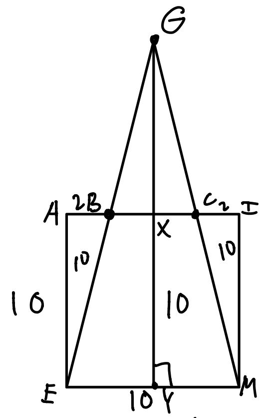
1.



$\triangle GEM \sim \triangle GBC$

$$\frac{GX}{XY+GX} = \frac{BG}{EB+BG}$$

$$\frac{GX}{GX+10} = \frac{BG}{2\sqrt{51}+BG} \Rightarrow \frac{3}{5} = \frac{GX}{GX+10}$$

$EB = \sqrt{104}$
$= 2\sqrt{51}$

$BC = 6$

$6:10$

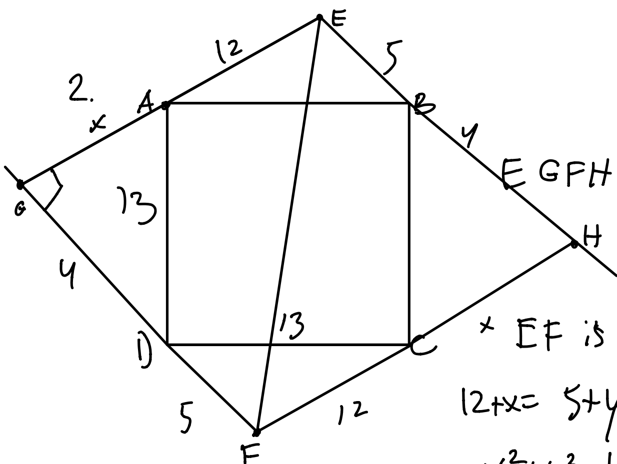
$3GX + 30 = 5GX$

$2GX = 30$

$GX = 15$

$\hookrightarrow 15 + 10 = 25$

2.



EGFH is a square

EF is diagonal

$12 + x = 5 + y \Rightarrow y = x + 7$

$x^2 + y^2 = 169$

$x^2 + x^2 + 14x + 49 = 169$

$2x^2 + 14x - 120 = 0$

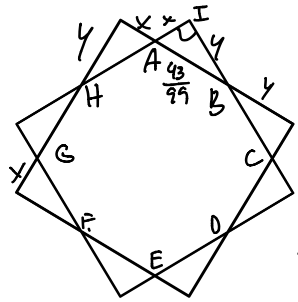
$x^2 + 7x - 60 = 0$

$(x-5)(x+12)$

$x = 5$

$y = 12 \Rightarrow EF^2 = 2 \cdot 17^2 = 2 \cdot 289 = 578$

3. $2 - 4 \cdot A$ of $\triangle ABI$



$$x^2+y^2=\left(\frac{43}{99}\right)^2$$

$$x+y+\frac{43}{99}=1$$

$$x+y=\frac{57}{99}$$

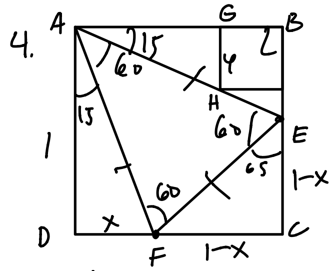
$$x^2+2xy+y^2+2\cdot\frac{43}{99}\cdot(x+y)+\left(\frac{43}{99}\right)^2=1$$

$$2\cdot\left(\frac{43}{99}\right)^2+2\cdot\frac{43}{99}\left(\frac{57}{99}\right)+2xy=1$$

$$1-2xy=\text{answer}=2\cdot\frac{43}{99}\left(\frac{57}{99}+\frac{43}{99}\right)$$

$$=2\cdot\frac{43}{99}\cdot 1=\frac{86}{99}\Rightarrow 86+99=185$$

4.



