The Torus Geometric Refinement Transform (TGRT)

A constant-Jacobian multiresolution transform on the solid ring torus

Abstract. We construct an equal-volume (constant-Jacobian) parameterization of the solid ring torus and build on it a multiresolution transform—the *Torus Geometric Refinement Transform* (TGRT). The TGRT combines dyadic refinement in a radial variable with Fourier analysis in two angular variables, yielding orthonormal bases, Parseval identities, and simple cell-count formulas. The associated discrete exterior calculus (DEC) inherits Kronecker structure, enabling near-diagonalization of the discrete Hodge Laplacian; leakage off the block-diagonal scales with torus thickness $\varepsilon = a/R$. The framework mirrors the SGRT on $\mathbb{B}^3$ but for the solid torus $\mathcal{T}_{R,a}$, and is designed for fast, stable analysis and PDE discretization.

1 Geometry of the ring torus

Let $R > a > 0$ and define the *ring torus* (solid) as the image of coordinates (ρ, θ, φ) under

$$\Phi_0(\rho,\theta,\varphi) = \big((R+\rho\cos\theta)\cos\varphi,\ (R+\rho\cos\theta)\sin\varphi,\ \rho\sin\theta\big), \tag{1}$$

with $\rho \in [0,a]$, $\theta \in [0,2\pi)$ (poloidal), $\varphi \in [0,2\pi)$ (toroidal). The standard metric in these orthogonal coordinates is

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\theta^2 + (R+\rho\cos\theta)^2\,\mathrm{d}\varphi^2, \tag{2}$$

with volume form

$$\mathrm{d}V = \rho\,(R+\rho\cos\theta)\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi. \tag{3}$$

The centerline (core circle) is $\mathbf{C}(\varphi) = (R\cos\varphi,\ R\sin\varphi,\ 0)$.

2 Equal-volume parameterization and constant Jacobian

Define the radial area variable $u = \frac{1}{2}\rho^2$ so $\mathrm{d}u = \rho\,\mathrm{d}\rho$, and define a ρ-dependent angular warp $\psi = \psi(\theta;\rho)$ by

$$\psi \ :=\ \theta + \frac{\rho}{R}\sin\theta \quad \Longleftrightarrow \quad \mathrm{d}\psi = \frac{R+\rho\cos\theta}{R}\,\mathrm{d}\theta. \tag{4}$$

Theorem 1 (Constant Jacobian map)**.** *For any $R > a > 0$, the change of variables $(\rho,\theta,\varphi) \mapsto (u,\psi,\varphi)$ given by (4) is a C^∞ diffeomorphism of the parameter rectangle, and the induced volume form is* constant*:*

$$\mathrm{d}V \ =\ R\,\mathrm{d}u\,\mathrm{d}\psi\,\mathrm{d}\varphi. \tag{5}$$

Proof. From (4), $\partial_\theta\psi = 1 + (\rho/R)\cos\theta \in (1-\rho/R, 1+\rho/R) \subset (0,2)$ for all $\rho \in [0,a)$ with $a < R$, hence $\theta \mapsto \psi$ is strictly increasing and 2π-periodic, yielding a diffeomorphism on the circle for each fixed ρ. The Jacobian determinant of $(\rho,\theta,\varphi) \mapsto (u,\psi,\varphi)$ equals $\det\big(\partial(u,\psi,\varphi)/\partial(\rho,\theta,\varphi)\big) = \rho\,\frac{R+\rho\cos\theta}{R}$, so substituting into (3) gives $\mathrm{d}V = R\,\mathrm{d}u\,\mathrm{d}\psi\,\mathrm{d}\varphi$. $\square$

Corollary 2 (Equal-volume dyadic bricks)**.** *Uniform splits $u \in [0,U]$ with $U = \frac{1}{2}a^2$, $\psi \in [0,2\pi)$, $\varphi \in [0,2\pi)$ into 2^k bins per axis produce $N_k = 8^k$ bricks, each of volume*

$$\Delta V_k \ =\ R\,\frac{U}{2^k}\,\frac{2\pi}{2^k}\,\frac{2\pi}{2^k} \ =\ \frac{2\pi^2 R a^2}{8^k}. \tag{6}$$

