1. $z = a + 164i$

$$\frac{z}{z+n} = 4i$$

$$a + 164i = 4i(z+n)$$

$$a+164i = 4i(a+n+164i)$$

$$= -656 + 4i(a+n)$$

$a=-656$ $\quad 164 = 4(a+n)$

$$41 = a+n$$

$$n = 41+656 = 697$$

2. $c=(a+bi)^3 - 107i$

$$(a+bi)^2 = a^2+2abi - b^2$$

$$= (a^2-b^2) + (2ab)i$$

$$(a+bi)^3 = \left((a^2-b^2)+(2ab)i\right)(a+bi)$$

$$= a(a^2-b^2) + 2a^2bi + bi(a^2-b^2) - 2ab^2$$

$$= a^3 - b^2a - 2ab^2 + (a^2b - b^3 + 2a^2b)i$$

$$a^2b - b^3 + 2a^2b = 107$$

$$b(3a^2-b^2) = 107$$

$a^3-3ab^2 = c$ $\qquad$ 107 is prime

$a(a^2-3b^2)=c$ $\qquad$ so either $b=107$ or $3a^2-b^2=107$

Case 1. $b=107$

$$3a^2-b^2=1$$

$$a^2 = \frac{108}{3} = 36$$

$$a=6$$

$c= 6(36-3\cdot 107) \Rightarrow$ negative

Case 2. $3a^2-b^2=107$

$b=1 \quad 3a^2=108$

$a^2=36 \quad a=6$

$$c=6(36-3)=6\cdot 33=198$$

3. $p(x) = x^{2001} + (\frac{1}{2} - x)^{2001}$

degree 2000

$p(\frac{1}{2}-r) = p(r)$

roots in pair $\Rightarrow \frac{1}{2}-r+r=\frac{1}{2}$

1000 pairs $\Rightarrow \frac{1}{2}\cdot 1000 = 500$

4. $P_0(x) = x^3+313x^2-77x-8$

$P_n(x) = P_{n-1}(x-n)$

$P_{20}(x) = P_{19}(x-20) = P_0(x-(20+19+\cdots+1))$

$= P_0(x-210)$

$= (x-210)^3 + 313(x-210)^2 - 77(x-210) - 8$

$3\cdot(210)^2 \qquad 313\cdot(2\cdot-210) \qquad -77$

$3\cdot 210\cdot 210 - 626\cdot 210 - 77$

$(630-626)\cdot 210 - 77 = 4\cdot 210 - 77$

$= 840 - 77 = 763$

5. $x^4-4x^3+6x^2-4x=2005$

$(x-1)^4 = 2006 \qquad 2\cdot 1003$

$(x-1)^2(x-1)^2 \qquad 17\cdot 118$

$y^4=2006$

$y = \pm\sqrt{2006}, \pm i\sqrt{2006}$

$x = 1\pm\sqrt[4]{2006}, 1\pm i\sqrt[4]{2006}$

$P = (1+i\sqrt[4]{2006})(1-i\sqrt[4]{2006})$

$= 1+\sqrt{2006} \quad \lfloor 1+\sqrt{2006}\rfloor \Rightarrow 1+44 = 45$

$\lfloor \sqrt{2006} \rfloor = 44$

6. $P(z) = z^3 + az^2 + bz + c$

$w+3i,\ w+9i,\ 2w-4$

$w = x+yi$

$w+3i$

$-a = w+3i + w+9i + 2w-4$

$a = -4w + 4 - 12i$

$b = (w+3i)(w+9i) + (w+3i)(2w-4) + (w+9i)(2w-4)$

$= w^2 + 12wi - 27 + 2w^2 + 6wi - 4w - 12i + 2w^2 + 18wi - 4w - 36i$

$= 5w^2 + 36wi - 8w - 48i - 27$

Since a is real $y = -$ $\therefore w = x-3i$

$5(x-3i)^2 + 36(x-3i)i - 8(x-3i) - 48i - 27$

$5(x^2 - 6xi - 9) + 36(x-3i)i - 8(x-3i)$

Since b real, $-30x + 36x + 24 - 48 = 0$

$6x - 24 = 0$

$x = 4$

Thus $w = 4-3i$

$w+3i = 4,\ w+9i = 4+6i,\ 2w-4 = 4-6i$

$a = -[4 + 4+6i + 4-6i] = -12$

$b = 4(4+6i) + 4(4-6i) + (4-6i)(4+6i)$

$= 16 + 16 + 16 + 36 = 48+36 = 84$

$c = -(4)(4-6i)(4+6i) = -4(16+36) = -4 \cdot 52 = -208$

$|a+b+c| = |-220 + 84| = 136$

7. $Q_1(x) = x^2 + (k-29)x - k$

$Q_2(x) = 2x^2 + (2k-43)x + k$

$P(x) = (x-a) \cdot Q_1(x) = x^3 + x^2(k-29) - kx - ax^2 - ax(k-29) - ak$

$= x^3 + (k-29-a)x^2 - (ka - 29a + k)x - ka$

$P(x) = \frac{1}{2}(x-b) \cdot Q_2(x) = \frac{1}{2}(2x^3 + x^2(2k-43) + kx - 2bx^2 - (2k-43)bx - bk)$

