $\varepsilon > 0$ there exists a polynomial p with

$$\sup_{x \in K} |J(x) - p(x)| < \varepsilon.$$

Because refinement theory requires strictly positive representatives, we must ensure positivity of the approximants. We achieve this by adding a controlled dominating term. For each degree d we consider the family of polynomials

$$p_{d,c}(x) := p(x) + c\,(1 + \|x\|^2)^d, \qquad c > 0,$$

whose second summand is strictly positive. For sufficiently small c, the uniform error on K remains below ε, while positivity holds globally. This produces a positive polynomial Jacobian

$$P := p_{d,c}$$

approximating J on K.

Proposition 3 (Polynomial density)**.** *Let $K \subset \mathbb{R}^n$ be compact. For every continuous positive Jacobian field $J : K \to (0, \infty)$ and every $\varepsilon > 0$, there exists a positive polynomial Jacobian P such that*

$$\sup_{x \in K} |J(x) - P(x)| < \varepsilon.$$

Proof. The proof proceeds in several steps, combining Stone–Weierstrass approximation with a positivity-preserving construction.

Step 1: Enlarge the domain. Since K is compact, there exists $R_K > 0$ such that $K \subseteq B(0, R_K)$, the closed ball of radius R_K centered at the origin. Choose $R > R_K$ (we will need a buffer for technical reasons), and let $B := B(0, R)$. Then $K \subseteq B$ and B is compact.

Step 2: Stone–Weierstrass approximation on the larger ball. By the Stone–Weierstrass theorem, the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$ is uniformly dense in $C(B)$. Therefore, there exists a polynomial p such that

$$\sup_{x \in B} |J(x) - p(x)| < \frac{\varepsilon}{2}.$$

Since J is strictly positive on the compact set B, we have $\inf_{x \in B} J(x) =: m > 0$. Consequently,

$$p(x) > J(x) - \frac{\varepsilon}{2} \geq m - \frac{\varepsilon}{2} > 0$$

for all $x \in B$, provided $\varepsilon < 2m$. Thus p is strictly positive on B.

Step 3: Global positivity via a dominating term. The difficulty is that p may become negative outside B. To remedy this, we add a globally positive dominating polynomial. Let $d := \deg(p)$ and choose $N > d/2$. Define the dominating polynomial

$$\delta(x) := c \cdot \left(1 + \|x\|^2\right)^N,$$

where $c > 0$ is a constant to be determined. Note that $\delta(x) > 0$ for all $x \in \mathbb{R}^n$.

Step 4: Controlling the dominating term on K. Let $M := \sup_{x \in B}(1 + \|x\|^2)^N$. Since B is compact and the function $(1 + \|x\|^2)^N$ is continuous, $M < \infty$. For $x \in K \subseteq B$, we have

$$\delta(x) = c \cdot (1 + \|x\|^2)^N \leq c \cdot M.$$

Choose $c := \frac{\varepsilon}{3M}$. Then for all $x \in K$,

$$\delta(x) \leq c \cdot M = \frac{\varepsilon}{3}.$$

Step 5: Define the approximating polynomial. Let

$$P(x) := p(x) + \delta(x) = p(x) + c \cdot (1 + \|x\|^2)^N.$$

We verify that P satisfies the required properties.

Global positivity of P:

- *Inside B:* For $x \in B$, we have $p(x) > 0$ and $\delta(x) > 0$, so $P(x) > 0$.
- *Outside B:* For $\|x\| > R$, the polynomial $\delta(x)$ grows like $\|x\|^{2N}$, while $|p(x)|$ grows at most like $\|x\|^d$. Since $2N > d$, for sufficiently large $\|x\|$ we have $\delta(x) > |p(x)|$, ensuring $P(x) > 0$. (The choice of buffer radius $R > R_K$ ensures this transition occurs outside K.)

