Exponentially Convergent Multiscale Methods for High Frequency Heterogeneous Helmholtz Equation

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Section 1

Helmholtz Equation

Setting of Helmholtz Equation

Helmholtz equation with mixed boundary conditions:

$$\begin{cases} -\nabla \cdot (A \nabla u) - k^2 V^2 u = f, \text{ in } \Omega, \\ u = 0, \text{ on } \Gamma_D, \\ A \nabla u \cdot \nu = T_k u, \text{ on } \Gamma_N \cup \Gamma_R, \end{cases} \tag{1}$$

where $A_{\min} \le A(x) \le A_{\max}$, $\beta_{\min} \le \beta(x) \le \beta_{\max}$, $V_{\min} \le V(x) \le V_{\max}$, $T_k u = 0$ for $x \in \Gamma_N$, and $T_k u = ik\beta u$ for $x \in \Gamma_R$.

- Bilinear form:

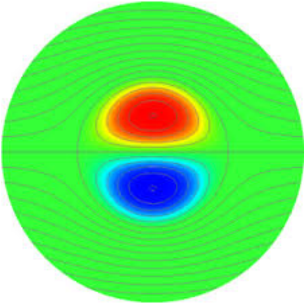
$$a(u, v) := (A \nabla u, \nabla v)_\Omega - k^2 (V^2 u, v)_\Omega - (T_k u, v)_{\Gamma_N \cup \Gamma_R}. \tag{2}$$

- Associated norm:

$$\|u\|_{\mathcal{H}(\Omega)} := \int_\Omega A |\nabla u|^2 + k^2 |Vu|^2. \tag{3}$$

Applications of Helmholtz Equation

1. Wave mechanics
2. Electrostatics
3. Seismology
4. Acoustics



Pollution Effect

I. Babuska, SINUM 1997.

- Mesh size sufficient to address the wave length: $O(1/k)$.

- For standard FEM: $h = O(1/k^2)$.

- Ideal method: $H = O(1/k)$!
  - $hp-$FEM with local polynomial of order $O(\log k)$. Melenk, Math. Comp., 2011.
  - Localizable orthogonal decompositions (LOD) with basis of support size $O(H \log(1/H))$. Peterseim, Math. Comp., 2014.
  - Multiscale edge basis with exponential rate of convergence.

- Fast solver with preconditioner: Ying, CPAM, 2011.

Sketch of Contributions

Our result: on a mesh of lengthscale $H = O(1/k)$, u can be computed by

$$u = \underbrace{\sum_{i\in I_1} c_i\psi_i^{(1)}}_{\text{(I)}} + \underbrace{\sum_{i\in I_2} d_i\psi_i^{(2)}}_{\text{(II)}} + \underbrace{\sum_{i\in I_3} \psi_i^{(3)}}_{\text{(III)}} + O\left(\exp(-m^{\frac{1}{d+1}-\epsilon})\right)$$

(Energy norm)

where: $\psi_i^{(1)}, \psi_i^{(2)}, \psi_i^{(3)}$ all have *local* support of size H.

- $\psi_i^{(1)}$ obtained by *local* SVD of $\mathcal{L}_\theta$ $\quad \#I_1 = O(m/H^d)$
- $\psi_i^{(2)}, \psi_i^{(3)}$ obtained by solving *local* $\mathcal{L}_\theta u = f$ $\quad \#I_2, \#I_3 = O(1/H^d)$
- $c_i, d_i \in \mathbb{R}$ obtained by Galerkin's methods with basis functions $\psi_i^{(1)}, \psi_i^{(2)}$
- (II),(III)$= O(H)$ (Energy norm)

A data-adaptive coarse-fine scale decomposition

Adjoint Operator and Stability

- Continuity estimate:

$$|a(u,v)| \le C_c \|u\|_{\mathcal{H}(\Omega)} \|v\|_{\mathcal{H}(\Omega)}. \tag{4}$$

- Stability: Let $N_k f := u$ be the solution operator.

$$\sup_{f \in L^2(\Omega) \backslash \{0\}} \frac{\|N_k f\|_{\mathcal{H}}}{\|f\|_{L^2(\Omega)}} =: C_{\text{stab}} < \infty. \tag{5}$$

  Assumption on the stability constant: $C_{\text{stab}} \le C_0 k^{\alpha}$.

