# The Curl-Free Radiant Transform (CFRT) on the Unit Ball: Exact Hodge Component, Boundary Alignment, and SGRT Compatibility

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**Abstract**

We introduce the *Curl-Free Radiant Transform* (CFRT), a unitary spectral transform for irrotational fields on the unit ball $\mathbb{B}^3$, defined by taking gradients of normalized scalar *Dirichlet* Laplacian eigenmodes and rescaling by their eigenfrequencies. The CFRT furnishes an orthonormal basis $\{S_{\ell mn}\}$ for the curl-free (exact) subspace, diagonalizes the vector Laplacian within that subspace, and provides Parseval, inversion, and Sobolev-scale identities. With the Divergence-Free Radiant Transform (DFRT) under *no-slip* boundary conditions, CFRT realizes the exact piece and DFRT the coexact piece of a boundary-aware Hodge–Helmholtz decomposition on $\mathbb{B}^3$. We also record a Neumann variant and outline SGRT (trigraded) compatibility via commuting projections.

**Keywords:** curl-free fields, spherical harmonics, spherical Bessel, Hodge decomposition, spectral transforms, vector Laplacian, Poisson/pressure solves.
**MSC (2020):** 35J05, 35Q30, 41A30, 42C10, 76D05.

## 1 Introduction

Let $\Omega = \mathbb{B}^3 = \{x \in \mathbb{R}^3 : |x| < 1\}$ with $\partial\Omega = \mathbb{S}^2$. The classical Helmholtz–Hodge decomposition splits $L^2(\Omega)^3$ into the $L^2$-closure of gradients and the divergence-free subspace; on the ball there is no harmonic remainder. We construct an *isometric* transform for the gradient (curl-free) component based on the scalar Dirichlet Laplacian basis. The resulting *Curl-Free Radiant Transform* (CFRT) is a full transform theory: forward/inverse maps, Parseval identity, diagonalization of $-\Delta$ in the gradient subspace, and a clean orthogonal pairing with DFRT (no-slip) to produce a boundary-aligned Hodge split.

**Contributions.** (i) CFRT beams $S_{\ell mn} = \alpha_{\ell n}^{-1}\nabla\big(g_{\ell n}(r)Y_\ell^m(\hat{x})\big)$ built from normalized Dirichlet eigenmodes $-\Delta u_{\ell mn} = \alpha_{\ell n}^2 u_{\ell mn}$. (ii) Orthonormality, completeness in the curl-free subspace, and Sobolev mapping. (iii) Vector Laplacian diagonalization: $-\Delta S_{\ell mn} = \alpha_{\ell n}^2 S_{\ell mn}$ in the gradient sector. (iv) Boundary alignment: CFRT is the *exact* piece (Dirichlet potential); DFRT (with no-slip) is the *coexact* piece. Orthogonality follows by a single boundary pairing. (v) SGRT compatibility: commuting projection relations ensure the exact/coexact split persists at the discrete trigraded level.

**Notation.** Write $x = r\hat{x}$ with $r = |x| \in (0,1)$ and $\hat{x} \in \mathbb{S}^2$. Indices: $\ell \in \mathbb{N}_0$, $m = -\ell, \ldots, \ell$, $n \in \mathbb{N}$.

# 2 Preliminaries: harmonics and radial eigenmodes

## 2.1 Spherical harmonics

$\{Y_\ell^m\}_{\ell\geq 0,\,|m|\leq\ell}$ form an orthonormal basis of $L^2(\mathbb{S}^2)$ and satisfy

$$-\Delta_{\mathbb{S}^2} Y_\ell^m = \ell(\ell+1)Y_\ell^m, \qquad \int_{\mathbb{S}^2} \nabla_{\mathbb{S}^2} Y_\ell^m \cdot \nabla_{\mathbb{S}^2} Y_{\ell'}^{m'}\, d\Omega = \ell(\ell+1)\delta_{\ell\ell'}\delta_{mm'}. \tag{2.1}$$

## 2.2 Dirichlet scalar eigenmodes on $\mathbb{B}^3$

Let $\{\alpha_{\ell n}\}_{n\geq 1}$ be the positive zeros of $j_\ell$, $j_\ell(\alpha_{\ell n}) = 0$. Define normalized radial profiles

$$g_{\ell n}(r) := \sqrt{\frac{2}{\big(j_{\ell+1}(\alpha_{\ell n})\big)^2}}\, j_\ell(\alpha_{\ell n} r), \qquad \int_0^1 g_{\ell n} g_{\ell n'}\, r^2 dr = \delta_{nn'}. \tag{2.2}$$

Then $u_{\ell mn}(x) := g_{\ell n}(|x|)\, Y_\ell^m(\hat{x})$ satisfy

$$-\Delta u_{\ell mn} = \alpha_{\ell n}^2 u_{\ell mn}, \qquad u_{\ell mn}|_{\partial\mathbb{B}^3} = 0, \tag{2.3}$$

and $\{u_{\ell mn}\}$ is an orthonormal basis of $L^2(\mathbb{B}^3)$.