$= x^3 + \frac{1}{2}(2k-43-2b)x^2 - \frac{1}{2}(2kb - 43b - k)x - \frac{1}{2}kb$

$a = -\frac{1}{2}b$

① $k - 29 - a = \frac{1}{2}(2k - 43 - 2b)$

② $ka - 29a + k = \frac{1}{2}(2kb - 43b - k)$

$\Rightarrow$ ① $2k - 58 - 2a = 2k - 43 - 2b$

② $2ka - 58a + 2k = 2kb - 43b - k$

$-58 - 2a = -43 - 2b$

$-2a + 2b = 15$

$3b = 15$

$b = 5$

$a = -\frac{5}{2}$

$10k - 215 - k = -5k + 145 + 2k$

$12k = 360$

$k = 30$

8. $(\sin t + i\cos t)^n = \sin(nt) + i\cos(nt)$

$(\sin t + i\cos t)^n = (\cos(90° - t) + i\sin(90° - t))^n$

$= \cos(n(90° - t)) + i\sin(n(90° - t))$

$\cos(n(90° - t)) = \sin(nt)$

$\cos(90n - nt) = \cos 90n \cdot \cos nt + \sin 90n \cdot \sin nt = \sin nt$

possible $\Rightarrow 90 \pm 360n$

$1 \pm 4n$

$1, 5, 9 \ldots, 997$

$0, 1, 2, \ldots, 249 \Rightarrow$ 250 numbers

9. $w_1 = 32 + 170i$

$w_2 = -7 + 64i$ $\quad y = mx + 3 \quad$ let $z_k = x_k + iy_k$

$w_3 = -9 + 200i$ $\quad z_1 + z_2 + z_3 + z_4 + z_5 = w_1 + \cdots + w_5 = 3 + 504i$

$w_4 = 1 + 27i$ $\quad = x_1 + x_2 + \cdots + x_5 + i(m(x_1 + \cdots + x_5) + 15)$

$w_5 = -14 + 43i$ $\quad x_1 + \cdots + x_5 = 3$

$m(x_1 + \cdots + x_5) + 15 = 504$

$3m = 489$

$m = 163$

10. $S_9 = S_8 +$ Sets of length 1–9 including 9

$9 \cdot i^9 = 9i$

$S_8 + \overset{9i}{} \; 2^8 \cdot 9i + S_8$

Length 1: $9i^1 = 9i$ $\quad 72 \cdot 9i - (28 + 28) \cdot 9i$

2: $8 \cdot 9i^2 = -72$ $\quad 16 \cdot 9i = 144i$

3: $\binom{8}{2} \cdot 9i^3 = 28 \cdot -9i$ $\quad -72 + 7 \cdot 72 + 72 - 7 \cdot 72 = 0$

4: $\binom{8}{3} \cdot 9i^4 = 56 \cdot 9 = 7 \cdot 72$

5: $\binom{8}{4} \cdot 9i^5 = \frac{\cancel{8}^{2} \cdot 7 \cdot \cancel{6} \cdot 5}{\cancel{4}} \cdot 9i = 70 \cdot 9i$ $\quad 144i + 2 \cdot S_8 = -352 - 128i + 144i$

6: $\binom{8}{5} \cdot 9i^6 = 56 \cdot -9$ $\quad = -352 + 16i$

7: $\binom{8}{6} \cdot 9i^7 = 28 \cdot -9i$ $\quad \Rightarrow |-352| + |16|$

8: $\binom{8}{7} \cdot 9i^8 = 8 \cdot 9$ $\quad = 368$

9: $\binom{8}{8} \cdot 9i^9 = 9i$

...

11. $z_1 = F(z_0) = \dfrac{z_0+i}{z_0-i}$

$$z_2 = \frac{\dfrac{z_0+i+z_0 i+1}{z_0-i}}{\dfrac{z_0+i-z_0 i-1}{z_0-i}} = \frac{z_0+i+i(z_0-i)}{z_0+i-i(z_0-i)}$$

$$z_3 = \frac{\dfrac{z_0+i+i(z_0-i)}{z_0+i-i(z_0-i)}+i}{\dfrac{z_0+i+i(z_0-i)}{z_0+i-i(z_0-i)}-i} = \frac{z_0+\cancel{i}+i(z_0-i)+i(z_0+i)+z_0\cancel{-i}}{\cancel{z_0}+i+i(z_0-i)-i(z_0+i)-\cancel{z_0}+i}$$

$$= \frac{2z_0+2z_0 i}{2+2i} = z_0$$

$z_3 = z_0$

So $z_0 = z_3 = z_6 = \dots = z_{2001}$

Thus $z_{2002} = z_1 = \dfrac{z_0+i}{z_0-i} = \dfrac{\frac{1}{137}+2i}{\frac{1}{137}} = 1+274i \Rightarrow 275$

