Orbiter User's Guide

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Abstract

Orbiter is a program package devoted to the field of Algebraic Combinatorics. Specifically, it is aimed at the problem of classifying combinatorial objects and orbit computations, hence the name. Orbiter provides algorithms for effective handling of finite permutation groups in various actions. This guide is targeted for command line interface usage. Programmers who want to use the Orbiter class library in their own C++ program should consult the programmer's guide.

1 Introduction

Orbiter is a computer algebra system for the classification of combinatorial objects. Orbiter contributes to the knowledge base of combinatorial structures, and to provide useful tools to investigate structures from various points of view, including their symmetry properties. Orbiter is optimized for efficiency in terms of memory and execution speed. Orbiter is a library of C++ classes, together with a command line driven front end. There is no graphical user interface. The system offers two modes of use, programming or command line interface. This manual is about the command line interface. Readers who are interested in the Orbiter C++ class library should consult the programmer's guide. A makefile with all commands used in this guide can be found in the `examples` subdirectory.

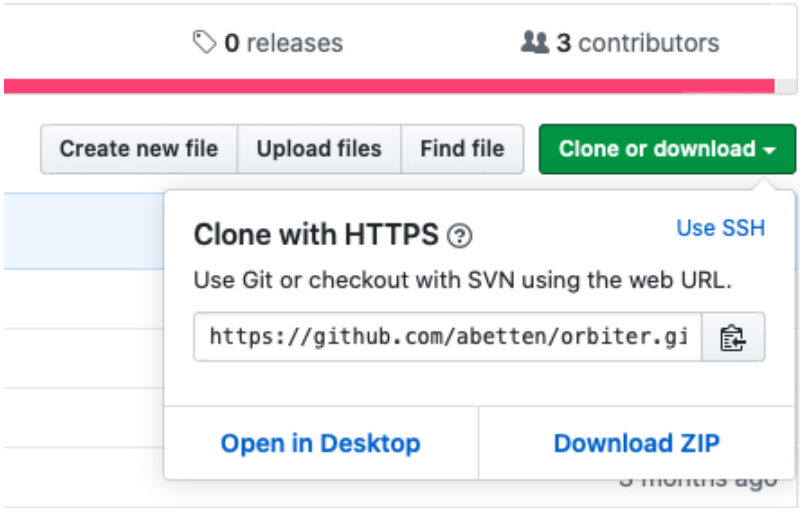


Figure 1: GitHub Clone or Download button

2 Installation

The installation of Orbiter requires the following steps:

(a) Ensure that `git` and the C++ development suite are installed (`gnuc` and `make`). Windows users may have to install `cygwin` (plus the extra packages git, make, gnuc). Macintosh users may have to install the xcode development tools from the appstore (it is free). Linux users may have to install the development packages. Orbiter often produces latex reports. In order to compile these files, make sure you have latex installed

(b) Clone the Orbiter source tree from github (abetten/orbiter). The commands are:

```
git clone <github-orbiter-path>
```

where `<github-orbiter-path>` has to be replaced by the actual address provided by github. To obtain this path, find Orbiter on github, then click on the green box that says "Clone or download" and copy the address into the clipboard by clicking the clipboard symbol (see Figure 1). Back in the terminal, you can paste this text after the `git clone` command.

(c) Issue the following commands to complete the download of submodules:

```
cd orbiter
git submodule init
git submodule update
```

(d) Issue the following commands to compile Orbiter using recursive makefiles:

```
make
make install
```

These two commands compile the Orbiter source tree and copy the executables to the subdirectory bin inside the Orbiter source tree. The orbiter executable is called `orbiter.out`.

3 Finite Fields

Finite fields and projective spaces over finite fields play an important role in Orbiter. The elements of the field $\mathbb{F}_q$ are represented in different ways. Suppose that $q = p^e$ for some prime p and some integer $q \geq 1$. The elements of $\mathbb{F}_q$ are mapped bijectively to the integers in the interval $[0, q-1]$, using the base-p representation. If $e = 1$, the map takes the residue class a mod p with $0 \leq a < p$ to the integer a. Otherwise, we write the field element as

$$\sum_{h=0}^{e-1} a_i \alpha^i$$

where α is the root of some irreducible polynomial $m(X)$ of degree e over $\mathbb{F}_p$ amd $0 \leq a_i < p$ for all i. The associated integer is obtained as

$$\sum_{h=0}^{e-1} a_i p^i.$$

This is the numerical rank of the polynomial. This representation takes 0 in $\mathbb{F}_q$ to the integer 0. Likewise, $1 \in \mathbb{F}_q$ is mapped to the integer 1. Arithmetic is done by considering the polynomials over $\mathbb{F}_p$ and modulo the irreducible polynomial $m(X)$ with root α. For instance, the field $\mathbb{F}_4$ is created using the polynomial $m(X) = X^2 + X + 1$. The elements are

$$0, \quad 1, \quad 2 = \alpha, \quad 3 = \alpha + 1.$$

Orbiter maintains a small database of primitive (irreducible) polynomials for the purposes of creating finite fields. This means that the residue class of α is a primitive element of the field, where α is a root of the polynomial.

The command

```
orbiter.out -cheat_sheet_GF 4
```

creates a report for the field $\mathbb{F}_4$.

Unlike other computer algebra systems (GAP [10] and Magma [7]) Orbiter does not use Conway polynomials. However, Orbiter provides the option to override the polynomial used to create the finite field. For subfield relationships, the cheat sheet for the large field will indicate the irreducible polynmials of the subfield. For instance, Table 1 shows the subfields of $\mathbb{F}_{64}$ generated by the polynomial $X^6 + X^5 + 1$ whose numerical rank is 97.

Subfield	Polynomial	Numerical rank
$\mathbb{F}_4$	$X^2 + X + 1$	7
$\mathbb{F}_8$	$X^3 + X + 1$	11

Table 1: The subfields of $\mathbb{F}_{64}$

4 Finite Projective Spaces

Finite projective spaces and their groups are essential objects in Orbiter. The projective space $\mathrm{PG}(n,q)$ is the set of non-zero subspaces of $\mathbb{F}_q^{n+1}$ ordered with respect to inclusion. The projective dimension of a subspace is always one less than the vector space dimension. So, a projective point is a vector subspace of dimension one. A projective line is a vector subspace of dimension two, etc. A point is written as $P(\mathbf{x})$ for some vector $\mathbf{x} = (x_0, \ldots, x_n)$ with $x_0, \ldots, x_n \in \mathbb{F}_q$, not all zero. For any non-zero element $\lambda \in \mathbb{F}_q$, $P(\lambda\mathbf{x})$ is the same point as $P(\mathbf{x})$. For $\mathbf{a} = (a_0, \ldots, a_n) \in \mathbb{F}_q$, not all zero, the symbol $[\mathbf{a}]$ represents the line

$$\{P(\mathbf{x}) \mid \mathbf{a} \cdot \mathbf{x} = \sum_{i=0}^{n} a_i x_i = 0\}.$$

For any non-zero element $\lambda \in \mathbb{F}_q$, $[\lambda\mathbf{a}] = [\mathbf{a}]$.

The command

```
orbiter.out -cheat_sheet_PG 3 2
```

creates a report for the projective geometry $\mathrm{PG}(3,2)$. Orbiter has enumerators for points and subspaces of $\mathrm{PG}(n,q)$. The point enumerator allows to represent the points using the integer interval $[0, \theta_n(q) - 1]$, where

$$\theta_n(q) = \frac{q^{n+1} - 1}{q - 1}.$$

The points in projective geometry are the one-dimensional subspaces.

The permutation representation of various groups acting on projective spave is based on an enumeration of the elements of $\mathfrak{G}r_1$. By enumerating $\mathfrak{G}r_1$ we mean that we choose a fixed bijection between the set $\mathfrak{G}r_1$ and the integer interval $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$ where $N = \theta_n(q)$. In order to facilitate the bijection, we enumerate generating vectors. The conditions on the vectors are summarized below:

1. The vector is not the zero vector.
2. The rightmost nonzero entry in the vector is one. If it is not, we nomalize the vector so that the rightmost nonzero vector is indeed one. This operation does not change the projective point which is associated with the vector.

The second condition ensures that we list each projective point exactly once. We require two functions, RANK and UNRANK. The function RANK takes a vector $\mathbf{x} \in \mathbb{F}_q^n$, not zero, and maps it to the element in $\mathbb{Z}_N$ representing the projective point $\mathbf{P}(\mathbf{x})$. A frame in $\mathrm{PG}(n,q)$ is a set of $n+2$ points, no $n+1$ in a hyperplane. We assume that the coordinates of a vector are indexed by the elements of $\mathbb{Z}_n$. Also, we let $\mathbf{e}_i$ be the i-th unit vector. A frame for $\mathrm{PG}(n,q)$ is

$$\mathbf{e}_0, \ldots, \mathbf{e}_{n-1}, \mathbf{e}_0 + \cdots + \mathbf{e}_{n-1}.$$

This is the *standard frame.* We start the labeling of points with the standard frame. After these $n+2$ points, we list the remaining points in lexicographic ordering (of the right normalized representative). Thus, for $\mathrm{PG}(2,2)$ the ordering is

$a = \textsc{Rank}(x)$	$\mathbf{x} = \textsc{Unrank}(a)$
0	$(1,0,0)$
1	$(0,1,0)$
2	$(0,0,1)$
3	$(1,1,1)$
4	$(1,1,0)$
5	$(1,0,1)$
6	$(0,1,1)$

Table 2: Representatives of the points of PG$(2,2)$

$$(1,0,0),\ (0,1,0),\ (0,0,1),\ (1,1,1),\ (1,1,0),\ (1,0,1),\ (0,1,1).$$

Let us describe the two functions rank and unrank to perform the actual mappings between PG(n,q) and $\mathbb{Z}_N$, where $N = \theta_n(q)$. For this, assume that ranking and unranking functions have already been defined for the elements of the finite field $\mathbb{F}_q$. Thus, we assume that for $x \in \mathbb{F}_q$, $\textsc{Rank}(\mathbb{F}_q, x)$ is a number b in $\mathbb{Z}_q$. Also, for $b \in \mathbb{Z}_q$, we assume that $\textsc{Unrank}(\mathbb{F}_q, b)$ is the corresponding $x \in \mathbb{F}_q$. So, we assume that Rank and Unrank are mutually inverse functions. Consider the group PGL$(3,2)$ acting on PG$(2,2)$, for instance. The points of

Algorithm 1 Rank

```
1: procedure RANK(vector : x, field : F_q, int : n)
2:     assert x is a nonzero vector in F_q^n.
3:     if x = e_i then
4:         return i
5:     if x = one then
6:         return n
7:     i ← max{j ∈ Z_n | x_j ≠ 0}
8:     x ← (1/x_i) x
9:     a := 0
10:    for j = i − 1, ..., 1, 0 do
11:        a ← a+RANK(F_q, x_j)
12:        if j > 0 then
13:            a ← a · q
14:    if i = n − 1 and a ≥ Σ_{j=0}^{i−1} q^j then
15:        a ← a − 1
16:    a ← a + n − i + Σ_{j=0}^{i−1} q^j
17:    return a
```

PG$(2,2)$ are listed in 2.

Let V be a finite dimensional vector space and let $\mathfrak{G}r_k(V)$ be the Grassmannian of k-dimensional subspaces of V. If $\dim(V) = n$, the notation $\mathfrak{G}r_{n,k}$ is used for $\mathfrak{G}r_k(V)$. The size

Algorithm 2 Unrank

1: **procedure** UNRANK(**int** : a, **field** : $\mathbb{F}_q$, **int** : n)
2: **assert** $a \in \mathbb{Z}_N$ where $N = \theta_{n-1}(q)$.
3: **if** $a < n$ **then**
4: **return** $\mathbf{e}_a$
5: $a \leftarrow a - n$
6: **if** $a = 0$ **then**
7: **return one**
8: $a \leftarrow a - 1$
9: $\mathbf{x} \leftarrow \mathbf{0}$
10: **for** $i = 1, \ldots, n-1$ **do**
11: **if** $a \geq \sum_{j=1}^{i-1} q^j$ **then**
12: $a \leftarrow a - \sum_{j=1}^{i-1} q^j$
13: **else**
14: $x_i \leftarrow 1$
15: **break**
16: **for** $k = i+1, \ldots, n-1$ **do**
17: $x_k \leftarrow 0$
18: $a \leftarrow a + 1$
19: **if** $i = n-1$ and $a \geq \sum_{j=0}^{i-1} q^j$ **then**
20: $a \leftarrow a + 1$
21: $j \leftarrow 0$
22: **while** $a > 0$ **do**
23: $r \leftarrow a \mod q$
24: $x_j \leftarrow$ UNRANK($\mathbb{F}_q, r$)
25: $j \leftarrow j + 1$
26: $a \leftarrow (a - r)/q$
27: **for** $h = j, \ldots, i-1$ **do**
28: $x_h \leftarrow 0$
29: **return** $\mathbf{x}$

$$L_0 = \begin{bmatrix} 100 \\ 010 \end{bmatrix} \qquad L_4 = \begin{bmatrix} 101 \\ 010 \end{bmatrix} \qquad L_8 = \begin{bmatrix} 102 \\ 010 \end{bmatrix} \qquad L_{12} = \begin{bmatrix} 010 \\ 001 \end{bmatrix}$$
$$L_1 = \begin{bmatrix} 100 \\ 011 \end{bmatrix} \qquad L_5 = \begin{bmatrix} 101 \\ 011 \end{bmatrix} \qquad L_9 = \begin{bmatrix} 102 \\ 011 \end{bmatrix}$$
$$L_2 = \begin{bmatrix} 100 \\ 012 \end{bmatrix} \qquad L_6 = \begin{bmatrix} 101 \\ 012 \end{bmatrix} \qquad L_{10} = \begin{bmatrix} 102 \\ 012 \end{bmatrix}$$
$$L_3 = \begin{bmatrix} 100 \\ 001 \end{bmatrix} \qquad L_7 = \begin{bmatrix} 110 \\ 001 \end{bmatrix} \qquad L_{11} = \begin{bmatrix} 120 \\ 001 \end{bmatrix}$$

Table 3: The 13 lines of PG$(2, 3)$

of the set $\mathfrak{G}r_{n,k}$ can be computed as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} \frac{q^{n-i} - 1}{q^{k-i} - 1},$$

using the q-binomial coefficient. A system of representatives of the k-dimensional subspaces is obtained from the $k \times n$ RREF (row-reduced echelon form) matrices of rank k. The elements of PG$(n-1, q)$ are the k-dimensional subspaces of $V = \mathbb{F}_q^n$. It is convenient to identify a subspace with a matrix whose rows contain a basis for it. In coding theory, such a matrix is called a generator matrix. For instance, the 13 lines of PG$(2, 3)$ are shown using 2×3 RREF generator matrices as shown in Table 3.

h	monomial	vector
0	X_0^2	$(2,0,0)$
1	X_1^2	$(0,2,0)$
2	X_2^2	$(0,0,2)$
3	X_0X_1	$(1,1,0)$
4	X_0X_2	$(1,0,1)$
5	X_1X_2	$(0,1,1)$

h	monomial	vector
0	X_0^3	$(3,0,0)$
1	X_1^3	$(0,3,0)$
2	X_2^3	$(0,0,3)$
3	$X_0^2X_1$	$(2,1,0)$
4	$X_0^2X_2$	$(2,0,1)$
5	$X_0X_1^2$	$(1,2,0)$
6	$X_1^2X_2$	$(0,2,1)$
7	$X_0X_2^2$	$(1,0,2)$
8	$X_1X_2^2$	$(0,1,2)$
9	$X_0X_1X_2$	$(1,1,1)$

h	monomial	vector
0	X_0^4	$(4,0,0)$
1	X_1^4	$(0,4,0)$
2	X_2^4	$(0,0,4)$
3	$X_0^3X_1$	$(3,1,0)$
4	$X_0^3X_2$	$(3,0,1)$
5	$X_0X_1^3$	$(1,3,0)$
6	$X_1^3X_2$	$(0,3,1)$
7	$X_0X_2^3$	$(1,0,3)$
8	$X_1X_2^3$	$(0,1,3)$
9	$X_0^2X_1^2$	$(2,2,0)$
10	$X_0^2X_2^2$	$(2,0,2)$
11	$X_1^2X_2^2$	$(0,2,2)$
12	$X_0^2X_1X_2$	$(2,1,1)$
13	$X_0X_1^2X_2$	$(1,2,1)$
14	$X_0X_1X_2^2$	$(1,1,2)$

Table 4: The Orbiter ordering of monomials of degree $2,3$ and 4 in a plane

5 Algebraic Sets

A set of points V in $\mathrm{PG}(n,q)$ is algebraic if there is a set of homogeneous polynomials $p_1,\ldots,p_r$ whose roots over $\mathbb{F}_q$ are the given set. In this case, we write $V=\mathbf{v}(p_1,\ldots,p_r)$. The set V is often called the variety of $p_1,\ldots,p_r$.

Conversely, given a set of points V in $\mathrm{PG}(n,q)$, the ideal $I(V)$ is the set of homogeneous polynomials in $\mathbb{F}_q[X_0,\ldots,X_n]$ which vanish on all of V. This set is an ideal in the polynomial ring. In fact, it is a pricipal ideal, meaning that it is generated by one element only. Orbiter has ways to compute the variety of a polynomial ideal and to compute a generator for the ideal of a set.

Interestingly, in $\mathrm{PG}(n,q)$, every set is algebraic of degree at most $(n+1)(q-1)$ [11]. The associated polynomial is unique and known as the algebraic normal form of the set.

Table 4 shows the Orbiter monomial orderings for degrees $2,3$ and 4 in a plane. Suppose we are interested in $\mathbb{F}_{11}$ rational points of the elliptic curve $y^2=x^3+x+3$. We write $x^3+3-y^2+x=0$. Homogenizing yields $X^3+3Z^3-Y^2Z+XZ=0$. Using X_0,X_1,X_2 instead of X,Y,Z yields

$$X_0^3+3X_2^3+10X_1^2X_2+X_0X_2^2=0.$$

Using the indexing of monomials from Table 4, we record the following pairs (a,i) where a

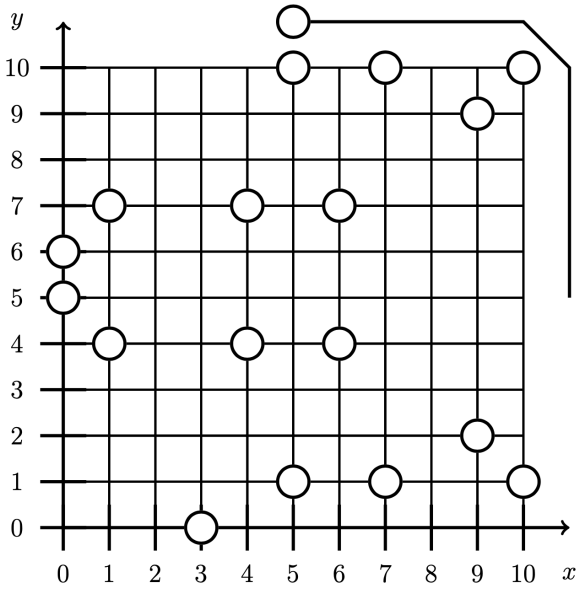


Figure 2: Elliptic curve $y^2 \equiv x^3 + x + 3 \bmod 11$

is the coefficient and i is the index of the monomial

$$(1,0),\ (3,2),\ (10,6),\ (1,7).$$

This is concatenated to the sequence $1, 0, 3, 2, 10, 6, 1, 7$. The Orbiter command

```
orbiter.out -v 2 -create_combinatorial_object -q 11 -n 2 \
    -projective_variety "EC" 3 "1,0,3,2,10,6,1,7"
```

creates the algebraic set associated to the cubic curve $y^2 = x^3 + x + 3$ in $\mathrm{PG}(2, 11)$. It turns out that there are exactly 18 points over $\mathbb{F}_{11}$ (cf. Figure 2).