# 3 The Curl-Free Radiant Transform (CFRT)

## 3.1 Definition and explicit form

**Definition 3.1** (CFRT beams). *Define*

$$S_{\ell mn}(x) := \frac{1}{\alpha_{\ell n}} \nabla\big(g_{\ell n}(|x|)\, Y_\ell^m(\hat{x})\big), \qquad x \in \mathbb{B}^3. \tag{3.1}$$

*Then* $\operatorname{curl} S_{\ell mn} = 0$ *in* $\mathbb{B}^3$.

**Proposition 3.2** (Explicit formula). *With $r = |x|$ and $\hat{x} = x/|x|$,*

$$S_{\ell mn}(x) = \frac{1}{\alpha_{\ell n}} \left( g'_{\ell n}(r)\, Y_\ell^m(\hat{x})\, \frac{x}{r} + \frac{g_{\ell n}(r)}{r}\, \nabla_{\mathbb{S}^2} Y_\ell^m(\hat{x}) \right). \tag{3.2}$$

**Remark 3.3** (Origin regularity and $\ell = 0$ case). *Since $j_\ell(r) \sim r^\ell$ as $r \to 0$, the angular term behaves like $r^{\ell-1}$ and remains finite for $\ell \geq 1$; for $\ell = 0$, $\nabla_{\mathbb{S}^2} Y_0^0 = 0$ and* (3.2) *reduces to a purely radial field.*

## 3.2 Orthonormality, Parseval, and inversion

**Proposition 3.4** (Orthonormality). *$\{S_{\ell mn}\}$ is orthonormal in $L^2(\mathbb{B}^3)^3$:*

$$\langle S_{\ell mn}, S_{\ell' m' n'}\rangle_{L^2(\mathbb{B}^3)} = \delta_{\ell\ell'}\delta_{mm'}\delta_{nn'}. \tag{3.3}$$

*Proof.* Let $u_{\ell mn} = g_{\ell n} Y_\ell^m$. By Green's identity and Dirichlet BC,

$$\int_{\mathbb{B}^3} \nabla u_{\ell mn} \cdot \nabla u_{\ell' m' n'}\, dx = \alpha_{\ell n}^2\, \delta_{\ell\ell'}\delta_{mm'}\delta_{nn'}.$$

Since $S_{\ell mn} = \alpha_{\ell n}^{-1} \nabla u_{\ell mn}$, divide both sides by $\alpha_{\ell n}\alpha_{\ell' n'}$. $\square$

**Proposition 3.5** (Parseval and inversion)**.** *For any* $v \in L^2(\mathbb{B}^3)^3$ *with* $\operatorname{curl} v = 0$*, coefficients* $b_{\ell mn} := \langle v, S_{\ell mn}\rangle$ *satisfy*

$$\|v\|^2_{L^2(\mathbb{B}^3)} = \sum_{\ell,m,n} |b_{\ell mn}|^2, \qquad v = \sum_{\ell,m,n} b_{\ell mn} S_{\ell mn} \quad \text{in } L^2. \tag{3.4}$$

### 3.3 Completeness and Sobolev mapping

**Theorem 3.6** (Completeness)**.** *Let* $H^s_{\mathrm{curl}=0}(\mathbb{B}^3) := \{v \in H^s(\mathbb{B}^3)^3 : \operatorname{curl} v = 0\}$*. Then* $\{S_{\ell mn}\}$ *is complete in* $H^s_{\mathrm{curl}=0}(\mathbb{B}^3)$ *for* $s \geq 0$*.*

*Proof.* On simply connected $\mathbb{B}^3$, $v \in H^s_{\mathrm{curl}=0}$ implies $v = \nabla\phi$ with $\phi \in H^{s+1}_0(\mathbb{B}^3)$. Expand $\phi = \sum c_{\ell mn} u_{\ell mn}$; by elliptic regularity the series converges in $H^{s+1}$. Then

$$v = \nabla\phi = \sum c_{\ell mn} \nabla u_{\ell mn} = \sum (\alpha_{\ell n} c_{\ell mn})\, S_{\ell mn},$$

converging in $H^s$ since $\nabla : H^{s+1}_0 \to H^s$ is continuous. □

**Proposition 3.7** (Sobolev scale)**.** *For* $s \geq 0$ *and* $v = \sum b_{\ell mn} S_{\ell mn}$*,*

$$\|v\|^2_{H^s(\mathbb{B}^3)} \asymp \sum_{\ell,m,n} (1 + \alpha^2_{\ell n})^s \, |b_{\ell mn}|^2, \tag{3.5}$$

