Computational Mathematics and Modeling, Vol. 11, No. 4, 2000

GENERALIZED HOPF FORMULAS FOR THE NONAUTONOMOUS HAMILTON–JACOBI EQUATION

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UDC 517.957, 517.977

Generalized Hopf formulas are provided for minimax (viscosity) solutions of Hamilton–Jacobi equations of the form $V_t + H(t, D_x V) = 0$ and $V_t + H(t, V, D_x V) = 0$ with the boundary condition $V(T, x) = \varphi(x)$, where φ is a convex function. The bounds within which these formulas apply are elucidated.

1. Introduction

In this article, we study the following Cauchy problem for a Hamilton–Jacobi equation:

$$\frac{\partial V(t,x)}{\partial t} + H\left(t, \frac{\partial V(t,x)}{\partial x}\right) = 0, \qquad (t,x) \in G = (0,T) \times R^n,$$
$$V(T,x) = \varphi(x), \qquad x \in R^n. \tag{1.1}$$

Such equations arise, in particular, for problems of optimal control theory with an integral-terminal functional of the form

$$\dot{x} = f(\tau, u), \qquad u(\tau) \in P(\tau), \qquad 0 \le \tau, t \le T, \qquad x(t) = \xi,$$
$$\int_t^T h(\tau, u)\, d\tau + \varphi(x(T)) \to \inf_{u(\cdot)}, \tag{1.2}$$
$$V(t,\xi) = \inf_{u(\cdot)} \left\{ \int_t^T h(\tau, u)\, d\tau + \varphi(x(T)) \right\}.$$

The cost function $V(t, x)$ in these problems is a generalized solution of the Hamilton–Jacobi equation. There are many equivalent ways to define a generalized solution. Here we use the notion of minimax solutions [1, 2].

Problem (1.1) has been previously studied in [3, 4], where the solution is represented as a set-valued integral. In the present study we represent the solution of (1.1) as the solution of some finite-dimensional optimization problem. This representation is fairly convenient for both theoretical work and numerical calculations. The Hopf formula [5] is a classical example of this case. The Hopf formula is applicable only when the Hamiltonian $H(t, x, s)$ is independent of both time t and the space coordinate x. The formula considered in this article removes in a certain sense the restriction of time-independent Hamiltonian and is a natural generalization of the classical Hopf formula. This is achieved at the cost of sacrificing some generality: the class of functions φ in the boundary condition for which the corresponding formula is a solution of the Cauchy problem constitutes only a subset of the class of convex functions and is of course dependent on the Hamiltonian.

Thus, if the function φ is convex, then under certain conditions on the Hamiltonian $H(t, s)$ we have the formula

Translated from Prikladnaya Matematika i Informatika, No. 3, pp. 81–89, 1999.

1046–283X/00/1104–0391 $25.00

$$V(t, x) = \sup_{s \in R^n} \left((s, x) + \int_t^T H(\tau, s) d\tau - \varphi^*(s) \right), \qquad (t, x) \in \overline{G}, \tag{1.3}$$

where $V(t, x)$ is the solution of problem (1.1). Here φ^* is the Fenchel-conjugate of φ [6]. If $H(t, s) \equiv H^0(t, s)$, then (1.3) reduces to Hopf's classical formula [5]. Unfortunately, contrary to the classical case, problem (1.1) requires convexity of both the function φ and the Hamiltonian itself on the set of so-called maximizers, i.e., the set of s on which (1.3) attains a supremum. We will show below that in the classical case the latter requirement is satisfied automatically. Here we also give sufficient conditions when (1.3) defines a minimax solution of (1.1) for all convex functions φ. In particular, it is sufficient for the Hamiltonian $H(t, s)$ to be convex or concave in s. This case corresponds to optimal control problems of the form (1.2), and therefore formula (1.3) can be essentially derived by tools of convex analysis. Note that in [4] it is shown for a Hamiltonian $H(t, s)$ concave in s that the formula defined by a set-valued integral reduces to formula (1.3). In general, when the Hamiltonian $H(t, s)$ is arbitrary, formula (1.3) is only a lower solution, i.e., a lower bound of the solution of (1.1).