- Adjoint problem with solution operator N_k^*:

$$\begin{cases} -\nabla \cdot (A\nabla u) - k^2 V^2 u = f, \text{ in } \Omega, \\ u = 0, \text{ on } \Gamma_D, \\ A\nabla u \cdot \nu = \overline{T_k u}, \text{ on } \Gamma_N \cup \Gamma_R, \end{cases} \tag{6}$$

  One can check that $N_k^* \overline{f} = \overline{N_k f}$.

Section 2

Coarse-Fine Scale Decomposition

Detour on Elliptic PDEs

- **Problem formulation**:

$$\begin{cases} -\nabla\cdot(a\nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

$\Omega = [0,1]^2$ and $u \in H_0^1(\Omega), f \in L^2(\Omega)$.

- **Galerkin methods**: choose a finite-dim space $V_H \subset H_0^1(\Omega)$:

Find $u_H \in V_H$ such that $\int_\Omega a\nabla u_H \cdot \nabla v = \int_\Omega fv$ for any $v \in V_H$.

Optimality: (notation $\|u\|_{H_a^1(\Omega)} := \int_\Omega a|\nabla u|^2$)

$$\|u - u_H\|_{H_a^1(\Omega)} = \inf_{v\in V_H} \|u - v\|_{H_a^1(\Omega)}.$$

V_H needs to approximate the solution space well in the $H_a^1(\Omega)$ norm.

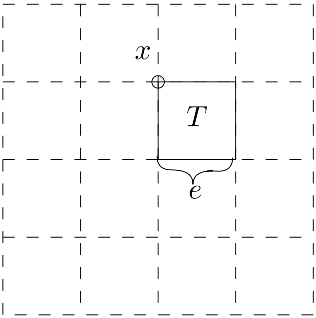
Explore the Solution Space

- **Mesh structure**:
  nodes, edges and elements.

- **Split** the solution locally:
  in each T, $u = u_T^{\mathsf{h}} + u_T^{\mathsf{b}}$.

$$\begin{cases} -\nabla \cdot (A \nabla u_T^{\mathsf{h}}) - k^2 V^2 u_T^{\mathsf{h}} = 0 \ \text{ in } T \\ u_T^{\mathsf{h}} = u \ \text{ on } \partial T, \end{cases}$$

$$\begin{cases} -\nabla \cdot (A \nabla u_T^{\mathsf{b}}) - k^2 V^2 u_T^{\mathsf{b}} = f \ \text{ in } T \\ u_T^{\mathsf{b}} = 0 \ \text{ on } \partial T. \end{cases}$$



$x \in \mathcal{N}_H, e \in \mathcal{E}_H, T \in \mathcal{T}_H$

- **Merge**: For each T, $u^{\mathsf{h}}(x) = u_T^{\mathsf{h}}(x)$ and $u^{\mathsf{b}}(x) = u_T^{\mathsf{b}}(x)$, when $x \in T$.

