1. $a+b+c=100 \Rightarrow a+ak+al=100$

$b=ak$ $\qquad a(1+k+l)=100$

$c=al$ $\qquad (1,100),\ (2,50),\ (4,25)\ (5,20)\ (10,10)$

a	$(k, l \geq 1) \Rightarrow$ (change variable)	
1	$k+l=99-2 \Rightarrow$	$\binom{98}{1}=98$
2	$k+l=49-2$	48
4	$k+l=24-2$	23
5	$k+l=19-2$	18
10	$k+l=9-2$	8
20	$k+l=4-2$	3
25	$k+l=3-2$	2
50	$\times$	
100	$\times$	

$100+51+31+18=100+100=200$

2. $1, 3, 5, \ldots, 199 = \dfrac{200!}{2\cdot4\cdot6\cdot8\cdots100} = \dfrac{200!}{2^{100}\cdot100!}$

$\dfrac{200!}{2^{100}\cdot100!}$

$\Downarrow$

$$\sum_{k=1}^{\infty}\left\lfloor\frac{200}{3^k}\right\rfloor=\left\lfloor\frac{200}{3}\right\rfloor+\left\lfloor\frac{200}{9}\right\rfloor+\left\lfloor\frac{200}{27}\right\rfloor+\left\lfloor\frac{200}{81}\right\rfloor$$

$$=66+22+7+2$$

$$=97$$

$$\sum_{k=1}^{\infty}\left\lfloor\frac{100}{3^k}\right\rfloor=\left\lfloor\frac{100}{3}\right\rfloor+\left\lfloor\frac{100}{9}\right\rfloor+\left\lfloor\frac{100}{27}\right\rfloor+\left\lfloor\frac{100}{81}\right\rfloor$$

$$=33+11+3+1$$

$$=48$$

$$\frac{3^{97}}{3^{48}}=3^{49}\Rightarrow 49$$

$$\begin{array}{r} 2008 \\ 42^2=\underline{1764} \\ 244 \end{array} \qquad 2^3\cdot251$$

3. $x^2+84x+2008=y^2$

$(x+42)^2+4=y^2$

$(x+42)^2=y^2-244$

4. $\frac{n}{810}$ $\quad n = d\cdot 10^{-1}(810) + .025(810) + 10^{-4}d(810)$

$$1000\frac{n}{810} = d25.\overline{d25}$$

$$\frac{n}{810} = .\overline{d25}$$

$$999\frac{n}{810} = d25$$

$$\frac{999}{810}n = \frac{111}{90}n = \frac{37}{30}n = d25$$

$$n = \frac{d25\cdot 30}{37}$$

$$\begin{array}{r} 37 \\ \times 25 \\ \hline 185 \\ 740 \\ \hline 925 \end{array} \rightarrow d=9$$

$$n = \frac{925\cdot 30}{37} = 25\cdot 30 = 750$$

5. factor of p (→prime) in $N! = \sum_{k=1}^{\infty}\left\lfloor \frac{N}{p^k}\right\rfloor$

$$\Rightarrow \sum_{N=1}^{100}\left(\sum_{k=1}^{\infty}\frac{N}{5^k}\right) = \sum_{N=1}^{100}\left(\left\lfloor\frac{N}{5}\right\rfloor + \left\lfloor\frac{N}{25}\right\rfloor\right)$$

$$5\cdot(1+2+\cdots+19)+20$$

$$= [5\cdot(1+2+\cdots+19)+20+25\cdot(1+2+3)+4]$$

$$= [5\cdot\frac{19\cdot 20}{2}+20+25\cdot(6)+4]$$

$$= [5\cdot 190+20+150+4]$$

$$= [970+154] = 1124 \Rightarrow 124$$

6.

$$1+2+3+\cdots+9 = 45$$

$$1+1.1+1.2+\cdots+1.9 = 46$$

$$p(1)+p(2)+\cdots+p(99) = 46\cdot(1+\underset{(20-29)}{2}+\underset{(30-39)}{3}+\cdots+9)+45 = 47.45$$

$$p(100)+p(101)+p(102)+\cdots+p(109) \Rightarrow 1+45 \Rightarrow 46$$

$p(100) + \cdots + p(199) = 46 \cdot (1 + 1 + 2 + 3 + 4 + \cdots + 9) = 46 \cdot 46$