Table 5 shows the Orbiter monomial orderings for degrees 2 and 3 in $\mathrm{PG}(3, q)$.
The command

```
orbiter.out -v 2 -create_combinatorial_object -n 3 -q 4 \
  -projective_variety "Hirschfeld_surface_q4" 3 "1,6,1,8,1,11,1,13" \
  -monomial_type_PART -end \
  -save ""
```

computes the points of the Hirschfeld surface in $\mathrm{PG}(3, 4)$. The equation

$$X_0^2 X_3 + X_1^2 X_2 + X_1 X_2^2 + X_0 X_3^2 = 0$$

consists of the monomials $6, 8, 11, 13$ in the partition-based numbering of Table 5. Each has a coefficient of one. Therefore, the equation is encoded as $1, 6, 1, 8, 1, 11, 1, 13$.

h	monomial	vector
0	X_0^2	$(2,0,0,0)$
1	X_1^2	$(0,2,0,0)$
2	X_2^2	$(0,0,2,0)$
3	X_3^2	$(0,0,0,2)$
4	X_0X_1	$(1,1,0,0)$
5	X_0X_2	$(1,0,1,0)$
6	X_0X_3	$(1,0,0,1)$
7	X_1X_2	$(0,1,1,0)$
8	X_1X_3	$(0,1,0,1)$
9	X_2X_3	$(0,0,1,1)$

h	monomial	vector
0	X_0^3	$(3,0,0,0)$
1	X_1^3	$(0,3,0,0)$
2	X_2^3	$(0,0,3,0)$
3	X_3^3	$(0,0,0,3)$
4	$X_0^2X_1$	$(2,1,0,0)$
5	$X_0^2X_2$	$(2,0,1,0)$
6	$X_0^2X_3$	$(2,0,0,1)$
7	$X_0X_1^2$	$(1,2,0,0)$
8	$X_1^2X_2$	$(0,2,1,0)$
9	$X_1^2X_3$	$(0,2,0,1)$
10	$X_0X_2^2$	$(1,0,2,0)$
11	$X_1X_2^2$	$(0,1,2,0)$
12	$X_2^2X_3$	$(0,0,2,1)$
13	$X_0X_3^2$	$(1,0,0,2)$
14	$X_1X_3^2$	$(0,1,0,2)$
15	$X_2X_3^2$	$(0,0,1,2)$
16	$X_0X_1X_2$	$(1,1,1,0)$
17	$X_0X_1X_3$	$(1,1,0,1)$
18	$X_0X_2X_3$	$(1,0,1,1)$
19	$X_1X_2X_3$	$(0,1,1,1)$

Table 5: The Orbiter ordering of monomials of degree 2 and 3 in PG$(3,q)$

Type	Perm. Domain	Degree
General linear $\mathrm{GL}(n,q)$	all vectors of V	q^n
Affine $\mathrm{AGL}(n,q)$	all vectors of V	q^n
Projective $\mathrm{PGL}(n,q)$	$\mathfrak{G}r_1(V)$	$\frac{q^n-1}{q-1}$
Wreath product $\mathrm{GL}(d,q) \wr \mathrm{Sym}(n)$	$\mathfrak{G}r_1((\mathbb{F}_q^d)^{\otimes n})$ extended	$n + nq^d + \frac{q^{d^n}-1}{q-1}$

Table 6: Basic actions

6 Linear Groups

Orbiter provides support for the classical groups of matrices over finite fields (cf. [21]). Any group is associated with a permutation action. There can be multiple actions for the same group though. New actions can be formed from old actions. Basic group actions are projective, affine, and general linear, as well as orthogonal, unitary and tensor product. Product actions can be defined also. In order to establish a permutation representation, the elements (aka points) of the permutation domain need to be made available. One way would be to make a table of all elements in the permutation domain. However, this would be time and memory intensive. For this reason, a different technique is used that creates points only when needed. The way this works is that the permutation domain is encoded implicitly, using a fixed bijection to a suitable integer interval (zero based), called the domain. Whenever we want the ith point in the domain, we can call a function that produces it. Conversely, whenever we have a point, we can call a function that tells us what the associated index in the domain. This is facilitated by two mutually inverse functions. The rank function turns a point into an index. The unrank function turns an index in the domain into a point. This way, we avoid storing tables and we make it easier for the computer to store points (namely by storing the associated indices in the domain instead).

Let $V \simeq \mathbb{F}_q^n$ be a finite dimensional vector space over $\mathbb{F}_q$. A group G can act on V in one of the types listed in Table 6. The elements of finite fields are represented as integers as described in Section 3. The elements of the various sets on which the group acts are encoded as integers. For instance,

```
orbiter.out -linear_group -PGL 4 2 -end
```

creates the group $\mathrm{PGL}(4,2)$ acting on the 15 elements of $\mathfrak{G}r_1(\mathbb{F}_2^4)$. The basic types of groups are listed in Table 7.

For instance,

```
orbiter.out -v 3 -linear_group -PGGL 3 4 -end
```

creates $\mathrm{P\Gamma L}(3,4)$ acting on the 21 points of $\mathrm{PG}(2,4)$.

The command

```
orbiter.out -v 3 -linear_group -GL_d_q_wr_Sym_n 2 2 4 -end
```

Command	Arguments	Group
`-GL`	$n\ q$	$\mathrm{GL}(n,q)$
`-GGL`	$n\ q$	$\Gamma\mathrm{L}(n,q)$
`-SL`	$n\ q$	$\mathrm{SL}(n,q)$
`-SSL`	$n\ q$	$\Sigma\mathrm{L}(n,q)$
`-PGL`	$n\ q$	$\mathrm{PGL}(n,q)$
`-PGGL`	$n\ q$	$\mathrm{P}\Gamma\mathrm{L}(n,q)$
`-PSL`	$n\ q$	$\mathrm{PSL}(n,q)$
`-PSSL`	$n\ q$	$\mathrm{P}\Sigma\mathrm{L}(n,q)$
`-AGL`	$n\ q$	$\mathrm{AGL}(n,q)$
`-AGGL`	$n\ q$	$\mathrm{A}\Gamma\mathrm{L}(n,q)$
`-ASL`	$n\ q$	$\mathrm{ASL}(n,q)$
`-ASSL`	$n\ q$	$\mathrm{A}\Sigma\mathrm{L}(n,q)$
`-GL_d_q_wr_Sym_n`	$d\ q\ n$	$\mathrm{GL}(d,q) \wr \mathrm{Sym}(n)$

Table 7: Basic types of Orbiter matrix groups

creates the group $\mathrm{GL}(2,2) \wr \mathrm{Sym}(4)$ acting on $\mathrm{PG}(15,2)$ extended by a set of 20 extra points. The extra points are associated with the actions of the components of the wreath product: Four points form a permutation domain for the permutation part Sym(4). An additional $16 = 4 \times 4$ points form four permutation domains of $\mathrm{GL}(2,2)$, one for each factor.

A collineation of a projective space π is a bijective mapping from the points of π to themselves which preserves collinearity. That is, a collineation φ maps any three collinear points P, Q, R to another collinear triple $\varphi(P), \varphi(Q), \varphi(R)$. The collineations form a group with respect to composition, the collineation group. If M is the matrix of an endomorphism, then Ψ_M is the induced map on projective space. By considering the homomorphism $M \mapsto \Psi_M$, the group $\mathrm{GL}(n+1,q)$ of invertible endomorphisms becomes a subgroup of the group of collineations of $\mathrm{PG}(n,q)$. This is the projectivity group $\mathrm{PGL}(n+1,q)$. It is isomorphic to $\mathrm{GL}(n+1,q)/\mathbb{F}_q^\times$. Another source of collineations is this: Let $\Phi \in \mathrm{Aut}(\mathbb{F}_q)$ be a field automorphism. Then Φ acts on projective space by sending $P(\mathbf{x})$ to $P(\mathbf{x}\Phi)$. This map is another type of collineation, called automorphic collineation. This way, $\mathrm{Aut}(\mathbb{F}_q)$ can be considered another subgroup of the group of collineations. If $q = p^h$ for some prime p and some integer h then

$$\Phi_0 : \mathbb{F}_q \to \mathbb{F}_q,\ x \mapsto x^p$$

is a generator for the cyclic group $C_h \simeq \mathrm{Aut}(\mathbb{F}_q)$. The collineation group of $\mathrm{PG}(n,q)$ $(n \geq 2)$ is isomorphic to the semidirect product of the projectivity group and the automorphism group of the field. The collineation group is $\mathrm{P}\Gamma\mathrm{L}(n+1,q) = \mathrm{PGL}(n+1,q) \ltimes \mathrm{Aut}(\mathbb{F}_q)$. We use the following notation for elements of $\mathrm{P}\Gamma\mathrm{L}(n+1,q)$. Let Φ_0 be a generator for $\mathrm{Aut}(\mathbb{F}_q)$ and let $M \in \mathrm{GL}(n+1,q)$. The map

$$(\Psi_M, \Phi_0^k) : \mathrm{PG}(n,q) \to \mathrm{PG}(n,q),\ P(\mathbf{x}) \mapsto P(\mathbf{y}), \quad \mathbf{y} = (\mathbf{x} \cdot M)^{\Phi_0^k}$$

is denoted as

$$M_k. \tag{1}$$

The identity element is I_0, where I is the identity matrix and 0 is the residue class modulo h. The rules for multiplication and inversion in the collineation group are given as

$$M_k \cdot N_l = \left(M \cdot N^{\Phi^{-k}}\right)_{k+l}, \tag{2}$$

$$\left(M_k\right)^{-1} = \left(\left(M^{-1}\right)^{\Phi^k}\right)_{-k}. \tag{3}$$

The affine group $\mathrm{AGL}(n,q)$ is the semidirect product of $\mathrm{GL}(n,q)$ with $\mathbb{F}_q^n$. The affine semilinear group $\mathrm{A\Gamma L}(n,q)$ is the semidirect product of $\mathrm{AGL}(n,q)$ with $\mathrm{Aut}(\mathbb{F}_q)$. The elements of $\mathrm{A\Gamma L}(n,q)$ are triples

$$M_{\mathbf{a},k} := (M, \mathbf{a}, k) \in \mathrm{GL}(n,q) \times \mathbb{F}_q^n \times \mathrm{Aut}(\mathbb{F}_q),$$

which act on $\mathbb{F}_q^n$:

$$\left(\mathbf{x}, (M, \mathbf{a}, k)\right) \mapsto \left(\mathbf{x} \cdot M + \mathbf{a}\right)^{\Phi^k}.$$

The multiplication in $\mathrm{A\Gamma L}(n,q)$ is

$$M_{\mathbf{a},k} \cdot N_{\mathbf{b},l} = (MN)_{\mathbf{a}N^{\Phi^{-k}} + \mathbf{b}^{\Phi^{-k}}, k+l}.$$

The inverse of an element is

$$\left(M_{\mathbf{a},k}\right)^{-1} = \left(M^{-1}\right)_{\mathbf{a}^{\Phi^k} M^{-1}, -k}.$$

A correlation is a one-to-one mapping between the set of points and the set of hyperplanes which reverses incidence. So, if ρ is a correlation and P is a point and ℓ is a hyperplane then P^ρ is a hyperplane and ℓ^ρ is a point and

$$\ell^\rho \in P^\rho \iff P \in \ell.$$

A correlation of order two is called polarity. The standard polarity is the map

$$\rho : \mathcal{P} \leftrightarrow \mathcal{L},\ P(\mathbf{x}) \leftrightarrow [\mathbf{x}].$$

There are many ways to create subgroups of a group. Table 8 lists some commands to do so. For instance, the command

```
orbiter.out -v 3 -linear_group -PGL 7 11 -Janko1 -end
```

creates the first Janko group as a subgroup of $\mathrm{PGL}(7,11)$.

The command

```
orbiter.out -v 3 -linear_group -PGL 3 11 -singer 1 -end
```

Modifier	Arguments	Meaning
`-Janko1`		first Janko group, needs PGL$(7, 11)$
`-monomial`		subgroup of monomial matrices
`-diagonal`		subgroup of diagonal matrices
`-null_polarity_group`		null polarity group
`-symplectic_group`		symplectic group
`-singer`	k	subgroup of index k in the Singer cycle
`-singer_and_frobenius`	k	subgroup of index k in the Singer cycle, extended by the Frobenius automorphism of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$
`-borel_upper`		Borel subgroup of upper triangular matrices
`-borel_lower`		Borel subgroup of lower triangular matrices
`-identity_group`		identity subgroup
`-subgroup_from_file`	f l	read subgroup from file f and give it the label l
`-orthogonal`	ϵ	orthogonal group $O^\epsilon(n, q)$, with $\epsilon \in \{\pm 1\}$ when n is even
`-subgroup_by_generators`	l o n s_1 ... s_n	Generate a subgroup from generators. The label "l" is used to denote the subgroup; o is the order of the subgroup; n is the number of generators and s_1, ..., s_n are the generators for the subgroup in string representation.

Table 8: Commands for creating subgroups

Modifier	Arguments	Meaning
`-wedge`		action on the exterior square
`-PGL2OnConic`		induced action of $\mathrm{PGL}(2,q)$ on the conic in the plane $\mathrm{PG}(2,q)$
`-subfield_ structure_action`	s	action by field reduction to the subfield of index s
`-on_k_subspaces`	k	induced action on k dimensional subspaces
`-on_tensors`		induced action of $\mathrm{GL}(d,q) \wr \mathrm{Sym}(n)$ on the tensor space
`-on_rank_one_ tensors`		induced action of $\mathrm{GL}(d,q) \wr \mathrm{Sym}(n)$ on the tensor space
`-restricted_action`	s	restricted action on the set s

Table 9: Commands for creating induced or restricted group actions

can be used to create the Singer cycle. The Singer cycle in $\mathrm{GL}(d,q)$ is a generator for a subgroup of order $q^d - 1$. It induces an element of order $\frac{q^d-1}{q-1}$ on the associated projective geometry $\mathrm{PG}(d-1,q)$. The additional integer parameter k after the `-singer` command can be used to create the subgroup of index k of the Singer cycle.

The command

```
orbiter.out -v 3 -linear_group -PGL 3 11 -singer_and_frobenius 19 -end
```

can be used to create a subgroup of index 19 of the Singer cycle of $\mathrm{PG}(2,11)$, extended by a group of order 3 that arises from the field extension $\mathbb{F}_{11}^3$ over $\mathbb{F}_{11}$. The group created by this command has order 21.

It is possible to create new group actions from old. Table 9 lists some commmands to do so. For instance, the command

```
orbiter.out -v 4 \
  -linear_group -GL_d_q_wr_Sym_n 2 2 4 -on_tensors -end
```

creates the group $\mathrm{GL}(2,2) \wr \mathrm{Sym}(4)$ acting on the 65535 elements of $\mathrm{PG}(15,2)$. By restricting the action to the points of $\mathrm{PG}(15,2)$, the auxilliary points in the permutation domain are hidden.

It is possible to perform group theoretic tasks using the `-group_theoretic_activities` option. Tables 10-12 list the possible commands.

The command

```
orbiter.out -v 5 -linear_group -PGL 3 4  -end \
  -group_theoretic_activities -find_singer_cycle
```

Modifier	Arguments	Meaning
-orbits_on_subsets	k	Compute orbits on k-subsets.
-orbits_on_points		Compute orbits in the action that was created.
-orbits_of	i	Compute orbit of point i in the action that was created.
-stabilizer		Compute the stabilizer of the orbit representative (needs -orbits_on_points).
-draw_poset		Draw the poset of orbits (needs -orbits_on_subsets).
-classes		Compute a report of the conjugacy classes of elements (this command relies on Magma [7]).
-group_table		Stores the group table as csv-file.
-centralizer_of_element	label coding	Compute the centralizer of the coded group element, using label to create file names (this command relies on Magma [7]).
-normalizer		Compute the normalizer (this command relies on Magma [7]; needs a group with a subgroup).
-report		Produce a latex report about the group.
-sylow		Include Sylow subgroups in the report (needs -report).
-print_elements		Produce a printout of all group elements.
-print_elements_tex		Produce a latex report of all group elements.
-orbits_on_set_system_from_file	fname f l	reads the csv file "fname" and extract sets from columns $[f, ..., f + l - 1]$.
-orbit_of_set_from_file	fname	reads a set from the text file "fname" and computes orbits on the elements of the set.
-order_of_products	$g_1 \dots g_n$	Creates a table of the orders of all products $g_i g_j$, $1 \le i, j \le n$.
-multiply	s_1 s_2	Creates group elements from s_1 and s_2 and multiplies.
-inverse	s	Creates a group element from s and computes its inverse.
-search_element_of_order	s	Finds all elements of order s in the group.
-find_singer_cycle		Finds all Singer cycles whose matrix is a companion matrix.

Table 10: Group theoretic activities (Part 1)

Modifier	Arguments	Meaning
`-classify_arcs`	s r	Classify (s,r)-arcs in $\mathrm{PG}(2,q)$. If $r=0$, classify all subsets of size s in $\mathrm{PG}(2,q)$. See Section 11.
`-classify_nonconical_arcs`	s r	Classify nonconical (s,r)-arcs in $\mathrm{PG}(2,q)$. If $r=0$, classify all subsets of size s in $\mathrm{PG}(2,q)$. See Section 11.
`-linear_codes`	d t	Classify linear codes with redundancy r, minimum distance at least d and length at most n using an r-dimensional linear group. See Section 14.
`-surface_classify`		Classify cubic surfaces. The group must be $\mathrm{P\Gamma L}(4,q)$ (or any subgroup) in the wedge action. See Section 12.
`-surface_report`		Produce a report (needs `surface_classify`). See Section 12.
`-surface_identify_Sa`		Identifies the isomorphism type of the Hilbert Cohn-Vossen surface $\mathcal{S}_a$. See Section 12.
`-surface_isomorphism_testing`	surface-descr-1 surface-descr-2	Computes an isomorphism between two given surfaces or concludes that none exists. See Section 12.
`-surface_recognize`	surface-descr	Identifies the isomorphism type of the given surface. See Section 12.
`-classify_surfaces_through_arcs_and_trihedral_pairs`		Classifies the surfaces using the associated arcs. The group must be $\mathrm{P\Gamma L}(4,q)$ (or any subgroup) in the standard action. See Section 12.
`-create_surface`	surface-descr	Creates a surface from a description. See Section 12.
`-spread_classify`	k	Classifies spreads of $\mathrm{PG}(k-1,q)$ in $\mathrm{PG}(n-1,q)$. The group must be $\mathrm{P\Gamma L}(n,q)$ of any subgroup. See Section 18.
`-packing_classify`	k spread-iso path	Classifies packings in $\mathrm{PG}(3,q)$ consisting of spreads whose isomorphism type belongs to the given list. The spreads are stored in files with prefix `path`. See Section 18.