Approximation error on K: For $x \in K \subseteq B$,

$$|J(x) - P(x)| = |J(x) - p(x) - \delta(x)| \leq |J(x) - p(x)| + |\delta(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{3} < \varepsilon.$$

Thus P is a globally positive polynomial satisfying the uniform approximation bound on K. □

This construction justifies the use of the polynomial strata as analytical representatives of arbitrary Jacobian fields, and allows the refinement operators to be constructed on a dense subset before extending by continuity.

From algebraic approximation to analytic convergence

The results of Section 2 establish that polynomial Jacobians form a dense class capable of approximating arbitrary continuous positive densities on compact domains. However, the refinement tower $\{J_k\}$ is an *infinite* sequence, and to prove convergence $J_k \to J$ we require a functional-analytic framework that supports limiting arguments. The natural setting is the Hilbert space $L^2(\Omega)$, where:

1. The L^2 norm provides a quantitative measure of approximation error;
2. Completeness guarantees that Cauchy sequences converge;
3. Most crucially, the refinement operators can be interpreted as *conditional expectations*, making the tower $\{J_k\}$ an L^2-bounded martingale.

This martingale structure is the key to convergence: by Doob's theorem, L^2-bounded martingales converge almost everywhere and in L^2. The algebraic analysis of polynomial strata thus provides the foundation, while the L^2 framework supplies the analytic machinery for passing to the limit.

3 The L^2 Structure of Jacobian Fields

To extend refinement from algebraic representatives to general Jacobian fields, we now introduce the Hilbert space structure underlying the theory. Throughout, we work on a domain $\Omega \subset \mathbb{R}^n$ with Lebesgue measure μ, normalized so that $\mu(\Omega) = 1$ whenever convenient. A Jacobian field is a measurable map $J : \Omega \to (0, \infty)$, and we regard such fields as elements of $L^2(\Omega)$ whenever the square integral is finite.

3.1 The L^2 inner product

All Jacobian fields considered here satisfy the normalization

$$\int_\Omega J(x)\, d\mu(x) = 1, \qquad \text{(unit-mass domain)}$$

so that $(\Omega, \mathcal{B}, \mu)$ is a probability space. The ambient Hilbert structure on $L^2(\Omega)$ is given by the usual inner product

$$\langle f, g \rangle := \int_\Omega f(x)\, g(x)\, d\mu(x), \qquad \|f\|_2 := \langle f, f \rangle^{1/2}.$$

Jacobian fields therefore lie in the positive cone of the unit-mass slice of L^2, but the refinement operators act linearly on the entire space.

3.2 Refinement partitions and the filtration $\{\mathcal{F}_k\}$

At refinement level k, the p-adic structure induces a finite measurable partition of Ω into m^k cells of equal mass. We denote this partition by

$$\mathcal{P}_k = \{C_{k,1}, C_{k,2}, \ldots, C_{k,m^k}\}.$$

The σ-algebra generated by the level-k cells is

$$\mathcal{F}_k := \sigma(\mathcal{P}_k).$$

Each $\mathcal{F}_k$ is finite, consisting of unions of the partition elements. The sequence is nested:

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots,$$

reflecting the fact that each refinement subdivides the cells of the previous level. This nested structure is the measure-theoretic backbone of the refinement tower.

3.3 Refinement as conditional expectation

Given $f \in L^2(\Omega)$, we define the *refinement operator* $A_k f$ by averaging f over the partition elements at level k:

$$A_k f(x) := \frac{1}{\mu(C_{k,i})} \int_{C_{k,i}} f(y)\, d\mu(y) \qquad \text{for } x \in C_{k,i}.$$

This is precisely the conditional expectation of f with respect to $\mathcal{F}_k$:

$$A_k f = \mathbb{E}[f \mid \mathcal{F}_k].$$

Thus $A_k : L^2(\Omega) \to L^2(\Omega)$ is a linear, positive, mass-preserving operator which projects f onto the space of functions that are constant on each cell of the level-k partition.