$p(200) + \cdots + p(299) = 2 \cdot (46 \cdot 46)$

$46 \cdot 46 \cdot (1 + 2 + \cdots + 9) = 46^2 \cdot 45 \Rightarrow 46^2 \cdot 45 + 47 \cdot 45$

$\Rightarrow 45 \cdot (46^2 + 47) = 45 \cdot (2163)$

$\frac{2163}{3} = 721 \Rightarrow 7 \cdot (\underline{103}) \Rightarrow 103$

7. $a =$ 1 - 100 except for $(10, 20, 30, 40, \ldots, 90 - 99, 100) \Rightarrow 100 - 10 - 9 = 81$ (for $1 \le a \le 100$, $900 \le a \le 999$)

$101 \to 200 \Rightarrow$ All except $101 - 110$, $120, 130, \ldots, 200 \Rightarrow 100 - 10 - 9 - 9 = 72$

$9 + 9 = 18$

8. $2 \cdot 81 = 738$

8. $2^x 3^y 5^z$

$0 \le x \le 9, \quad 0 \le y \le 9, \quad 0 \le z \le 9$

$8 \cdot 81 = 648$ switches are multiples of 4

Switches $(10 - x)(10 - y)(10 - z)$

Case 1 x, y, z all odd

Then never multiple of 4

Case 2. 2 of x, y, z odd

3 choices for x, y, z is even

$\to (2, 6)$

$2 \cdot 5 \cdot 5 = 50 \to (1, 3, 5, 7, 9)$

$50 \cdot 3 = 150$

Case 3. $x, y, z \Rightarrow 1$ odd

$\binom{3}{2} = 3$ choices $\quad x$ odd $\Rightarrow (1, 3, 5, 7, 9)$

$5 \cdot 5 \cdot 5 = 125 \Rightarrow 3 \cdot 125 = 375$

Case 4. x, y, z all even

$5 \cdot 5 \cdot 5 = 125$

so $125 + 375 + 150 = 650$

9. $1 \le x^2 \le 99999$

$1 \le x \le 316 \quad (316^2 < 99999,\ 317 > 99999)$

$(n+1)^2 - n^2 \ge 100 \Rightarrow$ then each belong to diff sets

$(n+1)^2 - n^2 = 2n+1 \ge 100$

$n \ge \lceil \frac{99}{2} \rceil \simeq 50$

$50 \to 316 \Rightarrow 0 \cdots 266 = 267$ sets

$+ \quad < 2500$

$\hookrightarrow (1 - \ldots - 2499) = 25$ sets

$267 + 25 = 292$

$1000 - 292 = 708$

10. $2004 = 2 \cdot 2 \cdot 501 = 2 \cdot 2 \cdot 3 \cdot 167 = 2^2 \cdot 3 \cdot 167$

$(2^2 \cdot 3 \cdot 167)^{2004} = 2^{4008} \cdot 3^{2004} \cdot 167^{2004}$

$d = 2^a \cdot 3^b \cdot 167^c$, where $(a,b,c) \Rightarrow 2004$

Theorem 4.2.2 $\Rightarrow (a+1)(b+1)(c+1) = 2004$

$$\left(\binom{3}{2}_{\hookrightarrow (2)} + \binom{3}{2}_{\hookrightarrow (3)} + \binom{3}{2}_{\hookrightarrow (167)} + \binom{3}{2}_{\hookrightarrow (4)} + \binom{3}{2}_{\hookrightarrow (6)} + \binom{3}{2}_{\hookrightarrow (334)} \right) \cdot 3$$

$= 6 \cdot 3 \cdot 3 = 54$

11. modulo question

$\frac{n(n+1)}{2} - 1 \le 2000$

$\frac{1993\,(1994)}{2} - 1 \equiv \frac{k(k+1)}{2} - 1$

$\hookrightarrow$ Congruence $1993 \cdot 1994 \equiv k(k+1) \mod 4000$

```
 2 2 1
  1994
  1993
 -----
  5982
```

$$\begin{array}{r} 179460 \\ 1794600 \\ 1994000 \\ \hline 3{,}974{,}042 \end{array}$$

$$\begin{array}{r} 1987 \\ 2000\,\overline{)\,3974042} \\ \underline{2000} \\ 19740 \\ \underline{18000} \\ 17404 \\ \underline{16000} \\ 14042 \\ \underline{14000} \\ 42 \end{array}$$