Table 11: Group theoretic activities (Part 2)

Modifier	Arguments	Meaning
`-packing_with_ assumed_symmetry`	description	Classifies packings in PG$(3,q)$ consisting of spreads whose isomorphism type belongs to the given list. A group of symmetries H is assumed. The normalizer N of H is used to classify the packings. See Section 18.
`-tensor_classify`	d	Classify tensors of tensor-rank at most d.
`-tensor_ permutations`		Compute the permutation representation of generators of wreath product.

Table 12: Group theoretic activities (Part 3)

finds all Singer cycles in PGL$(3,4)$ whose matrix is in the companion matrix form. The first one found is the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

whose projective order is 21. Here, we are using the numeric form of field elements, so 2 is ω and 3 is $\omega + 1$.

The command

```
orbiter.out -v 4 \
  -linear_group -GL_d_q_wr_Sym_n 2 2 4 -on_tensors -end \
  -group_theoretic_activities \
  -orbits_on_points
pdflatex GL_2_2_wreath_Sym4_res65535_orbits.tex
open GL_2_2_wreath_Sym4_res65535_orbits.pdf
```

create the group GL$(2,2) \wr$Sym(4) acting on the 65535 elements of PG$(15,2)$. The command also computes the orbit on points. The latex file `GL_2_2_wreath_Sym4_res65535_orbits.tex` is created. It contains a list of the generators of the group and the orbits.

The quaternion group is a group of order 8 generated by the following matrices over $\mathbb{R}$:

$$i = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad j = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad k = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

It is isomorphic to a subgroup of SL$(2,3)$. To perform the embedding, we need to replace the real number -1 by the corresponding field element in $\mathbb{F}_3$. Recalling the convention from Section 3 that Orbiter field elements are integers in the interval $[0,\ldots,q-1]$, we replace -1 by 2. The Orbiter command

```
orbiter.out -v 3 -linear_group -SL 2 3 \
    -subgroup_by_generators "quaternion" "8" 3 \
    "1,1,1,2" \
    "2,1,1,1" \
    "0,2,1,0" \
    -end \
    -group_theoretic_activities  \
    -print_elements_tex \
    -group_table \
    -report \
-end
```

creates the group. The command produces a list of group elements.

The group of the cube can be created over the field $\mathbb{F}_3$ like so:

```
orbiter.out -v 3 -linear_group -GL 3 3 \
  -subgroup_by_generators "cube" "48" 3 \
    "0,1,0,2,0,0,0,0,1" \
    "0,0,1,0,1,0,2,0,0" \
    "2,0,0,0,1,0,0,0,1" \
    -end \
    -group_theoretic_activities \
    -print_elements_tex \
    -report
```

The tetrahedral subgroup can be created like so:

```
orbiter.out -v 3 -linear_group -GL 3 3 \
    -subgroup_by_generators "tetra" "12" 2 \
    "0,1,0,0,0,1,1,0,0" \
    "0,0,1,2,0,0,0,2,0" \
    -end \
    -group_theoretic_activities \
    -print_elements_tex \
    -report
```

Sometimes, the generators depend on specific choices made for the finite field. For instance, if the field if a true extension field over its prime field, the choice of the polynomial matters. This is particularly relevant if generators are taken from other sources. For instance, the electronic Atlas of finite simple groups [22] lists generators for $U_3(3)$ as 3×3 matrices over the field $\mathbb{F}_9$ using the following short Magma [7] program:

```
F<w>:=GF(9);
x:=CambridgeMatrix(1,F,3,[
"164",
```

```
"506",
"851"]);
y:=CambridgeMatrix(1,F,3,[
"621",
"784",
"066"]);
G<x,y>:=MatrixGroup<3,F|x,y>;
```

The generators are given using the Magma command `CambridgeMatrix`, which allows for more efficient coding of field elements. The field elements are coded as base-3 integers (like in Orbiter) with respect to the Magma version of $\mathbb{F}_9$. Magma uses Conway polynomials to generate finite fields which are not of prime order. The Conway polynomial for $\mathbb{F}_9$ can be determined using the following Magma command (which can be typed into the Magma online calculator at [20])

```
F<w>:=GF(9);
print DefiningPolynomial(F);
```

which results in

```
$.1^2 + 2*$.1 + 2
```

which is the Magma way of printing the polynomial $X^2 + 2X + 2$. To have Orbiter use this polynomial, the `-override_polynomial` option can be used. First, the polynomial is identified with the vector of coefficients $(1, 2, 2)$ which is then read as base-3 representation of an integer as

$$(1, 2, 2) = 1 \cdot 3^2 + 2 \cdot 3 + 2 = 17.$$

The Orbiter command

```
orbiter.out -v 3 -linear_group -override_polynomial "17" -PGL 3 9 \
    -subgroup_by_generators "U_3_3" "6048" 2 \
    "1,6,4, 5,0,6, 8,5,1" \
    "6,2,1, 7,8,4, 0,6,6" \
    -end  \
-group_theoretic_activities \
    -report -end
```

can then be used to create the group. Notice how the generators are encoded almost like in the Magma command, except that commas are used to separate entries.

Through its interface to Magma [7], Orbiter can perform group theoretic computations. For instance, the command sequence

```
orbiter.out -v 6 \
  -linear_group -PGGL 4 4 -end \
  -group_theoretic_activities \
  -classes
```

```
/usr/local/magma/magma PGGL_4_4_classes.magma
orbiter.out -v 6 \
  -linear_group -PGGL 4 4 -end \
  -group_theoretic_activities \
  -classes
pdflatex PGGL_4_4_classes_out.tex
open PGGL_4_4_classes_out.pdf
```

computes the classes of elements in $\mathrm{P\Gamma L}(4,4)$ together with the normalizers of the associated cyclic subgroups generated by the class representatives. Note that the orbiter command is repeated twice, with one Magma command in between. Upon the first invocation, the file `PGGL_4_4_classes.magma` is written. The magma command produces the file `PGGL_4_4_classes_out.txt`, which is used during the second invocation of the Orbiter command to create the latex file containing the conjugacy classes in the group. This command sequence may have to be changed depending on the location of magma on the system. Also, the open command to display a pdf file is Macintosh specific.

The command sequence

```
orbiter.out -v 6 \
  -linear_group -PGGL 4 4 -end \
  -group_theoretic_activities \
  -centralizer_of_element "2A" "1,0,0,0, 0,1,0,0, 0,0,1,0, 0,0,0,1, 1"
/usr/local/magma/magma element_2A_centralizer.magma
orbiter.out -v 6 \
  -linear_group -PGGL 4 4 -end \
  -group_theoretic_activities \
  -centralizer_of_element "2A" "1,0,0,0, 0,1,0,0, 0,0,1,0, 0,0,0,1, 1"
```

computes the centralizer of the element

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_1$$

which is a Baer involution. The command finds the centralizer to be a group of order 40320.

Orbiter produces a list of generators in coded form, shown below:

$$
\begin{array}{c}
1,0,0,0,0,1,0,0,0,0,1,0,0,0,0,1,1,\\
1,0,0,0,0,1,0,0,0,0,1,0,1,1,0,1,1,\\
1,0,0,0,0,1,0,0,0,0,1,0,0,1,1,1,1,\\
1,0,0,0,0,1,0,0,0,0,1,0,1,1,1,1,0,\\
1,0,0,0,0,1,0,0,1,1,0,1,1,1,1,0,0,\\
1,0,0,0,0,1,0,0,1,0,1,0,0,1,0,1,1,\\
1,0,0,0,0,1,0,0,0,0,1,1,1,0,1,0,0,\\
1,0,0,0,1,1,1,1,1,0,1,0,1,1,1,0,0,\\
1,0,0,0,0,1,1,0,0,0,1,1,1,0,1,0,0,\\
0,1,0,0,0,1,0,1,1,1,0,1,0,1,1,1,1,
\end{array}
$$

The group is isomorphic to $\mathrm{PGL}(4,2).Z_2$, though this structure description cannot be obtained from Orbiter. The generators computed previously can be used to recreate the group using the following command:

```
orbiter.out -v 6 \
  -linear_group -PGGL 4 4 \
  -subgroup_by_generators "centralizer_2A" "40320" 10 \
    "1,0,0,0,0,1,0,0,0,0,1,0,0,0,0,1,1," \
    "1,0,0,0,0,1,0,0,0,0,1,0,1,1,0,1,1," \
    "1,0,0,0,0,1,0,0,0,0,1,0,0,1,1,1,1," \
    "1,0,0,0,0,1,0,0,0,0,1,0,1,1,1,1,0," \
    "1,0,0,0,0,1,0,0,1,1,0,1,1,1,1,0,0," \
    "1,0,0,0,0,1,0,0,1,0,1,0,0,1,0,1,1," \
    "1,0,0,0,0,1,0,0,0,0,1,1,1,0,1,0,0," \
    "1,0,0,0,1,1,1,1,1,0,1,0,1,1,1,0,0," \
    "1,0,0,0,0,1,1,0,0,0,1,1,1,0,1,0,0," \
    "0,1,0,0,0,1,0,1,1,1,0,1,0,1,1,1,1," \
  -end
```

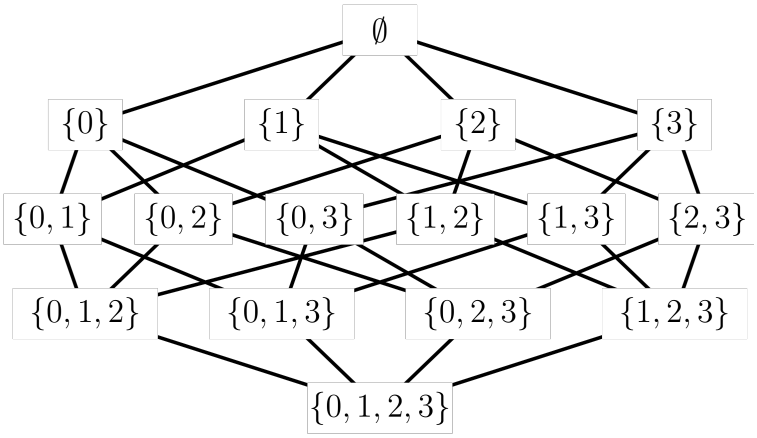


Figure 3: The lattice of subsets of a 4-element set

7 Orbits on Subsets

The lattice of subsets of a set X is $\mathfrak{P}(X)$, the set of all subsets of X, ordered with respect to inclusion. For instance, Figure 3 shows the lattice of subsets of a 4-element set. Assume that a group G acts on X, and hence on the lattice by means of the induced action on subsets. The orbits of G on subsets clump together nodes in the lattice. The set of G-orbits form a new poset, the poset of orbits. Poset classification is the process of computing the poset of orbits. Orbiter has an algorithm to perform poset classification. In many cases, we are not interested in the full lattice of subsets $\mathfrak{P}(X)$ but rather in a subposet of it. We require that the subposet is closed under the group action and that the following property holds:

$$x, y \in \mathfrak{P}(X) \text{ and } x \leq y \;\Rightarrow\; \Big(y \in \mathcal{P} \rightarrow x \in \mathcal{P}\Big).$$

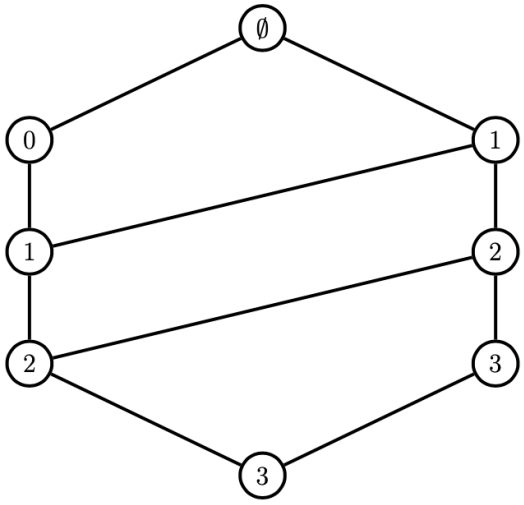
The join of two subsets in the poset may or may not belong to the poset. Let us consider the poset of subsets of the 4-element set under the action of a group of order 3. We take the 4 points to be the vectors of $X = \mathbb{F}_2^2$. Let G be the group generated by the Singer cycle in $\mathrm{GL}(2, 2)$, so

$$G = \langle \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \rangle \simeq \langle (0)(1, 3, 2) \rangle,$$

the latter being the permutation representation on the set X. Thus, G is a group of order 3 acting with one fixed point. The command

```
orbiter.out -v 3 -linear_group -GL 2 2 -singer 1 -end  \
    -group_theoretic_activities \
    -orbits_on_subsets 4 \
    -draw_poset \
    -report
```

computes the orbits of G on the poset of subsets. The poset of orbits is shown in Figure 4. All nodes except for the root node are labeled by elements of $X = \{0, 1, 2, 3\}$. In order to



$\{\}_3$
$\{0\}_3$
$\{1\}_1$
$\{0,1\}_1$
$\{1,2\}_1$
$\{0,1,2\}_1$
$\{1,2,3\}_3$
$\{0,1,2,3\}_3$

Figure 4: The poset of orbits under the Singer group

determine the set that is associated to a node, we follow the unique leftmost path to the root node and collect the node lables. This produces the set associated to the node. The orbit representatives are indicated next to the diagram. By convention, the order of the stabilizer is written as a subscript. Since this example is small enough, it is possible to show the complete orbits, as in Figure 5. Elements belonging to an orbit are grouped together. The leftmost element in each group is the orbit representative.

The lex-least spanning tree of a poset is obtained by joining each non-root node to its unique lex-least ancestor. Figure 6 shows the spanning tree for the poset of subsets of a 4 element set.

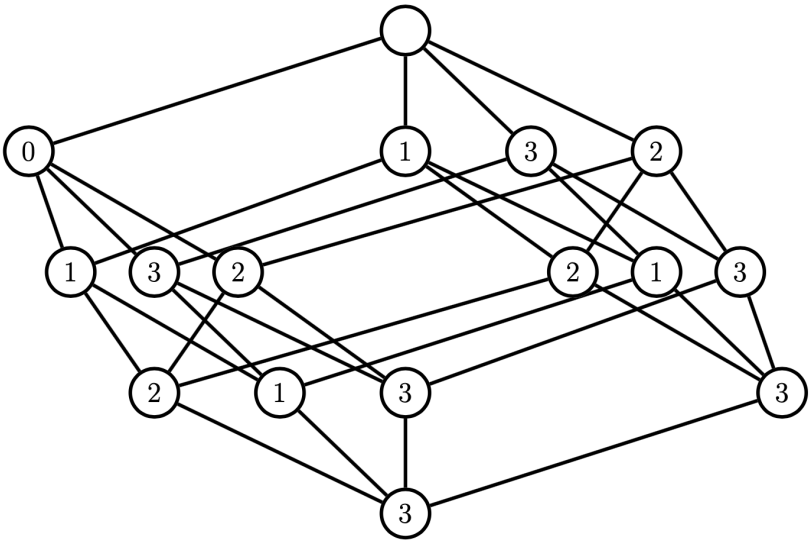


Figure 5: The poset with orbits indicated by grouping

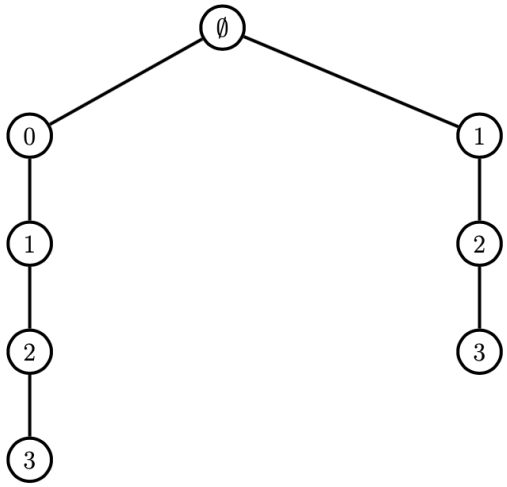


Figure 6: The lex-least spanning tree for the poset of orbits

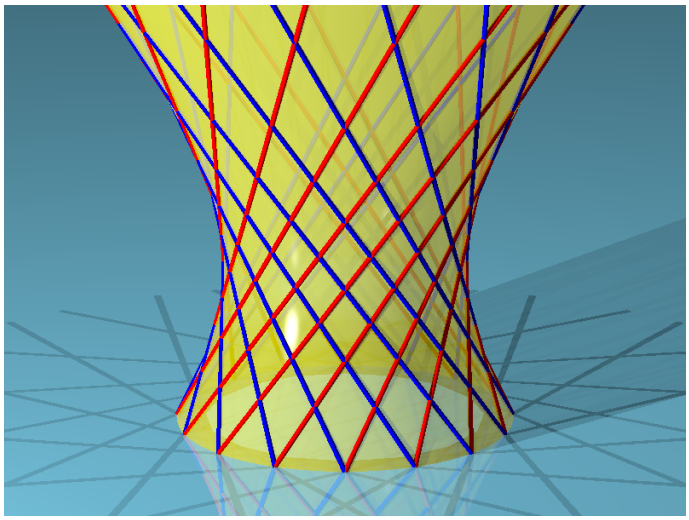


Figure 7: The hyperbolic quadric in affine space $\mathbb{R}^3$

8 Orbits on Subspaces

Orbiter can compute the orbits of a group on the lattice of subspaces of a finite vector space.