3 The TGRT: spaces, basis, and transform

Let $\mathcal{T}_{R,a} = \Phi_0([0,a]\times\mathbb{T}\times\mathbb{T})$ denote the solid ring torus. Via the diffeomorphism $(u,\psi,\varphi)\mapsto x = \Phi(u,\psi,\varphi) := \Phi_0(\rho(u),\theta(u,\psi),\varphi)$, pull a function $f\in L^2(\mathcal{T}_{R,a})$ back to $g(u,\psi,\varphi) = f\big(\Phi(u,\psi,\varphi)\big)$. By Theorem 1, the L^2 inner product is flat:

$$\langle f,h\rangle_{L^2(\mathcal{T})} = \int_{\mathcal{T}_{R,a}} f\,\overline{h}\,\mathrm{d}V = R\int_0^U\int_0^{2\pi}\int_0^{2\pi} g\,\overline{g_h}\,\mathrm{d}u\,\mathrm{d}\psi\,\mathrm{d}\varphi. \tag{7}$$

Scaling and wavelet factors in u; Fourier in ψ,φ

Fix a dyadic level $j\in\mathbb{Z}_{\geq 0}$ and let $\{I_{j,\ell}\}_{\ell=0}^{2^j-1}$ be the partition of $[0,U]$ into intervals of length $U/2^j$. Define the normalized *u-scaling* and *Haar wavelet* functions

$$\phi_{j,\ell}(u) := \frac{1}{\sqrt{U/2^j}}\,\mathbf{1}_{I_{j,\ell}}(u), \tag{8}$$

$$\psi_{j,\ell}(u) := \frac{1}{\sqrt{U/2^j}}\Big(\mathbf{1}_{I_{j+1,2\ell}}(u) - \mathbf{1}_{I_{j+1,2\ell+1}}(u)\Big), \tag{9}$$

which are orthonormal in $L^2([0,U],\mathrm{d}u)$. For angular variables, use the standard Fourier system $\{(2\pi)^{-1/2}\mathrm{e}^{\mathrm{i}m\psi}\}_{m\in\mathbb{Z}}$ and $\{(2\pi)^{-1/2}\mathrm{e}^{\mathrm{i}n\varphi}\}_{n\in\mathbb{Z}}$.

Definition 3 (TGRT basis)**.** For each j,ℓ and integers m,n, define the *TGRT scaling* and *TGRT wavelet* atoms on $\mathcal{T}_{R,a}$ by

$$\Phi^{\mathrm{sc}}_{j,\ell,m,n}(u,\psi,\varphi) := \frac{1}{\sqrt{R}}\,\phi_{j,\ell}(u)\,\frac{\mathrm{e}^{\mathrm{i}m\psi}}{\sqrt{2\pi}}\,\frac{\mathrm{e}^{\mathrm{i}n\varphi}}{\sqrt{2\pi}}, \tag{10}$$

$$\Phi^{\mathrm{w}}_{j,\ell,m,n}(u,\psi,\varphi) := \frac{1}{\sqrt{R}}\,\psi_{j,\ell}(u)\,\frac{\mathrm{e}^{\mathrm{i}m\psi}}{\sqrt{2\pi}}\,\frac{\mathrm{e}^{\mathrm{i}n\varphi}}{\sqrt{2\pi}}. \tag{11}$$

The $1/\sqrt{R}$ factor makes the family orthonormal in the inner product (7).

Proposition 4 (Orthonormality and Parseval)**.** *The TGRT atoms form an orthonormal family in $L^2(\mathcal{T}_{R,a})$. For any $f\in L^2(\mathcal{T}_{R,a})$ with pulled-back g, the TGRT analysis coefficients*

$$\widehat{f}^{\mathrm{sc}}_{j,\ell}[m,n] = \langle g,\Phi^{\mathrm{sc}}_{j,\ell,m,n}\rangle, \qquad \widehat{f}^{\mathrm{w}}_{j,\ell}[m,n] = \langle g,\Phi^{\mathrm{w}}_{j,\ell,m,n}\rangle \tag{12}$$

obey the Parseval identity: the sum of squared magnitudes equals $\|f\|^2_{L^2(\mathcal{T})}$.