$\triangle AGH \sim \triangle ABE$

$$\frac{AG}{GH}=\frac{AB}{BE}:\frac{1-y}{y}=\frac{1}{x}$$

$$AE=\sqrt{1+x^2}=EF=\sqrt{2(1-x)^2}$$

$$1+x^2=2(1-x^2)$$

$$1+x^2=2-4x+2x^2$$

$$x^2-4x+1=0$$

$$\frac{4\pm\sqrt{16-4}}{2}=2\pm\sqrt{3}$$

$\Rightarrow x=2-\sqrt{3}$ since $x<1$

Thus $\frac{1}{2-\sqrt{3}}=\frac{1-y}{y}$

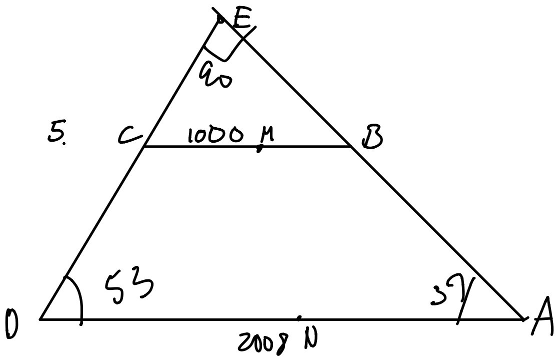
$$y=2-\sqrt{3}-2y+\sqrt{3}y$$

$$y(3-\sqrt{3})=2-\sqrt{3}$$

$$y=\frac{2-\sqrt{3}}{3-\sqrt{3}}=\frac{(2-\sqrt{3})(3+\sqrt{3})}{9-3}$$

$$= \frac{6-3-\sqrt{3}}{6}$$
$$= \frac{3-\sqrt{3}}{6} \Rightarrow 12$$

5.



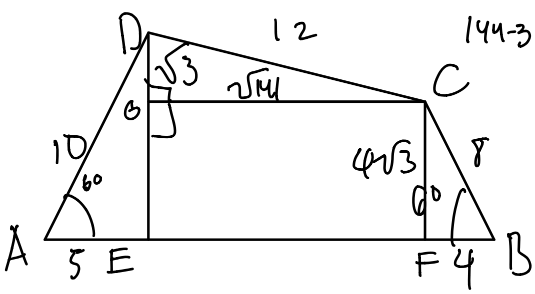
$$\triangle ECB \sim \triangle EDA$$

$$\frac{EM}{EN} = \frac{EC}{ED} = \frac{1000}{2008}$$

$$ED = 2008 \cos 53$$

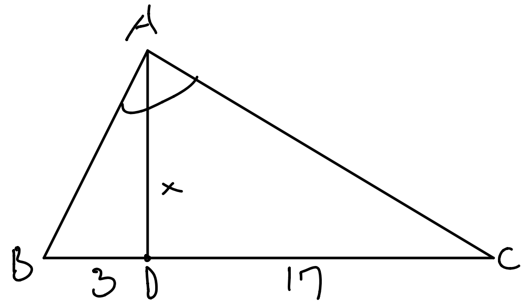
$$EN^2 = ND^2 + ED^2 - 2ND \cdot ED \cdot \cos 53$$
$$= 1004^2 + 2008\cos 53 - 2008 \cos 53$$
$$= 1004^2$$
$$EN = 1004$$
$$EM = 500$$
$$MN = EN - EM = 504$$

6.



$$AB = AE + EF + BF$$
$$AB = 9 + EF = 9 + \sqrt{141} \Rightarrow 150$$

7.



$\tan(CAB) = \frac{22}{7}$

$$\tan(CAD + BAD) = \frac{\tan(CAD) + \tan(BAD)}{1 - \tan CAD \cdot \tan BAD}$$

$$= \frac{\frac{17}{x} + \frac{3}{x}}{1 - \frac{51}{x^2}} = \frac{\frac{20}{x}}{\frac{x^2-51}{x^2}} = \frac{20}{\cancel{x}} \cdot \frac{x^{\cancel{2}}}{x^2-51} = \frac{20x}{x^2-51}$$