The orthogonal group is the stabilizer of a non-degenerate quadric. Suppose we want to classify the subspaces in PG$(3,2)$ under the action of the orthogonal group. In PG$(3,2)$ there are two distinct nondegenerate quadrics, $\mathcal{Q}^+(3,2)$ and $\mathcal{Q}^-(3,2)$. The $\mathcal{Q}^+(3,2)$ quadric is a finite version of the quadric given by the equation

$$x_0x_1 + x_2x_3 = 0,$$

and depicted over the real numbers in Figure 7. PG$(3,2)$ has 15 points:

$P_0 = (1,0,0,0)$	$P_4 = (1,1,1,1)$	$P_8 = (1,1,1,0)$	$P_{12} = (0,0,1,1)$
$P_1 = (0,1,0,0)$	$P_5 = (1,1,0,0)$	$P_9 = (1,0,0,1)$	$P_{13} = (1,0,1,1)$
$P_2 = (0,0,1,0)$	$P_6 = (1,0,1,0)$	$P_{10} = (0,1,0,1)$	$P_{14} = (0,1,1,1)$
$P_3 = (0,0,0,1)$	$P_7 = (0,1,1,0)$	$P_{11} = (1,1,0,1)$	

The $\mathcal{Q}^+(3,2)$ quadric given by the equation above consists of the nine points

$$P_0, P_1, P_2, P_3, P_4, P_6, P_7, P_9, P_{10}.$$

The quadric is stabilized by the group PGO$^+(4,2)$ of order 72. The command

```
orbiter.out -v 5 -linear_group -PGL 4 2 -orthogonal 1 -end \
  -group_theoretic_activities -orbits_on_subspaces 4 \
  -draw_poset
```

produces a classification of all subspaces of PG$(3,2)$ under PGO$^+(4,2)$. The option `-draw_poset` creates a Hasse diagram of the classification as shown Figure 8. The Hasse diagram is the poset of orbits of the group on the subspace lattice.

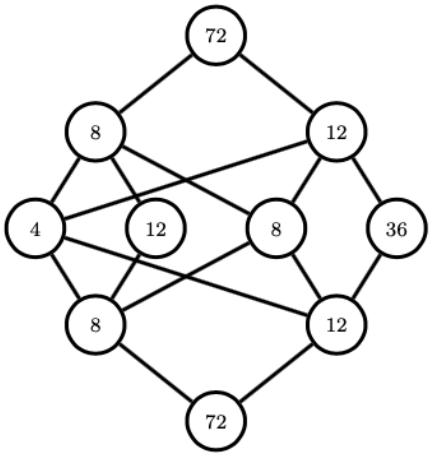


Figure 8: Hasse-diagram for the orbits of the orthogonal group $O^+(4,2)$ on subspaces of PG$(3,2)$

Key	Arguments	Meaning
`-load_from_file`	filename	Read a graph from file
`-edge_list`	n list-of-edges	Create a graph on n vertices from a list of edges as ranked pairs.
`-edges_as_pairs`	n edges-as-pairs	Create a graph on n vertices from a list of edges as pairs.
`-Johnson`	$n\ k\ s$	Johnson graph
`-Paley`	q	Paley graph
`-Sarnak`	$p\ q$	Lubotzky-Phillips-Sarnak graph [14]
`-Schlaefli`	q	Schlaefli graph
`-Shrikhande`		Shrikhande graph
`-Winnie_Li`	$q\ i$	Winnie-Li graph [13]
`-Grassmann`	$n\ k\ q\ r$	Grassmann graph
`-coll_orthogonal`	$\epsilon\ d\ q$	Collinearity graph of $O^\epsilon(d,q)$

Table 13: Types of graphs

9 Graph Theory

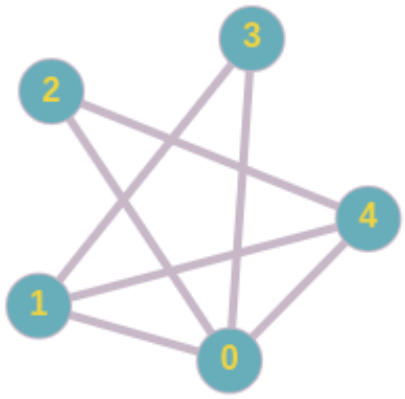
Orbiter can construct certain algebraically defined graphs. It can also construct and classify small graphs and tournaments up to isomorphism. Table 13 shows some Orbiter commands to create graphs. For instance,

```
orbiter.out -v 2 -create_graph -Johnson 5 2 0 -end -save graph_J520.bin
```

creates $J(5,2,0)$, also known as the Petersen graph.

```
orbiter.out -v 2 -create_graph  -Paley 13 -end -save graph_P13.bin
```

creates the Paley graph of order 13. Very small graphs can be encoded manually. For instance, the graph



can be created using the command

Key	Arguments	Meaning
-find_cliques	options	Find all cliques.
-export_magma		Export to Magma.
-export_maple		Export to Maple.
-export_csv		Export to csv-file.
-print		Print the graph.
-sort_by_colors		Sort the vertices by color classes.
-split	filename	Split the graph.

Table 14: Graph Theoretic Activities

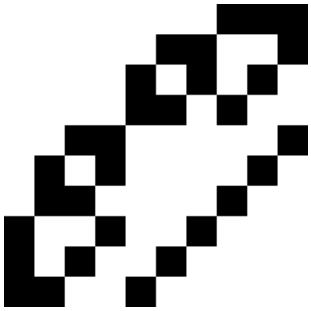


Figure 9: Adjacency Matrix of the Petersen graph

```
orbiter.out -v 2 -create_graph -edges_as_pairs 5 \
  "0,1,0,2,0,3,0,4,1,3,1,4,2,4" -end
```

The graph is stored as file `graph_v5_e7.colored_graph`.

In Table 14, the commands for graph theoretic activities are shown. For instance,

```
orbiter.out -v 2 -create_graph -Johnson 5 2 0 -end \
  -graph_theoretic_activity -export_csv graph_J520.bin -end
orbiter.out -v 2 -draw_matrix Johnson_5_2_0.csv 20
```

creates the Petersen graph and writes the adjacency matrix to file. The second command creates a bitmap drawing of the adjacency matrix, shown in Figure 9.

The clique finding command allows for additional commands as shown in Table 15. For instance, the cliques of size 3 in the graph `graph_v5_e7.colored_graph` can be found using

```
orbiter.out -v 2 -create_graph -load_from_file graph_v5_e7.colored_graph \
  -end -graph_theoretic_activity -find_cliques -target_size 3 -end -end
```

This command finds three cliques of size 3.

Table 16 lists the commands to classify small graphs and tournaments. For instance,

Key	Arguments	Meaning
`-rainbow`		Find all rainbow cliques. The size of the cliques is the number of vertex colors.
`-target_size`	s	Find all cliques of size s.
`-weighted`	s	Find weighted cliques.
`-Sajeeb`		Use the implementation by Sajeeb Chowdhury.
`-output_file`	fname	Write cliques to the named file.
`-restrictions`	$l\ r\ m$	Restricted search: At level l, do all branches congruent to r moulo m only. Here, $0 \leq r < m$.

Table 15: Clique Finding Options

Option	Arguments	Meaning
`-girth`	d	Girth at least d
`-regular`	r	Regular of degree r
`-no_transmitter`		Tournament without transmitter (requires `-tournament`)

Table 16: Options for classifying graphs

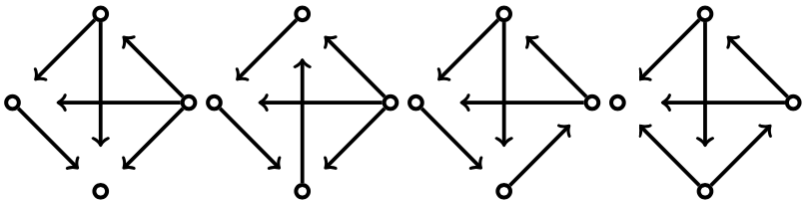


Figure 10: The four isomorphism types of tournaments on 4 vertices

```
orbiter.out -v 2 -graph_classify -n 4
```

classifies all graphs with 4 vertices. For tournaments, the option `-tournament` can be added. For example,

```
orbiter.out -v 2 -graph_classify -n 4 -v 2 -tournament -draw_graphs_at_level 6
```

classifies the tournaments on 6 vertices. The `-draw_graphs_at_level 6` option instructs Orbiter to draw all representatives at level 6. Figure 10 shows the resulting list of 4 tournaments.

10 Projective Geometry

Orbiter can create objects in projective space using the `-create_combinatorial_object` command. The possible secondary commands are shown in Tables 17 and 18. Modifier options that apply to multiple options are listed in Table 19.

Here is an example. The following command sequence creates the elliptic curve

$$y^2 \equiv x^3 + x + 3 \bmod 11.$$

over the field $\mathbb{F}_{11}$ and computes the canonical form and automorphism group:

```
orbiter.out -v 5 -create_combinatorial_object -q 11 \
  -elliptic_curve 1 3 -end \
  -save "./"
orbiter.out -v 2 -process_combinatorial_objects -q 11 -n 2  \
  -draw_points_in_plane EC_11_1_3 \
  -fname_base_out EC_11_1_3 -embedded \
  -input -file_of_points elliptic_curve_b1_c3_q11.txt -end \
  -end
orbiter.out -v 2 -canonical_form_PG  2 11 \
  -input -file_of_points elliptic_curve_b1_c3_q11.txt -end \
  -classification_prefix elliptic_curve_b1_c3_q11 \
  -report \
  -end
```

The first command creates the 18 points of the curve in $\mathrm{PG}(2, 11)$. The second command produces a picture of the point set in $\mathrm{PG}(2, 11)$, shown in Figure 2. The third command computes the collineation stabilizer. This is done by using the techniques of canonical forms in graphs, using the Nauty [15] package which is included in Orbiter. Orbiter shows that the curve has a collineation stabilizer of order 6, generated by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 8 \\ 5 & 9 & 5 \\ 8 & 1 & 1 \end{bmatrix}.$$

The purpose of the `-process_combinatorial_objects` option is to perform various jobs for data streams. A data stream represents a possibly large number of integer data sets, stored in files. Using the rank and unrank encoding from Section 4, integer data can encode sets of points in projective space, for instance. The types of jobs are listed in Table 20.

Key	Arguments	Purpose
-hyperoval		To create a hyperoval
-subiaco_oval	f_short	Create points of the Subiaco oval
-subiaco_hyperoval		Create points of the Subiaco hyperoval
-adelaide_hyperoval		Create points of the Adalaide hyperoval
-translation	i	Create points of the translation hyperoval with exponent i
-Segre		Create points of the Segre hyperoval
-Payne		Create points of the Payne hyperoval
-Cherowitzo		Create points of the Cherowitzo hyperoval
-OKeefe_Penttila		Create points of the O'Keefe, Penttila hyperoval
-BLT_database	k	The kth BLT-set of order q from the database ($k = 0, 1, \ldots$)
-ovoid		Create points of an ovoid
-Baer		Create points of the (standard) Baer subgeometry
-orthogonal	ϵ	Create points of the $Q^{\epsilon}(n, q)$ quadric
-hermitian		Create points of Hermitian variety given by $\sum_{i=0}^{n} X_i^{\sqrt{q}+1} = 0$
-cubic		Create points of a cubic
-twisted_cubic		Create points of the twisted cubic
-elliptic_curve	a b	Create points of the elliptic curve $y^2 = x^3 + ax + b$
-ttp_construction_A		Create points of the twisted tensor product code of type A [3]
-ttp_construction_A_hyperoval		Create points of the twisted tensor product code of type A
-ttp_construction_B		Create points of the twisted tensor product code of type B
-monomial_type_LEX		Monomials are in lexicographical ordering.
-monomial_type_PART		Monomials are in partition ordering.

Table 17: Orbiter Objects (Part 1)

Key	Arguments	Purpose
-unital_ XXq_YZq_ZYq		Create points of the unital with equation $XX^q + YZ^q + ZY^q = 0$
-desarguesian_ line_spread_ in_PG_3_q		Create the desarguesian line spread in $\mathrm{PG}(3, q)$ as a set of 2-subspaces
-Buekenhout_Metz		Create points of the Buekenhout Metz unital
-Uab	a b	Create points of the Buekenhout Metz unital in the form of Barwick and Ebert [2]
-whole_space		Create points of the whole space
-hyperplane	pt	Create points of the hyperplane given by dual coordinates associated with the given point
-segre_variety	a b	Create points of the Segre variety
-Maruta_Hamada_arc		Create points of the Maruta Hamada arc
-projective_variety	l d $\mathcal{C}$	Create points of the projective variety of degree d with label l, with coefficient vector $\mathcal{C}$
-projective_curve	l r d $\mathcal{C}$	Create points of the projective curve of degree d with label l, with coefficient vector $\mathcal{C}$ in r variables

Table 18: Orbiter Objects (Part 2)

Key	Arguments	Purpose
-q	q	The size of the finite field $\mathbb{F}_q$
-Q	Q	The field size of the extension field $\mathbb{F}_Q$
-n	n	The projective dimension
-poly	r	Use polynomial with rank r to create the field $\mathbb{F}_q$
-poly_Q	r	Use polynomial with rank r to create the field $\mathbb{F}_Q$
-embedded_in_PG_4_q		
-BLT_in_PG		Create points of the BLT-set with ranks in $\mathrm{PG}(n, q)$ instead of orthogonal point ranks

Table 19: Orbiter Objects: Modifiers

Job key	Purpose
-dualize_hyperplanes_ to_points	Turns ranks of hyperplanes into ranks of points
-dualize_points_ to_hyperplanes	Turns ranks of points into ranks of hyperplanes
-ideal_LEX	Compute the ideal of a set of points, using lexicographic ordering of monomials
-ideal_PART	Compute the ideal of a set of points, using partition ordering of monomials
-homogeneous_ polynomials_LEX	Prints the equation whose coefficient vector is the input vector using lexicographic ordering of monomials.
-homogeneous_ polynomials_PART	Prints the equation whose coefficient vector is the input vector using partition ordering of monomials.
-canonical_form	Computes the canonical form of a set
-draw_points_in_plane	Produces a drawing of a set of points in a projective plane
-klein	Applies the Klein correspondence
-line_type	Computes the line type
-plane_type	Computes the plane type
-conic_type	Computes the conic type
-hyperplane_type	Computes the hyperplane type
-intersect_with_ set_from_file	Computes the intersection with a set specified in a file
-arc_with_given_ set_as_s_lines_ after_dualizing	Finds arcs with the given set as s-lines
-arc_with_two_ given_sets_of_lines_ after_dualizing	Finds arcs with the two given sets as s-lines and t-lines, respectively
-arc_with_three_ given_sets_of_lines_ after_dualizing	Finds arcs with the three given sets as s-lines and t-lines and u-lines, respectively

Table 20: Orbiter Jobs in PG(n, q)

11 Arcs in Projective Planes

A (k, d)-arc in a projective plane π is a set S of k points such that very line intersects S in at most d points. Arcs are related to linear codes and other structures. Two arcs S_1 and S_2 are equivalent if there is a projectivity Φ such that $\Phi(A) = B$. The problem of classifying arcs is the problem of determining the orbits of the projectivity group on arcs. At times, we consider the larger group of collineations. In that case, the problem of classifying arcs is the problem of determining the orbits of the collineation group on arcs. Orbiter can solve such classification problems, at least for small parameter cases. Here is an example. A hyperoval in a plane $\mathrm{PG}(2, 2^e)$ is a $(2^e + 2, 2)$-arc. It is interesting to classify the hyperovals up to collineation equivalence under the group $\mathrm{P\Gamma L}(3, 2^e)$. The command

```
orbiter.out -v 4 \
  -linear_group -PGGL 3 16 -end \
  -group_theoretic_activities \
    -poset_classification_control -problem_label arcs_q16_2 -W -end \
    -classify_arcs 18 2 \
    -end
```

performs the classification of hyperovals in $\mathrm{PG}(2, 16)$. There are exactly two hyperovals in this plane. Orbiter also finds the stabilizers of these arcs. They have orders 16320 and 144, respectively.

12 Cubic Surfaces

Orbiter can classify cubic surfaces with 27 lines over finite fields. There are several different approaches to classify cubic surfces over finite fields with 27 lines under the collineation group $\mathrm{P\Gamma L}(4,q)$. One approach is described in [6] and relies on Schlaefli's notion of a double six as a substructure [18]. Another approach is through non-conical six-arcs in a plane, as described in [12]. Both approaches have been implemented in Orbiter. The purpose of the construction algorithm is to produce the equations of surfaces. In order to do so, the notion of a double six of lines in $\mathrm{PG}(3,q)$ is used. A double six determines a unique surfaces but a surface may have several double sixes associated to it. The classification algorithms sorts out the relationship between the isomorphism types of double sixes and the isomorphism types of cubic surfaces. In order to classify all double sixes, yet another substructure is considered. These are the five-plus-ones. They consist of 5 lines with a common transversal. The poset classification algorithm is used to classify the five-plus-ones. Also, Orbiter will sort out the isomorphism classes of double sixes based on their relation to the five-plus-ones. In order to classify the five-plus-ones, the related Klein quadric is considered. Lines in $\mathrm{PG}(3,q)$ correspond to points on the Klein quadric. Thus, the five-plus-one configurations of lines correspond to certain configurations of points on the Klein quadric.

The command

```
orbiter.out -v 3 -linear_group -PGL 4 7 -wedge -end \
  -group_theoretic_activities -surface_classify -end
```

classifies all cubic surfaces over the field $\mathbb{F}_7$ under the projective linear group. If desired, it is possible to use

```
orbiter.out -v 3 -linear_group -PGGL 4 4 -wedge -end \
  -group_theoretic_activities -surface_classify -end
```

to perform the same classification with respect to the collineation group $\mathrm{P\Gamma L}(4,4)$. The `-report` option can be used to create a report of the classified surfaces. So, for instance

```
orbiter.out -v 3 -linear_group -PGGL 4 4 -wedge -end \
  -group_theoretic_activities -surface_classify -report
```

produces a latex report of the surface in $\mathrm{PG}(3,4)$.

The commands in Table 21 can be used to create a known surface.

The `-surface_recognize` option can be used to identify a given surface in the list produced by the classification. For instance,

```
orbiter.out -v 3 -linear_group -PGGL 4 8 -wedge -end \
    -group_theoretic_activities -surface_recognize -q 8 \
    -by_coefficients "1,6,1,8,1,11,1,13,1,19" -end -end
```

Command	Arguments	Description
`-q`	q	Specify the order of the field. The surface will be defined in PG$(3, q)$.
`-catalogue`	i	Create the ith surface in the catalogue. Here, i is an index variable used to index all surfaces in PG$(3, q)$. The index i is zero-based.
`-by_coefficients`	list-of-coeff-pairs	Create a surface from a list of coefficient-monomial pairs.
`-family_S`	a	Create the Hilbert, Cohn-Vossen surface with parameter a, see [6] (use $b = 1$).
`-arc_lifting`	$a_1\ a_2\ a_3\ a_4\ a_5\ a_6$	Create the surface associated with the arc $a_1, \ldots, a_6$ in PG$(2, q)$ by means of the Clebsch map. Each of the a_i is the rank of a point in PG$(2, q)$. Use the trihedral pair algorithm.
`-arc_lifting_with_two_lines`	$a_1\ a_2\ a_3\ a_4\ a_5\ a_6$	Create the surface associated with the arc $a_1, \ldots, a_6$ in PG$(2, q)$ by means of the Clebsch map. Each of the a_i is the rank of a point in PG$(2, q)$. Use the two-lines algorithm.