Proof. Follows immediately from constant measure (7) and orthonormality of Haar in u and complex exponentials in ψ,φ. □

Synthesis. With the usual multiresolution convention (fix a base scale j_0),

$$g(u,\psi,\varphi) = \sum_{\ell,m,n}\widehat{f}^{\mathrm{sc}}_{j_0,\ell}[m,n]\,\Phi^{\mathrm{sc}}_{j_0,\ell,m,n} + \sum_{j\geq j_0}\sum_{\ell,m,n}\widehat{f}^{\mathrm{w}}_{j,\ell}[m,n]\,\Phi^{\mathrm{w}}_{j,\ell,m,n}, \tag{13}$$

with convergence in L^2. Composing with $\Phi(u,\psi,\varphi)$ yields f on $\mathcal{T}_{R,a}$.

4 Dyadic cells, counts, and boundaries

Let $N = 2^k$. The TGRT grid at level k has N^3 bricks (cells) with constant volume $\Delta V_k = 2\pi^2 R a^2/8^k$. The interval edges are

$$u_j = \frac{j}{N}U, \quad \psi_m = \frac{2\pi m}{N}, \quad \varphi_n = \frac{2\pi n}{N}, \qquad j, m, n \in \{0, \ldots, N\}. \tag{14}$$

The corresponding radii are $\rho_j = \sqrt{2u_j} = a\sqrt{j/N}$. The level-1 bricks are the eight blocks $(\mathrm{I/O}) \times (\mathrm{T/B}) \times (\mathrm{L/R})$.

5 Discrete Exterior Calculus on the TGRT grid

We outline mass-lumped Hodge stars consistent with the metric induced by (1). Incidence (coboundary) matrices factor as Kronecker products across the three axes, ensuring $d^2 = 0$ exactly on the grid.

Primal edge lengths and dual face areas

Let a brick be indexed by its cell-centered values (u_c, ψ_c, φ_c), with $\rho_c = \sqrt{2u_c}$ and θ_c determined implicitly by $\psi_c = \theta_c + (\rho_c/R)\sin\theta_c$. Using the embedding (1) and Jacobian $J = R$ from Theorem 1, one finds the following *cell-centered* expressions (exact in the continuous metric, applied in lumped form):

$$\begin{aligned}
\ell_u &= \rho_{j+1} - \rho_j, && (15)\\
\ell_\psi &= \frac{\rho_c R}{R + \rho_c\cos\theta_c}\,\Delta\psi, && (16)\\
\ell_\varphi &= (R + \rho_c\cos\theta_c)\,\Delta\varphi, && (17)
\end{aligned}$$

$$\begin{aligned}
A_{\psi\varphi} &= \rho_c R\,\Delta\psi\,\Delta\varphi, && (18)\\
A_{u\varphi} &= \frac{R + \rho_c\cos\theta_c}{\rho_c}\,\Delta u\,\Delta\varphi, && (19)\\
A_{u\psi} &= \frac{R}{R + \rho_c\cos\theta_c}\,\Delta u\,\Delta\psi. && (20)
\end{aligned}$$

These follow from $J = |\partial_u\Phi \cdot (\partial_\psi\Phi \times \partial_\varphi\Phi)| = R$ and the identities $|\partial_u\Phi| = 1/\rho$, $|\partial_\varphi\Phi| = R + \rho\cos\theta$, $|\partial_\psi\Phi| = \rho R/(R + \rho\cos\theta)$.

Mass-lumped Hodge stars

With (15)–(18), diagonal Hodge stars are

$$(\star_0)_{\mathrm{cell}} = \underbrace{R\,\Delta u\,\Delta\psi\,\Delta\varphi}_{\Delta V}, \tag{21}$$

$$(\star_1)_u = \frac{A_{\psi\varphi}}{\ell_u}, \quad (\star_1)_\psi = \frac{A_{u\varphi}}{\ell_\psi}, \quad (\star_1)_\varphi = \frac{A_{u\psi}}{\ell_\varphi}, \tag{22}$$

$$(\star_2)_{u\psi} = \frac{\ell_\varphi}{A_{u\psi}}, \quad (\star_2)_{u\varphi} = \frac{\ell_\psi}{A_{u\varphi}}, \quad (\star_2)_{\psi\varphi} = \frac{\ell_u}{A_{\psi\varphi}}, \tag{23}$$

$$(\star_3)_{\mathrm{cell}} = \Delta V^{-1}. \tag{24}$$

Thus the DEC Hodge Laplacian on 0-forms is $\Delta_0 = \delta\, \mathrm{d} = \star_0^{-1} d_0^\top\ \star_1\ d_0$; analogous formulas hold for 1- and 2-forms.