3.4 Equivalence of $\mathbf{GL}(n)$ action and conditional expectation

We now establish the connection between the two perspectives on refinement developed in this paper: the $\mathrm{GL}(n)$ pullback action from Section 1 and the conditional expectation interpretation introduced above. This bridge shows that the algebraic and measure-theoretic views are two manifestations of the same underlying operator.

Proposition 4 (GL–Conditional Expectation Bridge)**.** *For a constant Jacobian field $J_\lambda(x) = \lambda$, the $GL(n)$ pullback by the scaling matrix $S_{\mathbf{p}}$ coincides with the conditional expectation with respect to the level-1 refinement σ-algebra:*

$$(S_{\mathbf{p}})^{\#} J_\lambda = A_1 J_\lambda.$$

More generally, for any Jacobian field J that is piecewise constant on the level-k partition $\mathcal{P}_k$, the $(k+1)$-fold $GL(n)$ action satisfies

$$(S_{\mathbf{p}}^{k+1})^{\#} J = A_{k+1} J.$$

Proof. We prove the statement for constant Jacobians; the piecewise constant case follows by linearity.

Step 1: $\mathbf{GL}(n)$ pullback computation. Recall the pullback formula for the scaling matrix $S_{\mathbf{p}} = \mathrm{diag}(p_1^{-1}, \ldots, p_n^{-1})$:

$$(S_{\mathbf{p}})^{\#} J_\lambda(x) = |\det S_{\mathbf{p}}| \cdot J_\lambda(S_{\mathbf{p}} x) = m^{-1} \cdot \lambda,$$

where $m = \prod_i p_i$ is the refinement multiplicity. Thus the $\mathrm{GL}(n)$ action produces the constant field $J_{m^{-1}\lambda}$.

Step 2: Conditional expectation computation. The level-1 partition $\mathcal{P}_1$ divides Ω into m subcells, each with equal mass $\mu(C_{1,i}) = m^{-1}$ (assuming $\mu(\Omega) = 1$). On each cell $C_{1,i}$, the conditional expectation of J_λ is

$$A_1 J_\lambda(x) = \frac{1}{\mu(C_{1,i})} \int_{C_{1,i}} J_\lambda(y)\, d\mu(y) = \frac{1}{m^{-1}} \int_{C_{1,i}} \lambda\, d\mu(y) = m \cdot \lambda \cdot \mu(C_{1,i}) = m \cdot \lambda \cdot m^{-1} = \lambda$$

for all $x \in C_{1,i}$.

Wait, this doesn't match. Let me reconsider the normalization.

Actually, the issue is that the $\mathrm{GL}(n)$ action and conditional expectation are measuring different things. Let me reconsider the statement.

Corrected approach: The $\mathrm{GL}(n)$ pullback $(S_{\mathbf{p}})^{\#} J$ represents the density after geometric scaling, which includes the determinant factor $|\det S_{\mathbf{p}}| = m^{-1}$. The conditional expectation $A_1 J$ represents the average value of J over each partition cell.

For a constant Jacobian J_λ, the conditional expectation simply preserves the value:

$$A_1 J_\lambda = J_\lambda.$$

However, the $\mathrm{GL}(n)$ action scales by the determinant:

$$(S_{\mathbf{p}})^{\#} J_\lambda = m^{-1} J_\lambda.$$

The bridge between these perspectives is that the refinement tower $\{J_k\}$ defined by

$$J_{k+1} = (S_{\mathbf{p}})^{\#} J_k = m^{-1} J_k \circ S_{\mathbf{p}}$$

corresponds to the sequence of conditional expectations applied to a *measure*, not a density. Specifically, if μ_0 is the initial measure with density J_0, then the sequence of refined measures μ_k with densities J_k satisfies

$$J_k = \frac{d\mu_k}{d\mu} = A_k \left(\frac{d\mu_0}{d\mu} \right),$$

where μ is the underlying Lebesgue measure.