Coarse-fine Scale Decomposition

- **Poincaré inequality**: $\|v\|_{L^2(T)} \leq C_P H \|\nabla v\|_{L^2(T)}$.
- **Mesh assumption**: $H \leq A_{\min}^{1/2} / (\sqrt{2} C_P V_{\max} k)$.
- **Decomposition**: $u = u^{\mathsf{h}} + u^{\mathsf{b}} \in V^{\mathsf{h}} + V^{\mathsf{b}}$.

$$V^{\mathsf{h}} := \{v \in \mathcal{H}(\Omega) : -\nabla \cdot (A \nabla v) - k^2 V^2 v = 0 \text{ in each } T \in \mathcal{T}_H,$$
$$A \nabla v \cdot \nu = T_k v, \text{ on } \Gamma_N \cup \Gamma_R\} \quad (\textit{harmonic part})$$
$$V^{\mathsf{b}} := \{v \in \mathcal{H}(\Omega) : v = 0 \text{ on each } e \in \mathcal{E}_H\} \quad (\textit{bubble part})$$

For $v \in V^{\mathsf{h}}$ and $w \in V^{\mathsf{b}}$, it holds that $a(v, w) = 0$.

This decomposition makes sense by the C^α estimate of the solution.

Small Bubble Part

Bubble part is *local* and *small*:

- *local*: $u^{\mathsf{b}} = \sum_{i\in I_3} \psi_i^{(3)}$ (term (III)),
  each $\psi_i^{(3)}$ solves an elliptic equation inside each T.
- *small*: elliptic estimate,

$$\|u^{\mathsf{b}}\|_{\mathcal{H}(\Omega)} \leq \frac{3C_P}{A_{\min}^{1/2}} H \|f\|_{L^2(\Omega)}.$$

i.e. u^{b} oscillates at a frequency larger than $O(1/H)$.

Bubble part *is* the *fine scale* part.

Approximation of Harmonic Part

- **Idea**: choose $V_H \subset V^{\mathsf{h}}$ in Galerkin's method, yielding

$$\|u^{\mathsf{h}} - u_H\|_{\mathcal{H}(\Omega)} \approx \inf_{v \in V_H} \|u^{\mathsf{h}} - v\|_{\mathcal{H}(\Omega)}.$$

(Recall: bubble part is small)

Galerkin's solution u_H now *only* approximates the *harmonic* part.

- **Observation**: V^{h} is isomorphic to an edge space:

$$V^{\mathsf{h}} := \{v \in \mathcal{H}(\Omega) : -\nabla \cdot (A\nabla v) - k^2 V^2 v = 0 \text{ in each } T \in \mathcal{T}_H, \\ A\nabla v \cdot \nu = T_k v, \text{ on } \Gamma_N \cup \Gamma_R\}$$

Functions in V^{h}, locally solving Helmholtz-harmonic problems, only depend on values of v on edges.

Section 3

Exponentially Efficient Edge Basis

Localization to Edge Functions

- **Edge function**: $u^{\mathsf{h}} : \Omega \to \mathbb{R}$ restricted to edges: $\tilde{u}^{\mathsf{h}} : E_H \to \mathbb{R}$.

Task: find edge basis functions to approximate $\tilde{u}^{\mathsf{h}}$.

- **Localization to each edge**: $(\tilde{u}^{\mathsf{h}} - I_H \tilde{u}^{\mathsf{h}})|_e$ vanishes at nodal points where I_H is nodal interpolation operator, e.g., by linear tent functions.

Next: find edge basis functions to approximate $(\tilde{u}^{\mathsf{h}} - I_H \tilde{u}^{\mathsf{h}})|_e$ for each e.

The edge residual $R_e \tilde{u}^{\mathsf{h}} := (\tilde{u}^{\mathsf{h}} - I_H \tilde{u}^{\mathsf{h}})|_e$ lies in the Lions-Magenes space, i.e. functions $v \in H^{1/2}(e)$ s.t. $\frac{v(x)}{\mathrm{dist}(x, \partial e)} \in L^2(e)$, by the C^α estimate.

Local Norm for Approximation

- **The $\mathcal{H}^{1/2}(e)$ norm**: (connect back to energy norms)

$$\|\tilde{\psi}\|^2_{\mathcal{H}^{1/2}(e)} := \int_\Omega A|\nabla\psi|^2 + k^2|V\psi|^2.$$

where ψ is the harmonic extension of $\tilde{\psi}$ to neighboring elements.