$$\frac{22}{7} = \frac{20x}{x^2-51} \Rightarrow 140x = 22(x^2-51)$$

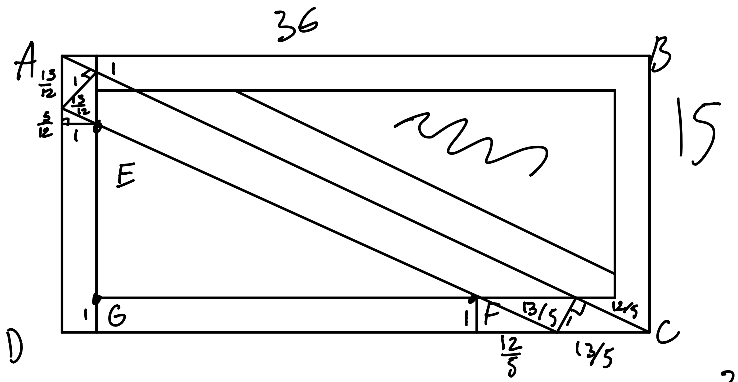
$$22x^2 - 140x - 1122 = 0$$

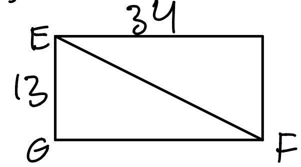
$$\frac{140 \pm 344}{44} = \pm 11$$

$$x = 11$$

$$\frac{11 \cdot 20}{2} = 110$$

8.





Center in 13×34 rectangle

$\triangle ACD \sim \triangle EFG$

5-12-13

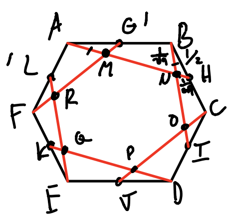
$EG = 15 - 1 - \frac{18}{12} = \frac{25}{2}$ $\qquad 25 : 60$

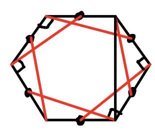
$GF = 36 - \frac{25}{5} - 1 = 36 - 6 = 30$ $\qquad 5 : 12$

$$2\left(\frac{\frac{25}{2}\cdot 30}{2}\right) = 15 \cdot 25 = 375$$

$$\frac{375}{13 \times 34} = \frac{375}{442} \Rightarrow 817$$

9.





$\frac{720}{6} = 120$

$1^2 : MN^2$

$MN = AH - AM - NH$ $\qquad 1 : \frac{1}{\sqrt{7}}$

$AH^2 = 1^2 + \frac{1}{2}^2 - 2 \cdot 1 \cdot \frac{1}{2}\cos(120°)$ $\qquad \frac{\sqrt{7}}{2} : \frac{1}{2}$

$= 1 + \frac{1}{4} + \frac{1}{2} = \frac{7}{4}$ $\qquad \sqrt{7} : 1$

$AH = \frac{\sqrt{7}}{2}$

$MN = \frac{\sqrt{7}}{2} - \frac{1}{\sqrt{7}} - \frac{1}{2\sqrt{7}}$

$= \frac{\sqrt{7}}{2} - \frac{3}{2\sqrt{7}}$

$= \frac{7}{2\sqrt{7}} - \frac{3}{2\sqrt{7}} = \frac{2}{\sqrt{7}} = \frac{2\sqrt{7}}{7}$

$MN^2 = \frac{4 \cdot 7}{49} = \frac{4}{7} \Rightarrow 11$

10. $d = DP + PE$

$\triangle DPG \sim \triangle ABC$

$\triangle PEH \sim \triangle ABC$

$\frac{DP}{425} = \frac{DG}{510}$

$\frac{PE}{425} = \frac{HE}{450} \Rightarrow \frac{EP}{EH} = \frac{425}{450} = \frac{17}{18} \Rightarrow EP = \frac{17}{18} EH$

