The Divergence-Free Radiant Transform on the Unit Ball: Coexact Hodge Component, Boundary Alignment, and Poloidal Determinant

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Abstract

We present a comprehensive spectral analysis of divergence-free vector fields on the unit ball $\mathbb{B}^3$ via the *Divergence-Free Radiant Transform* (DFRT). The transform furnishes an orthogonal basis of the coexact subspace $L^2_\sigma(\mathbb{B})$ (the divergence-free component in the Hodge–Helmholtz decomposition) as a direct sum of toroidal and poloidal branches. We impose a single global boundary model—no-slip Dirichlet $u|_{\partial\mathbb{B}} = 0$—chosen to be orthogonal to the curl-free exact component generated by Dirichlet scalar potentials (CFRT), yielding a clean exact/coexact split. On each angular frequency $\ell \geq 1$, toroidal eigenmodes arise from a second-order Sturm–Liouville problem with zeros of j_ℓ, while poloidal modes arise from a fourth-order bi-Laplacian reduction with a *two-condition* no-slip boundary that selects eigenvalues via a transcendental 2×2 determinant (Appendix A). We establish Plancherel, inversion, Sobolev-scale energy identities, and an exact/coexact identification compatible with trigraded SGRT calculus.

1 Introduction

The spectral analysis of vector fields on bounded domains is central in mathematical physics and numerical PDE. On the unit ball $\mathbb{B}^3$, the Hodge–Helmholtz decomposition splits $L^2(\mathbb{B})^3$ into the L^2-closure of gradients and the divergence-free subspace. With the *no-slip Dirichlet* boundary condition $u|_{\partial\mathbb{B}} = 0$, there are no nontrivial harmonic vector fields, so the decomposition reduces to an orthogonal sum of an *exact* part (gradients of Dirichlet scalar modes, i.e. CFRT) and a *coexact* part (divergence-free fields satisfying no-slip, i.e. DFRT).

The DFRT we develop gives a complete orthonormal basis of $L^2_\sigma(\mathbb{B})$ built from toroidal and poloidal vector spherical harmonics with radial profiles determined by Sturm–Liouville problems. The poloidal branch involves a bi-Laplacian structure and enforces *two* boundary constraints at $r = 1$ under no-slip, leading to an explicit 2×2 determinant condition that distinguishes it from the toroidal branch's single Bessel zero condition.

Contributions.

1. Orthogonal toroidal/poloidal spectral decomposition of $L^2_\sigma(\mathbb{B})$ with no-slip boundary.
2. Exact/coexact identification compatible with a trigraded (SGRT) calculus: CFRT spans exact 1-forms; DFRT spans coexact 1-forms.
3. Plancherel, inversion, and H^s energy identities in terms of Stokes eigenvalues.
4. A concrete poloidal 2×2 determinant condition (Appendix A) capturing the coupled no-slip constraints.

2 Spaces, spherical calculus, and boundary model

Let $\mathbb{B} = \{x \in \mathbb{R}^3 : |x| < 1\}$ with spherical coordinates $x = r\,\omega$, $r \in (0,1)$ and $\omega \in \mathbb{S}^2$. Define

$$L^2_\sigma(\mathbb{B}) := \overline{\{u \in C^\infty_c(\mathbb{B};\mathbb{R}^3) : \operatorname{div} u = 0\}}^{L^2}, \qquad V := \{u \in H^1_0(\mathbb{B};\mathbb{R}^3) : \operatorname{div} u = 0\}.$$

We fix the global boundary model:

$$u|_{\partial\mathbb{B}} = 0 \quad \text{(no-slip Dirichlet)}, \tag{2.1}$$

so that the curl-free exact subspace is $\overline{\nabla H^1_0(\mathbb{B})}^{L^2}$ (CFRT) and the coexact subspace is $L^2_\sigma(\mathbb{B})$ (DFRT); see Section 7.