Theorem (Edge Coupling)

If on each edge, there is $\tilde{v}_e$ such that the local error satisfies

$$\|\tilde{u}^{\mathsf{h}} - I_H\tilde{u}^{\mathsf{h}} - \tilde{v}_e\|_{\mathcal{H}^{1/2}(e)} \leq \epsilon_e,$$

then the global error satisfies

$$\|u^{\mathsf{h}} - I_H u^{\mathsf{h}} - \sum_{e\in\mathcal{E}_H} v_e\|^2_{\mathcal{H}(\Omega)} \leq C_{\text{mesh}} \sum_{e\in\mathcal{E}_H} \epsilon_e^2.$$

Local Approximation via Oversampling

- **Oversampling**: $e \subset \omega_e := \overline{\bigcup\{T \in \mathcal{T}_H : \overline{T} \cap e \neq \emptyset\}}$.

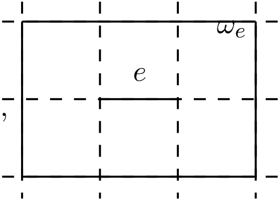
$$\text{on } e: \; u^{\mathsf{h}} - I_H u^{\mathsf{h}} = (u^{\mathsf{h}}_{\omega_e} - I_H u^{\mathsf{h}}_{\omega_e}) + (u^{\mathsf{b}}_{\omega_e} - I_H u^{\mathsf{b}}_{\omega_e}).$$

$u^{\mathsf{h}}_{\omega_e}, u^{\mathsf{b}}_{\omega_e}$: oversampling harmonic / bubble part.

Recall the definition:

$$\begin{cases} -\nabla \cdot (A \nabla u^{\mathsf{h}}_{\omega_e}) - k^2 V^2 u^{\mathsf{h}}_{\omega_e} = 0 \text{ in } \omega_e \\ u^{\mathsf{h}}_{\omega_e} = u \text{ on } \partial\omega_e, \end{cases}$$

$$\begin{cases} -\nabla \cdot (A \nabla u^{\mathsf{b}}_{\omega_e}) - k^2 V^2 u^{\mathsf{b}}_{\omega_e} = f \text{ in } \omega_e \\ u^{\mathsf{b}}_{\omega_e} = 0 \text{ on } \partial\omega_e. \end{cases}$$



interior edge

Next: *Restrictions* of harmonic part are of *low* complexity!

Low Complexity: Restrictions of Harmonic Part

Theorem (Y. Chen, T.Y. Hou, Y. Wang, 2021)

For any $\epsilon > 0$, there exists a C_ϵ, such that for all m, we can find an $(m-1)$ dimensional space $W_e^m = \mathrm{span}\ \{\tilde{v}_e^k\}_{k=1}^{m-1}$ so that for any Helmholtz-harmonic functions v in ω_e,

$$\min_{\tilde{v}_e \in W_e^m} \|v - I_H v - \tilde{v}_e\|_{\mathcal{H}^{1/2}(e)} \leq C_\epsilon \exp\left(-m^{\left(\frac{1}{d+1}-\epsilon\right)}\right) \|v\|_{\mathcal{H}(\omega_e)}.$$

- W_e^m obtained by left singular vectors of the operator $R_e v = v - I_H v$.
- Proof technique combines [Babuska, Lipton 2011] and C^α estimates.
- Essentially Helmholtz operator resembles an elliptic operator locally.