$DP = 425 = 5$ $\quad DP = 5\ DG$

$\frac{DP}{DG} = \frac{\ }{510} = \frac{\ }{6},\ DP = \frac{5}{6}DG$

$AC = AD + DG + GC$ $\qquad\qquad$ $BC = EB + EH + CH$

$= IP + DG + PH$ $\qquad\qquad$ $= d + EH$

$= d + DG$ $\qquad\qquad$ $EH = 450 - d$

$\qquad\qquad$ $EP = \frac{17}{18}(450 - d)$

$DG = 510 - d \Rightarrow DP = \frac{5}{6}(510 - d)$

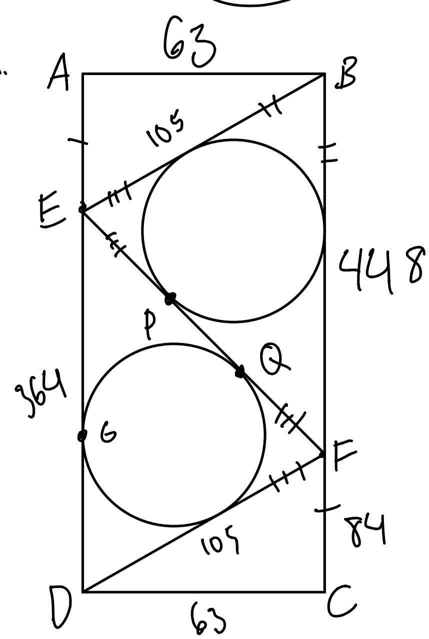
$d = \frac{5}{6}(510 - d) + \frac{17}{18}(450 - d)$

$\frac{32}{18}d + d = 425 + 425$

$\frac{50}{18}d = 850$

$d = 306$

11.



$\triangle BEF \cong \triangle DFE$

$$\begin{array}{r} 448 \\ -\ 84 \\ \hline 364 \end{array}$$

$PQ = QE - EP = 364 - 105 + FQ - FQ$

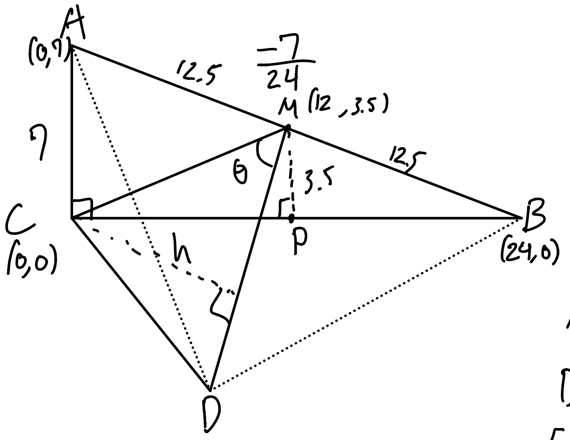
$= 259$

$EQ = EG = 364 - DG$

$DG = 105 - FQ$

$EP = FQ$ since $\triangle BEF \cong \triangle DFE$

12.



$M = (x_1, y_1) = (12, 3.5)$

$D = (x_2, y_2)$

$[CDM] = \frac{1}{2}|x_1 y_2 - x_2 y_1|$

$$b = DM = \sqrt{AD^2 - AM^2} = \sqrt{15^2 - \left(\frac{25}{2}\right)^2}$$

$$= \sqrt{\frac{275}{4}} = \frac{5}{2}\sqrt{11}$$

$h = CM \sin\theta$

$$CM = \sqrt{3.5^2 + 12^2} = \sqrt{\frac{49}{4} + \frac{576}{4}}$$

$$= \frac{25}{2}$$

$$BC^2 = CM^2 + BM^2 - 2CM \cdot BM \cos(\theta + 90)$$

$$= 2 \cdot \left(\frac{25}{2}\right)^2 + 2 \cdot \left(\frac{25}{2}\right)^2 \sin\theta$$