We use scalar spherical harmonics $\{Y_\ell^m\}$ ($\ell \ge 0$, $|m| \le \ell$) and the vector spherical harmonics (VSH)

$$\boldsymbol{Y}_{\ell m} := Y_\ell^m\,\hat{r}, \qquad \boldsymbol{\Psi}_{\ell m} := \nabla_S Y_\ell^m, \qquad \boldsymbol{\Phi}_{\ell m} := \hat{r}\times\nabla_S Y_\ell^m, \qquad \boldsymbol{X}_{\ell m} := \frac{1}{\sqrt{\ell(\ell+1)}}\,\boldsymbol{\Phi}_{\ell m} \quad (\ell \ge 1).$$

Key curl identities (for smooth radial f) on $\mathbb{B} \setminus \{0\}$:

$$\nabla \times \big(f\,\boldsymbol{Y}_{\ell m}\big) = -\frac{f}{r}\,\boldsymbol{\Phi}_{\ell m}, \tag{2.2}$$

$$\nabla \times \big(f\,\boldsymbol{\Phi}_{\ell m}\big) = -\frac{\ell(\ell+1)}{r}\,f\,\boldsymbol{Y}_{\ell m} - \frac{1}{r}\,\partial_r(rf)\,\boldsymbol{\Psi}_{\ell m}. \tag{2.3}$$

Regularity at the origin imposes r^ℓ-type behavior in radial profiles.

3 DFRT ansatz: toroidal and poloidal branches

(T) Toroidal modes. For $\ell \ge 1$, define $\boldsymbol{T}_{\ell mn}(r,\omega) := \phi_{\ell n}(r)\,\boldsymbol{X}_{\ell m}(\omega)$. These are tangential and surface-divergence free, hence $\operatorname{div} \boldsymbol{T}_{\ell mn} = 0$.

(P) Poloidal modes. Let $h_{\ell n}$ be a radial scalar potential and set

$$\boldsymbol{P}_{\ell mn}(r,\omega) := \nabla \times \nabla \times \big(h_{\ell n}(r)\,r\,\boldsymbol{Y}_{\ell m}(\omega)\big). \tag{3.1}$$

Using (2.2)–(2.3) with $f(r) = h_{\ell n}(r)\,r$,

$$\boldsymbol{P}_{\ell mn}(r,\omega) = \frac{\ell(\ell+1)}{r}\,h_{\ell n}(r)\,\boldsymbol{Y}_{\ell m}(\omega) + \frac{1}{r}\,\partial_r\big(rh_{\ell n}(r)\big)\,\boldsymbol{\Psi}_{\ell m}(\omega), \tag{3.2}$$

hence $\operatorname{div} \boldsymbol{P}_{\ell mn} = 0$.

3.1 Boundary and origin regularity

Under (2.1), no-slip enforces both radial and tangential components to vanish at $r = 1$. Regularity at $r = 0$ selects the regular branches.

Lemma 3.1 (Boundary/regularity)**.** *For each (ℓ, m, n):*

(i) (Toroidal) $\boldsymbol{T}_{\ell mn}|_{r=1} = 0$ *iff* $\phi_{\ell n}(1) = 0$. *Regularity:* $\phi_{\ell n}(r) = \mathcal{O}(r^\ell)$.

(ii) (Poloidal) *Using* (3.2)*, no-slip at* $r = 1$ *is equivalent to*

$$h_{\ell n}(1) = 0, \qquad \partial_r\big(rh_{\ell n}(r)\big)\big|_{r=1} = 0,$$

and regularity: $h_{\ell n}(r) = \mathcal{O}(r^\ell)$.

4 Stokes operator and radial Sturm–Liouville reductions

Let $A : V \to V'$ be the Stokes operator $Au := -\mathbb{P}\Delta u$ with domain $D(A) = \{u \in H^2(\mathbb{B})^3 \cap V : u|_{\partial\mathbb{B}} = 0\}$; A is positive self-adjoint with compact inverse on $L^2_\sigma(\mathbb{B})$.

