1. equidistant primes

$a_1 + 4k$ is prime

$+3k$ is prime
$+2k$ is prime

So a_1 not a factor of 2, 3, or 4 (can't be odd b/c then $a_n + k$ = even / not prime)

Thus maybe $k=6$?

5 11 17 23 29 $\Rightarrow$ 29

2. $N = 100^2 + 99^2 - 98^2 - (97^2 - 96^2) + (95^2 - 94^2) \cdots + (3^2 - 2^2) - 1^2$

$= 100^2 + (99-98)\cdot(99+98) - (97-96)(97+96) + (95-94)\cdot(95+94) + \cdots + (3-2)(3+2) - 1^2$

$= 100^2 + (\underbrace{99+98-97-96}_{4} + \underbrace{95+94-93-92}_{4} + \cdots \underbrace{3+2-1}_{4})$

$= 100^2 + 25\cdot 4$

$= 10000 + 100 = 10\,100 \Rightarrow 100$

3. $S = 2005$

$\frac{a}{1-r} = 2005 \qquad a = 2005 - 2005r$

$\frac{a^2}{1-r^2} = 10 \cdot \frac{a}{1-r} = 20050$

$\frac{a}{(1-r)} \cdot \frac{a}{(1+r)} = 10 \cdot \frac{a}{1-r}$

$\frac{a}{1+r} = 10$

$a = 10 + 10r$

$10 + 10r = 2005 - 2005r$

$2015r = 1995$

$r = \frac{1995}{2015} = \frac{399}{403} \Rightarrow 802$

4. $x_1 = 211$

$x_2 = 375$

$x_3 = 420$

$x_4 = 523$

$x_n = x_{n-1} - x_{n-2} + x_{n-3} - x_{n-4}$ when $n \geq 5$

$x_{531}\ x_{753} +\ x_{975}$

$x_5 = 267$

$x_6 = 267 - 523 + 420 - 375$

$= -256 + 35 = -211$

$x_7 = -211 - 267 + 523 - 420$

$= -478 + 103 = -375$

$x_8 = -420$

$x_9 = -520 + 485 - 221 - 267$

$= -35 - 488 = -523$

$x_{10} = -523 + 520 - 485 + 221$

$= -3 - 264 = -267$

$x_{11} = -267 + 523 - 520 + 485$

$= 256 - 35 = 221$

$x_{12} = 375$

$x_{13} =$

$= 420$

$x^4 = x^3 - x^2 + x - 1$

$x^4 - x^3 + x^2 - x + 1 = 0$

$$\begin{array}{r} x^3 + x \\ x-1 \overline{) x^4 - x^3 + x^2 - x + 1} \\ \underline{x^4 - x^3} \\ 0 \quad x^2 - x \\ x^2 - x \end{array}$$

$(x-1)(x^3+x) + 1 = 0$

$531 \% 10 = 1$

$\hookrightarrow 211$

$753 \% 10 = 3$

$\hookrightarrow 420$

$975 \% 10 = 5$

$\hookrightarrow 267$

$\Downarrow$

898

-a.

5. $\underset{1}{a_1} + \underset{2}{a_2} + \underset{3}{(a_2 - a_1)} + (a_2 - \underset{4}{a_1} - a_2) + (-\underset{5}{a_1} - a_2 + a_1) + \underset{6}{(a_1 - a_2)} + (a_1 - \underset{7}{a_2} + a_2)$

$\underset{8}{a_1 - a_1 + a_2} \quad \underset{9}{a_2 - a_1}$

every 6 terms cancel out

(repeats every 6)

first 1492 = 1985

248 4

first 1985 = 1492

36 5

$a_1 + a_2 + (a_2 - a_1) - a_1 = 2a_2 - a_1 = 1985$

$a_1 + a_2 + (a_2 - a_1) - a_1 - a_2 = a_2 - a_1 = 1492$

2001 $\quad a_2 = 493$

333 $\quad a_1 = -999$

3

$a_1 + a_2 + (a_2 - a_1) = 2a_2 = 986$

6. $1000 \cdot \sum_{n=3}^{10000} \frac{1}{n^2-4}$

$\frac{1}{n^2-4} = \frac{1}{4}\left(\frac{1}{n-2} - \frac{1}{n+2}\right)$

$250 \cdot \sum_{n=3}^{10000}\left(\frac{1}{n-2} - \frac{1}{n+2}\right)$

$250\left(1 - \cancel{\frac{1}{5}} + \frac{1}{2} - \cancel{\frac{1}{6}} + \frac{1}{3} - \cancel{\frac{1}{7}} + \frac{1}{4} - \cancel{\frac{1}{8}} + \cancel{\frac{1}{5}} - \cancel{\frac{1}{9}} + \cancel{\frac{1}{6}} - \cancel{\frac{1}{10}} + \cancel{\frac{1}{7}} - \cancel{\frac{1}{11}} + \cdots + \cancel{\frac{1}{9998}} - \frac{1}{10002}\right)$

$250 \cdot \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{9999} - \frac{1}{10000} - \frac{1}{10001} - \frac{1}{10002}\right)$