$$= 2 \cdot \left(\frac{25}{2}\right)^2 (1 + \sin\theta)$$

$$576 = \frac{625}{2}(1 + \sin\theta)$$

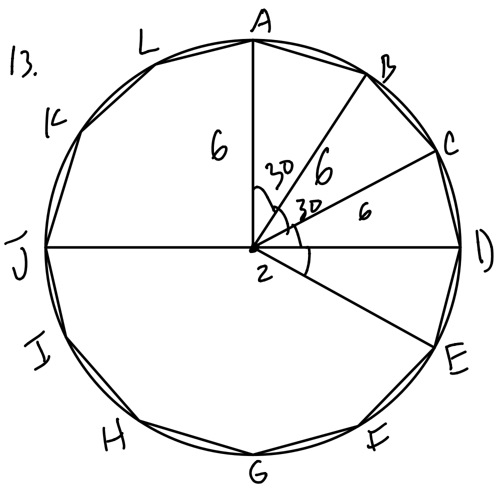
$$\sin\theta + 1 = \frac{1152}{625}$$

$$\sin\theta = \frac{527}{625}$$

Then $h = CM \cdot \sin\theta = \frac{\cancel{25}}{2} \cdot \frac{527}{\cancel{625}_{25}} = \frac{527}{50}$

$$[CDM] = \frac{1}{2} bh = \frac{1}{2} \cdot \frac{\cancel{5}}{2}\sqrt{11} \cdot \frac{527}{\cancel{50}_{10}} = \frac{527}{40}\sqrt{11} \Rightarrow 578$$

13.



$\binom{12}{2} = \frac{12 \cdot 11}{2} = 66$

$66 - 12 = 54$ diagonals

11

10

$AB^2 = 288 - 288\cos(30) = 288 - 288 \cdot \frac{\sqrt{3}}{2}$

$= 288 - 144\sqrt{3}$

$AB = \sqrt{288 - 144\sqrt{3}} = 12\sqrt{2 - \sqrt{3}}$

$= 12 \cdot \left(\frac{\sqrt{6} - \sqrt{2}}{2}\right) = 6(\sqrt{6} - \sqrt{2})$

$AC^2 = 288 - 288\cos(60) = 144$

$AC = 12$

$AD^2 = 288$

$AD = 12\sqrt{2}$

$AE^2 = 288 - 288\cos(120)$

$= 288 + 288 \cdot \frac{\sqrt{3}}{2} \Rightarrow AE = 12\sqrt{2 + \sqrt{3}} = 6(\sqrt{6} + \sqrt{2})$

$AF^2 = 288 + 144 = 532$

$AF = \sqrt{144 \cdot 3} = 12\sqrt{3}$

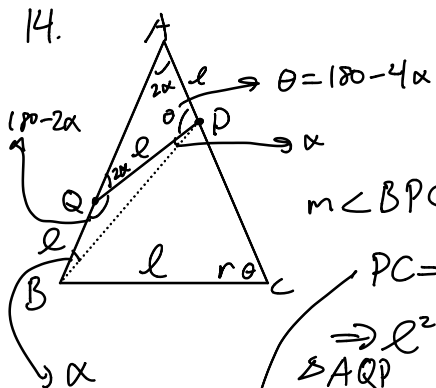
$AG = 24$

$12 \cdot AB + 12 \cdot AC + 12 \cdot AD + 12 \cdot AE + 12 \cdot AF + 6 \cdot AG$

$12(6(\sqrt{6} - \cancel{\sqrt{2}}) + 12 + 12\sqrt{2} + 6(\sqrt{6} + \cancel{\sqrt{2}}) + 12\sqrt{3}) + 24 \cdot 6$

$144\sqrt{6} + 144 + 144\sqrt{2} + 144\sqrt{3} + 144 \Rightarrow 5 \cdot 144 = 720$

14.



$m\angle BPC = 3\alpha$

$PC = AC - AP = AC - \ell = AB - \ell = AQ$

$\Rightarrow \ell^2 = AQ^2 + \ell^2 - 2AQ\ell\cos(2\alpha)$

$\triangle AQP$

$2\cancel{AQ}\cdot\ell\cos(2\alpha) = AQ^{\cancel{2}}$

$\Rightarrow AQ = 2\ell\cos(2\alpha) = PC$

Law of Sines in $\triangle BPC$

$$\frac{\ell}{\sin(3\alpha)} = \frac{2\ell\cos(2\alpha)}{\sin(\angle PBC)}$$

$\alpha + \angle PBC + 2\alpha + r\theta = 180$

$\alpha + \angle PBC = r\theta$ since $\triangle ABC$ isosceles

$\Rightarrow 4\alpha + 2\angle PBC = 180$

$2\alpha + \angle PBC = 90$

$\Rightarrow 2\alpha = 90 - \angle PBC$

$$\frac{\ell}{\sin(3\alpha)} = \frac{2\ell\,\cancel{\cos(90-\angle PBC)}}{\cancel{\sin(\angle PBC)}}$$