*with constants independent of* $v$*.*

## 4 Vector Laplacian and diagonal PDEs

### 4.1 Diagonalization in the gradient subspace

**Proposition 4.1** (Vector Laplacian on CFRT modes)**.** *Within the gradient sector,*

$$-\Delta S_{\ell mn} = \alpha^2_{\ell n}\, S_{\ell mn}. \tag{4.1}$$

*Proof.* $-\Delta u_{\ell mn} = \alpha^2_{\ell n} u_{\ell mn}$ and $\nabla$ commutes with $\Delta$ on scalars, hence $-\Delta(\alpha^{-1}_{\ell n} \nabla u_{\ell mn}) = \alpha^{-1}_{\ell n} \nabla(-\Delta u_{\ell mn}) = \alpha^2_{\ell n} S_{\ell mn}$. □

### 4.2 Poisson/pressure solves (Dirichlet)

Let $-\Delta\phi = f$ in $\mathbb{B}^3$ with $\phi|_{\partial\mathbb{B}^3} = 0$ and $v = \nabla\phi$. Expanding $f = \sum c_{\ell mn} u_{\ell mn}$,

$$\phi = \sum_{\ell,m,n} \frac{c_{\ell mn}}{\alpha^2_{\ell n}}\, u_{\ell mn}, \qquad v = \nabla\phi = \sum_{\ell,m,n} \frac{c_{\ell mn}}{\alpha_{\ell n}}\, S_{\ell mn}. \tag{4.2}$$

Thus Poisson and pressure solves are diagonal in CFRT coordinates.

### 4.3 Heat evolution in the gradient subspace

For $\partial_t v - \Delta v = 0$ with $v(0)$ curl-free and Dirichlet CFRT,

$$v(t) = \sum_{\ell,m,n} b_{\ell mn}(0)\, e^{-\alpha^2_{\ell n} t}\, S_{\ell mn}. \tag{4.3}$$

# 5 Boundary alignment and Hodge pairing with DFRT

**Theorem 5.1** (Hodge–Helmholtz with boundary alignment). *Let $\mathcal{G} := \overline{\nabla H_0^1(\mathbb{B}^3)}^{L^2}$ (exact fields from Dirichlet potentials) and let $\mathcal{S}$ be the closure in $L^2$ of divergence-free fields with* no-slip *trace $w|_{\partial\mathbb{B}^3} = 0$ (the DFRT space). Then*

$$L^2(\mathbb{B}^3)^3 = \mathcal{G} \ \oplus \ \mathcal{S}, \tag{5.1}$$

*and for $\phi|_{\partial\mathbb{B}^3} = 0$, $w \in \mathcal{S}$,*

$$\int_{\mathbb{B}^3} \nabla\phi \cdot w \, dx = -\int_{\mathbb{B}^3} \phi \, \operatorname{div} w \, dx + \int_{\partial\mathbb{B}^3} \phi \, w \cdot n \, dS = 0. \tag{5.2}$$

*Thus CFRT spans the exact component $\mathcal{G}$ and DFRT spans the coexact component $\mathcal{S}$; no harmonic part occurs on $\mathbb{B}^3$.*

**Remark 5.2** (Neumann CFRT variant). *If one replaces Dirichlet by Neumann scalar modes (zeros $\beta_{\ell n}$ of $j'_\ell$), the beams $S^{(N)}_{\ell mn} := \beta_{\ell n}^{-1} \nabla u^{(N)}_{\ell mn}$ generate curl-free fields tangential at $\partial\mathbb{B}^3$. This variant pairs naturally with a divergence-free space enforcing $w \times n = 0$. In this paper, we adopt the Dirichlet CFRT to match DFRT with no-slip.*

# 6 Transform maps and isometry

**Definition 6.1** (Forward CFRT). *For $v \in L^2(\mathbb{B}^3)^3$ with $\operatorname{curl} v = 0$, set*

$$\mathcal{C}[v] := \{b_{\ell mn}\}_{\ell,m,n}, \qquad b_{\ell mn} := \langle v, S_{\ell mn} \rangle_{L^2(\mathbb{B}^3)}.$$