$\approx 250 \cdot \left(\frac{18}{12} + \frac{4}{12} + \frac{3}{12}\right)$

$= 250 \cdot \left(\frac{25}{12}\right) = \frac{6250}{12} = 520\frac{10}{12} \approx 521$

7. $36 + k = (a_1 + m)^2 = a_1^2 + 2a_1m + m^2 \qquad 2a_1m + 3m^2 = 264$

$300 + k = (a_1 + 2m)^2 = a_1^2 + 4a_1m + 4m^2 \qquad 2a_1m + 5m^2 = 296$

$596 + k = (a_1 + 3m)^2 = a_1^2 + 6a_1m + 9m^2 \qquad 2m^2 = 32$

$36 + k = a_1^2 + 216 + 16 = a_1^2 + 232 \qquad m^2 = 16$

$$300+k = a_1^2 + 432+64 = a_1^2 + 496$$

$$2a_1 m + 48 = 264$$

$$k = a_1^2 + 196$$

$$= 729+196 = 925$$

$$2a_1 m = 216$$

$$a_1 = \frac{216}{8} = 27$$

8. $1, 3, 4, 9, 10, 12, 13$

$$a_n = c_n \cdot 3^n + c_{n-1} \cdot 3^{n-1} + \cdots + c_1 \cdot 3 + c_0$$

where $c_0, c_1, \ldots, c_n$ either 0 or 1

$$100 = 1100100_2$$

$$\ 64\ 32\ 16\ 8\ 4\ 2\ 1$$

$$3^2 + 3^5 + 3^6 = 9 + 243 + 729 = 972 + 9 = 981$$

9. $a_0 = 37,\ a_1 = 72,\ a_m = 0$

$$a_{k+1} = a_{k-1} - \frac{3}{a_k}$$

$$a_2 = 37 - \frac{3}{72} = 37 - \frac{1}{24} = 36\tfrac{23}{24}$$

$$a_3 = 72 - 36\tfrac{23}{24} = 35\tfrac{1}{24}$$

$$a_4 = 36\tfrac{23}{24} - \frac{3}{\left(35\frac{1}{24}\right)}$$

$$a_m = 0 = a_{m-2} - \frac{3}{a_{m-1}} \qquad a_{m-1} \cdot a_{m-2} = 3$$

$$a_{m-1} = a_{m-3} - \frac{3}{a_{m-2}} \qquad a_{m-1} \cdot a_{m-2} = a_{m-3} \cdot a_{m-2} - 3$$

$$a_{m-2} = a_{m-4} - \frac{3}{a_{m-3}} \qquad 6 = a_{m-2} \cdot a_{m-3}$$

$$a_{m-3} \cdot a_{m-2} = a_{m-3} \cdot a_{m-4} - 3$$

$$9 = a_{m-3} \cdot a_{m-4}$$

$$\Longrightarrow a_{m-k} \cdot a_{m-(k+1)} = 3k$$

$k = m-1$

$$\Rightarrow a_1 \cdot a_0 = 37 \cdot 72 = 3 \cdot (37 \cdot 24)$$

$$= 3 \cdot (888)$$

$$888 = m-1$$

$$m = 889$$

10. $x_1 = x_2 + \cdots + x_{100} - 1$

$x_2 = x_1 + x_3 + \cdots + x_{100} - 2$

$x_3 = x_1 + x_2 + x_4 + \cdots + x_{100} - 3$

$\vdots$

$x_{100} = x_1 + \cdots + x_{99} - 100$

$\Rightarrow x_1 + x_2 + \cdots + x_{100} = 99(x_1 + \cdots + x_{100}) - \frac{100 \cdot 101}{2}$

$x_1 + \cdots + x_{100} = 99(x_1 + \cdots + x_{100}) - 5050$

$98(x_1 + \cdots + x_{100}) = 5050$

$x_1 + \cdots + x_{100} = \frac{5050}{98} = \frac{2525}{49}$

$x_{50} = x_1 + \cdots + x_{49} + x_{51} + \cdots + x_{100} - 50$

$2x_{50} = x_1 + \cdots + x_{100} - 50 = \frac{2525}{49} - 50$

$2x_{50} = \frac{2525}{49} - \frac{2450}{49} = \frac{75}{49}$

$x_{50} = \frac{75}{98} \Rightarrow 173$

11. $a_1 = a_2 = a_3 = 1 \quad a_4 = 3 \quad a_5 = 5 \quad a_6 = 9 \quad a_7 = 17$

$a_{n+3} = a_{n+2} + a_{n+1} + a_n$ $\quad a_8 = 31 \quad a_9 = 57$

$a_{30} = 20603361$ $\quad \sum_{k=1}^{28} a_k$ $\quad a_{10} = 105$

$a_{29} = 11201821$

$a_{28} = 6090307$

$a_{n+3} = a_{n+1} + a_n + a_{n-1} + a_{n+1} + a_n$

$= 2a_{n+1} + 2a_n + a_{n-1}$

$= 2(a_n + a_{n-1} + a_{n-2}) + 2a_n + a_{n-1}$

$= 4a_n + 3a_{n-1} + 2a_{n-2}$

$a_{n+3} = a_{n+2} + a_{n+1} + a_n$

$a_{n+3} - a_{n+2} = a_{n+1} + a_n$

$\Downarrow$

$\sum_{k=1}^{28} a_{k+3} - a_{k+2} = a_{k+1} + a_k$

$a_4 + \cdots + a_{31} - (a_3 + \cdots + a_{30}) = (a_2 + \cdots + a_{29}) + (a_1 + \cdots + a_{28})$