Define the radial operator

$$\mathcal{L}_\ell := -\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \frac{\ell(\ell+1)}{r^2}. \tag{4.1}$$

Lemma 4.1 (Radial eigen-ODEs). *Fix $\ell \geq 1$.*

(a) ***Toroidal:*** *$\phi_{\ell n}$ solves*

$$\mathcal{L}_\ell\phi_{\ell n} = \lambda^T_{\ell n}\,\phi_{\ell n}, \qquad \phi_{\ell n}(1) = 0, \quad \phi_{\ell n}(r) = \mathcal{O}(r^\ell), \tag{4.2}$$

with $\phi_{\ell n}(r) = c_{\ell n} j_\ell(\alpha^T_{\ell n} r)$, $j_\ell(\alpha^T_{\ell n}) = 0$, and $\lambda^T_{\ell n} = (\alpha^T_{\ell n})^2$.

(b) ***Poloidal:*** *If $\boldsymbol{P}_{\ell mn}$ is given by (3.1), set $q_{\ell n} := \mathcal{L}_\ell h_{\ell n}$. Then*

$$\mathcal{L}_\ell q_{\ell n} = \lambda^P_{\ell n}\, q_{\ell n} \quad \text{on } (0,1), \tag{4.3}$$

with $q_{\ell n}(r) = c_{\ell n} j_\ell(\alpha^P_{\ell n} r)$ (regular branch), and $h_{\ell n}$ is the unique regular solution of $\mathcal{L}_\ell h_{\ell n} = q_{\ell n}$ subject to the two *no-slip conditions $h_{\ell n}(1) = 0$ and $\partial_r(rh_{\ell n})(1) = 0$. The eigenvalues $\lambda^P_{\ell n} = (\alpha^P_{\ell n})^2$ are thus selected by a* determinant *condition (Appendix A).*

Remark 4.2 (SO(3)-equivariance). A with no-slip is $SO(3)$-equivariant; the decomposition into $(2\ell+1)$-dimensional isotypic components yields separation of variables. Toroidal and poloidal types are orthogonal across ℓ, m and across type.

5 Normalization, orthonormality, and completeness

We normalize toroidal radial profiles by

$$\int_0^1 \phi_{\ell n}(r)\,\phi_{\ell n'}(r)\,r^2\,dr = \delta_{nn'}.$$

Poloidal modes are normalized at the *velocity* level: choose the normalization of $h_{\ell n}$ so that $\langle \boldsymbol{P}_{\ell mn}, \boldsymbol{P}_{\ell mn'}\rangle_{L^2(\mathbb{B})} = \delta_{nn'}$.

Proposition 5.1 (Orthonormality). *For all indices,*

$$\langle \boldsymbol{T}_{\ell mn}, \boldsymbol{T}_{\ell' m' n'}\rangle_{L^2(\mathbb{B})} = \delta_{\ell\ell'}\delta_{mm'}\delta_{nn'}, \qquad \langle \boldsymbol{P}_{\ell mn}, \boldsymbol{P}_{\ell' m' n'}\rangle_{L^2(\mathbb{B})} = \delta_{\ell\ell'}\delta_{mm'}\delta_{nn'},$$

and $\langle \boldsymbol{T}_{\ell mn}, \boldsymbol{P}_{\ell' m' n'}\rangle_{L^2(\mathbb{B})} = 0$.

Theorem 5.2 (Stokes spectral basis). *The collection*

$$\mathcal{B} = \{\boldsymbol{T}_{\ell mn} : \ell \geq 1, |m| \leq \ell, n \geq 1\} \;\cup\; \{\boldsymbol{P}_{\ell mn} : \ell \geq 1, |m| \leq \ell, n \geq 1\}$$

is an $L^2(\mathbb{B})$-orthonormal basis of $L^2_\sigma(\mathbb{B})$, with

$$A\boldsymbol{T}_{\ell mn} = \lambda^T_{\ell n}\,\boldsymbol{T}_{\ell mn}, \qquad A\boldsymbol{P}_{\ell mn} = \lambda^P_{\ell n}\,\boldsymbol{P}_{\ell mn}.$$