We also consider the following Cauchy problem for the Hamilton–Jacobi equation:

$$\begin{gathered} \frac{\partial V(t, x)}{\partial t} + H\left(t, V(t, x), \frac{\partial V(t, x)}{\partial x}\right) = 0, \qquad (t, x) \in G, \\ V(T, x) = \varphi(x), \qquad x \in R^n. \end{gathered} \tag{1.4}$$

Such equations arise for optimal control problems with a Chebyshev functional of the form

$$\dot{x} = f(\tau, u), \qquad u(\tau) \in P(\tau), \qquad 0 \le \tau, t \le T, \qquad x(t) = \xi,$$

$$\operatorname{ess} \sup_{t \le \tau \le T} h(\tau, u) \vee \varphi(x(T)) \to \inf_{u(\cdot)},$$

$$V(t, \xi) = \inf_{u(\cdot)} \{ \operatorname{ess} \sup_{t \le \tau \le T} h(\tau, u) \vee \varphi(x(T)) \}.$$

Here $a \vee b$ stands for $\max\{a, b\}$. For more details about problems of this type see [7]. Our formula is a generalization of the Lax–Hopf formula derived in [8]. The conditions when the corresponding Hopf formula defines a solution of problem (1.4) are similar to the conditions for formula (1.3).

2. Generalized Hopf formulas

Consider the Hamilton–Jacobi equation (1.1):

$$\frac{\partial V(t, x)}{\partial t} + H\left(t, \frac{\partial V(t, x)}{\partial x}\right) = 0, \qquad (t, x) \in G,$$

$$V(T, x) = \varphi(x), \qquad x \in R^n,$$

where $G = (0, T) \times R^n$.

Let us state the main assumptions regarding $H(t, s)$ and $\varphi(x)$:

(A) The function $\varphi \colon R^n \to R$ is convex;

(B) The function $H\colon (0,T)\times R^n \to R$ is continuous in t on $(0,T)$ for every $s\in R^n$ and

(i) for all $(t,s)\in(0,T)\times S$, where $S=\left\{s\in R^n\colon \|s\|=1\right\}$, the limit $\lim_{r\downarrow 0} rH(t,s/r)=H_0(t,s)$ exists and $H_0(\cdot,s)$ is continuous on $(0,T)$ for every $s\in S$;

(ii) for all $t\in(0,T)$, $(s',r'),(s'',r'')\in\overline{B}_+$, we have

$$\left|r'H(t,s'/r') - r''H(t,s''/r'')\right| \le L\cdot\left(\|s'-s''\|^2 + (r'-r'')^2\right)^{1/2},$$

where $\overline{B}_+ = \{(s,r)\in R^n\times R\colon \|s\|^2 + r^2 \le 1,\ r>0\}$;

(iii) for all $s\in R^n$, the function $H(\cdot,s)$ is summable on $(0,T)$.

Before proceeding with the main results, we give the definition of minimax solutions from [1]. We introduce the set-valued mappings

$$\overline{F}_{\hat{A}}(t,q) = \{(f,g)\in\sqrt{2}L\cdot\overline{B}\colon (q,f) + g \ge H(t,q)\},$$

$$\overline{F}_{\hat{I}}(t,p) = \{(f,g)\in\sqrt{2}L\cdot\overline{B}\colon (p,f) + g \le H(t,p)\}.$$

Here $\overline{B} = \{(s,r)\in R^n\times R\colon \|s\|^2 + r^2 \le 1\}$. Then we know [1] that for all $p,q\in R^n$ the mappings $\overline{F}_{\hat{A}}(t,q)$ and $\overline{F}_{\hat{I}}(t,p)$ are upper semicontinuous, take nonempty convex compact values, and

$$\sup_{q\in R^n}\min\left\{(s,f) + g \,\middle|\, (f,g)\in\overline{F}_{\hat{A}}\right\} = H(t,s),$$

$$\inf_{p\in R^n}\max\left\{(s,f) + g \,\middle|\, (f,g)\in\overline{F}_{\hat{I}}\right\} = H(t,s). \tag{2.1}$$

