Blowup analysis for a 1D quasi-exact model of Navier-Stokes

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Self-similar Blowup

- Millennium prize problem: global well-posedness or finite time blowup of Navier-Stokes equation from smooth initial data.
- Self-similar blowup:
$$\mathbf{u}(t,\mathbf{x}) = (T-t)^{-c_{\mathbf{u}}}\mathbf{U}(\mathbf{x}\cdot(T-t)^{c_l})\,. \tag{1}$$
T: blowup time; $c_{\mathbf{u}}>0$: blowup rate.
- Hou-Luo (2013) [3]: numerical evidence of self-similar blowup for smooth initial data of 3D axisymmetric Euler equation with boundary.
- Chen-Hou (2022) [1]: rigorous proof of blowup.

Hou-Li Model

- 3D axisymmetric Navier-Stokes equation:
$$\begin{aligned} u_{1,t} - r\psi_{1,z}u_{1,r} + (2\psi_1 + r\psi_{1,r})\,u_{1,z} &= 2u_1\psi_{1,z} + \nu\Delta u_1\,,\\ \omega_{1,t} - r\psi_{1,z}\omega_{1,r} + (2\psi_1 + r\psi_{1,r})\,\omega_{1,z} &= \left(u_1^2\right)_z + \nu\Delta\omega_1\,,\\ -\left[\partial_r^2 + (3/r)\partial_r + \partial_z^2\right]\psi_1 &= \omega_1\,. \end{aligned}$$
- Hou-Li (2006) [2] constant approximation in r:
$$\begin{aligned} u_t + 2\psi u_z &= 2u\psi_z + \nu u_{zz}\,,\\ \omega_t + 2\psi\omega_z &= \left(u^2\right)_z + \nu\omega_{zz}\,,\\ -\psi_{zz} &= \omega\,. \end{aligned} \tag{2}$$
- Periodic setting in z; the model is well-posed for C^1 data by L^∞ estimate.

Weak Convection Model

- Based on numerical observation for the 3D Euler potential blowup scenario, $\psi_{1,r}<0$ near maximal point of u. Local convection is weaker in 3D.
- Weak convection model with $a<1$:
$$\begin{aligned} u_t + 2a\psi u_z &= 2u\psi_z + \nu u_{zz}\,,\\ \omega_t + 2a\psi\omega_z &= \left(u^2\right)_z + \nu\omega_{zz}\,,\\ -\psi_{zz} &= \omega\,. \end{aligned} \tag{3}$$

Dynamic Rescaling Formulation

- Idea: the self-similar profile would solve a steady state equation. We use DRF to compute the profile and study the dynamic stability.
- For the inviscid case, we let
$$\tilde{u}(x,\tau) = C_u(\tau)u(x,t(\tau))\,,\ \tilde{\omega}(x,\tau) = C_u(\tau)\omega(x,t(\tau))\,,\ \tilde{\psi}(x,\tau) = C_u(\tau)\psi(x,t(\tau))\,,$$
where
$$C_u(\tau) = \exp\left(\int_0^\tau c_u(s)ds\right)\,,\quad t(\tau) = \int_0^\tau C_u(s)ds\,.$$
Rescaled variables solve the following dynamic rescaling equation
$$\begin{aligned} \tilde{u}_\tau + 2a\tilde{\psi}\tilde{u}_x &= 2\tilde{u}\tilde{\psi}_x + c_u\tilde{u}\,,\\ \tilde{\omega}_\tau + 2a\tilde{\psi}\tilde{\omega}_x &= \left(\tilde{u}^2\right)_x + c_u\tilde{\omega}\,,\\ -\tilde{\psi}_{xx} &= \tilde{\omega}\,. \end{aligned} \tag{4}$$

Dynamic stability of equation (4) close to an approximate steady state with corresponding $c_u < -\epsilon < 0$ uniformly in time implies blowup.

Our Results

- For $a<1$ close to 1: there exists a smooth self-similar blowup profile of the inviscid model. The profile has dynamic stability: i.e. for initial data close to the profile it would converge to the profile and blow up. Blowup of the viscous model is also established.
- For $a=1$: for any $\alpha<1$, there exists a C^α self-similar blowup profile of the inviscid model with dynamic stability.

Roadmap of Establishing Self-similar Blowup

1. Approximate profile: explicit construction; solving DRF/ profile.
2. Stability: all initial data close to the approximate profile would develop finite-time blowup, i.e. the blowup is stable. Weighted L^2 estimate or L^∞ estimate using characteristics.
3. Rigorous proof: interval algorithms of numerical verifications.
4. Characterization of the blowup: rate, regularity, asymptotics...

Key Ingredients in Stability

- Explicit approximate profiles: from the steady state for $a=1$.
- L^2-estimate with weighted singular norm: Linear stability using integration by parts, with singular weights $\rho_0 = \frac{1}{1-\cos x}$, $\rho_k = (1+\cos x)^k$.
- Extract damping using exact computation in Fourier basis.
- Computer-assisted proof to establish a quadratic form to be postive-definite.
- H^k estimate for the viscous term spits out
$$-(\omega^{(k+1)},\omega^{(k+1)}\rho_k) + C(k)(\omega^{(k)},\omega^{(k)}\rho_{k-1})\,.$$
Use a combination of a cascade of norms to close the estimate.

References

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