Summary of Approximations

- $u = u^{\mathsf{h}} + \overbrace{u^{\mathsf{b}}}^{\text{(III) local\&small}}$ (harmonic-bubble splitting)

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- $(u^{\mathsf{h}} - I_H u^{\mathsf{h}})|_e = \overbrace{(u^{\mathsf{h}}_{\omega_e} - I_H u^{\mathsf{h}}_{\omega_e})|_e}^{\text{restriction of harmonic part}} + \overbrace{(u^{\mathsf{b}}_{\omega_e} - I_H u^{\mathsf{b}}_{\omega_e})|_e}^{\text{basis functions in (II), small}}$

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- $(u^{\mathsf{h}}_{\omega_e} - I_H u^{\mathsf{h}}_{\omega_e})|_e = \overbrace{\sum_{j=1}^{m-1} c_j v_e^j}^{\text{basis functions in (I)}} + O\left(\exp(-m^{\frac{1}{d+1}-\epsilon})\|u\|_{\mathcal{H}(\omega_e)}\right)$

(I) basis functions not dependent on f, but on $\mathcal{L}_\theta$ (*local*)
(II) basis functions adapted to $\mathcal{L}_\theta$ and f (*local* and *small*)
(III) bubble part (*local* and *small*)

Section 4

Multiscale Basis

Overall Exponential Accuracy in Approximation

By the local to global error estimate, we have the overall approximation accuracy using $V_{H,m}$ consisting of basis functions (I)+(II):

Theorem (Global Approximation)

$$\min_{v \in V_{H,m}} \|u^{\mathsf{h}} - v\|_{\mathcal{H}(\Omega)} \leq C(C_{\text{stab}}(k) + H) \exp\left(-m^{(\frac{1}{d+1} - \epsilon)}\right) \|f\|_{L^2(\Omega)},$$

where C is a generic constant independent of k, m, H.

Multiscale Framework for Galerkin Methods

Handle coarse part u^{h} and fine part u^{b} separately.
Choose a finite-dim trial space $S \subset V^{\mathsf{h}}$, compute locally u^{b}, and then:

$$\text{find } u_S \in S \quad \text{such that} \quad a(u_S, v) = (f, v)_\Omega - a(u^{\mathsf{b}}, v) \quad \text{for any } v \in S_{\text{test}}.$$

- $S_{\text{test}} = S$: Ritz-Galerkin;
- $S_{\text{test}} = \overline{S}$: Petrov-Galerkin.

Approximation Implies Accuracy

- **Approximation Ability**:

$$\eta^{\mathsf{h}}(S) := \sup_{f \in L^2(\Omega)\backslash\{0\}} \inf_{v \in S} \frac{\left\|u^{\mathsf{h}} - v\right\|_{\mathcal{H}(\Omega)}}{\|f\|_{L^2(\Omega)}} \quad \text{with} \quad u = N_k f. \qquad (7)$$

- Given that $k\eta^{\mathsf{h}}(S) \leq 1/(4C_c V_{\max})$, for the Ritz-Galerkin method:
  - **Discrete Stability**:

$$\inf_{v \in S} \sup_{v' \in S\backslash\{0\}} \frac{|a(v, v')|}{\|v\|_{\mathcal{H}(\Omega)} \|v'\|_{\mathcal{H}(\Omega)}} \geq \frac{1}{2 + C_c^{-1} + 4kV_{\max} C_{\text{stab}}(k) + 6\sqrt{2}}.$$

  - **Quasi-optimal Approximation**:

$$\|u^{\mathsf{h}} - u_S\|_{\mathcal{H}(\Omega)} \leq 2C_c \inf_{v \in S} \|u^{\mathsf{h}} - v\|_{\mathcal{H}(\Omega)},$$

$$\|u^{\mathsf{h}} - u_S\|_{L^2(\Omega)} \leq C_c \eta^{\mathsf{h}}(S) \|u^{\mathsf{h}} - u_S\|_{\mathcal{H}(\Omega)}.$$

Ritz-Galerkin Method

Theorem (Galerkin Exponential Accuracy)