Definition 2.1. *The lower semicontinuous function $V(t,x)$ is called an upper minimax solution of (1.1) if $V(T,x)\ge\varphi(x)$ and*

$$\sup_{q\in R^n}\min\left\{\partial^-_{1,f}V(t,x) + g \,\middle|\, (f,g)\in\overline{F}_{\hat{A}}(t,q)\right\} \le 0, \qquad (t,x)\in G.$$

The upper semicontinuous function $V(t,x)$ is called a lower minimax solution of (1.1) if $V(T,x)\le\varphi(x)$ and

$$\inf_{p\in R^n}\max\left\{\partial^+_{1,f}V(t,x) + g \,\middle|\, (f,g)\in\overline{F}_{\hat{I}}(t,p)\right\} \ge 0, \qquad (t,x)\in G.$$

The continuous function $V(t,x)$ is called a minimax solution of (1.1) if it is a lower and an upper solution of (1.1) at the same time.

Here $\partial^-_{1,f}V(t,x)$ and $\partial^+_{1,f}V(t,x)$ denote the lower and upper derivatives with respect to the direction $(1,f)$ respectively. These derivatives are defined as

$$\partial^-_{1,f} V(t,x) = \liminf_{\delta \to 0,\, g \to f} [V(t+\delta, x+\delta g) - V(t,x)]/\delta,$$

$$\partial^+_{1,f} V(t,x) = \limsup_{\delta \to 0,\, g \to f} [V(t+\delta, x+\delta g) - V(t,x)]/\delta.$$

Let us now consider the generalized Hopf formula. We introduce the following definition.

Definition 2.2. *Given is the set $S \subset R^n$. The function $\sigma\colon \mathrm{co}\, S \to R$ is called pseudoconvex on S, where $\mathrm{co}\, S$ is the convex hull of S, if for any $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \ge 0$ such that $\sum_{i=1}^{n+1} \alpha_i = 1$ and also any $s_1, s_2, \ldots, s_{n+1} \in S$ we have the inequality*

$$\sigma\left(\sum_{i=1}^{n+1} \alpha_i s_i\right) \le \sum_{i=1}^{n+1} \alpha_i \sigma(s_i).$$

Remark 2.1. In Definition 2.2 the set S is not necessarily convex. If S is convex, then the definitions of convexity and pseudoconvexity are equivalent.

Caratheodory's theorem easily establishes the following proposition.

Lemma 2.1. *If the function $\sigma\colon \mathrm{co}\, S \to R$ is pseudoconvex on $S \subset R^n$, then for every natural N, any $\alpha_1, \alpha_2, \ldots, \alpha_N \ge 0$ such that $\sum_{i=1}^{N} \alpha_i = 1$, and also any $s_1, s_2, \ldots, s_N \in S$ we have*

$$\sigma\left(\sum_{i=1}^{N} \alpha_i s_i\right) \le \sum_{i=1}^{N} \alpha_i \sigma(s_i).$$

Remark 2.2. If in Definition 2.2 $S \subset R^n$ is a compact set and σ a continuous function, then for any probability measure π with a carrier in S we have the inequality

$$\int_S \sigma(s)\pi(ds) \ge \sigma\left(\int_S s\pi(ds)\right).$$

Theorem 2.1. *Assume that conditions (A) and (B) are satisfied.*
(I) Let φ be a Lipschitzian function. Define the maximizer set

$$S_0(t,x) = \left\{ s \in R^n \colon (s,x) + \int_t^T H(\tau, s)\, d\tau - \varphi^*(s) \right.$$

$$\left. = \sup_{s \in R^n} \left((s,x) + \int_t^T H(\tau, s)\, d\tau - \varphi^*(s) \right) \right\}.$$