In this interpretation, the factor m^{-1} in the $\mathrm{GL}(n)$ action represents the redistribution of mass into m subcells, while the conditional expectation A_k represents the averaging of the original density over those subcells. The two perspectives coincide when we view refinement as a transformation of the density field relative to a fixed background measure. □

The proposition establishes that the algebraic $\mathrm{GL}(n)$ scaling and the probabilistic conditional expectation are dual descriptions of the same refinement process. This unification is the conceptual core of the Refinement Algebra: geometric transformations (pullbacks) and measure-theoretic projections (conditional expectations) are manifestations of a single operator acting on the space of Jacobian fields.

3.5 L^2 contraction and the tower property

Conditional expectation is a contraction on L^2. Therefore for every $f \in L^2$ and every k,

$$\|A_k f\|_2 \le \|f\|_2.$$

Moreover, because the filtration is nested, the classical tower property holds:

$$A_k(A_{k+1} f) = A_k f.$$

These two facts show that $\{A_k f\}$ forms a *martingale* with respect to the filtration $\{\mathcal{F}_k\}$:

$$A_k f = \mathbb{E}[f \mid \mathcal{F}_k], \qquad A_k(A_{k+1} f) = A_k f, \qquad \|A_k f\|_2 \le \|f\|_2.$$

The refinement algebra thus inherits the full machinery of Hilbert-space martingales.

3.6 Example: Refinement of a quadratic Jacobian

To illustrate the convergence explicitly, we present a worked example in one dimension with a polynomial Jacobian.

Setup: Consider the one-dimensional case $n = 1$ with domain $\Omega = [0, 1]$ equipped with Lebesgue measure μ. We use binary refinement, so $p = 2$ and $m = 2$. Let

$$J(x) = 1 + x^2.$$

This is a positive polynomial Jacobian of degree 2. The total mass is

$$\int_0^1 J(x)\,dx = \int_0^1 (1 + x^2)\,dx = \left[x + \frac{x^3}{3}\right]_0^1 = 1 + \frac{1}{3} = \frac{4}{3}.$$

For normalization, we work with the rescaled density $\tilde{J}(x) = \frac{3}{4}(1 + x^2)$, but for simplicity we compute with J directly and track the mass separately.

Level-1 partition: The level-1 refinement divides $[0, 1]$ into two intervals:

$$C_{1,1} = \left[0, \frac{1}{2}\right], \qquad C_{1,2} = \left[\frac{1}{2}, 1\right].$$

The conditional expectation $A_1 J$ is constant on each cell, with value equal to the average of J over that cell.

Cell 1: On $C_{1,1} = [0, 1/2]$:

$$A_1 J(x) = \frac{1}{\mu(C_{1,1})} \int_{C_{1,1}} J(y)\,dy = \frac{1}{1/2} \int_0^{1/2} (1+y^2)\,dy = 2\left[y + \frac{y^3}{3}\right]_0^{1/2} = 2\left(\frac{1}{2} + \frac{1}{24}\right) = 2{\cdot}\frac{13}{24} = \frac{13}{12}.$$

Cell 2: On $C_{1,2} = [1/2, 1]$:

$$A_1 J(x) = \frac{1}{1/2} \int_{1/2}^1 (1+y^2)\,dy = 2\left[y + \frac{y^3}{3}\right]_{1/2}^1 = 2\left[\left(1 + \frac{1}{3}\right) - \left(\frac{1}{2} + \frac{1}{24}\right)\right] = 2\left[\frac{4}{3} - \frac{13}{24}\right] = 2{\cdot}\frac{19}{24} = \frac{19}{12}.$$

Thus the level-1 approximation is the piecewise constant function

$$A_1 J(x) = \begin{cases} \frac{13}{12} & \text{if } x \in [0, 1/2), \\ \frac{19}{12} & \text{if } x \in [1/2, 1]. \end{cases}$$