Suppose $Ck(C_{\text{stab}}(k)+H)\exp\left(-m^{\left(\frac{1}{d+1}-\epsilon\right)}\right) \le 1/(4C_cV_{\max})$*, then using* $S = V_{H,m} + \overline{V_{H,m}}$ *in Ritz-Galerkin method leads to a solution* u_S *such that*

$$\|u^{\mathsf{h}} - u_S\|_{\mathcal{H}(\Omega)} \le 2C_cC(C_{\text{stab}}(k)+H)\exp\left(-m^{\left(\frac{1}{d+1}-\epsilon\right)}\right)\|f\|_{L^2(\Omega)},$$

$$\|u^{\mathsf{h}} - u_S\|_{L^2(\Omega)} \le 2(C_cC)^2\left(C_{\text{stab}}(k)+H\right)^2\exp\left(-2m^{\left(\frac{1}{d+1}-\epsilon\right)}\right)\|f\|_{L^2(\Omega)}.$$

- $m \sim \log^{d+2}(k)$ suffices for an exponential rate of convergence.
- $V_{H,m}$ and $\overline{V_{H,m}}$ only differ on the edges connected to the boundary, where Robin boundary conditions make the operator non-Hermitian.
- $a(u^{\mathsf{b}}, v)$ needed only for v on edges connected to the boundary.

Petrov-Galerkin Method

Theorem (L^2 Galerkin Exponential Accuracy)

Suppose $Ck(C_{\mathrm{stab}}(k)+H)\exp\left(-m^{(\frac{1}{d+1}-\epsilon)}\right) \leq 1/(4C_c V_{\max})$*, then using* $S = V_{H,m}$ *in Petrov-Galerkin method leads to a solution* u_S *such that*

$$\left\|u^{\mathsf{h}} - u_S\right\|_{L^2(\Omega)} \leq 2C_c C(C_{\mathrm{stab}}(k)+H)\exp\left(-m^{(\frac{1}{d+1}-\epsilon)}\right)\|u^{\mathsf{h}} - u_S\|_{\mathcal{H}(\Omega)}.$$

- $m \sim \log^{d+2}(k)$ suffices for an exponential rate of convergence.
- The trial and test spaces are smaller compared to the Ritz counterpart.
- $a(u^{\mathsf{b}}, v)$ vanishes by the decomposition.

Section 5

Numerical Experiments

Online and Offline Implementations

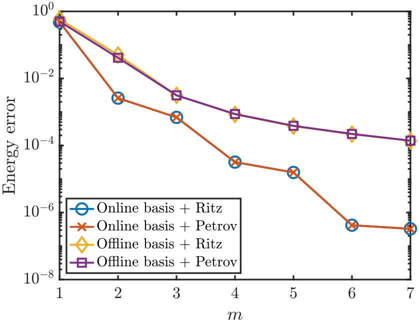
- **Offline Basis**: Only use (I) in the Galerkin methods.
  - $O(H)$ accuracy in the energy norm.
  - Same set of basis for multiple right hand sides.
- **Online Basis**: Use (I) and (II) in the Galerkin methods.
  - Exponential accuracy in the energy norm.
  - $O(1/H^d)$ local online basis associated to the right hand side.

We can choose adaptively the number of basis associated to each edge.

In practice, all of the four methods exhibit *exponential* convergence rates!

High Wavenumber Example

- $A = V = \beta = 1$, $k = 2^7$, fine mesh $h = 2^{-10}$, coarse mesh $H = 2^{-5}$.
- Exact solution: $u(x_1, x_2) = \exp(-ik(0.6x_1 + 0.8x_2))$.