Under the above conditions, formula (1.3)

$$V(t,x) = \sup_{s \in R^n} \left((s,x) + \int_t^T H(\tau, s)\, d\tau - \varphi^*(s) \right), \qquad (t,x) \in \overline{G},$$

defines a continuous lower solution of Eq. (1.1). Formula (1.3) is an upper solution (and thus simply a solution) if and only if for all $(t, x) \in G$ *the function* $H(t, \cdot)$ *is pseudoconvex on* $S_0(t, x)$. *Formula (1.3) can be written in the form*

$$V(t, x) = \left(\varphi^*(s) - \int_t^T H(\tau, s)\, d\tau \right)^*(x).$$

(II) For an arbitrary convex φ, *formula (1.3) defines a continuous lower solution. Formula (1.3) is a solution of (1.1) for arbitrary convex* φ *if it defines a solution of (1.1) for all convex Lipschitzian* φ.

Proof. To prove Theorem 2.1 we follow [1]. By the duality theorem, $V(T, x) = \varphi(x)$. We start with a proof of the first part of the theorem, when φ is Lipschitzian and convex. By Theorem (13.3.3) [6], dom φ^* is bounded. Then, by closure of φ^*, $S_0(t, x)$ is a nonempty compactum. By Lemma 5.1 [1], we can show that $V(t, x)$ is continuous, Lipschitzian for any bounded convex region $D \subset G$, and

$$\partial_{1, f} V(t, x) = \max[(s, f) - H(t, s) \mid s \in S_0(t, x)].$$

We will show that formula (1.3) defines a lower solution. For all $(t, x) \in G$,

$$\inf_p \max\left\{ \partial_{1, f} V(t, x) + g \mid (f, g) \in \overline{F}_l(t, p) \right\}$$

$$\geq \inf_p \max\left\{ (s, f) + g - H(t, s) \mid (f, g) \in \overline{F}_l(t, p) \right\} = 0.$$

The last holds by (2.1).

We will now derive a necessary and sufficient condition when formula (1.3) defines an upper solution. Thus, $S_0(t, x)$ is a nonempty compactum. Let $\Pi = \Pi(S_0(t, x))$ be the class of probability measures with a carrier in $S_0(t, x)$. It is easy to show that given a continuous function $f\colon S_0(t, x) \to R$, we have

$$\max_{s \in S_0(t, x)} f(s) = \max_{\pi \in \Pi} \left\{ \int_{S_0(t, x)} f(s) \pi(ds) \right\}.$$

Then from the above

$$\sup_q \min\left\{ \partial_{1, f} V(t, x) + g \mid (f, g) \in \overline{F}_{\hat{A}}(t, q) \right\}$$

$$= \sup_q \min_{(f, g) \in \overline{F}_{\hat{A}}} \max_{\pi \in \Pi} \left\{ \int_{S_0(t, x)} ((s, f) + g - H(t, s)) \pi(ds) \right\}$$

$$= \sup_q \min_{(f, g) \in \overline{F}_{\hat{A}}} \max_{\pi \in \Pi} \left\{ (s_\pi, f) + g - \int_{S_0(t, x)} H(t, s) \pi(ds) \right\},$$

where

$$s_\pi = \int_{S_0(t,x)} s\pi(ds).$$

Let

$$\Theta((f, g), \pi) = (s_\pi, f) + g - \int_{S_0(t,x)} H(t, s)\pi(ds),$$

$(f, g) \in \overline{F}_{\hat{A}}(t, q)$, $\pi \in \Pi$. Then Θ is linear in (f, g) and π, and the sets $\overline{F}_{\hat{A}}(t, q)$ and Π are convex compacta. The minimax theorem is thus applicable, and so

$$\sup_q \min\left\{ \partial_{1,f} V(t, x) + g \,\middle|\, (f, g) \in \overline{F}_{\hat{A}}(t, q) \right\}$$