Hence any $u_0 \in L^2_\sigma(\mathbb{B})$ has the unique DFRT expansion

$$u_0 = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\left(\sum_{n=1}^{\infty}\widehat{u}^T_{\ell mn}\,\boldsymbol{T}_{\ell mn} + \sum_{n=1}^{\infty}\widehat{u}^P_{\ell mn}\,\boldsymbol{P}_{\ell mn}\right) \quad \text{in } L^2(\mathbb{B}).$$

Moreover, $u_0 \in V$ iff $\sum_{\ell,m,n}\left(\lambda^T_{\ell n}|\widehat{u}^T_{\ell mn}|^2 + \lambda^P_{\ell n}|\widehat{u}^P_{\ell mn}|^2\right) < \infty$.

6 DFRT coefficients, inversion, and energy identities

Definition 6.1 (DFRT)**.** For $u \in L^2_\sigma(\mathbb{B})$, define

$$\widehat{u}^T_{\ell mn} := \langle u, \boldsymbol{T}_{\ell mn} \rangle_{L^2(\mathbb{B})}, \qquad \widehat{u}^P_{\ell mn} := \langle u, \boldsymbol{P}_{\ell mn} \rangle_{L^2(\mathbb{B})}.$$

Theorem 6.2 (Plancherel, inversion, Sobolev scale)**.** *For $u \in L^2_\sigma(\mathbb{B})$,*

$$\|u\|^2_{L^2(\mathbb{B})} = \sum_{\ell,m,n} \Big(|\widehat{u}^T_{\ell mn}|^2 + |\widehat{u}^P_{\ell mn}|^2 \Big),$$

and u is recovered by the series in Theorem 5.2. If $u \in V$, then

$$\|\nabla u\|^2_{L^2(\mathbb{B})} = \sum_{\ell,m,n} \Big(\lambda^T_{\ell n} |\widehat{u}^T_{\ell mn}|^2 + \lambda^P_{\ell n} |\widehat{u}^P_{\ell mn}|^2 \Big).$$

More generally, for $s \in [0,1]$,

$$\|u\|^2_{H^s(\mathbb{B})} \simeq \sum_{\ell,m,n} (1+\lambda^T_{\ell n})^s |\widehat{u}^T_{\ell mn}|^2 + \sum_{\ell,m,n} (1+\lambda^P_{\ell n})^s |\widehat{u}^P_{\ell mn}|^2.$$

7 Exact/coexact Hodge split and CFRT pairing

Let $\mathcal{G} := \nabla H^1_0(\mathbb{B})$ (curl-free exact fields) and $\mathcal{S} := L^2_\sigma(\mathbb{B})$ (coexact fields) under the boundary model (2.1). Then:

Theorem 7.1 (Hodge–Helmholtz with boundary alignment)**.** *With $u|_{\partial\mathbb{B}} = 0$,*

$$L^2(\mathbb{B})^3 = \overline{\mathcal{G}}^{L^2} \oplus \mathcal{S}, \qquad \overline{\mathcal{G}}^{L^2} = \overline{\nabla H^1_0(\mathbb{B})}^{L^2}, \quad \mathcal{S} = L^2_\sigma(\mathbb{B}).$$

The CFRT (gradients of Dirichlet scalar eigenmodes) forms an orthonormal basis of $\overline{\mathcal{G}}^{L^2}$; the DFRT basis $\mathcal{B}$ forms an orthonormal basis of $\mathcal{S}$. The orthogonality is given by $\langle \nabla\varphi, w \rangle_{L^2(\mathbb{B})} = 0$ for all $\varphi \in H^1_0(\mathbb{B})$ and $w \in L^2_\sigma(\mathbb{B})$.

