1. $2 = 1$

$4 = 2$ $\quad 4 - 7 = 4$

$8 = 3$

$16 \quad 4$ $\quad 16 - 31 = 16$

$32 \quad 5$ $\quad 20 + 320 = 340$

$64 \quad 6$ $\quad 64 - 127 \Rightarrow 64$

$126 \quad 7$

$256 \quad 8$ $\quad 256 - 511 \Rightarrow 256$

$512 \quad 9$

2. $1995^{1/2} x^{\log_{1995}(x)} = x^2$

$\frac{1}{2} + (\log_{1995}(x))^2 = 2\log_{1995}(x)$

$(\log_{1995}(x))^2 - 2\log_{1995}(x) + \frac{1}{2} = 0$

$(\log_{1995}(x) - 1)^2 = \frac{1}{2}$

$\log_{1995}(x) - 1 = \pm\frac{1}{4}$

$\log_{1995}(x) = 1 \pm \frac{1}{4}$

$x = 1995^{5/4}$ or $1995^{3/4}$

$1995^{5/4} \cdot 1995^{3/4} = 1995^2 \Rightarrow$

$$\begin{array}{r} 995 \\ \times\ 995 \\ \hline 4975 \\ 8955\,0 \\ 55\,00 \\ \hline 025 \end{array}$$

025

3. $\tan(x) + \tan(y) = 25$

$\frac{1}{\tan(x)} + \frac{1}{\tan(y)} = 30$

$30\tan x \tan y = 25 \Rightarrow \tan x \tan y = \frac{25}{30} = \frac{5}{6}$

$\underline{\tan x + \tan y} = 30\tan x \tan y$

$30 - (\tan x + \tan y) = 30 - 30\tan x \tan y$

$30 - (\tan x + \tan y)$

$$\frac{\quad}{1-\tan x\tan y}=30$$

$$\frac{30}{1-\tan x\tan y}-\tan(x+y)=30$$

$$\tan(x+y)=\frac{30}{1-\tan x\tan y}-30$$

$$=\frac{30\tan x\tan y}{1-\tan x\tan y}=\frac{25}{1-\frac{5}{6}}=\frac{25}{\left(\frac{1}{6}\right)}$$

$$=25\cdot 6$$

$$=150$$

4. $\log_a b+6\log_b a=5,\quad 2\le a\le 2005,\ 2\le b\le 2005$

$$\log_a b=\frac{1}{\log_b a}$$

$$1+6\log_b a\cdot\log_b a=5\cdot\log_b a$$

$$6(\log_b a)^2-5\cdot\log_b a+1=0$$

$$(3\log_b a-1)(2\log_b a-1)=0$$

$$\log_b a=\frac{1}{3}\quad\text{or}\quad\log_b a=\frac{1}{2}$$

So $a=b^{1/3}$ or $a=\sqrt{b}$ $\qquad a=2-44\Rightarrow 43$ pairs

largest perfect cube < 2005

$$\begin{array}{r} 169\\ \times\ 13\\ \hline 507\\ 1690\\ \hline 2197 \end{array}\qquad \begin{array}{r} 144\\ \times\ 12\\ \hline 288\\ 1440\\ \hline 1728 \end{array}\qquad a=2-12\Rightarrow 11 \text{ pairs}\qquad \begin{array}{r} 44\\ \times\ 44\\ \hline 176\\ 1760\\ \hline 1936 \end{array}$$

54 pairs

5. $\log_{225}x+\log_{64}y=4$ ①

$\log_x 225-\log_y 64=1$ ②

$$\frac{1}{\log_{225}x}-\frac{1}{\log_{64}}=1$$

$$\frac{1}{a}-\frac{1}{b}=1$$

$$a+b=4$$

$b = 4 - a$

$\frac{1}{a} - \frac{1}{4-a} = 1$

$\frac{4-a-a}{a(4-a)} = 1$

$4 - 2a = 4a - a^2$

$a^2 - 6a + 4$

$\frac{6 \pm \sqrt{36-16}}{2} = 3 \pm \sqrt{5}$

$\log_{225} x = 3 \pm \sqrt{5}$ $\qquad \log_{64} y \quad 1 \pm \sqrt{5}$

$225^{3+\sqrt{5}} \cdot 225^{3-\sqrt{5}} \cdot 64^{1+\sqrt{5}} \cdot 64^{1-\sqrt{5}}$

$\log_{30} (225^{3+\sqrt{5}} \cdot 225^{3-\sqrt{5}} \cdot 64^{1+\sqrt{5}} \cdot 64^{1-\sqrt{5}})$

$\left(\left(3^2 \cdot 5^2\right)^3\right)^2 \cdot \left(2^6\right)^2 = 30^{12} \Rightarrow \log_{30}(30^{12}) = 12$