Table 21: Commands to create a known cubic surface

identifies the surface (cf. Table 5)

$$X_0^2X_3 + X_1^2X_2 + X_1X_2^2 + X_0X_3^2 + X_1X_2X_3 = 0 \tag{4}$$

in the classification of surfaces over the field $\mathbb{F}_8$. This means that an isomorphism from the given surface to the surface in the list is computed. Also, the generators of the automorphism group of the given surface are computed, using the known generators for the automorphism group of the surface in the classification. For instance, executing the command above creates an isomorphism between the given surface and the surface in the catalogue:

$$\begin{bmatrix} 1 & 4 & 4 & 0 \\ 6 & 0 & 0 & 0 \\ 6 & 2 & 0 & 1 \\ 7 & 0 & 4 & 0 \end{bmatrix}_0 . \tag{5}$$

Orbiter can compute isomorphism between two given surfaces. The surfaces must have 27 lines. For instance, the command

```
orbiter.out -v 3 -linear_group -PGGL 4 8 -wedge -end \
    -group_theoretic_activities -surface_isomorphism_testing \
      -q 8 -by_coefficients \
        "5,5,5,8,5,9,5,10,5,11,5,12,4,14,4,15,1,18,1,19" -end \
      -q 8 -by_coefficients "1,6,1,8,1,11,1,13,1,19" -end
```

computes an isomorphism between the two $\mathbb{F}_8$-surfaces

$$\begin{aligned} 0 &= \alpha^3X_0^2X_2 + \alpha^3X_1^2X_2 + \alpha^3X_1^2X_3 + \alpha^3X_0X_2^2 + \alpha^3X_1X_2^2 + \alpha^3X_2^2X_3 \\ &\quad + \alpha^2X_1X_3^2 + \alpha^2X_2X_3^2 + X_0X_2X_3 + X_1X_2X_3, \\ 0 &= X_0^2X_3 + X_1^2X_2 + X_1X_2^2 + X_0X_3^2 + X_1X_2X_3. \end{aligned}$$

The isomorphism is given as a collineation:

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 7 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 2 & 3 & 2 & 4 \end{bmatrix}_2 .$$

In here, the numerical representation of elements of $\mathbb{F}_8$ as integers in the interval $[0, 7]$ is used. The exponent of the Frobenius automorphism is listed as a subscript.

A second algorithm to classify cubic surfaces has been described in [5] and in [12]. This algorithm is available in Orbiter also. For instance, the command

```
orbiter.out -v 4 -linear_group -PGGL 4 4 -end \
  -group_theoretic_activities \
  -trihedra1_control -problem_label tri1_q4 -end \
  -trihedra2_control -problem_label tri2_q4 -end \
  -control_six_arcs -problem_label sixarcs_q4 -end \
  -classify_surfaces_through_arcs_and_trihedral_pairs \
  -end
```

classifies all cubic surfaces with 27 lines over the field $\mathbb{F}_4$ using this algorithm (there is just one, the Hirschfeld surface). A report of the classification is produced in the file `arc_lifting_q4.tex`.

Besides classification, there are two further ways to create surfaces in Orbiter. The first is a built-in catalogue of cubic surfaces with 27 lines for small finite fields $\mathbb{F}_q$ (at the moment, $q \leq 101$ is required). The second is a way of creating members of known infinite families. Both are facilitated using the `-create_surface` option. For instance,

```
orbiter.out -v 3 -linear_group -PGL 4 13 -wedge -end \
    -group_theoretic_activities \
    -create_surface -family_S 3 -q 13 -end
```

creates the member of the Hilbert, Cohn-Vossen surface described in [6] with parameter $a = 3$ and $b = 1$ over the field $\mathbb{F}_{13}$. The command

```
orbiter.out -v 3 -linear_group -PGGL 4 8 -wedge -end \
    -group_theoretic_activities \
    -create_surface -q 4 -catalogue 0 -end
```

creates the unique cubic surface with 27 lines over the field $\mathbb{F}_4$ which is stored under the index 0 in the catalogue. It is possible to apply a transformation to the surface. Suppose we are interested in the surface over $\mathbb{F}_8$ created in (4). The command

```
orbiter.out -v 3 -linear_group -PGGL 4 8 -wedge -end \
    -group_theoretic_activities \
    -create_surface -q 8 -catalogue 0 -end \
    -transform_inverse "1,4,4,0,6,0,0,0,6,2,0,1,7,0,4,0,0"
```

creates surface 0 over $\mathbb{F}_8$ and applies the inverse transformation to recover the surface whose equation was given in (4). The surface number 0 over $\mathbb{F}_8$ is created, and the transformation (5) is applied in inverse. The commands `-transform` and `-transform_inverse` accept the transformation matrix in row-major ordering, with the field automorphism as additional element. It is possible to give a sequence of transformations. In this case, the transformations are applied in the order in which the commands are given on the command line.

13 Cryptography and Number Theory

In Table 22, some number theoretic commands are shown. For instance,

```
orbiter.out -v 2 -inverse_mod  18059241 58014043
```

conputes the inverse of 18059241 modulo 58014043. The command

```
orbiter.out -v 5 -jacobi 2221 7817
```

computes the Jacobi symbol

$$\left(\frac{2221}{7817}\right).$$

The denominator p has to be a positive odd integer.

In Table 23, some cryptographic commands are shown. For instance,

```
orbiter.out -v 2 -EC_add 11 1 3 "1,4" "1,4"
```

adds the point $(1, 4)$ on the curve $y^2 = x^3 + x + 3 \mod 11$ to itself. The command

```
orbiter.out -v 2 -EC_cyclic_subgroup 11 1 3 "1,4"
```

computes the cyclic subgroup generated by the point $(1, 4)$ on the curve $y^2 = x^3 + x + 3 \mod 11$. The command

```
orbiter.out -v 2 -EC_points 199 5 7
```

computes all points on the curve $y^2 = x^3 + 5x + 7 \mod 199$. The command

```
orbiter.out -v 6 -seed 17 -EC_Koblitz_encoding 199 5 7 67 "147,164" "DEADBEEF"
```

encode the message "DEADBEEF" on the curve $y^2 = x^3 + 5x + 7 \mod 199$ using the base point $(147, 164)$ and the secret key 67. The ith input character is encoded as two points (R_i, T_i) on the curve using the Elgamal scheme. A random round key is generated for each plaintext symbol. As seen in this example, the `-seed` command can be used to seed the random number generator with an arbitrary integer (here 17). The command

```
orbiter.out -v 2 -EC_bsgs 199 5 7 "147,164" 212 \
  "172,158,45,195,50,22,10,103,55,33,50,22,145,105,31,74,73,155,67,60,25,6"
```

performs a baby-step-giant-step brute force attack on the ciphertext sequence

$$R_i = (172, 158), (45, 195), (50, 22), (10, 103), (55, 33),$$
$$(50, 22), (145, 105), (31, 74), (73, 155), (67, 60), (25, 6),$$

using the base point $(147, 164)$ on the curve $y^2 = x^3 + 5x + 7 \mod 199$, assuming a group order of 212. The command

Command	Arguments	Description
`-trace`	q	Computes the absolute trace function for all elements in $\mathbb{F}_q$
`-norm`	q	Computes the absolute norm function for all elements in $\mathbb{F}_q$
`-jacobi`	a p	Computes the Jacobi symbol $\left(\frac{a}{p}\right)$
`-power_mod`	a n p	Raises a to the power n modulo p
`-primitive_ root`	p	Computes a primitive root modulo p
`-discrete_log`	b a p	Computes n such that $a^n \equiv n \mod p$
`-square_ root_mod`	a p	computes a square root of a modulo p
`-inverse_mod`	a p	computes the modular inverse of a modulo p
`-sift_smooth`	a n primes	Computes all smooth numbers in the interval $[a, a+n-1]$. Smooth means that they factor completely over the list of primes given.
`-solovay_ strassen`	a n	Performs n Solovay / Strassen tests on the number a
`-miller_rabin`	a n	Performs n Miller / Rabin tests on the number a
`-fermat`	a n	Performs n Fermat tests on the number a
`-find_pseudoprime`	a n_1 n_2 n_3	Computes a pseudoprime which survives n_1 Fermat tests, n_2 Miller Rabin tests, n_3 Solovay Strassen tests
`-find_strong_ pseudoprime`	a n_1 n_2	Computes a pseudoprime which survives n_1 Fermat tests and n_2 Miller Rabin tests
`-random`	n fname	Creates n random numbers and writes them to the csv file `fname`
`-random_last`	n	Creates n random numbers prints the last one
`-affine_sequence`	a b p	Splits the interval $[0, p-1]$ into affine sequences of the form $x_{n+1} = ax_n + b \mod p$

Table 22: Number Theoretic Commmands

Command	Arguments	Description
`-RSA_encrypt_text`	d n b text	Using blocks of b letters at a time, encrypt "text" using RSA with exponent d modulo n
`-RSA`	d n list-of-integers	encrypt the given sequence of integers using RSA with exponent d modulo n
`-EC_add`	p a b i_1 i_2	On the elliptic curve $y^2 \equiv x^3 + ax + b \mod p$, add the points with ibdices i_1 and i_2, each given as a pair x, y
`-EC_points`	p a b	Computes all points over $\mathbb{F}_p$ of the elliptic curve $y^2 \equiv x^3 + ax + b \mod p$
`-EC_multiple_of`	p a b pt n	Computes the n fold multiple of the given point pt on the elliptic curve $y^2 \equiv x^3 + ax + b \mod p$
`-EC_cyclic_subgroup`	p a b pt	Computes the cyclic subgroup generated by the given point pt on the elliptic curve $y^2 \equiv x^3 + ax + b \mod p$
`-EC_Koblitz_encoding`	p a b s pt plain	Computes the Koblitz encoding of "plain" (all caps) on the elliptic curve $y^2 \equiv x^3 + ax + b \mod p$ using the base point pt and the secret exponent s
`-EC_bsgs`	p a b pt n cipher	Prepare the baby-step giant-step tables for the ciphertext "cipher" on the elliptic curve $y^2 \equiv x^3 + ax + b \mod p$ using the base point pt of order n
`-EC_bsgs_decode`	p a b pt n cipher round-keys	Decodes the ciphertext "cipher" on the elliptic curve $y^2 \equiv x^3 + ax + b \mod p$ using the base point pt of order n and the round keys "keys"
`-EC_discrete_log`	p a b pt base-pt	Computes the elliptic curve discrete log analogue of pt with respect to base-pt on the elliptic curve $y^2 \equiv x^3 + ax + b \mod p$

Table 23: Cryptographic Commmands

```
orbiter.out -v 2 -EC_bsgs_decode 199 5 7 "129,176" 212 \
  "127,188,51,141,85,29,106,90,41,105,179,71,171,2,16,197,183,72,27,129,37,10" \
  "50,179,169,13,153,169,115,116,188,110,176"
```

performs a decoding of the ciphertext sequence

$$T_i = (127, 188), (51, 141), (85, 29), (106, 90), (41, 105), (179, 71),$$
$$(171, 2), (16, 197), (183, 72), (27, 129), (37, 10),$$

assuming round keys

$$k_i = 50, 179, 169, 13, 153, 169, 115, 116, 188, 110, 176,$$

using the base point $(147, 164)$ on the curve $y^2 = x^3 + 5x + 7 \mod 199$, and assuming a group order of 212.

14 Coding Theory

Orbiter can classify linear codes with prescribed minimum distance. Recall that a linear $[n, k; q]$-code $\mathcal{C}$ is a k-dimensional subspace of $\mathbb{F}_q^n$. The Hamming space $H(n, q)$ is the space $\mathbb{F}_q^n$ equipped with the Hamming metric. For two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n$, the Hamming distance is

$$d(\mathbf{x}, \mathbf{y}) = \#\, i : x_i \neq y_i.$$

The weight of a vector is

$$\mathrm{wt}(\mathbf{x}) = d(\mathbf{x}, \mathbf{0}) = \#\, i : x_i \neq 0.$$

The minimum distance of a subset $\mathcal{C}$ of $H(n, q)$ is

$$d(\mathbf{C}) = \min_{\substack{\mathbf{c}, \mathbf{c}' \in \mathcal{C} \\ \mathbf{c} \neq \mathbf{c}'}} d(\mathbf{c}, \mathbf{c}').$$

We say that $\mathcal{C}$ is a linear $[n, k, d; q]$-code if $\mathcal{C}$ is $[n, k; q]$ with $d(\mathcal{C}) = d$. The minimum weight of a subset $\mathcal{C}$ of $H(n, q)$ is

$$\mathrm{wt}(\mathcal{C}) = \min_{\mathbf{c} \in \mathcal{C} \setminus \{\mathbf{0}\}} \mathrm{wt}(\mathbf{c}).$$

For a linear code $\mathcal{C}$, the two quantities agree, and we have

$$d(\mathcal{C}) = \mathrm{wt}(\mathcal{C}) = \min_{\mathbf{c} \in \mathcal{C} \setminus \{\mathbf{0}\}} \mathrm{wt}(\mathbf{c}).$$

It is desirable to have linear codes for which both k and d are reasonably large with respect to n. However, these are contradicting aims: A large value of k limits d from above. Likewise, a large value of d limits k. It is interesting to have construction methods for linear codes which bound d from below. This is one of the main topics of coding theory.

There is a notion of isometry with respect to the Hamming metric. This leads to a notion of equivalence of codes. Two codes are equivalent if the coordinates of the vectors in one code can be computed (simultaneously) so as to obtain the second code. The automorphism group is the set of isometry maps from one code to itself.

In Table 24, some coding theoretic commands of Orbiter are shown. For instance, the command

```
orbiter.out -BCH 15 2 3
```

creates a binary BCH-code of length 15 with minimum distance at least 3. The generator

Command	Arguments	Description
`-RREF`	q m n list-of-integers	Compute the RREF of the $m \times n$ matrix over $\mathbb{F}_q$
`-nullspace`	q m n list-of-integers	Compute a basis for the right nullspace of the given $m \times n$ matrix
`-normalize_from_the_right`		Normalizes the result of `-RREF` or `nullspace` from the right
`-weight_enumerator`	q m n list-of-integers	Computes the weight enumerator of the linear code generated by the given $m \times n$ matrix
`-BCH`	n q t	Creates the BCH-code of length n over the field $\mathbb{F}_q$ with designed distance t
`-Hamming_graph`	n q	Creates the distance matrix of the Hamming graph $H(n, q)$. The vertices are the elements of $\mathbb{F}_q^n$, and the i, j-entry is the distance between the vectors whose affine ranks are i and j, respectively. The matrix is written as csv-file.
`-draw_matrix`	csv-file w	Creates a colored bitmap drawing of the matrix in the csv file, using w pixels per entry. The color indicates the value of the matrix entry.

Table 24: Coding Theoretic Commmands

matrix produced by this command is

$$\left[\begin{array}{cccccccccccccccc}
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1
\end{array}\right].$$

The commands

```
orbiter.out -Hamming_graph 4 2
orbiter.out -draw_matrix Hamming_n4_q2.csv 20
```

create the csv-file `Hamming_n4_q2.csv` and produce the bitmap file `Hamming_n4_q2_draw.bmp` shown in Figure 11.

The classification problem of optimal codes in coding theory is the problem of determining the equivalence classes of codes for a given set of values of n and k and q with a lower bound on d. Orbiter can be used for solving this problem for small instances.

Orbiter can be used to classify linear codes with given redundancy and bounded minimum distance. The redundancy of a linear $[n, k]$ code is the parameter $r = n - k$. Codes with redundancy r can be identified with subsets of $\mathrm{PG}(r-1, q)$. Under this correspondence, a code with minimum distance at least d corresponds to a subset such that any $d-1$ elements are independent. We use the notation $\Lambda_{r-1,s}(q)$ to denote the poset of subsets of $\mathrm{PG}(r-1, q)$ for which any $d-1$-subset (if any) is independent. Under the correspondence, the action of $\mathrm{P\Gamma L}(r, q)$ on $\Lambda_{r-1,s}(q)$ corresponds to the orbits of equivalent linear codes. For this reason, we are interested in determining the orbits of $\mathrm{P\Gamma L}(r, q)$ on $\Lambda_{r-1,s}(q)$. An orbit of size n represents an isometry class of $[n, n-r, d; q]$ codes with $d \geq s+1$. The projective stabilizer of the subset is the automorphism group of the code. The Orbiter command

```
orbiter.out -v 3 -linear_group -PGL 4 2 -end \
  -group_theoretic_activities -poset_classification_control \
  -W -problem_label codes_r4_d4 -end -linear_codes 4 100 -end
```

can be used to classify linear codes with redundancy 4 and minimum distance at least 4. The extremal code that satisfies these conditions is the $[8, 4, 4]$ codes over $\mathbb{F}_2$. The Orbiter program confirms that there is exactly one such code. Orbiter computes the code together with the projective stabilizer. To do so, we first create the group $\mathrm{PGL}(4, 2)$ acting on the