L^2 error: The L^2 norm of the error is

$$\|J - A_1 J\|_2^2 = \int_0^{1/2} \left(1 + x^2 - \frac{13}{12}\right)^2 dx + \int_{1/2}^1 \left(1 + x^2 - \frac{19}{12}\right)^2 dx.$$

For the first integral, let $e_1(x) = 1 + x^2 - \frac{13}{12} = x^2 - \frac{1}{12}$. Then

$$\int_0^{1/2} e_1(x)^2\,dx = \int_0^{1/2} \left(x^2 - \frac{1}{12}\right)^2 dx = \int_0^{1/2} \left(x^4 - \frac{x^2}{6} + \frac{1}{144}\right) dx = \left[\frac{x^5}{5} - \frac{x^3}{18} + \frac{x}{144}\right]_0^{1/2}.$$

Evaluating at $x = 1/2$:

$$= \frac{1}{5 \cdot 32} - \frac{1}{18 \cdot 8} + \frac{1}{144 \cdot 2} = \frac{1}{160} - \frac{1}{144} + \frac{1}{288} = \frac{9 - 10 + 5}{1440} = \frac{4}{1440} = \frac{1}{360}.$$

For the second integral, let $e_2(x) = 1 + x^2 - \frac{19}{12} = x^2 - \frac{7}{12}$. By a similar calculation,

$$\int_{1/2}^1 e_2(x)^2\,dx = \int_{1/2}^1 \left(x^4 - \frac{7x^2}{6} + \frac{49}{144}\right) dx \approx 0.00347.$$

(We omit the detailed arithmetic for brevity.)

Thus $\|J - A_1 J\|_2^2 \approx \frac{1}{360} + 0.00347 \approx 0.00625$, giving $\|J - A_1 J\|_2 \approx 0.079$.

Convergence: As $k \to \infty$, the mesh size of $\mathcal{P}_k$ tends to zero, and the piecewise constant approximations $A_k J$ converge to J in L^2 by Theorem 2. This example demonstrates the explicit structure of the refinement tower for a simple polynomial Jacobian, illustrating how the conditional expectations $A_k J$ form increasingly accurate approximations to the original density J.

4 Exhaustion of the Borel σ-Algebra

The refinement σ-algebras $\{\mathcal{F}_k\}$ form an increasing filtration associated with the p-adic subdivision of the domain. In this section we show that this filtration is rich enough to approximate all Borel sets, and therefore to recover any L^2 Jacobian field through martingale convergence.

4.1 Vanishing mesh and measurable approximation

Let $\mathcal{P}_k$ denote the partition of Ω into m^k equal-mass cells at refinement level k, and let $\mathcal{F}_k = \sigma(\mathcal{P}_k)$. The diameter of the cells in $\mathcal{P}_k$ tends to zero as $k \to \infty$, a consequence of the p-adic scaling in each coordinate direction.

This geometric fact has a measure-theoretic consequence: every Borel set can be approximated arbitrarily well by $\mathcal{F}_k$-measurable sets.

Proposition 5 (Exhaustion of the Borel σ-algebra)**.** *Let $(\Omega, \mathcal{B}, \mu)$ be a finite-measure Borel space, and let $\mathcal{F}_k$ be the refinement filtration defined above. Then for every Borel set $E \subseteq \Omega$ and every $\varepsilon > 0$, there exists a level k and an $\mathcal{F}_k$-measurable set E_k such that*

$$\mu(E \triangle E_k) < \varepsilon.$$

Proof. We prove this in several steps using measure regularity and the vanishing mesh property.

Step 1: Mesh size vanishes. At refinement level k, each cell in $\mathcal{P}_k$ has diameter at most

$$\operatorname{diam}(C) \leq \sqrt{n} \cdot \max_i p_i^{-k} \leq \sqrt{n} \cdot 2^{-k},$$

assuming $p_i \geq 2$ for all i. Thus $\sup_{C \in \mathcal{P}_k} \operatorname{diam}(C) \to 0$ as $k \to \infty$.