6. $(1+\sin t)(1+\cos t) = \frac{5}{4}$

$(1-\sin t)(1-\cos t) = \frac{m}{n} - \sqrt{k}$

$(1+\sin t)(1-\sin t)(1+\cos t)(1-\cos t) = 5/4\left(\frac{m}{n} - \sqrt{k}\right)$

$(1-\sin^2 t)(1-\cos^2 t) = \frac{5}{4}\left(\frac{m}{n} - \sqrt{k}\right)$

$\cos^2 t \cdot \sin^2 t = (\sin t \cos t)^2 = \frac{5}{4}\left(\frac{m}{n} - \sqrt{k}\right)$

$\sin t \cos t = \frac{\sqrt{5\left(\frac{m}{n} - \sqrt{k}\right)}}{2}$

$2 + 2\sin t \cos t = \frac{5}{4} + \frac{m}{n} + \sqrt{k}$

$2 + \sqrt{5a} = \frac{5}{4} + a$

$\sqrt{5a} = a - \frac{3}{4}$

$5a = \left(a - \frac{3}{4}\right)^2$

$$5a = a^2 - \frac{3}{2}a + \frac{9}{16}$$

$$a^2 - \frac{13}{2}a + \frac{9}{16} = 0$$

$$\frac{\frac{13}{2} \pm \sqrt{\frac{169}{4} - \frac{9}{4}}}{2} = \frac{13}{4} \pm \frac{\sqrt{\frac{160}{4}}}{2} = \frac{13}{4} \pm \frac{4\sqrt{10}}{4}$$

$$= \frac{13}{4} \pm \sqrt{10}$$

$m = 13$ $\quad k = 10$
$n = 4$

$\hookrightarrow$ 27

7. $a = \pi/2008$

$$2\left[\cos a \sin a + \cdots \cos(n^2 a) \sin(na)\right]$$

$$\sin(k^2a + ka) = \cos(k^2a)\sin(ka) + \cos(ka)\sin(k^2a)$$

$$\sin(-k^2a + ka) = \cos(-k^2a)\sin(ka) + \cos(ka)\sin(-k^2a)$$

$$= \cos(k^2a)\sin(ka) - \cos(ka)\sin(-k^2a)$$

Thus $\sin(k^2a + ka) + \sin(-k^2a + ka) = 2\cos(k^2a)\sin(ka)$

$$\sum_{k=1}^{n} 2\cos(k^2a)\sin(ka) = \sum_{k=1}^{n} \sin(k^2a + ka) + \sin(-k^2a + ka)$$

$$= \sin(2a) + \sin(0) + \sin(6a) + \sin(-2a)$$

$$\sin(12a) + \sin(-6a)$$

$$\sin(0) + \sin(n^2a + na) = \sin(n^2a + na)$$

$$\frac{(n^2+n)\pi}{2008} = \frac{m\pi}{2}$$

$$2(n^2+n) = 2008m$$

$$n^2 + n = 1004m$$

$$n(n+1) = 2 \cdot 502m$$

$$= 4 \cdot 251m$$

252 $\quad$ 63 $\longrightarrow$ 251

8. $\log_{24\sin x}(24\cos x) = \frac{3}{2}$

$$24\cos x = (24\sin x)^{3/2} = 24^{3/2} \cdot \sin x\sqrt{\sin x}$$

$$24\cot x = 24^{3/2}\sqrt{\sin x}$$

$$24^2\cot^2 x = 24^3\sin x$$

$$24\cot^2 x = 24^2\sin x$$

$$24\cot^2 x = 24^2 \cdot \frac{1}{3} = 8 \cdot 24 = 192$$

$$\frac{\cos^2 x}{\sin^2 x} = 24\sin x$$

$$1 - \sin^2 x = 24\sin^3 x$$

$$24\sin^3 x + \sin^2 x - 1 = 0$$

$a = \sin x$

$$24a^3 + a^2 - 1 = 0$$

$$\begin{array}{r|l} & 8a^2+3a+1 \\ 3a-1 & 24a^3+a^2-1 \\ & \underline{24a^3-8a^2} \\ & 9a^2-1 \\ & \underline{9a^2-3a} \\ & 3a-1 \end{array}$$