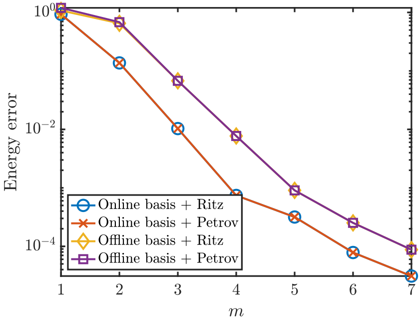


Varying Wavelength Example

$A = 1$, $k = 2^7, \beta = 1$. The function V takes values 1 or 2, where $V = 2$ in some periodically spaced blocks which form the domain

$$D = \bigcup_{j \in \{2,3,\cdots,8\}^2} 0.1\left(j + (-0.25, 0.25)^2\right)$$

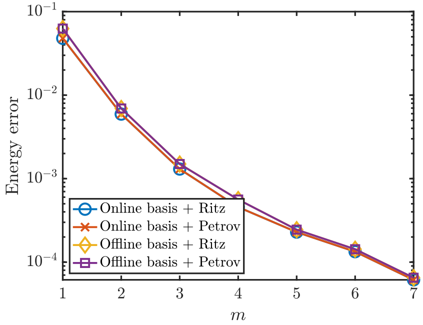
.



High Contrast Example

$$X := \{(x_1, x_2) \in [0,1]^2, x_1, x_2 \in \{0.2, 0.3, ..., 0.8\}\} \subset [0,1]^2$$

$A(x) = 1$, if dist$(x, X) \geq 0.025$ and 2^{12} if else. $\beta = 1, V = 1, k = 2^5$.



Mixed Boundary and Rough Field Example

Rough media with mixed boundary conditions. (Artificial)

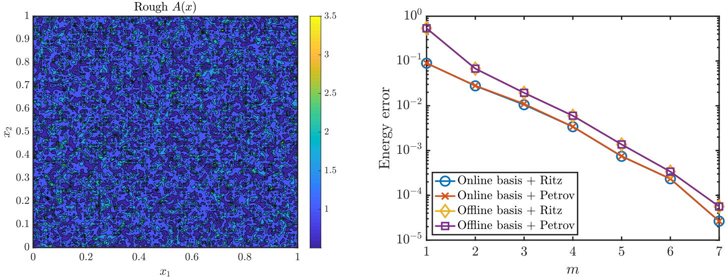


Figure: Left: the contour of A; right: relative errors in the energy norm.

Section 6

Conclusions

Summary of the Framework

1. Galerkin solution as a quasi-optimal approximation.
2. Harmonic-bubble decomposition to avoid non positive definiteness in the whole domain.
3. Local nodal/edge basis construction for global error estimate.
4. Exponential decay of the error by oversampling method to achieve optimal design.
5. Extensive numerical experiments to corroborate the exponential rate of convergence.

Sketch of Contributions

Our result: on a mesh of lengthscale $H = O(1/k)$, u can be computed by

$$u = \underbrace{\sum_{i\in I_1} c_i\psi_i^{(1)}}_{\text{(I)}} + \underbrace{\sum_{i\in I_2} d_i\psi_i^{(2)}}_{\text{(II)}} + \underbrace{\sum_{i\in I_3} \psi_i^{(3)}}_{\text{(III)}} + O\left(\exp(-m^{\frac{1}{d+1}-\epsilon})\right)$$

(Energy norm)

where: $\psi_i^{(1)}, \psi_i^{(2)}, \psi_i^{(3)}$ all have *local* support of size H.

- $\psi_i^{(1)}$ obtained by *local* SVD of $\mathcal{L}_\theta$ $\qquad \#I_1 = O(m/H^d)$
- $\psi_i^{(2)}, \psi_i^{(3)}$ obtained by solving *local* $\mathcal{L}_\theta u = f$ $\qquad \#I_2, \#I_3 = O(1/H^d)$
- $c_i, d_i \in \mathbb{R}$ obtained by Galerkin's methods with basis functions $\psi_i^{(1)}, \psi_i^{(2)}$
- (II),(III)$= O(H)$ (Energy norm)

A data-adaptive coarse-fine scale decomposition

Future Work

1. Generalization to other non-elliptic (time-dependent) problems, e.g. the Schrödinger equation, where the non-elliptic term could be treated as a perturbation term.
2. Generalization to higher-order operators and higher-dimensions.

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Thanks!