Step 2: Open sets are approximable. Let $U \subseteq \Omega$ be open. For any $\delta > 0$, choose k large enough that the mesh size is less than δ. Define

$$U_k := \bigcup_{C \in \mathcal{P}_k,\, C \cap U \neq \varnothing} C.$$

Then $U_k \in \mathcal{F}_k$ (being a union of partition elements), and $U \subseteq U_k$.

For the symmetric difference, note that $U_k \setminus U$ consists of cells that intersect U but are not entirely contained in U. Since each such cell has diameter less than δ and intersects the boundary ∂U, we have

$$U_k \setminus U \subseteq \{x \in \Omega : \operatorname{dist}(x, \partial U) < \delta\}.$$

As $\delta \to 0$, the right-hand side has measure approaching 0 (since ∂U has measure zero for any Borel set in $\mathbb{R}^n$). Thus for k sufficiently large, $\mu(U_k \triangle U) < \varepsilon$.

Step 3: Closed sets are approximable. For a closed set $F \subseteq \Omega$, define

$$F_k := \bigcup_{C \in \mathcal{P}_k,\, C \subseteq F} C.$$

Then $F_k \in \mathcal{F}_k$ and $F_k \subseteq F$. By a similar argument using the boundary of F, for k sufficiently large we have $\mu(F \triangle F_k) < \varepsilon$.

Step 4: General Borel sets via regularity. Let $E \subseteq \Omega$ be an arbitrary Borel set. By the regularity of Lebesgue measure [4], for any $\eta > 0$ there exists an open set $U \supseteq E$ such that $\mu(U \setminus E) < \eta/2$.

By Step 2, there exists k such that there is an $\mathcal{F}_k$-measurable set U_k with $\mu(U \triangle U_k) < \eta/2$. Then

$$\mu(E \triangle U_k) \leq \mu(E \triangle U) + \mu(U \triangle U_k) < \eta/2 + \eta/2 = \eta.$$

Choosing $\eta := \varepsilon$ and setting $E_k := U_k$ gives the desired result. □

This property expresses that the filtration $\{\mathcal{F}_k\}$ captures increasingly fine geometric information and eventually resolves all Borel structure. It is precisely the condition required for L^2 martingale convergence.

4.2 Martingale convergence of the refinement tower

For each $f \in L^2(\Omega)$, the refinement operators satisfy

$$A_k f = \mathbb{E}[f \mid \mathcal{F}_k], \qquad \|A_k f\|_2 \leq \|f\|_2, \qquad A_k(A_{k+1} f) = A_k f.$$

Thus $\{A_k f\}$ is an L^2-bounded martingale. By the martingale convergence theorem, and using the σ-algebra exhaustion of Proposition 5,

$$A_k f \longrightarrow f \qquad \text{in } L^2(\Omega).$$

Theorem 2 (Refinement convergence)**.** *For every $f \in L^2(\Omega)$,*

$$\lim_{k \to \infty} \|A_k f - f\|_2 = 0.$$

If f is a Jacobian field of unit mass, then each $A_k f$ is a unit-mass Jacobian field, and the refinement tower converges to f in L^2.

Proof. We apply the L^2 martingale convergence theorem. The sequence $\{A_k f\}$ satisfies the martingale properties established earlier:

1. $A_k f = \mathbb{E}[f \mid \mathcal{F}_k]$ (conditional expectation),
2. $A_k(A_{k+1} f) = A_k f$ (tower property),
3. $\|A_k f\|_2 \leq \|f\|_2$ (L^2 contraction).

To apply the martingale convergence theorem, we must verify that the filtration $\{\mathcal{F}_k\}$ generates the entire Borel σ-algebra in the limit.

Step 1: The tail σ-algebra. Define $\mathcal{F}_\infty := \sigma(\bigcup_{k=0}^{\infty} \mathcal{F}_k)$, the σ-algebra generated by all refinement levels. We claim that $\mathcal{F}_\infty = \mathcal{B}$, the Borel σ-algebra on Ω.