12. $k = a_3(-3+i)^3 + a_2(-3+i)^2 + a_1(-3+i) + a_0$

$a_0, a_1, a_2, a_3 \in \{0, 1, 2, \dots, 9\}$

$(-3+i)^2 = 9-6i-1 = 8-6i$

$(-3+i)(8-6i) = -24+26i+6 = -18+26i$

$a_0 - 3a_1 + 8a_2 - 18a_3$

$a_1 - 6a_2 + 26a_3 = 0$

$a_1 = 6a_2 - 26a_3$

$k = a_3(-3+i)^3 + a_2(-3+i)^2 + (6a_2-26a_3)(-3+i) + a_0$

Some calculations redundant since i term guaranteed to cancel

$= (-18+26i)a_3 + (8-6i)a_2 + a_2(-18+6i) - 26a_3(-3+i) + a_0$

$= a_0 + -10a_2 + 60a_3$

$6a_2 = 26a_3 + a_1$

$26a_3 + a_1 \le 54$

a_3 can be $= 1$ or 2

Case 1. $a_3 = 1$. Then $a_2 = 5$

then $k = a_0 + -10a_2 + 60$

So $10 + \cdots + 19 = 100 + \frac{9 \cdot 10}{2} = 100 + 45 = 145$

Case 2. $a_3 = 2$. Then $a_2 = 9$

Then $k = a_0 - 90 + 120 = a_0 + 30$

$30 + \cdots + 39 = 300 + 45 = 345$

$345 + 145 = 490$

13. $x^2 - x - 1$

$(x^2 - x - 1)$

$$
\begin{array}{r l}
 & ax^{15} + ax^{14} + bx^{14} \ldots = q(x) \\
x^2 - x - 1 \,\big)\!\! & ax^{17} + bx^{16} + 1 \\
 & \underline{ax^{17} - ax^{16} - ax^{15}} \\
 & ax^{16} + bx^{16} + ax^{15} \\
 & \underline{ax^{16} - ax^{15} - ax^{14}} \\
 & bx^{16} + 2ax^{15} + ax^{14} \\
 & \underline{bx^{16} - bx^{15} - bx^{14}} \\
 & 2ax^{15} + 2b^{15} + ax^{14} + bx^{14}
\end{array}
$$

Coefficients of x in $q(x)$

$r(x)$ = remainder

$q(x) = ax^{15} + (a+b)x^{14} + (2a+b)x^{13} + (3a+2b)x^{12}$

$+ \cdots + (987a + 510b)$

$r(x) = (1597a + 987b) + (987a + 610b + 1)$

So $1597a + 987b = 0$

$987a + 610b = -1$

$\Rightarrow (\underbrace{1597 \cdot 610 - 987^2}_{=1})a = 987$

$\Rightarrow a = 987$

14. $P(n) = n+3$ has 2 roots

$P(17) = 10$

$P(24) = 17$

$S(x) = (x-n_1)(x-n_2) - x - 3$

$= Q(x)(x-n_1)(x-n_2)$

$S(17) = P(17) - 20 = -10$

$S(24) = P(24) - 27 = -10$

Thus $Q(17)(17-n_1)(17-n_2) = Q(24)(24-n_1)(24-n_2) = -10$

Thus $n_1 - 17$, $n_2 - 17$, $n_1 - 24$, $n_2 - 24$ divisors of 10

$n_1 - 17 - n + 24 = 7$

So $(5,-2)$, $(2,-5)$

$n_1 - 17 = 2$ or 5

$n_1 = 19, 22$

Since $n_1 \neq n_2$, $n_1, n_2 \in \{19, 22\}$

Thus $n_1 \cdot n_2 = 19 \cdot 22 = 418$

$$\begin{array}{r} 22 \\ \underline{} \\ 38 \\ \underline{380} \\ 418 \end{array}$$

15. $x^3 - 2011x + m$

$(x-a)(x-b)(x-c)$

$= (x^2 - (a+b)x + ab)(x-c)$

$= x^3 - (a+b+c)x^2 + (ab + ac + bc)x - abc$

$a+b+c = 0$ $\quad$ $ab + ac + bc = -2011$

$-abc = m$ $\qquad -2011 = a(b+c) + bc = -a^2 + bc$

$\Downarrow$

$a^2 = bc + 2011$

So $a > \sqrt{2011} > 0$ since assume $bc > 0$

$a^2 = bc + 2011 \le \left(-\frac{a}{2}\right)^2 + 2011$

$\Downarrow$

$\frac{3a^2}{4} \le 2011$

$a^2 \le \frac{8044}{3} < 2704$

$\Downarrow$

$44 < a < 52$

Since $bc = a^2 - 2011$

and $b + c = -a$

a	bc	$b+c$
45	14	-45
46	105 $7 \cdot 15$ $7 \cdot 3 \cdot 5$	-46
47	198 $2 \cdot 3^2 \cdot 11$	-47
48	293 prime	-48
49	390 $2 \cdot 5 \cdot 39$	-49 $-39 - 10$

$49 + 49 = 98$