$$= \max_{\pi\in\Pi} \sup_q \min_{(f,g)\in\overline{F}_{\hat{A}}} \left\{ (s_\pi, f) + g - \int_{S_0(t,x)} H(t, s)\pi(ds) \right\}$$

$$= \max_{\pi\in\Pi} \left\{ H(t, s_\pi) - \int_{S_0(t,x)} H(t, s)\pi(ds) \right\}$$

$$= \max_{\pi\in\Pi} \left\{ H(t, \int_{S_0(t,x)} s\pi(ds)) - \int_{S_0(t,x)} H(t, s)\pi(ds) \right\}.$$

But by Remark 2.2,

$$\max_{\pi\in\Pi} \left\{ H(t, \int_{S_0(t,x)} s\pi(ds)) - \int_{S_0(t,x)} H(t, s)\pi(ds) \right\} \le 0$$

if and only if $H(t, \cdot)$ is pseudoconvex on $S_0(t, x)$.

Let us now prove the second part of the theorem. Take the sequence of functions

$$\varphi_k^*(s) = \varphi^*(s) + \delta(s \mid B_k(0)),$$

where

$$\delta(s \mid A) = \begin{cases} 0, & s \in A, \\ \infty, & s \notin A, \end{cases}$$

and $B_k(0) = \left\{ s \in R^n : \|s\| \le k \right\}$. Then φ_k^* corresponds to a sequence of functions $\varphi_k(x) = (\varphi_k^*)^*(x)$, which by Theorem (13.3.3) [6] are convex, Lipschitzian, and monotone increasing approaching φ from below. Therefore by Dini's theorem, for any compactum M from R^n the sequence φ_k uniformly converges to φ. Moreover, by the assumptions of the theorem,

$$V(t, x) = \sup_{s\in B_k(0)} \left((s, x) + \int_t^T H(\tau, s)d\tau - \varphi^*(s) \right), \quad (t, x) \in \overline{G},$$

is a solution (a lower solution) of (1.1) with the boundary condition φ_k. By the theorem of stability of the solution in the Hamiltonian and in initial values (see Theorem 4.3 [9], Theorem 1.2 [9], Theorem 1.4 [10]), $V_k(t, x)$ tends to $V(t, x)$ defined by formula (1.3), which is a solution (a lower solution) of the Cauchy problem (1.1). Q.E.D.

Let us determine what information is available on the behavior of $H(t, s)$ on the maximizer set $S_0(t, x)$.

Lemma 2.2. *Assume that conditions (A) and (B) are satisfied. Then for any* $(t, x) \in G$, *any* $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \geq 0$ *such that* $\sum_{i=1}^{n+1} \alpha_i = 1$, *and all* $s_1, s_2, \ldots, s_{n+1} \in S_0(t, x)$,

$$\int_t^T \sum_{i=1}^{n+1} \alpha_i H(\tau, s_i) d\tau \geq \int_t^T H\left(\tau, \sum_{i=1}^{n+1} \alpha_i s_i\right) d\tau. \tag{2.2}$$

Proof. For $i = \overline{1, n+1}$ and $s = \sum_{i=1}^{n+1} \alpha_i s_i$, we have the inequality

$$(s_i, x) + \int_t^T H(\tau, s_i) d\tau - \varphi^*(s_i) \geq (s, x) + \int_t^T H(\tau, s) d\tau - \varphi^*(s).$$

It remains to multiply these inequalities by α_i and add them up, using finiteness of φ^* in $S_0 \subset \operatorname{dom} \varphi^*$ and the inequality

$$\varphi^*\left(\sum_{i=1}^{n+1} \alpha_i s_i\right) \leq \sum_{i=1}^{n+1} \alpha_i \varphi^*(s_i)$$

which is true by convexity of φ^*. Q.E.D.

We will now establish sufficient conditions when (1.3) is a solution of problem (1.1) for all convex functions φ.