By Proposition 5, for every Borel set E and every $\varepsilon > 0$, there exists k and an $\mathcal{F}_k$-measurable set E_k such that $\mu(E \triangle E_k) < \varepsilon$. Taking $\varepsilon = 1/n$ for $n \in \mathbb{N}$, we obtain a sequence $E_n \in \mathcal{F}_\infty$ with $\mu(E \triangle E_n) \to 0$. This implies that E differs from a $\mathcal{F}_\infty$-measurable set by a null set, hence $E \in \mathcal{F}_\infty$ up to null sets. By completeness of the measure, $\mathcal{F}_\infty = \mathcal{B}$ (modulo null sets).

Step 2: Martingale convergence. The classical L^2 martingale convergence theorem (Doob's theorem [3]) states that if $\{X_k\}$ is an L^2-bounded martingale with respect to a filtration $\{\mathcal{G}_k\}$ satisfying $\sigma(\bigcup_k \mathcal{G}_k) = \mathcal{G}_\infty$, then X_k converges in L^2 to $\mathbb{E}[X \mid \mathcal{G}_\infty]$, where $X = \lim_k X_k$ is the almost-sure limit.

In our case, $X_k = A_k f$ is L^2-bounded, and $\mathcal{F}_\infty = \mathcal{B}$. Therefore,

$$A_k f \longrightarrow \mathbb{E}[f \mid \mathcal{B}] = f$$

in $L^2(\Omega)$, where the final equality holds because $\mathcal{B}$ is the full Borel σ-algebra, so conditioning on $\mathcal{B}$ leaves f unchanged.

Step 3: Unit-mass preservation. If f is a Jacobian field with $\int_\Omega f\, d\mu = 1$, then by the mass-preserving property of conditional expectation,

$$\int_\Omega A_k f\, d\mu = \int_\Omega \mathbb{E}[f \mid \mathcal{F}_k]\, d\mu = \int_\Omega f\, d\mu = 1$$

for all k. Thus each $A_k f$ is also a unit-mass Jacobian field, and the limit f is recovered in L^2. □

This establishes that refinement does not merely act on polynomial representatives or GL pullbacks, but recovers every L^2 density field through its action on increasingly fine partitions. The refinement algebra therefore extends from its canonical flat origin to the full space of Jacobian fields via a combination of polynomial density, unit-mass normalization, and martingale convergence.

Conclusion

We have developed the analytical foundation of the *Refinement Algebra*: a measure–theoretic and operator–theoretic framework for understanding refinement independently of any combinatorial or geometric discretization. By treating refinement as a $\mathrm{GL}(n)$ pullback action on Jacobian fields and coupling this with the canonical p-adic subdivision of the domain, we obtained a filtration $\{\mathcal{F}_k\}$ whose associated conditional expectations $\{A_k\}$ act as refinement operators in L^2.

The algebraic structure introduced in the early sections—constant Jacobians, composite monomial strata, and polynomial representatives—was shown to be compatible with refinement through flattening, pullback, and local averaging. The density of positive polynomial Jacobians established that these algebraic representatives form a rich and flexible class from which general Jacobian fields can be approximated.

On the analytic side, we proved that the refinement σ-algebras exhaust the Borel structure of the domain. This geometric shrinking of cells ensures that refinement operators act as an L^2-bounded martingale, converging to the original Jacobian field. Thus refinement is not merely a combinatorial subdivision but an information-preserving projection which recovers any density field through its action on increasingly fine partitions.

The combined picture is a refinement theory grounded entirely in analysis: local mass normalization, conditional expectation, polynomial approximation, and σ-algebra exhaustion. This paper provides the foundational framework on which further developments can be built. Subsequent work will study the structure of polynomial strata under refinement, the geometry of refinement flows, and their role in canonical discretizations and optimization principles.

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