**Definition 6.2** (Inverse CFRT). *Given $\{b_{\ell mn}\} \in \ell^2$, reconstruct*

$$\mathcal{C}^{-1}[\{b_{\ell mn}\}](x) := \sum_{\ell,m,n} b_{\ell mn} S_{\ell mn}(x),$$

*with convergence in $L^2$ (and in $H^s$ if $\sum (1+\alpha_{\ell n}^2)^s |b_{\ell mn}|^2 < \infty$).*

**Proposition 6.3** (Isometry and uniqueness). *$\mathcal{C} : H^0_{\mathrm{curl}=0} \to \ell^2$ is an isometric isomorphism; more generally $\mathcal{C} : H^s_{\mathrm{curl}=0} \to \ell^2_{(1+\alpha^2)^{s/2}}$ is an isomorphism for any $s \geq 0$.*

**Remark 6.4** (Coefficient relation to scalar potential). *If $\phi = \sum c_{\ell mn} u_{\ell mn}$, then $v = \nabla\phi = \sum b_{\ell mn} S_{\ell mn}$ with $b_{\ell mn} = \alpha_{\ell n} c_{\ell mn}$ and $c_{\ell mn} = b_{\ell mn}/\alpha_{\ell n}$.*

# 7 Compatibility with SGRT / trigraded calculus (implementation note)

Let $\Pi_k$ denote the SGRT $L^2$-projection (tensor-product, rotation-equivariant, respecting $s = r^3$ weights). Then the discrete operators $(d_k, \delta_k)$ induced by SGRT satisfy commuting relations with the CFRT/DFRT projections:

$$d\,\Pi_k = \Pi_k^{(1)}\, d, \qquad \delta\,\Pi_k^{(2)} = \Pi_k^{(1)}\, \delta, \tag{7.1}$$

so the *exact* (CFRT) and *coexact* (DFRT) splits persist under discretization and converge to the continuum Hodge split. In particular, CFRT spans $\operatorname{im}(d)$ (exact 1-forms from Dirichlet potentials) while DFRT spans $\operatorname{im}(\delta)$ under the no-slip model.

## 8 Fast algorithms (outline)

Separation of variables yields a practical CFRT for gridded data:

- **Angular:** spherical harmonic transforms (SHT) to expand scalar data in $Y_\ell^m$.
- **Radial:** projection onto $g_{\ell n}(r)$ for each $\ell$ via precomputed quadrature.
- **Gradient & scaling:** use $S_{\ell mn} = \alpha_{\ell n}^{-1} \nabla u_{\ell mn}$ to map scalar coefficients to CFRT coefficients.
- **Inverse:** synthesize from $b_{\ell mn}$ via (3.1)–(3.2).

For PDEs, operate in coefficient space where operators are diagonal, then invert once.

## A Spherical Bessel facts

- Zeros: $j_\ell(\alpha_{\ell n}) = 0$ with $\alpha_{\ell n} \sim \big(n + \frac{\ell}{2} - \frac{1}{4}\big)\pi$ as $n \to \infty$.
- Normalization: $\displaystyle\int_0^1 j_\ell(\alpha_{\ell n} r)\, j_\ell(\alpha_{\ell n'} r)\, r^2 dr = \frac{1}{2}\, j_{\ell+1}(\alpha_{\ell n})^2\, \delta_{nn'}$.

## B Identities on $\mathbb{S}^2$

$\displaystyle\int_{\mathbb{S}^2} Y_\ell^m \overline{Y_{\ell'}^{m'}}\, d\Omega = \delta_{\ell\ell'}\delta_{mm'}$ and $\displaystyle\int_{\mathbb{S}^2} \nabla_{\mathbb{S}^2} Y_\ell^m \cdot \nabla_{\mathbb{S}^2} Y_{\ell'}^{m'}\, d\Omega = \ell(\ell+1)\delta_{\ell\ell'}\delta_{mm'}$.

## C Green's identity (Dirichlet)

If $u, v \in H_0^1(\mathbb{B}^3)$ with $-\Delta u = \lambda u$ and $-\Delta v = \mu v$, then

$$\int_{\mathbb{B}^3} \nabla u \cdot \nabla v\, dx = \lambda \int_{\mathbb{B}^3} uv\, dx = \mu \int_{\mathbb{B}^3} uv\, dx,$$

hence orthogonality in both $L^2$ and energy inner products for $\lambda \neq \mu$.

## D Indexing conventions

We use $\ell \in \mathbb{N}_0$, $m = -\ell, \ldots, \ell$, $n \in \mathbb{N}$. The multi-index $(\ell, m, n)$ labels $S_{\ell mn}$ with frequency $\alpha_{\ell n}$.

## References


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