Corollary 2.1. *Formula (1.3) defines a solution of (1.1) if conditions (A) and (B) are satisfied and* $H(t, s)$ *is concave (convex) in s for any* $t \in (0, T)$.

Proof. The case of convex $H(t, \cdot)$ is obvious. Now let $H(t, \cdot)$ be concave for $t \in (0, T)$. Then $S_0(t, x)$ is convex. For arbitrary $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \geq 0$ such that $\sum_{i=1}^{n+1} \alpha_i = 1$ and arbitrary $s_1, s_2, \ldots, s_{n+1} \in R^n$, we have by concavity for any $t \in (0, T)$

$$\sum_{i=1}^{n+1} \alpha_i H(t, s_i) \leq H\left(t, \sum_{i=1}^{n+1} \alpha_i s_i\right). \tag{2.3}$$

It suffices to verify that for $s_1, s_2, \ldots, s_{n+1} \in S_0(t, x)$ we have the inequality

$$\sum_{i=1}^{n+1} \alpha_i H(t, s_i) \geq H(t, \sum_{i=1}^{n+1} \alpha_i s_i).$$

Suppose that this is not so, i.e., there are $(t_0, x_0) \in G$ and also $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \geq 0$, where $\sum_{i=1}^{n+1} \alpha_i = 1$, and $s_1, s_2, \ldots, s_{n+1} \in S_0(t_0, x_0)$ such that

$$\sum_{i=1}^{n+1} \alpha_i H(t_0, s_i) - H(t_0, \sum_{i=1}^{n+1} \alpha_i s_i) = -\varepsilon < 0.$$

But then by continuity of $H(\cdot, s)$ for any $s \in R^n$ on $(0, T)$ and also by (2.3) we obtain the inequality

$$\int_{t_0}^{T} \sum_{i=1}^{n+1} \alpha_i H(\tau, s_i)\, d\tau - \int_{t_0}^{T} H\left(\tau, \sum_{i=1}^{n+1} \alpha_i s_i\right) d\tau < 0$$

which contradicts inequality (2.2). Q.E.D.

Remark 2.3. As we have noted previously, if the Hamiltonian is concave in s, then the set $S_0(t, x)$ is convex. Since the function $H(t, s)$ on the set $S_0(t, x)$ is simultaneously convex and concave in s, it is affine on $S_0(t, x)$.

Lemma 2.2 easily leads to the following proposition.

Corollary 2.2. *Assume that conditions (A) and (B) are satisfied for the Hamiltonian* $H(t, s) = H^0(s)\psi(t)$ *and for the function* φ, *where* $\psi(t) > 0$ *(or* < 0*) for all* $t \in (0, T)$. *Then the solution of the corresponding Cauchy problem (1.1) is provided by the function*

$$V(t, x) = \sup_{s \in R^n} \left((s, x) + H^0(s) \int_t^T \psi(\tau) d\tau - \varphi^*(s) \right), \qquad (t, x) \in \overline{G}.$$

It is easy to see that for $\psi(t) \equiv 1$ we obtain the classical Hopf formula

$$V(t, x) = \sup_{s \in R^n} \big((s, x) + (T - t) H^0(s) - \varphi^*(s) \big), \qquad (t, x) \in \overline{G}.$$

Remark 2.4. The classical Hopf formula produces a solution which is convex in all variables. The more general formula (1.3) gives a solution which is convex only in x for every fixed $t \in [0, T]$.

Unfortunately, in general, formula (1.3) is not a solution of (1.1) for all convex functions φ. We have the following proposition.

Corollary 2.3. *Formula (1.3) defines a solution of problem (1.1) if* $H(t, s)$ *and* φ *satisfy conditions (A) and (B), and* $|S_0(t, x)| = 1$ *for all* $(t, x) \in G$, *i.e., the maximizer in (1.3) is unique for all* (t, x).

Corollary 2.3 implies that (1.3) is a solution if φ is an affine function.