$$(3a-1)(8a^2+3a+1) = 0$$

$$3a = 1$$

$$a = \frac{1}{3}$$

$$\sin x = \frac{1}{3}$$

9. $\sec x + \tan x = \frac{22}{7}$

$$\tan^2 x + 1 = \sec^2 x$$

$$\left(\frac{22}{7} - \tan x\right)^2 = \tan^2 x + 1$$

$$\left(\frac{22}{7}\right)^2 - \frac{44}{7}\tan x + \tan^2 x = \tan^2 x + 1$$

$$\frac{484 - 49}{49} = \frac{44}{7}\tan x$$

$$\frac{435}{49} = \frac{44}{7}\tan x$$

$$\tan x = \frac{\frac{435}{49}}{7} \cdot \frac{7^1}{44} = \frac{435}{308} \qquad \sec x = \frac{22}{7} - \frac{435}{308} = \frac{533}{308}$$

$$\cot x = \frac{308}{435} \qquad \cot x \cdot \sec x = \csc x = \frac{533}{435}$$

$$\frac{533}{435} + \frac{308}{435} = \frac{841}{435} = \frac{29^2}{15\cdot 29} = \frac{29}{15} = 44$$

10. $2\log_x(2y) = 2\log_{2x}(4z) = \log_{2x^4}(8yz) = k \neq 0$

xy^5z

$$x^k = (2y)^2 = 4y^2$$

$$2x^k = 16z^2$$

$$(2x^4)^k = 8yz$$

$$x^k \cdot 2^k x^k = \left((2x^4)^k\right)^2 = (2x^4)^{2k} = 2^{2k}x^{8k}$$

$$2^k x^{2k} = 2^{2k}x^{8k}$$

$$2^k x^{6k} = 1$$

$$x^{6k} = 2^{-k}$$

$$x^6 = \frac{1}{2}$$

$$x = \left(\frac{1}{2}\right)^{1/6}$$

$$2^{-k/6} = 4y^2$$

$$2^{5k/6} = 16z^2$$

$$2^k x^{4k} = 2^{k/3} = 8yz$$

$$1 = 16y^4 \cdot 8yz = 128y^5z$$

$$y^5z = 2^{-7}$$

$$xy^5z = 2^{-7}\cdot 2^{-1/6} = 2^{-43/6} = \frac{1}{2^{43/6}}$$

$\Rightarrow 49$

11. $$x = \frac{\sum_{n=1}^{44}\cos n}{\sum_{n=1}^{44}\sin n} = \frac{\sum_{n=1}^{44}\sin(90^\circ - n)}{\sum_{n=1}^{44}\sin n} = \frac{\sum_{n=46}^{89}\sin n}{\sum_{n=1}^{44}\sin n} = \frac{\sum_{n=1}^{45}\sin(n+45^\circ)}{\sum_{n=1}^{44}\sin n}$$

$$\sin(n+45^\circ) = \frac{\sqrt{2}}{2}\left(\sin n + \cos n\right)$$

thus $$x = \frac{\frac{\sqrt{2}}{2}\left(\sum_{n=1}^{45}\sin n + \sum_{n=1}^{45}\cos n\right)}{44} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(\frac{\sum_{n=1}^{44}\cos n}{\sum_{n=1}^{44}\sin n}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}x$$

$\sum_{n=1} \sin n$ $\left(\sum_{n=1} \sin n \right)$

$$\left(\frac{2-\sqrt{2}}{2}\right)x = \frac{\sqrt{2}}{2}$$

$$(2-\sqrt{2})x = \sqrt{2}$$

$$x = \frac{\sqrt{2}}{2-\sqrt{2}} = \frac{\sqrt{2}(2+\sqrt{2})}{4-2} = \frac{2\sqrt{2}+2}{2} = \sqrt{2}+1$$

$$100x = 100 + 100\sqrt{2} = 241.42$$

$$241$$

12. (m,n) $m \geq n$

$|\log m - \log k| < \log n$

$|\log \frac{m}{k}| < \log n$

$-\log n < |\log \frac{m}{k}| < \log n$

$\frac{1}{n} < \frac{m}{k} < n$

$\frac{1}{n} < \frac{k}{m} < n$

$\frac{m}{n} < k < mn$

exactly 50 integers

$mn - 51 \leq \frac{m}{n} < mn - 50$

$-51 \leq \frac{m}{n} - mn < -50$

$50 < mn - \frac{m}{n} \leq 51$

$50n < mn^2 - m \leq 51n$

$50n < m(n-1)(n+1) \leq 51n$

$\frac{50n}{n^2-1} < m \leq \frac{51n}{n^2-1}$

$n \leq \frac{51n}{n^2-1}$ (since $m \geq n$

$n^2 \leq 52$, so $1 \leq n \leq 7$

$n=1$: $\frac{50}{0} < m \leq$ $\Rightarrow$ no solutions

$n=2$ $\frac{100}{3} < m \leq \frac{102}{3} = 34$

$m = 34$

$n=3$

$\frac{150}{8} < m \leq \frac{153}{8}$

$m = \frac{152}{8} = 19$

$19 \cdot 3 + 34 \cdot 2 = 57 + 68 = 125$

$n=4$

$\frac{200}{15} < m \leq \frac{204}{15}$

no sol

$n=5$

$\frac{250}{24} < m \leq \frac{255}{24}$

no sol

$n=6$

$\frac{300}{35} < m \leq \frac{306}{35}$

no sol

$n=7$

$\frac{350}{48} < m \leq \frac{357}{48}$

no sol

13. $0 < x \le 1$ $0 < y \le 1$

$\lfloor \log_2(\frac{1}{x}) \rfloor$ $\lfloor \log_5(\frac{1}{y}) \rfloor$ both even