Remark 7.2 (Exact/coexact identification for trigraded calculus)**.** In a trigraded discrete exterior calculus on $\mathbb{B}$ (e.g. SGRT), the CFRT spans $\mathrm{im}(d)$ (exact 1-forms) with Dirichlet potential, and the DFRT spans $\mathrm{im}(\delta)$ (coexact 1-forms) with no-slip boundary. Since $H^1(\mathbb{B}) = 0$ under these conditions, $\Omega^{(1)} = \mathrm{im}(d) \oplus \mathrm{im}(\delta)$ is an *orthogonal* split at the continuum limit.

8 Remarks on alternative boundary models

For no-penetration (free slip) $u \cdot n|_{\partial\mathbb{B}} = 0$ with tangential slip, toroidal modes are still tangential and automatically satisfy the normal condition, while poloidal modes reduce to a single Robin-style condition resulting from the vanishing radial component in (3.2). The corresponding solenoidal space and spectral theory are analogous and can be treated in the same framework.

9 Compatibility with SGRT/trigraded calculus (implementation note)

Let Π_k denote the SGRT L^2-projection (tensor-product, rotation-equivariant, and respecting the $s = r^3$ uniformization weights). Then the DFRT/CFRT pair is compatible with the trigraded calculus in the sense that

$$\Pi_k : \overline{\mathcal{G}}^{L^2} \to \overline{\mathcal{G}}^{L^2}, \qquad \Pi_k : \mathcal{S} \to \mathcal{S}, \qquad \langle \Pi_k u, \Pi_k v \rangle_{L^2} \to \langle u, v \rangle_{L^2} \quad (k \to \infty),$$

and the discrete d_k, δ_k induced by SGRT satisfy the commuting-projection relations

$$d\,\Pi_k = \Pi_k^{(1)}\, d, \qquad \delta\,\Pi_k^{(2)} = \Pi_k^{(1)}\, \delta,$$

ensuring the exact/coexact split survives discretization and converges to the continuum Hodge split. (A detailed proof follows from tensor-product stencils and the diagonal metric weights in s, μ, φ; see your SGRT note.)

10 Applications and computational aspects

For incompressible Navier–Stokes with no-slip boundary, the DFRT basis automatically enforces $\operatorname{div} u = 0$ and $u|_{\partial\mathbb{B}} = 0$, enabling spectrally accurate Galerkin schemes. Toroidal/poloidal eigenvalues exhibit quadratic growth in the radial index n at fixed ℓ, $\lambda_{\ell n}^T \sim c\,n^2$ and $\lambda_{\ell n}^P \sim c\,n^2$, and display branch-dependent interlacing patterns due to the distinct boundary selection (single j_ℓ zero vs. a determinant root).

11 Conclusions

We have constructed the DFRT as the coexact Hodge component on $\mathbb{B}$ under no-slip Dirichlet boundary, furnished by orthogonal toroidal and poloidal families. The poloidal branch is governed by a bi-Laplacian reduction with two boundary constraints, leading to a 2×2 determinant condition distinct from the toroidal Bessel-zero rule. The resulting Plancherel, inversion, and energy identities give a turnkey spectral tool for theory and computation, and the exact/coexact identification is aligned with a trigraded (SGRT) calculus and its discrete Hodge operators.

A Poloidal determinant under no-slip

We outline a compact formulation of the determinant selecting poloidal eigenvalues.

Fix $\ell \geq 1$ and write $q = \mathcal{L}_\ell h$. The Stokes eigenproblem reduces to

$$\mathcal{L}_\ell q = \lambda q, \qquad \mathcal{L}_\ell h = q, \qquad h(1) = 0, \quad \partial_r(rh)(1) = 0, \quad h(r) = \mathcal{O}(r^\ell).$$