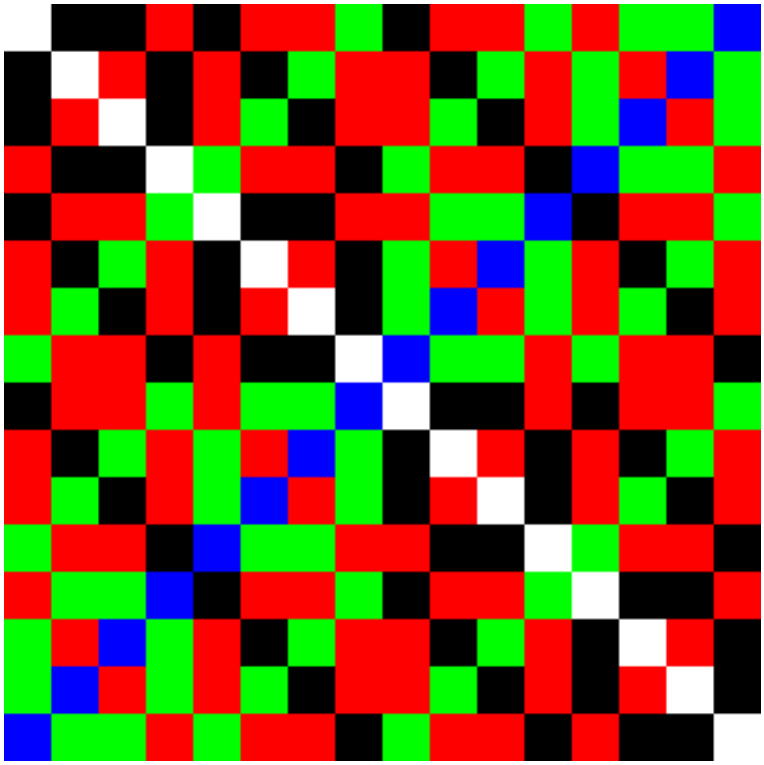


Figure 11: The color-coded distance matrix of the Hamming graph $H(4, 2)$

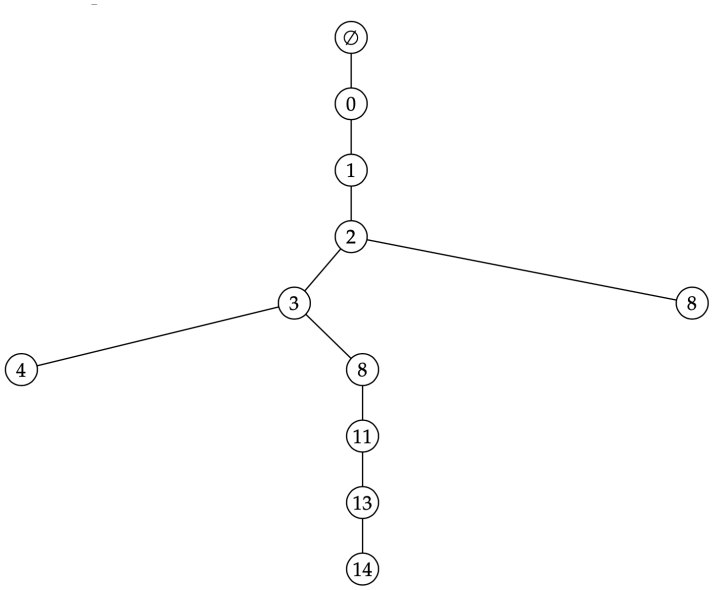


Figure 12: Orbits of $\mathrm{PGL}(4, 2)$ on the poset $\Lambda_{3,3}(2)$

poset $\Lambda_{3,3}(2)$. Orbiter then produces the poset of orbits shown in Figure 12. In this diagram, the numbers stand for Orbiter ranks of points in PG$(3,2)$. All nodes except for the root node have a number attached to it. The nodes represent subsets. In order to determine the set associated to a node, follow the path from the root node to the node and collect the points according to their labels. The root node represents the empty set. The $[8,4,4;2]$-code is represented by the set $\{0,1,2,3,8,11,13,14\}$. The fact that there is only one node at level 8 in the poset of orbits tells us that the code is unique up to equivalence. Let us look at the code. The elements of the set $\{0,1,2,3,8,11,13,14\}$ are points in PG$(3,2)$. We write the coordinate vectors in the columns of a matrix H:

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

This matrix is the parity check matrix H of the code $\mathcal{C}$. This means that the words of the code are the vectors $\mathbf{c}$ such that $\mathbf{c} \cdot H^\top = 0$. Observe that the vectors that we put in the columns of H all have odd weight. They are in fact the points of the hyperplane $x+y+z+w=0$. This shows that the stabilizer of the code which is the stabilizer of the set is equal to AGL$(3,2)$, a group of order 1344.

15 Diophantine Systems

Diophantine systems of equations and inequalities arise frequently in Combinatorics. Suppose we want all partitions of an integer n as

$$n = a_1 + a_2 + \ldots + a_k, \quad a_1 \geq a_2 \geq \cdots \geq a_k \geq 1,$$

with $a_i \in \mathbb{Z}_{>0}$. For $1 \leq j \leq n$, let

$$c_j = \#\{i \mid a_i = j\}.$$

The following diophantine equation holds for any partition

$$\sum_{j=1}^{n} j c_j = n \tag{6}$$

Conversely, any partition is uniquely determined by a solution of this equation. Therefore, counting partitons of n is the same as counting nonnegative integer solutions of (6). Let p_n be the number of partitions of n. Suppose we wish to compute p_{10}. In this case, the extended coefficient matrix of the system is

$$[10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1 \ \mid 10\ ].$$

Orbiter creates this diophantine system using the command

```
orbiter.out -v 4 -diophant -from_scratch part10 1 10 \
  -coefficient_matrix "10,9,8,7,6,5,4,3,2,1" \
  -RHS "10,10,1" -x_min_global 0 -x_max_global 10
```

It also creates three csv-files: one for the coefficient matrix, one for the RHS information, and one for the bounds on the variables. With these files, it is possible to recreate the diophantine system using the command

```
orbiter.out -v 4 -diophant -from_scratch part10 1 10 \
  -coefficient_matrix_csv part10_coeff_matrix.csv \
  -RHS_csv part10_RHS.csv \
  -x_bounds_csv part10_x_bounds.csv
```

The command

```
orbiter.out -v 4 -diophant_activity -input_file part10.diophant -solve_mckay
```

solves the system and finds that $p_{10} = 42$. The sequence p_n is recorded under the key A000041 in Sloane's Handbook of integer sequences [19].

A linear space is a pair $(S, \mathcal{L})$ where S is a set and $\mathcal{L}$ is a set of subsets of S such that each set $L \in \mathcal{L}$ satisfies $|L| \geq 2$ and morover for any two $a, b \in S$ there is exactly one element $L \in \mathcal{L}$ such that both a and b belong to L. The usual notions of isomorphism and automorphism apply. For finite linear spaces, a first combinatorial property is the number a_i which counts

the number of sets $L \in \mathcal{L}$ of size i. The vector $(a_2, \ldots, a_n)$ is the line type of $(S, \mathcal{L})$. The equation

$$\binom{n}{2} = \sum_{j=2}^{n} a_j \binom{j}{2} \tag{7}$$

is satisfied. The equation can be used to generate all possible line types of a putative linear space. Here is an example. For $|S| = 6$, (7) becomes

$$x_0\binom{6}{2} + x_1\binom{5}{2} + x_2\binom{4}{2} + x_3\binom{3}{2} + x_4\binom{2}{2} = \binom{6}{2}.$$

Here, $(x_0, x_1, \ldots, x_4)$ is the line type of a putative linear space on 6 points. That is, $x_i = a_{6-i}$ is the number of lines of size $6 - i$. The extended coefficient matrix of the system is

$$[15\ 10\ 6\ 3\ 1\ \ |\ 15\ ].$$

The Orbiter command

```
orbiter.out -v 4 -diophant -from_scratch linsp6 1 5 \
  -coefficient_matrix "15,10,6,3,1" -RHS "15,15,1" \
  -x_min_global 0 -x_max_global 15
```

creates this system and stores it in the file `linsp6.diophant` using the name specified. The command

```
orbiter.out -v 3 -diophant_activity -input_file linsp6.diophant -solve_mckay
```

solves the system using McKay's program possolve [16]. The program finds 15 solutions, written to the file `linsp6.sol`. In Table 25, Orbiter commands for creating diophantine systems are shown. In Table 26, Orbiter activities for diophantine systems are shown.

Let us consider a problem from [4]. Suppose we are interested in linear spaces on 30 points with line type $(7, 5^{27}, 4^{24})$. This notation means that we assume one 7-lines, 27 5-lines and 24 4-lines. The type of a point P is the set of integers

$$p_j = \#j\text{-lines though } P.$$

We are trying to precompute the matrix of point types

$$(p_{ij})$$

where $j = 7, 5, 4$ and i belongs to an index set of all possible point types. Fixing a point P, counting points $Q \neq P$ collinear with P yields

$$6p_7 + 4p_5 + 3p_4 = 29, \quad p_7 \leq 1,\ p_5 \leq 27, p_4 \leq 24.$$

Using the Orbiter commands

Command	Arguments	Description
-from_scratch	label n m	Create diophantine system of size $n \times m$.
-coefficient_ matrix	list-of-integers	Set the $n \times m$ coefficient matrix.
-coefficient_ matrix_csv	fname	Read the coefficient matrix from the given csv-file.
-RHS	list-of-integers	$3n$ values: (RHS-low, RHS-high, RHS-type) for each row of the system.
-RHS_csv	fname	Read the RHS information from the given csv file.
-x_max_global	a	Set the upper bound for all variables to a
-x_min_global	a	Set the lower bound for all variables to a
-x_bounds	list-of-values	Set the lower and upper bounds for all variables.
-x_bounds_csv	fname	Read the lower and upper bounds for all variables from the given file.
-has_sum	s	For the sum of the variables to be s.
-maximal_arc	s d secants subset	Create system for a maximal arc of size s and degree d in PG$(2, q)$. Use the given set of two pencil lines. The subset picks the lines from the given pencils which are external.
-q	q	Use PG$(2, q)$ for maximal arcs.
-override_ polynomial	a	Use polynomial numerically coded as a for creating $\mathbb{F}_q$.

Table 25: Orbiter Commmands to create Diophantine systems

Command	Arguments	Description
-input_file	file	Specify the input file
-print		Print the system
-solve_mckay		Solve the system using McKay's possolve
-solve_standard		Solve the system using the standard solver
-draw		Produce a metapost drawing of the coefficient matrix of the system
-draw_as_bitmap	w	Produce a bitmap drawing of the coefficient matrix of the system, using w pixel per entry. The output is a bmp-file.
-perform_ column_reductions		Eliminate variables which must be zero and write a reduced system
-test_single _equation		For each row of the system, compute the number of solutions of the system restricted to the nonzero coefficients.
-project_to_ single_equation _and_solve	i j	Solve the system assuming the jth solution to the restricted system consisting of the ith row.
-project_to_ two_equations _and_solve	i j r m	Solve the system assuming any solution to the restricted system consisting of the ith and the j-th row whose number is congruent to r mod m.

Table 26: Orbiter activities for Diophantine systems

```
orbiter.out -v 4 -diophant -from_scratch linsp30_pt_types 1 3 \
  -coefficient_matrix "6,4,3" -RHS "29,29,1" -x_bounds "0,1,0,27,0,24"
orbiter.out -v 4 -diophant_activity -input_file \
  linsp30_pt_types.diophant -solve_mckay
```

we determine the possibilities

$$(p_7, p_5, p_4) = \begin{Bmatrix} 1 & 5 & 1 \\ 1 & 2 & 5 \\ 0 & 5 & 3 \\ 0 & 2 & 7 \end{Bmatrix}$$

The rows in this matrix are called the point types ($i = 0, 1, 2, 3$). Let b_i be the number of points of type i. By counting points, incident (point,line) pairs by j-lines and pairs of intersecting j-lines, we arrive at the following system:

$$\begin{aligned} b_0 + b_1 + b_2 + b_3 &= 30 \\ b_0 + b_1 &= 7 \\ 5b_0 + 2b_1 + 5b_2 + 2b_3 &= 135 = 27 \cdot 5 \\ b_0 + 5b_1 + 3b_2 + 7b_3 &= 96 = 24 \cdot 4 \\ 10b_0 + b_1 + 10b_2 + b_3 &\le 351 = \binom{27}{2} \\ 10b_1 + 3b_2 + 21b_3 &\le 276 = \binom{24}{2} \end{aligned}$$

Using the Orbiter commands

```
orbiter.out -v 4 -diophant -from_scratch linsp30_pt_distribution 6 4 \
  -coefficient_matrix "1,1,1,1,1,1,0,0,5,2,5,2,1,5,3,7,10,1,10,1,0,10,3,21" \
  -RHS "30,30,1,7,7,1,135,135,1,96,96,1,0,351,2,0,276,2" \
  -x_min_global 0 -x_max_global 30
orbiter.out -v 4 -diophant_activity -input_file \
  linsp30_pt_distribution.diophant -solve_mckay
```

we determine the possibilities

$$(b_0, b_1, b_2, b_3) = \begin{Bmatrix} 2 & 5 & 23 & 0 \\ 3 & 4 & 22 & 1 \\ 4 & 3 & 21 & 2 \\ 5 & 2 & 20 & 3 \\ 6 & 1 & 19 & 4 \\ 7 & 0 & 18 & 5 \end{Bmatrix}$$

16 Design Theory

We use the convention of design theory that the incidence matrix of a design has rows indexed by points and columns indexed by blocks (also called lines). A decomposition is a partition of the points and blocks of the geometry such that each class consists either exclusively of points or exclusively of blocks.

A decomposition is point-tactical if for all points, the number of incident lines in the jth block class depends only on the class of the point. It the point belongs to class i, this number is denoted as a_{ij}. A decomposition is block-tactical if for all blocks, the number of incident points in the ith point class depends only on the class of the block. It the block belongs to class j, this number is denoted as b_{ij}.

A projective plane of order n is a design with $n^2 + n + 1$ points and equally many blocks (also called lines), each of size $n + 1$ such that any two points lie in exactly one block and any two blocks have exactly one point in common. Projective planes are known to exist for all $n = q$ which are a power of a prime. This follows from a construction which utilizes the projective geometry $\mathrm{PG}(2, q)$. Points are the one-dimensional subspaces of $\mathbb{F}_q^3$, blocks are the two-dimensional subspaces of $\mathbb{F}_q^3$, and incidence is natural (inclusion of subspaces). The automorphism group of this design is the collineation group of the projective space. Projective planes other than these exist, though none are known when n is not a prime power. The number of lines through a point equals the number of points on a line. The fact that these numbers exist imply that there is a tactical decomposition. Namely, the trivial decomposition with two classes, one containing all points and one containing all lines. The structure constants of the decomposition are the numbers just described.

The command

```
orbiter.out -v 8  -create_design -q 3 -family PG_2_q  -end
```

creates the design $\mathrm{PG}(2, 3)$. The blocks of the design are encoded in the lexicographic ordering of k-subsets (here $k = 4$) as

$$\{19, 79, 126, 219, 256, 284, 371, 392, 465, 541, 619, 627, 653\}.$$

It also creates the automorphism group of the design (of order 5616). The group is created as matrix group $\mathrm{PGL}(3, 3)$, defined using strong generators:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The program also prints out the tactical decomposition schemes of the design, which are

$\rightarrow$	13_1
13_0	4

$\downarrow$	13_1
13_0	4

17 Linear Spaces and Tactical Decompositions

Suppose we want to study the projective plane of order 16. So, the geometry is is a linear space with $16^2 + 16 + 1 = 273$ points and equally many lines. Each point lies on 17 lines and each line contains 17 points. Any two points lie on exactly one line and any two lines intersect in exactly one point. Of course, the linear space could be the desarguesian plane $\mathrm{PG}(2, 16)$, but it could also be any of the other projective planes of orfer 16. At this point, we are only working with the parameters of the geometry as a linear space, and the isomorphism type of the plane is as yet undecided.

We decide to study maximal arcs of degree 4 in this plane (the degree has to divide the order of the plane). A maximal arc of degree d is a set of points so that each line intersects in either d or zero points. A line which intersects in d points is called a secant. A line which intersects in no point is called an external line. The command

```
orbiter.out -v 4 -maximal_arc_parameters 16 4
```

creates a decomposition stack for the parameters of the arc and writes the file `max_arc_q16_r4.stack`

```
<HTDO type=pt ptanz=2 btanz=2 fuse=simple>
             221       52
      52      17        0
     221      13        4

  1  1
</HTDO>
```

This is a point-tactical decomposition with 2 point-classes and 2 block-classes. The point classes are associated with the rows. The block-classes are associated with the columns. The first row and column indicates the size of the classes. The entries a_{ij} count the number of blocks in the column class j that are incident with a given point in the ith row class. The fuse information at the bottom (`1 1`) is a partition of the row classes which indicates the ancestor decomposition which was column tactical. The next step is to convert the stack file to a tdo file. The command

```
orbiter.out -v 4 -convert_stack_to_tdo max_arc_q16_r4.stack
```

does that. It creates the file `max_arc_q16_r4.tdo`. It also prints the decomposition stack:

```
lambda_scheme at level 2 :
is 1 x 1
          | 273_{  1}
====================
273_{  0} |

row_scheme at level 4 :
is 2 x 2
          | 221_{  1} 52_{  2}
```

```
==============================
 52_{  0} |        17        0
221_{  3} |        13        4

col_scheme at level 3 :
is 1 x 2
          | 221_{  1} 52_{  2}
==============================
273_{  0} |        17       17
```

Next, we can compute all coarsest column-tactical refinements of the decomposition. To this end, the command

```
orbiter.out -v 4 -tdo_refinement -input_file max_arc_q16_r4.tdo \
  -dual_is_linear_space -end
```

is used. Because the incidence structure is a projective plane, the dual is a linear space also. Hence the option `-dual_is_linear_space` can be used, which is helpful to reduce possibilities. As it turns out, there is exactly one refinement, and it is tactical. The file `max_arc_q16_r4r.tdo` is produced. Note the added letter `r` at the end of the file name (r for refinement). We can use the following command to display the decomposition stack in the file:

```
orbiter.out -v 4 -tdo_print max_arc_q16_r4r.tdo
```

This produces the following output:

```
decomposition 0.1:
lambda_scheme at level 2 :
is 1 x 1
          | 273_{  1}
====================
273_{  0} |

row_scheme at level 4 :
is 2 x 2
          | 221_{  1} 52_{  2}
==============================
 52_{  0} |        17        0
221_{  3} |        13        4

col_scheme at level 4 :
is 2 x 2
          | 221_{  1} 52_{  2}
==============================
 52_{  0} |         4        0
221_{  3} |        13       17
```

```
extra_col_scheme at level 3 :
is 1 x 2
          | 221_{  1} 52_{  2}
==============================
273_{  0} |        17       17
```

At the moment, the classes of the partitions are too large to be useful for generation of these objects. It would be helpful to investigate the block derived maximal arc. This would lead to finer partitions with more classes of smaller size. We start with the block tactical decomposition in the following stack file:

```
<HTDO type=bt ptanz=2 btanz=2 fuse=simple>
              221       52
      52       4         0
     221      13         17


  1  1
</HTDO>
```

This decomposition is block-tacktical (note the `type=bt` entry). The matrix entries are b_{ij} which is the number of points in point class i which lie on a line in block class j. The fuse partition is a partition of the block classes. And now for the block derived decomposition. TWe use the stack file `max_arc_q16_r4bd.stack`:

```
<HTDO type=bt ptanz=2 btanz=3 fuse=simple>
              221       51  1
      52       4         0  0
     221      13         17 17


  3
</HTDO>
```

This time, we split off one line from the class of 52 external lines, to yield two classes (of size 51 and 1, respectively). The block partition is three, indicating that all three block classes need to be fused in order to arrive at the row-tactical ancestor decomposition in the stack. We are using the command

```
orbiter.out -v 4 -convert_stack_to_tdo max_arc_q16_r4bd.stack
```

to convert the stack file into a tdo file. This produces the following output:

```
lambda_scheme at level 2 :
is 1 x 1
          | 273_{  1}
======================
273_{  0} |
```

```
row_scheme at level 4 :
is 2 x 2
          | 221_{  1} 52_{  2}
==============================
 52_{  0} |        17        0
221_{  3} |        13        4

col_scheme at level 5 :
is 2 x 3
          | 221_{  1} 51_{  2}  1_{  4}
======================================
 52_{  0} |         4        0        0
221_{  3} |        13       17       17
```

The command

```
orbiter.out -v 4 -tdo_refinement -input_file max_arc_q16_r4bd.tdo \
  -dual_is_linear_space -end
```

is used to compute the coarsest row-tactical refinements. It turns out that there is exactly one, obtained by splitting the point-class of size 221 into two, one of size 204 and one of size 17. The command

```
orbiter.out -v 4 -tdo_print max_arc_q16_r4bdr.tdo
```

can be used to print this decomposition as

	0.1		
$\rightarrow$	221_1	51_2	1_4
52_0	17	0	0
204_3	13	4	0
17_5	13	3	1

	0.1		
$\downarrow$	221_1	51_2	1_4
52_0	4	0	0
221_3	13	17	17

Using the command

```
orbiter.out -v 4 -tdo_refinement -input_file max_arc_q16_r4bdr.tdo \
  -dual_is_linear_space -end
```

we compute the coarsest column tactical refinements yet again. As it turns out, the decomposition does not split because it is already tactical. Another print command

```
orbiter.out -v 4 -tdo_print max_arc_q16_r4bdrr.tdo
```

yields the tactical decomposition

	0.1.1		
$\rightarrow$	221_1	51_2	1_4
52_0	17	0	0
204_3	13	4	0
17_5	13	3	1

	0.1.1		
$\downarrow$	221_1	51_2	1_4
52_0	4	0	0
204_3	12	16	0
17_5	1	1	17

How do we read this decomposition? We see that there is one external line of 17 points. We may call this the line at infinity. Through each of these 17 points at infinity, there are 16 further lines, coming in two types: Always 3 lines have the property that they are external, and the remaining 13 lines are secants. Taken together, all 51 external lines (distinct from the very first line) arises in this way and cover 204 points. Likewise, all 221 secants arise from the second type of lines. Each of the secants has the following type with respect to points: it intersects 4 points of the maximal arc, 12 points from the set of 204, and exactly one point at infinity.