if x in

$\left(\frac{1}{2^{2k+1}}, \frac{1}{2^k}\right]$

$$\frac{1}{2^k} - \frac{1}{2^{k+1}} = \frac{1}{2^{k+1}}$$

$$\sum_{k=0}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{4}} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}$$

$\left(\frac{1}{5^{2k+1}}, \frac{1}{5^{2k}}\right]$

$$\frac{1}{5^{2k}} - \frac{1}{5^{2k+1}} = \frac{4}{5^{2k+1}}$$

$$\frac{4}{5} \sum_{k=0}^{\infty} \left(\frac{1}{25}\right)^k = \frac{4}{5} \cdot \frac{25}{24} = \frac{5}{6}$$

Area of $S = \frac{2}{3} \cdot \frac{5}{6} = \frac{10}{18} = \frac{5}{9} \Rightarrow$ 14

14. $\sum_{n=0}^{7} \log_3(x_n) = 308$

$$56 \le \log_3\left(\sum_{n=0}^{7} x_n\right) \le 57$$

$x_0 = 3^a$

$x_n = 3^{a+bn}$

$\log_3(x_n) = a + bn$

Then

$$308 = \sum_{n=0}^{7} \log(x_n) = 8a + 28b \rightarrow 77 = 2a + 7b$$

$$\log_3\left(3^a + 3^{a+b} + 3^{a+b} + \cdots + 3^{a+7b}\right)$$
$$= \log_3\left(3^a + 3^b \cdot 3^a + \cdots + 3^{7b} \cdot 3^a\right)$$
$$= \log_3\left(3^a\left(1 + 3^b + \cdots + 3^{7b}\right)\right)$$
$$= \log_3\left(3^a\right) + \log_3\left(1 + 3^b + \cdots + 3^{7b}\right)$$
$$= a + \log_3\left(1 + 3^b + \cdots 3^{7b}\right)$$

Thus $56 \le a + \log_3\left(1 + 3^b + \cdots 3^{7b}\right) \le 57$

$$56 - a \le \log_3\left(1 + 3^b + \cdots 3^{7b}\right) \le 57 - a$$

Since $\log_3(x)$ increases

$$7b = \log_3\left(3^{7b}\right) < \log_3\left(1 + 3^b + \cdots 3^{7b}\right) < \log_3\left(8 \cdot 3^{7b}\right) = \log_3 8 + 7b < 2 + 7b$$

So $56 - a < 7b + 2$ and $7b < 57 - a$

$$7b < 57 - a < 7b + 3$$

Thus either $57 - a = 7b + 1$ or $57 - a = 7b + 2$

first case: $7b + a = 56$

$$77 = 2a + 7b$$
$$a = 21, \ b = 5$$

second case: $7b + a = 55$

$$\cancel{77 = 2a + 7b}$$
$$\cancel{a = 22}$$
$$\cancel{7b = 33}, \times \text{ (b not integer)}$$

Thus $\log_3(x_{14}) = a + bn = 21 + 5 \cdot 14 = 21 + 70 = 91$

15. $\frac{1}{\sin 45 \sin 46} + \cdots + \frac{1}{\sin 133} \cdot \frac{1}{\sin 134} = \frac{1}{\sin n}$

$\sin 1 \left(\frac{1}{\sin 45 \sin 46} + \cdots + \frac{1}{\sin 133} \cdot \frac{1}{\sin 134} \right) = \frac{\sin 1}{\sin n}$

$\sin 1 = \sin(x+1-x) = \sin(x+1)\cos x - \cos(x+1)\sin x$

$\frac{\sin 46 \cos 45 - \cos 46 \sin 45}{\sin 45 \sin 46} + \cdots + \frac{\sin 134 \cos 133 - \cos 134 \sin 133}{\sin 133 \sin 134}$

$= \cot 45 - \cot 46 + \cdots + \cot 133 \cot 134 = \cot 45 - \cot 90$

$= \cot 45 = 1$

$\frac{\sin 1}{\sin n} = 1, \quad \sin n = \sin 1 \Rightarrow n = 1$