$$\frac{1}{\sin(3\alpha)} = 2$$

$\sin(3\alpha) = \frac{1}{2}$

$3\alpha = 30$

$\alpha = 10$

$\angle PBC = 70$

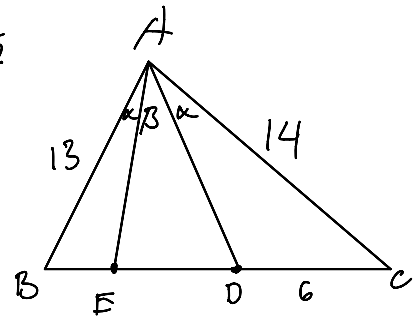
$\theta = 180 - 40 = 140$

$$140r = 70 + 10 = 80$$

$$r = \frac{80}{140} = \frac{4}{7} \Rightarrow \frac{4000}{7} = 571\ldots$$

$$\Rightarrow 571$$

15.



$$[ABC] = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = \sqrt{84 \cdot 84} = 84$$

Herons

$$\frac{1}{2}bh = [ABC]$$

$$h = \frac{2 \cdot [ABC]}{15} = \frac{2 \cdot 84}{15} = \frac{56}{5}$$

$$[BAD] = \frac{1}{\cancel{2}} \cdot \frac{\cancel{56}^{28}}{5} \cdot 9 = \frac{252}{5}$$

$$[ADC] = \frac{1}{\cancel{2}} \cdot \frac{56}{5} \cdot \cancel{6}^{3} = \frac{168}{5}$$

$$[ABE] = \frac{1}{2} \cdot 13 \cdot AE\sin(\alpha)$$

$$\frac{168}{5} = \frac{1}{2} \cdot 14 \cdot AD\sin(\alpha)$$

$$\sin\alpha = \frac{24}{5} \cdot \frac{1}{AD}$$

$$[ABE] = \frac{13}{2} \cdot \frac{AE}{AD} \cdot \frac{24}{5} = \frac{12 \cdot 13}{5} \frac{AE}{AD}$$

$$\frac{[ACE]}{[ABD]} = \frac{\frac{14}{2} \cdot AE\cancel{\sin(\alpha+\beta)}}{\frac{13}{2} \cdot AD\cancel{\sin(\alpha+\beta)}} = \frac{14}{13} \cdot \frac{AE}{AD}$$

$$\Rightarrow \frac{AE}{AD} = \frac{13}{14} \frac{[ACE]}{[ABD]}$$

$\cancel{28}$ 13

$$\hookrightarrow [ABE] = \frac{\cancel{156}}{\cancel{8}} \cdot \frac{13}{\underset{7}{\cancel{14}}} \cdot [ACE] \cdot \frac{\cancel{8}}{\underset{42}{\cancel{\cancel{252}\;126}}}$$

$$= \frac{169}{294}[ACE]$$

$$= \frac{169}{294}\left([ABC] - [ABE]\right)$$

$$\frac{463}{\cancel{294}}[ABE] = \frac{169}{\cancel{294}} \cdot 84$$

$$[ABE] = \frac{169 \cdot 84}{463} = \frac{1}{2} \cdot BE \cdot h = \frac{1}{2} \cdot BE \cdot \frac{56}{5}$$

$$B = \frac{169 \cdot 84 \cdot 2}{463} \cdot \frac{5}{56} = \frac{13^2 \cdot \cancel{7} \cdot \overset{3}{\cancel{24}} \cdot 5}{463 \cdot \cancel{8} \cdot \cancel{7}}$$

$$\hookrightarrow 463$$