18 Finite Geometry

Orbiter can classify spreads. For instance, the command

```
orbiter.out -v 6 \
  -linear_group -PGGL 4 4 -end \
  -group_theoretic_activities \
    -poset_classification_control \
      -W \
      -problem_label spreads_16_4 \
    -end \
  -spread_classify 2 \
  -end
```

classifies the line-spreads of $\mathrm{PG}(3,4)$ under the action of $\mathrm{P\Gamma L}(4,4)$. Under the André Bruck-Bose construction [1, 8], these spreads correspond to translation planes of order 16 with kernel $\mathbb{F}_4$. Up to isomorphism, there are exactly three line-spreads in $\mathrm{PG}(3,4)$. They are the dearguesian spread, the Hall spread, and the semifield spread. Spreads of $\mathrm{PG}(k-1,q)$ exist in $\mathrm{PG}(n-1,q)$ if and only if k divides n. However, constructing and classifying all spreads is difficult and possible only for very small parameters.

A packing of $\mathrm{PG}(3,q)$ is a set of pairwise line-disjoint spreads that contains all lines. Each line belongs to exactly ne spread in the packing. There are exactly q^2+q+1 spreads in a packing. Orbiter can classify packings. Suppose we want to find packings in $\mathrm{PG}(3,4)$. To this end, we can use the three types of spreads that we just discussed in the previous example. For instance, the command

```
orbiter.out -v 6 \
  -linear_group -PGGL 4 4 -end \
  -group_theoretic_activities \
    -poset_classification_control \
      -W \
      -problem_label packing_4 \
    -end \
  -packing_classify 2 "0,1,2" "SPREAD_TABLES_4/" \
  -end
```

sets up the line-spreads of isomorphism types 0, 1 and 2 in $\mathrm{PG}(3,4)$ and computes a table of all spreads in this space. A packing in $\mathrm{PG}(3,q)$ is a set of pairwise line-disjoint spreds of size q^2+q+1. Packings are difficult to find and classify. For this reason, the problem is often restricted to packings with some non-trivial symmetry assumption. For instance, we may assume that the packings are invariant under a (nontrivial) group H. The problem becomes much more tractable with this assumption. On the other hand, packings with trivial symmetry group cannot be found in this way.

Here is an example. We know that the group $\mathrm{P\Gamma L}(4,4)$ contains three conjugacy classes of involutions, called $2A$, $2B$ and $2C$, respectively. The class $2A$ may be taken to be the class

containing the Baer involutions

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_1 .$$

The following command can be used to construct and classify all packings which are invariant under the Baer involution:

```
orbiter.out -v 6 \
  -linear_group -PGGL 4 4 -end \
  -group_theoretic_activities \
    -poset_classification_control \
      -W \
      -problem_label packing_4 \
    -end \
    -packing_classify 2 "0,1,2" "SPREAD_TABLES_4/" \
    -packing_with_assumed_symmetry \
      -H -PGGL 4 4 -subgroup_by_generators "2A" 2 1 \
        "1,0,0,0, 0,1,0,0, 0,0,1,0, 0,0,0,1, 1" \
      -end \
      -N -PGGL 4 4 -subgroup_by_generators "centralizer_2A" "40320" 10 \
        "1,0,0,0,0,1,0,0,0,0,1,0,0,0,0,1,1," \
        "1,0,0,0,0,1,0,0,0,0,1,0,1,1,0,1,1," \
        "1,0,0,0,0,1,0,0,0,0,1,0,0,1,1,1,1," \
        "1,0,0,0,0,1,0,0,0,0,1,0,1,1,1,1,0," \
        "1,0,0,0,0,1,0,0,1,1,0,1,1,1,1,0,0," \
        "1,0,0,0,0,1,0,0,1,0,1,0,0,1,0,1,1," \
        "1,0,0,0,0,1,0,0,0,0,1,1,1,0,1,0,0," \
        "1,0,0,0,1,1,1,1,1,0,1,0,1,1,1,0,0," \
        "1,0,0,0,0,1,1,0,0,0,1,1,1,0,1,0,0," \
        "0,1,0,0,0,1,0,1,1,1,0,1,0,1,1,1,1," \
      -end \
    -cliques_on_fixpoint_graph 7 \
    -cliques_on_fixpoint_graph_control -W \
      -problem_label PGGL_4_4_2A_fixp_cliques -end \
    -end \
  -end
```

The option `-H` is used to specify the assumed symmetry group H. The option `-N` is used to specify the normalizer of H in $G = \mathrm{P\Gamma L}(4,4)$. We use the generators for the groups obtained in Section 6. The normalizer is used to help with the classification of the packings. The command will compute the orbits of H on spreads and test which of them consist of line-disjoint spreads. The orbits are classified by length. The orbits of length one (fixed spreads) are selected and a graph is defined. The vertices of the graph are the spreads that

are fixed. Two vertices are adjacent if the associated spreads are line-disjoint. A packing must be related to a clique of a certain size in this graph. In order to find these cliques, poset classification is applied to classify all cliques of size 7 under the action of N. Note that because $N \leq N_G(H)$, N is contained in the automorphism group of the graph and hence induces an action on cliques of any size in the graph. As it turns out, there are exactly 440 types of cliques of size 7.

A drawing of the adjacency matrix of the graph can be produced using the following command sequence:

```
orbiter.out -v 2 -create_graph -load_from_file \
  PGGL_4_4_Subgroup_2A_2_fixp_graph.bin -end \
  -graph_theoretic_activity -export_csv \
  PGGL_4_4_Subgroup_2A_2_fixp_graph.csv -end
orbiter.out -v 2 -draw_matrix PGGL_4_4_Subgroup_2A_2_fixp_graph.bin.csv 2
```

19 The Povray Interface

Orbiter can be used to create raytracing 3D-graphics. Orbiter serves as a front end for the raytracing software Povray [17]. This is a multi step process: A 3D scene is defined through orbiter commands. Next, Orbiter produces Povray files. After that, the povray files are processed through povray, and turned into graphics files (png), called frames. The frames can be turned into a video by using tools like ffmpeg. By default, an rotational animation is produced. Tables 27- 28 list the commands to control the 3D-povray frontend.
Tables 29 and 30 summarizes the Orbiter commands to build objects of a 3D scene. Building the scene itself does not create any graphical output. To this end, the commands in Table 31 are used. Each of these commands applies to a group of objects of the same kind. Groups of objects are created using the commands in Table 30 which start with `group_of`. Here is a simple example which combines scene building and graphical output. The example creates a cube with vertices, edges and faces:

```
1       orbiter.out -v 2 -povray \
2           -round 0 -nb_frames_default 30 -output_mask cube_%d_%03d.pov \

3           -video_options -W 1024 -H 768  -global_picture_scale 0.5 \
4             -default_angle 75 -clipping_radius 2.7 \
5           -end \
6           -scene_objects \
7             -obj_file cube_centered.obj \
8             -edge "0, 1" \
9             -edge "0, 2" \
10            -edge "0, 4" \
11            -edge "1, 3" \
12            -edge "1, 5" \
13            -edge "2, 3" \
14            -edge "2, 6" \
15            -edge "3, 7" \
16            -edge "4, 5" \
17            -edge "4, 6" \
18            -edge "5, 7" \
19            -edge "6, 7" \
20            -group_of_things_as_interval 0 8 \
21            -spheres 0 0.3 "texture{ Polished_Chrome pigment{quick_color
        White} }" \
22            -group_of_things_as_interval 0 6 \
23            -prisms 1 0.05 "texture{ pigment{ color Yellow transmit 0.7
       } finish {diffuse 0.9 phong 0.6} }" \
24            -group_of_things_as_interval 0 12 \
25            -cylinders 2 0.15 "texture{ pigment{ color Red  } finish {di
       ffuse 0.9 phong 0.6} }" \
26          -scene_objects_end \
27          -povray_end
28
```

Option	Arguments	Meaning
-do_not_rotate		No rotation.
-rotate_about_ z_axis		Rotate around z-axis.
-rotate_about_111		Rotate around $(1, 1, 1)$-axis (default).
-rotate_about_ custom_axis	axis	Rotate around a custom axis. The axis is specified as a vector of length 3.
-boundary_none		Remove the clipping.
-boundary_box		Clip using a box shape.
-boundary_sphere		Clip using a sphere (default).
-font_size	s	Set font size to s.
-stroke_width	s	Set text depth to s.
-omit_bottom_plane		Remove the bottom plane.
-W	w	Set output dimension to w pixels wide.
-H	h	Set output dimension to h pixels height.
-nb_frames	n	Set number of frames to n. One revolution around the axis is split into n frames.
-zoom	$r\ a_s\ a_t\ c_s\ c_t$	Set zoom angle and clipping with in round r to change from a_s to a_t and from c_s to c_t, respectively.
-pan	$r\ F\ T\ C$	In round r, pan the camera from location F to location T in a rotational movement with center at C. Each of F, T, C are three dimensional coordinates.
-pan_reverse	$r\ F\ T\ C$	Same as -pan, but camera movement is in opposite order.
-no_background		Remove background.
-no_bottom_plane	r	Remove bottom plane in round r.
-camera	$r\ S\ C\ L$	In round r, set camera location at C, sky at S and pointing towards L. Each of S, C, L are three-dimensional coordinate vectors.
-clipping	$r\ c$	In round r, set clipping radius to c.

Table 27: Options for Orbiter 3D-graphics (Part 1)

Option	Arguments	Meaning
`-text`	r a text	In round r, produce running text `text` with sustain value a.
`-label`	r s a g text	In round r, produce running text `text` with start value s, sustain s and gravity g.
`-latex`	r s a praeamble g text l fname	In round r, produce running latex text `text` with start value s, sustain s and gravity g. Put `praeamble` in the latex source code. Use `fname` for the latex file names (no extension).
`-global_picture_scale`	d	Set scaling factor to d.
`-picture`	r d fname options	In round r, place picture `fname` scaled by d using options.
`-picture`	r d fname options	In round r, place picture `fname` scaled by d using options.
`-look_at`	L	Override camera look-at value to L. L is a three-dimensional vector.
`-default_angle`	a	Set default camera angle to a.
`-clipping_radius`	f	Set default clipping radius to f.
`-scale_factor`	s	Set default scale factor to s.
`-line_radius`	s	Set default line radius to s.

Table 28: Options for Orbiter 3D-graphics (Part 2)

Command	Arguments	Purpose
`-cubic_lex`	coeffs	Cubic surface given by 20 coefficients in lexicographic ordering
`-cubic_orbiter`	coeffs	Cubic surface given by 20 coefficients in Orbiter ordering
`-cubic_Goursat`	$A\ B\ C$	Cubic surface with tetrahedral symmetry given by 3 Goursat coefficients as $Axyz + B(x^2 + y^2 + z^2) + C = 0$
`-quadric_lex_10`	coeffs	Quadric surface given by 10 coefficients in lexicographic ordering
`-quartic_lex_35`	coeffs	Quartic surface given by 35 coefficients in lexicographic ordering
`-octic_lex_165`	coeffs	Octic surface given by 165 coefficients in lexicographic ordering
`-point`	coeffs	Point given by three coordinates
`-point_list_from_csv_file`	fname	List of points with coordinates given in a csv file
`-line_through_two_points_recentered_from_csv_file`	fname	List of lines through two points with point coordinates given in a csv file
`-line_through_two_points_from_csv_file`	fname	List of lines through two points with point coordinates given in a csv file
`-point_as_intersection_of_two_lines`	$i_1\ i_2$	Create a point from the intersection of two lines i_1 and i_2
`-edge`	$i_1\ i_2$	Create an edge (line segment) between points i_1 and i_2
`-text`	$i_1\ s$	Create a label s located at the point i
`-triangular_face_given_by_three_lines`	$i_1\ i_2\ i_3$	Create a triangular face give by three lines i_1, i_2, i_3
`-face`	pts	Create a face through the vertices pts, ordered cyclically
`-quadric_through_three_skew_lines`	$i_1\ i_2\ i_3$	Create a quadric through three skew lines
`-plane_defined_by_three_points`	$i_1\ i_2\ i_3$	Create a plane through three noncollinear points
`-line_through_two_points_recentered`	pt-coords	Create a line through two points given by 6 coordinates, recentered

Table 29: Scene definition commands (part 1)

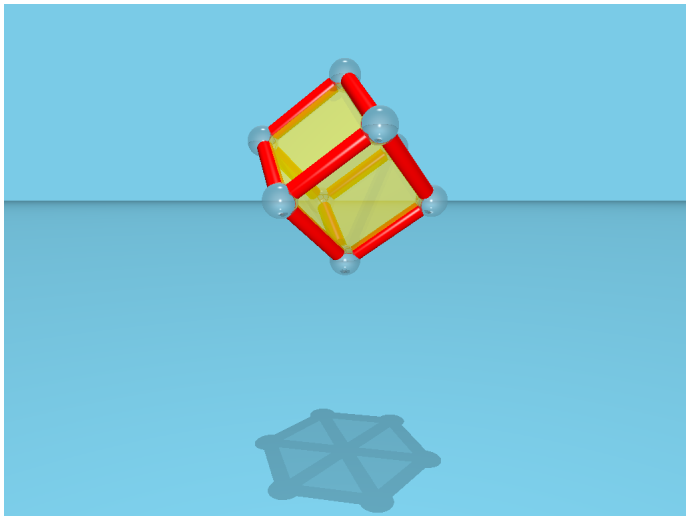
Command	Arguments	Purpose
`-line_through_two_points`	pt-coords	Create a line through two points given by 6 coordinates
`-line_through_two_existing_points`	i_1 i_2	Create a line through two points
`-line_through_point_with_direction`	coeffs6	Create a line through a point (x, y, z) with a given direction (u_x, u_y, u_z), where coeffs6 $= x, y, z, u_x, u_y, u_z$
`-plane_by_dual_coordinates`	coeffs4	Create a plane $ax + by + cz + d = 0$ give four dual coordinates as coeff4 $= a, b, c, d$
`-dodecahedron`		Create a Dodecahedron centered at the origin (20 points, 30 edges, 12 faces)
`-Hilbert_Cohn_Vossen_surface`		Create the Hilbert, Cohn-Vossen surface (1 cubic surface, 45 tritangent planes, 27 lines)
`-obj_file`	fname	Read points and faces from the given .obj file
`-group_of_ things`	list	Create a group of things from the given list
`-group_of_things_with_offset`	list offset	Create a group of things from the given list, each value is increase by offset
`-group_of_things_as_interval`	a b	Create a group of things from the interval $a, \ldots, a + b - 1$
`-group_of_things_as_interval_with_exceptions`	a b ex	Create a group of things from the interval $a, \ldots, a+b-1$ with the exceptional elements in the list ex removed
`-group_of_all_points`		Create a group of things from all points currently defined
`-group_of_all_faces`		Create a group of things from all faces currently defined
`-group_subset_at_random`	i f	Create a group of things from the existing group i by picking a random subset with probability f
`-create_regulus`	i N	Create a regulus for quadric i with N lines

Table 30: Scene definition commands (part 2)

Command	Arguments	Purpose
`-spheres`	i r prop	For each element in point group i, create a sphere of radius r with given Povray properties.
`-cylinders`	i r prop	For each element in edge group i, create a cylinder of radius r with given Povray properties.
`-prisms`	i d prop	For each element in face group i, create a prism of half-thickness d with given Povray properties.
`-planes`	i prop	For each element in plane group i, create a plane with given Povray properties.
`-lines`	i r prop	For each element in line group i, create a line of radius r with given Povray properties.
`-cubics`	i prop	For each element in group i of cubics, create a surface with given Povray properties.
`-quadrics`	i prop	For each element in group i of quadrics, create a surface with given Povray properties.
`-quartics`	i prop	For each element in group i of quartics, create a surface with given Povray properties.
`-octics`	i prop	For each element in group i of octics, create a surface with given Povray properties.
`-texts`	i d s prop	For each element in group i of labels, create a text element with half-thickness d and size s with given Povray properties.