$\Downarrow$ implies (congruence holds for any divisor of n)

$-7\cdot -6 \equiv k(k+1) \mod 2000$

$k(k+1) \equiv 42 \equiv 2042 \equiv 2\cdot 1021 \equiv 2\cdot(-979)$

$2042 \qquad \frac{979}{11} = 89$

$2\cdot 1021$

$$\begin{aligned} k(k+1) &\equiv 2\cdot 11\cdot -89 \equiv 2\cdot 11\cdot 1911 \equiv 2\cdot 11\cdot 3\cdot 637 \\ &\equiv 2\cdot 11\cdot 3\cdot 7\cdot 91 \\ &\equiv 2\cdot 11\cdot 3\cdot 7\cdot 7\cdot 13 \end{aligned}$$

$k(k+1) \equiv 42$

$k^2 + k - 42 \equiv 0$

$(k-6)(k+7) \equiv 0 \mod (2000)$

$2000 = 2^4\cdot 5^3$, $k-6$, $k+7$ differ by 13

(one divisible by 5, one even)

Case 1. $k+7$ divisible by 125, $k-6$ divisible by 16

$\Downarrow$

smallest $k \Rightarrow 118$, $k-6 = 112 = 7\cdot 16$

Case 2 $k+7$ even, $k-6$ divisible by 125

smallest $k = 131$ for divisible by 125, not smaller than 181

k smaller in Case 1, so $k = 118$

12. $1000 = 2^3\cdot 5^3$

2^x, 5^y, for $-3 \le x \le 3$, $-3 \le y \le 3$

7 choices for both $\Rightarrow$ 49 terms

$$S = \sum_{x=-3}^{3}\sum_{y=-3}^{3} 2^x 5^y = \left(\tfrac{1}{8}+\tfrac{1}{4}+\tfrac{1}{2}+1+2+4+8\right)\cdot\left(\tfrac{1}{125}+\tfrac{1}{25}+\tfrac{1}{5}+1+5+25+125\right)$$

$$= \left(15\tfrac{7}{8}\right)\cdot\left(156\tfrac{31}{125}\right)$$

$$\left(\frac{127}{8}\right)\cdot\frac{19531}{125} = \frac{2480437}{1000} > 2480 \Rightarrow \frac{2480}{10} = 248$$

13. 3^{11}

$n+1, n+2, n+3, \ldots, n+k$

$3^{11} = n+1+n+2+\cdots n+k = kn + \frac{k(k+1)}{2}$

$2 \cdot 3^{11} = 2kn + k(k+1) = k(2n+k+1)$

k must be of form $2^x 3^y$, where $0 \le x \le 1$, $0 \le y \le 11$

$\hookrightarrow$ ~~24 possible values~~ $2 \cdot 3^5(2 \cdot 3^5 - 1 + 3^5 + 1)$

Case 1. k even $(x=1)$

Then $3^{11-x} = 2n+k+1$

$x \le 5$ since otherwise $3^{11-x} < 2n+k+1$

if $x=5$, then $3^6 = 2n + 2 \cdot 3^5 + 1$

$\hookrightarrow 486$

Case 2. $x=0$

Then $2 \cdot 3^{11-x} = 2n + k + 1$

Again $x \le 5$, since otherwise right exceeds left

But $k = 3^5 \le 2 \cdot 3^5$, so $k = 2 \cdot 3^5 = 486$

14.

All primes

$\tau(n)$ only odd if n is perfect square (k s.t. $k^2 = n$ not counted twice)

So $S(n)$ odd if $1, 2, \ldots, n$ contains odd # of perfect squares, even otherwise

$S(1)$ odd, $S(2), (3)$ even $S(4) \to S(8)$ odd, $S(9) - S(15)$ even, $S(16) - S(24)$ odd

$S(25) - S(35)$ even, $S(36) - S(48)$ odd, $S(49) - S(63)$ even

$1, 4, 9, 16$ $\qquad 44^2$

$3, 5, 7, 9, 11, 13, 15, 17, 19, \ldots, 89$

$3, 5, 7, 9, 11, 13, \ldots, 89$

odd even

$5+9+13+17+\cdots+\cancel{89}\ (2005-44^2+1)=70$

$2005-990=1015$

$3+7+11+15+\cdots+87$

$90\cdot\frac{22}{2}=90\cdot 11=990$

$|990-1015|=|-25|=25$

15. 120