Let $q_\alpha(r) := j_\ell(\alpha r)$ with $\alpha = \sqrt{\lambda}$. Let G_ℓ denote the regular Green operator for $\mathcal{L}_\ell$ on $(0,1)$ with the origin condition $h(r) = \mathcal{O}(r^\ell)$. Then every regular solution has the form

$$h_\alpha = G_\ell[q_\alpha] + C_\ell\, h^{(0)}, \qquad \mathcal{L}_\ell h^{(0)} = 0, \quad h^{(0)}(r) = r^\ell,$$

for some constant C_ℓ. Enforcing the two no-slip constraints yields a homogeneous 2×2 linear system in the unknowns $(1, C_\ell)$:

$$\begin{pmatrix} \big(G_\ell[q_\alpha]\big)(1) & h^{(0)}(1) \\ \partial_r\big(r\, G_\ell[q_\alpha]\big)(1) & \partial_r\big(rh^{(0)}\big)(1) \end{pmatrix} \begin{pmatrix} 1 \\ C_\ell \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Nontrivial solutions exist iff

$$\Delta_\ell(\alpha) := \det\begin{pmatrix} \big(G_\ell[j_\ell(\alpha\,\cdot)]\big)(1) & 1 \\ \partial_r\big(r\, G_\ell[j_\ell(\alpha\,\cdot)]\big)(1) & \ell+1 \end{pmatrix} = 0. \tag{A.1}$$

Equivalently, expressing G_ℓ by variation of parameters (or integrating factors) yields a closed form in terms of j_ℓ and j_ℓ' at $r=1$, producing an explicit transcendental equation involving $j_\ell(\alpha)$ and $j_\ell'(\alpha)$.[1] The poloidal eigenvalues are the positive roots $\alpha_{\ell n}^P$ of $\Delta_\ell(\alpha) = 0$, and $\lambda_{\ell n}^P = (\alpha_{\ell n}^P)^2$.

[1] For implementation, one may write G_ℓ using the fundamental system of $\mathcal{L}_\ell$ and its Wronskian to obtain $h_\alpha(r) = a(\alpha)\, r^\ell + b(\alpha)\, r^{-\ell-1} + \int_0^r K_\ell(r,\rho)\, j_\ell(\alpha\rho)\, \rho^2\, d\rho$, discarding the singular branch $r^{-\ell-1}$ by regularity. The boundary constraints at $r=1$ then yield an explicit 2×2 determinant in $a(\alpha)$, $b(\alpha)$ that collapses to (A.1) once regularity is enforced.

Remark A.1 (Asymptotics)**.** For fixed ℓ, standard Sturm–Liouville/OLDE methods give $\alpha_{\ell n}^{P} \sim \pi n$ as $n \to \infty$, hence $\lambda_{\ell n}^{P} \sim \pi^2 n^2$. The interlacing with toroidal zeros of j_ℓ depends on ℓ and the boundary coupling.

References

[1] R.G. Barrera, G.A. Estevez, and J. Giraldo. Vector spherical harmonics and their application to magnetostatics. *European Journal of Physics*, 6(4):287–294, 1985.

[2] D.E. Edmunds and W.D. Evans. *Spectral Theory and Differential Operators*. Oxford University Press, 1987.

[3] C. Foias, O. Manley, R. Rosa, and R. Temam. *Navier-Stokes Equations and Turbulence*. Cambridge University Press, 2001.

[4] G.P. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problems*. Springer-Verlag, 2nd edition, 2011.

[5] V. Girault and P.-A. Raviart. *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*. Springer-Verlag, 1986.

[6] O.A. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Flow*. Gordon and Breach, 2nd edition, 1969.

[7] S. Rummler and G. Thäter. The Stokes eigenvalue problem on balls and annuli in three dimensions: Solutions with poloidal and toroidal fields. *arXiv preprint arXiv:2408.06948*, 2024.

[8] C. Schwab. *p- and hp-Finite Element Methods: Theory and Applications in Solid and Fluid Mechanics*. Oxford University Press, 1998.

[9] R. Temam. *Navier-Stokes Equations: Theory and Numerical Analysis*. AMS Chelsea Publishing, 2001.

[10] D.A. Varshalovich, A.N. Moskalev, and V.K. Khersonskii. *Quantum Theory of Angular Momentum*. World Scientific, 1988.