Table 31: Graphical output commands

This command will tell Orbiter to create 30 povray files (extension .pov), one for each frame of a rotating scene. The scene containes a cube whose vertices are shown in chrome, whose edges are in red, and whose faces are yellow and transparent. The cube turns around a vertical axis of symmetry. Here is the first frame of the result:



The coordinates of the cube are stored in an object file `cube_centered.obj`. The content of this file is:

```
v -1 -1 -1
v 1 -1 -1
v -1 1 -1
v 1 1 -1
v -1 -1 1
v 1 -1 1
v -1 1 1
v 1 1 1
f 1 2 4 3
f 1 2 6 5
f 1 3 7 5
f 2 4 8 6
f 3 4 8 7
f 5 6 8 7
```

Here is a simple example of a cubic surface, called the monkey saddle. The equation of the surface is

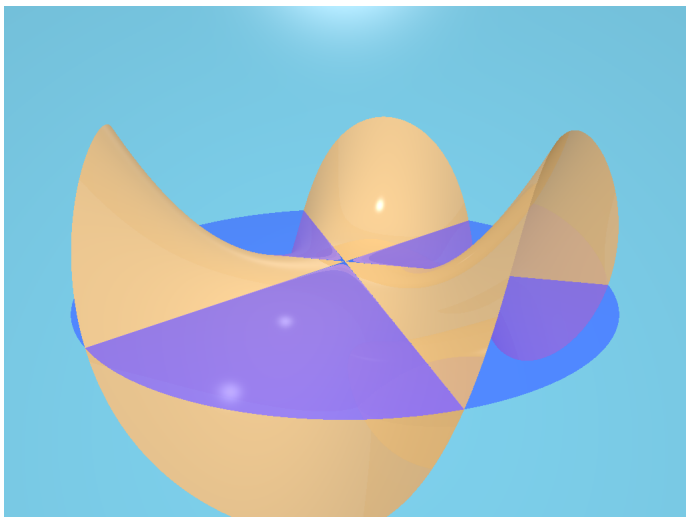
$$z = x^3 - 3xy^2$$

The example plots the surface together with the tangent plane at $(0, 0, 0)$, rotated around the z-axis.

```
1       orbiter.out -v 2 -povray \
2          -round 0 -nb_frames_default 30 -output_mask monkey_%d_%03d.pov \
```

```
3          -video_options -W 1024 -H 768  -global_picture_scale 0.8 \
4          -default_angle 75 -clipping_radius 0.8 \
5          -camera 0 "0,0,1" "1,1,0.5" "0,0,0"  \
6          -rotate_about_z_axis \
7          -end \
8          -scene_objects \
9             -cubic_lex "1,0,0,0,-3,0,0,0,0,0,0,0,0,0,0,0,0,0,-1,0" \
10            -plane_by_dual_coordinates "0,0,1,0" \
11            -group_of_things "0" \
12            -group_of_things "0" \
13            -cubics 0 "texture{ pigment{ Gold } finish {ambient 0.4 diffus
        e 0.5 roughness 0.001 reflection 0.1 specular .8} }" \
14            -planes 1 "texture{ pigment{ color Blue transmit 0.5 } finish
        { diffuse 0.9 phong 0.2}}" \
15         -scene_objects_end \
16         -povray_end
17
```

Here is one of the frames that are created:



Here is another cubic surface, called Hilbert Cohn-Vossen. The equation of the surface is

$$\frac{5}{2}xyz - (x^2 + y^2 + z^2) + 1 = 0.$$

```
1          orbiter.out -v 2 -povray \
2             -round 0 -nb_frames_default 30 -output_mask HCV_%d_%03d.pov \
3             -video_options -W 1024 -H 768  -global_picture_scale 0.9 \
4             -default_angle 75 -clipping_radius 2.4 \
5             -camera 0 "1,1,1" "-3,1,3" "0.12,0.12,0.12"  \
6             -end \
7             -scene_objects \
8                -Hilbert_Cohn_Vossen_surface \
9                -group_of_things "0" \
```

```
10              -cubics 0 "texture{ pigment{ White*0.5 transmit 0.5 } finish
        {ambient 0.4 diffuse 0.5 roughness 0.001 reflection 0.1 specular
        .8} }" \
11              -group_of_things_as_interval 0 6 \
12              -group_of_things_as_interval 6 6 \
13              -group_of_things_as_interval_with_exceptions 12 15 "14,19,23
        " \
14              -lines 1 0.02 "texture{ pigment{ color Red } finish { diffus
        e 0.9 phong 1}}" \
15              -lines 2 0.02 "texture{ pigment{ color Blue } finish { diffu
        se 0.9 phong 1}}" \
16              -lines 3 0.02 "texture{ pigment{ color Yellow } finish { dif
        fuse 0.9 phong 1}}" \
17              -label 0 "a1" -label 2 "a2" -label 4 "a3" \
18              -label 6 "a4" -label 8 "a5" -label 10 "a6" \
19              -label 12 "b1" -label 14 "b2" -label 16 "b3" \
20              -label 18 "b4" -label 20 "b5" -label 22 "b6" \
21              -label 24 "c12" -label 26 "c13" -label 30 "c15" \
22              -label 32 "c16" -label 34 "c23" -label 36 "c24" \
23              -label 40 "c26" -label 42 "c34" -label 44 "c35" \
24              -label 48 "c45" -label 50 "c46" -label 52 "c56" \
25              -group_of_things_as_interval 0 6 \
26              -texts 4 0.2 0.15 "texture{ pigment{Black} } no_shadow" \
27              -group_of_things_as_interval 6 6 \
28              -texts 5 0.2 0.15 "texture{ pigment{Black} } no_shadow" \
29              -group_of_things_as_interval 12 12 \
30              -texts 6 0.2 0.15 "texture{ pigment{Black} } no_shadow" \
31            -scene_objects_end \
32            -povray_end
33
```

Figure 13 shows the final product. The Schlaefli labeling of lines can be seen. Orbiter can plot functions using a built-in function tracker. The functions must be continuous apart from a finite number of poles. The function can have multiple components, each described using an expression. Each expression is specified in Reverse Polish Notation (RPN). Consider an example. A Lissajous curve is defined using coordinate functions of the form

$$x = r\sin\big(at + c\big), \quad y = r\sin\big(bt\big), \quad a, b, c, r \in \mathbb{R}.$$

The terms

$$r\sin\big(at + c\big), \quad r\sin\big(bt\big)$$

are the expressions of the two coordinate functions. RPN means that the operator is listed after the operands. A stack data structure is used to hold temporary values. Operators are pushed to the top of the stack using the push commands. A binary operator pops the two elements from the stack, performs the operation, and pushes the resulting value back onto the stack. For a unary operator, only one element is poped and replaced by the result. Here

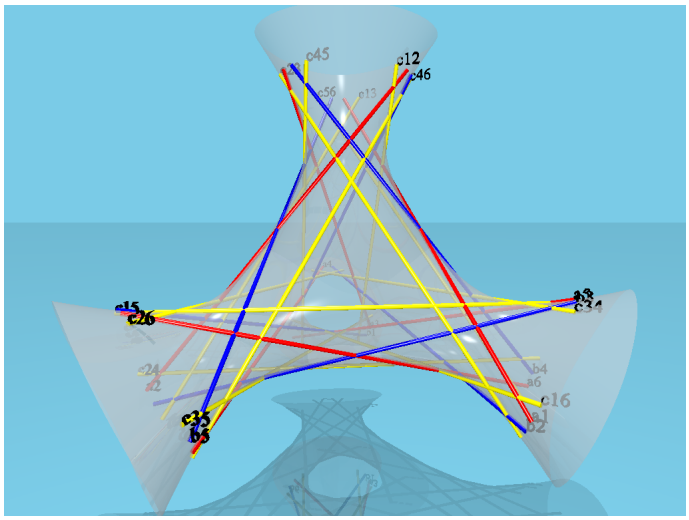


Figure 13: The Hilbert Cohn-Vossen surface

are some examples of expressions rewritten in RPN:

$$\begin{aligned} \sin(x) &\mapsto \text{ push } x \text{ sin}, \\ a+b &\mapsto \text{ push } a \text{ push } b \text{ add}, \\ a \cdot b &\mapsto \text{ push } a \text{ push } b \text{ mult}. \end{aligned}$$

The coordinate functions are enclosed between `-code` and `-code_end` commands. Each coordinate function is described in RPN and terminated using a `return` keyword. By the time the `return` keyword is reached, the RPN expression must have exactly one value on the stack which is considered the value of the expression. Constants are declared between the `-const` and `-const_end` keywords. Likewise, variables are declared between the `-var` and `-var_end` keywords. Picking $a = 3$, $b = 2$, $c = \pi/2$ and $r = 7$, the function is computed using

```
orbiter.out -v 2 -smooth_curve "lissajous" 0.07 2000 15 0 18.85 \
    -const a 3 b 2 c 1.57 r 7 -const_end \
    -var t -var_end \
    -code \
      push t push a mult push c add sin push r mult return \
      push t push b mult sin push r mult  return \
    -code_end
```

The sequence

```
push t push a mult push c add sin push r mult
```

is $r \sin(at + c)$ expressed in RPN. The constants are defined in the line

```
-const a 3 b 2 c 1.57 r 7 -const_end
```

The input variable is defined using the line

```
-var t -var_end
```

The sequence

```
-smooth_curve "lissajous" 0.07 2000 15 0 18.85
```

defines the name of the output file, the fact that two consequtive points are never further than $\epsilon = 0.07$ away, the fact that points that are 15 or more away from the origin should be ignored, and the fact that the variable t loops over the range $[0, 18.85]$ with a default of 2000 steps. The evaluator automatically reduces the step-size if consecutive image points are more than ϵ apart. The code to produce the plot is

```
1       orbiter.out -v 2 -povray \
2          -round 0 -nb_frames_default 1 -output_mask lissajous_%d_%03d.pov
        \
3          -video_options -W 1024 -H 768  -global_picture_scale 0.40 \
4          -default_angle 45 -clipping_radius 5 -omit_bottom_plane  \
5          -camera 0 "0,-1,0" "0,0,12" "0,0,0"  \
6          -rotate_about_z_axis \
7          -end \
8          -scene_objects \
9             -line_through_two_points_recentered_from_csv_file coordinate_g
       rid.csv \
10            -group_of_things "0" \
11            -group_of_things "1" \
12            -group_of_things "2" \
13            -lines 0 0.09 "texture{ pigment{ color Yellow } }" \
14            -lines 1 0.09 "texture{ pigment{ color Yellow } }" \
15            -lines 2 0.09 "texture{ pigment{ color Yellow } }" \
16            -group_of_things_as_interval 3 39 \
17            -lines 3 0.02 "texture{ pigment{ color Black } }" \
18            -point_list_from_csv_file function_lissajous_N2000_points.csv
       \
19            -group_of_things_as_interval 0 6524\
20            -spheres 4 0.1 "texture{ pigment{ color Red } finish { diffuse
        0.9 phong 1}}" \
21            -plane_by_dual_coordinates "0,0,1,0" \
22            -group_of_things "0" \
23            -planes 5 "texture{ pigment{ color Blue*0.5 transmit 0.5 } }"
       \
24         -scene_objects_end \
25         -povray_end
26
```

The plot is shown in Figure 14. We can turn it into a 3D plot by using the t value for the z coordinate. The code to produce the 3D plot is

```
1       orbiter.out -v 2 -povray \
2          -round 0 -nb_frames_default 30 -output_mask lissajous_3d_%d_%03d
```

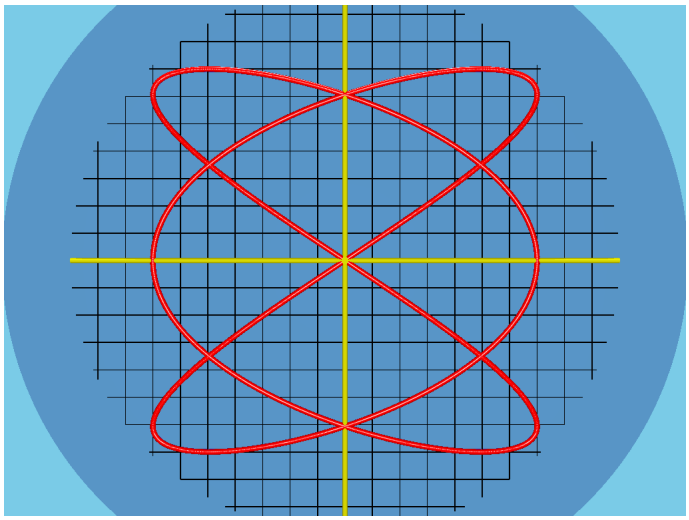


Figure 14: Lissajous figure

```
        .pov \
3          -video_options -W 1024 -H 768  -global_picture_scale 0.40 \
4          -default_angle 45 -clipping_radius 5 -omit_bottom_plane  \
5          -camera 0 "0,0,1" "7,7,5" "0,0,1"  \
6          -rotate_about_z_axis \
7          -end \
8          -scene_objects \
9             -line_through_two_points_recentered_from_csv_file coordinate_g
        rid.csv \
10            -group_of_things "0" \
11            -group_of_things "1" \
12            -group_of_things "2" \
13            -lines 0 0.09 "texture{ pigment{ color Yellow } }" \
14            -lines 1 0.09 "texture{ pigment{ color Yellow } }" \
15            -lines 2 0.09 "texture{ pigment{ color Yellow } }" \
16            -group_of_things_as_interval 3 39 \
17            -lines 3 0.02 "texture{ pigment{ color Black } }" \
18            -point_list_from_csv_file function_lissajous_3d_N2000_points.c
        sv \
19            -group_of_things_as_interval 0 6538\
20            -spheres 4 0.1 "texture{ pigment{ color Red } finish { diffuse
         0.9 phong 1}}" \
21            -plane_by_dual_coordinates "0,0,1,0" \
22            -group_of_things "0" \
23         -scene_objects_end \
24         -povray_end
25
```

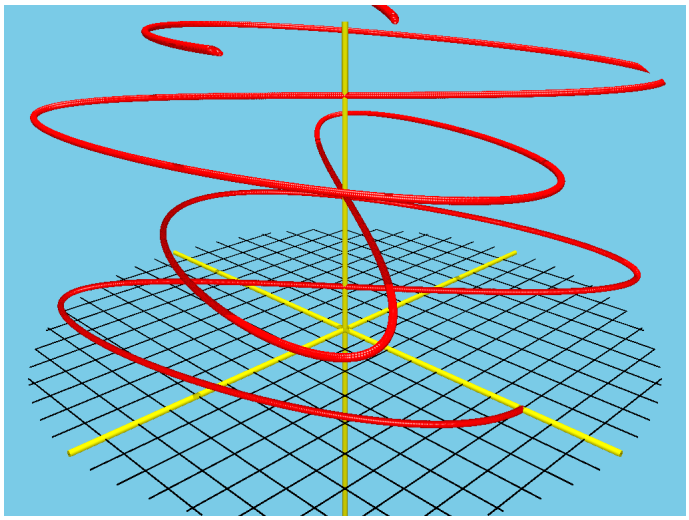


Figure 15: Lissajous Spacecurve

The function is computed using the command

```
orbiter.out -v 2 -smooth_curve "lissajous_3d" 0.07 2000 50 0 18.85 \
    -const a 3 b 2 c 1.57 r 7 -const_end \
    -var t -var_end \
    -code \
      push t push a mult push c add sin push r mult return \
      push t push b mult sin push r mult  return \
      push t return \
    -code_end \
```

The 3D curve is shown in Figure 15.

The Endrass octic [9] is the algebraic surface given by the equation

$$X8 := 64\left(-w^2+x^2\right)\left(-w^2+y^2\right)\left((x+y)^2-2w^2\right)\left((x-y)^2-2w^2\right)-\left(-4\left(1+\sqrt{2}\right)\left(x^2+y^2\right)^2+\left(8\left(2+\sqrt{2}\right)z^2+2\left(2+7\sqrt{2}\right)w^2\right)\left(x^2+y^2\right)-16z^4+8\left(1-2\sqrt{2}\right)z^2w^2-\left(1+12\sqrt{2}\right)w^4\right)^2$$

The following Orbiter command creates a povray graphics of the octic, shown in Figure 16:

```
1       orbiter.out -v 2 -povray \
2          -round 0 -nb_frames_default 30 -output_mask endrass_octic_%d_%03
        d.pov \
3          -video_options -W 1024 -H 768  -global_picture_scale 0.75 -defau
        lt_angle 75 -clipping_radius 3.7 -no_bottom_plane \
4          -camera 0 "1,1,1" "6,6,3" "0,0,0"  \
5          -rotate_about_111 \
6          -end \
7          -scene_objects \
```

```
8             -line_through_two_points_recentered_from_csv_file coordinate_g
      rid.csv \
9             -group_of_things "0" \
10            -group_of_things "1" \
11            -group_of_things "2" \
12            -group_of_things_as_interval 3 39 \
13            -lines 0 0.15 "texture{ pigment{ color Red } finish { diffuse
      0.9 phong 1}}" \
14            -lines 1 0.15 "texture{ pigment{ color Green } finish { diffus
      e 0.9 phong 1}}" \
15            -lines 2 0.15 "texture{ pigment{ color Blue } finish { diffuse
       0.9 phong 1}}" \
16            -lines 3 0.05 "texture{ pigment{ color Black } finish { diffus
      e 0.9 phong 1}}" \
17            -octic_lex_165 "-93.2548,0,0,0,-309.019,0,0,527.529,0,395.647,
      0,0,0,0,0,0,0,0,0,0,-687.529,0,0,1582.59,0,1186.94,0,0,0,0,-1055.0
      6,0,-1582.59,0,-593.47,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-
      309.019,0,0,1582.59,0,1186.94,0,0,0,0,-2110.12,0,-3165.17,0,-1186.
      94,0,0,0,0,0,0,874.039,0,1560.63,0,1677.92,0,343.362,0,0,0,0,0,0,0
      ,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-93.254
      8,0,0,527.529,0,395.647,0,0,0,0,-1055.06,0,-1582.59,0,-593.47,0,0,
      0,0,0,0,874.039,0,1560.63,0,1677.92,0,343.362,0,0,0,0,0,0,0,0,-256
      ,0,-468.077,0,-789.019,0,-525.726,0,0.941125" \
18            -plane_by_dual_coordinates "0,0,1,0" \
19            -group_of_things "0" \
20            -group_of_things "0" \
21            -octics 4 "texture{ pigment{ White*0.5 transmit 0.5 } finish {
      ambient 0.4 diffuse 0.5 roughness 0.001 reflection 0.1 specular .8
      } }" \
22            -planes 5 "texture{ pigment{ color Blue transmit 0.5 } finish
      { diffuse 0.9 phong 1}}" \
23         -scene_objects_end \
24         -povray_end
25
```

This illustration includes coordinate axes and the x, y-plane.

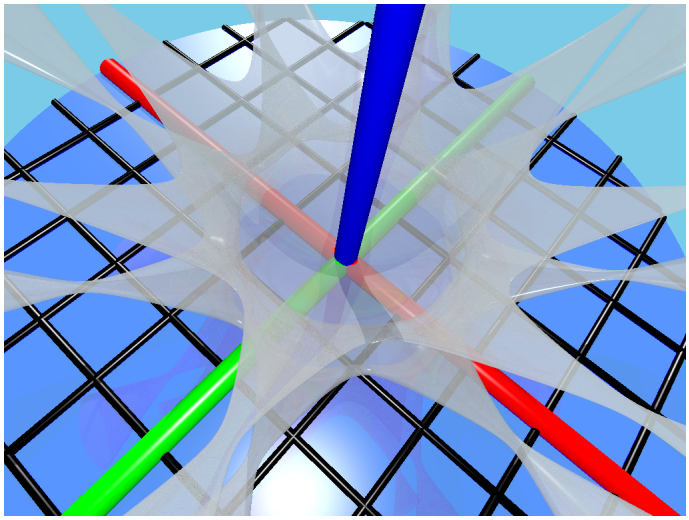


Figure 16: The Endrass Octic

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