We now generalize the Lax–Hopf formula from [8]. Consider the following equation (1.4):

$$\frac{\partial V(t, x)}{\partial t} + H\left(t, V(t, x), \frac{\partial V(t, x)}{\partial x}\right) = 0, \qquad (t, x) \in G,$$

$$V(T, x) = \varphi(x), \qquad x \in R^n.$$

Assume that the following conditions are satisfied:

(C) The function $\varphi\supseteq\colon R^n \to R$ is continuous and quasiconvex;

(D) The function $H\colon (0,T)\times R\times R^n \to R$ is continuous in all variables and

(iii) $H(t,\gamma,\lambda s) = \lambda H(t,\gamma,s)$ for all $(t,\gamma)\in(0,T)\times R$, $\lambda\ge 0$, $s\in R^n$;

(iv) for any $(t,\gamma)\in(0,T)\times R$, $s', s''\in R^n$,

$$|H(t,\gamma,s') - H(t,\gamma,s'')| \le L\cdot\|s'-s''\|;$$

(v) $\gamma\mapsto H(t,\gamma,s)$ is nonincreasing for all $(t,s)\in(0,T)\times R^n$;

(vi) for all $(\gamma,s)\in R\times R^n$ the function $H(\cdot,\gamma,s)$ is summable on $(0,T)$.

Introduce the following quasiconvex conjugates [8]:

$$f^{\#}(\gamma,s) = \sup\left\{(s,x)\,\middle|\, x\in R^n,\ f(x)\le\gamma\right\}, \qquad (\gamma,s)\in R\times R^n,$$

$$f^{\#\#}(x) = \inf\left\{\gamma\in R\,\middle|\, \sup_{s\in R^n}\left((s,x)-f^{\#}(\gamma,s)\right)\le 0\right\}, \qquad x\in R^n.$$

By convention, $\sup(\varnothing) = -\infty$, $\inf(\varnothing) = +\infty$. We have the following theorem.

Theorem 2.2. *When conditions (C) and (D) hold, formula (2.4) below is a continuous viscosity subsolution of problem (1.4). Formula (2.4) defines a continuous viscosity supersolution (and thus a solution) if for all* $(t,\gamma)\in(0,T)\times R$ *the function* $H(t,\gamma,\cdot)$ *is pseudoconvex on the set* $S_0(t,x,\gamma)$, *where*

$$S_0(t,x,\gamma) = \left\{s\in B\colon (s,x) - \varphi^{\#}(\gamma,s) + \int_t^T H(\tau,\gamma,s)\,d\tau \right.$$

$$\left. = \sup_{q\in B}\left((q,x) - \varphi^{\#}(\gamma,q) + \int_t^T H(\tau,\gamma,q)\,d\tau\right)\right\},$$

$$V(t,x) = \inf\left\{\gamma\in R\,\middle|\, \sup_{s\in R^n}\left((s,x) - \varphi^{\#}(\gamma,s) + \int_t^T H(\tau,\gamma,s)\,d\tau\right) \le 0\right\}. \tag{2.4}$$

Here $B = \left\{s\in R^n\colon \|s\|\le 1\right\}$. *In other words, formula (2.4) can be written in quasiconjugate form as*

$$V(t,x) = \left(\varphi^{\#}(\gamma,s) - \int_t^T H(\tau,\gamma,s)\,d\tau\right)^{\#}(x).$$

Remark 2.5. As with formula (1.3), we can prove analogues of Corollaries 2.1, 2.2, and 2.3.

Remark 2.6. Contrary to the formula in [8], (2.4) gives a solution which is quasiconvex only in x for any $t\in[0,T]\cdot$

The proof of Theorem 2.2 repeats the proof in [8], except for minor details.

In conclusion, I would like to thank Professor A. B. Kurzhanskii for statement of the problem and useful comments.

The study received financial support from the Ministry of General and Professional Education, Russia's Universities grant No. 5393.

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