6 Laplacian structure and thin-torus asymptotics

For scalar f, the continuous Laplacian in (ρ, θ, φ) is (*cf.* (2))

$$\Delta f = \partial_{\rho\rho} f + \frac{R + 2\rho\cos\theta}{\rho(R + \rho\cos\theta)}\, \partial_\rho f + \frac{1}{\rho^2}\, \partial_{\theta\theta} f - \frac{\sin\theta}{\rho(R + \rho\cos\theta)}\, \partial_\theta f + \frac{1}{(R + \rho\cos\theta)^2}\, \partial_{\varphi\varphi} f. \tag{25}$$

Under (u, ψ, φ), angle coefficients acquire a mild ψ-dependence that vanishes as $\varepsilon = a/R \to 0$.

Proposition 5 (Mode-coupling scales with thickness)**.** *Let $L_\psi g = \partial_\psi\big((1 + \varepsilon\cos\psi)\partial_\psi g\big)$ be the model angular operator at fixed (u, φ). For input $g(\psi) = \mathrm{e}^{\mathrm{i}m\psi}$, the energy leaked from the main Fourier mode m into sidebands $m \pm 1$ scales as $O(\varepsilon^2)$. Hence, in the TGRT Fourier basis, off-diagonal blocks induced by angular metric variation are $O(\varepsilon)$ in operator norm and* energy *leakage is $O(\varepsilon^2)$.*

Sketch. Expand $\cos\psi\, \mathrm{e}^{\mathrm{i}m\psi} = \frac{1}{2}(\mathrm{e}^{\mathrm{i}(m+1)\psi} + \mathrm{e}^{\mathrm{i}(m-1)\psi})$; a first-order Born approximation shows coefficients to $m \pm 1$ are $\propto \varepsilon/2$. Parseval then yields energy $\propto (\varepsilon/2)^2$. □

7 Algorithms and complexity

Forward TGRT (analysis). For a grid with $N_u \times N_\psi \times N_\varphi$ samples:

1) Pull back samples to $g(u, \psi, \varphi)$ via the diffeomorphism Φ.
2) Apply 1D Haar (or other compactly supported) transform along u (cost $O(N_u)$ per (ψ, φ) slice).
3) Apply FFTs in ψ and φ (cost $O(N_\psi \log N_\psi + N_\varphi \log N_\varphi)$ per u-coefficient).

Total cost $O(N_u N_\psi N_\varphi\ \log(N_\psi N_\varphi))$. Inverse TGRT reverses the steps.

Boundary conditions. Periodic in ψ, φ. In u, one may impose Dirichlet/Neumann/Robin at $u = U$ and regularity at $u = 0$ (axis). For vector fields, impose physical boundary conditions on tangential/normal components on the outer surface $\rho = a$.

8 Notes on novelty and scope

The constant-Jacobian map (4)–(5) packages classical ingredients (tubular coordinates and an equiareal poloidal warp) into a simple diffeomorphism that exactly flattens the measure on the *solid* torus. The TGRT then leverages this to build orthonormal multiresolution bases (Haar×Fourier×Fourier) and DEC discretizations with Kronecker structure. The approach extends to anisotropic refinements (non-dyadic) and to the surface torus $\rho = a$ by dropping the u-dimension.

Appendix A: Inverting ψ to θ

For fixed $\rho < R$, $\psi = \theta + (\rho/R)\sin\theta$ is strictly increasing, so each (ρ, ψ) determines a unique θ. Standard Newton iteration with initial guess $\theta^{(0)} = \psi$ converges quadratically:

$$\theta^{(k+1)} = \theta^{(k)} - \frac{\theta^{(k)} - \psi + (\rho/R)\sin\theta^{(k)}}{1 + (\rho/R)\cos\theta^{(k)}}. \tag{26}$$

Appendix B: Surface TGRT (T^2)

At $\rho = a$, the induced metric is $\mathrm{d}s^2 = a^2\,\mathrm{d}\theta^2 + (R + a\cos\theta)^2\,\mathrm{d}\varphi^2$ and the area form is $\mathrm{d}A = a(R + a\cos\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi$. The map $\psi = \theta + (a/R)\sin\theta$ yields $\mathrm{d}A = aR\,\mathrm{d}\psi\,\mathrm{d}\varphi$, i.e. a constant area element, thus a 2D TGRT with Haar in no radial direction and Fourier in both angles (or 2D Haar on (ψ, φ)).

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