$= a_{31} - a_3 = a_1 + 2(a_2 + \cdots + a_{28}) + a_{29}$

$$a_{31} - a_3 + a_1 = 2(a_1 + \cdots + a_{28}) + a_{29}$$

$$a_{31} - a_{29} = 2(a_1 + \cdots + a_{28})$$

$$3668 = 2(a_1 + \cdots + a_{28})$$

$$\begin{array}{r} 0387 \\ 1821 \\ \underline{3367} \\ 5489 \\ \underline{-1821} \\ 3668 \end{array}$$

$$a_1 + \cdots + a_{28} \bmod 1000 = \sum_{k=1}^{28} a_k \bmod 1000 = 1834 \bmod 1000$$

$$= 834$$

12. $a_1 = 1 \quad a_9 + a_{10} = 646$

$a_{2n-1} \quad a_{2n} \quad a_{2n+1}$

$a_{2n} \quad a_{2n+1} \quad a_{2n+2}$

$$a_1 = 1 \quad a_2 = r \quad a_3 = r^2$$

$$a_3 - a_2 = r^2 - r$$

$$a_4 = a_3 + a_3 - a_2 = r^2 + r^2 - r = 2r^2 - r$$

$$\frac{a_4}{a_3} = \frac{2r^2 - r}{r^2} = \frac{2r-1}{r}$$

$$a_5 = a_4 \cdot \frac{a_4}{a_3} = (2r^2 - r) \cdot \frac{(2r-1)}{r} = (2r-1)^2$$

$$a_6 = a_5 - a_4 = (2r-1)^2 + ((2r-1)^2 - r(2r-1))$$

$$= (2r-1)^2 + (2r-1) \cdot (2r-1-r)$$

$$= (2r-1)^2 + (2r-1) \cdot (r-1)$$

$$= (2r-1+r-1) \cdot (2r-1) = (3r-2)(2r-1)$$

$\vdots$

$$a_7 = (3r-2)^2$$

$$a_8 = (4r-3)(3r-2)$$

$$a_9 = (4r-3)^2 \qquad a_{10} = (5r-4) \cdot (4r-3)$$

$$\Rightarrow (4r-3)^2 + (5r-4) \cdot (4r-3) = 646$$

Then $a_{2n} = (nr - (n-1)) \cdot ((n-1)r - (n-2))$ $(4r-3) \cdot (4r-3+5r-4) = 646$

$a_{2n+1} = (nr - (n-1))^2$ $(4r-3) \cdot (9r-7) = 646$

$36r^2 - 27r - 28r + 21 - 646 = 0$

$36r^2 - 55r - 625 = 0$

1 5

36 125

$(36r+125)(r-5) = 0$

$r = 5$

$a_{2n} = (5n - n + 1) \cdot (5n - 5 - n + 2)$

$= (4n+1) \cdot (4n-3)$

$a_{2n+1} = (5n - n + 1)^2$

$= (4n+1)^2$

$\Downarrow$

$(4n+1)^2 \leq 1000$

$4n+1 \leq \approx 33$

$n \approx 8 \Rightarrow a_{16} = (8 \cdot 5 - 7) \cdot (7 \cdot 5 - 6)$

$= (33) \cdot (29) = 957 \longrightarrow 957 + 16 = 973$

$a_{17} = (8 \cdot 5 - 7)^2 = 33^2 = 1089$

13. $f_1(x) = \frac{2}{3} - \frac{3}{3x+1}$

$f_n(x) = f_1(f_{n-1}(x))$ for $n \geq 2$

$f_{1001}(x) = x - 3$

$x - 3 = f_1(f_{1000}(x))$

$f_2(x) = f_1(f_1(x)) = \frac{2}{3} - \frac{3}{3f(1)+1} = \frac{2}{3} - \frac{3}{3 \cdot \left(\frac{2}{3} - \frac{3}{3x+1}\right) + 1} = \frac{2}{3} - \frac{3}{3 - \frac{9}{3x+1}} = \frac{2}{3} - \frac{1}{1 - \frac{3}{3x+1}}$

$= \frac{2}{3} - \frac{3x+1}{3x-2}$

$= \frac{6x - 4 - (9x+3)}{3 \cdot (3x-2)}$

$= \frac{-3x-7}{9x-6}$

$f_3(x) = f_1(f_2(x)) = \frac{2}{3} - \frac{3}{3f(2)+1}$

$= \frac{2}{3} - \frac{3}{3 \cdot \left(\frac{-3x-7}{9x-6}\right) + 1}$

$= \frac{2}{3} - \frac{3}{\frac{-3x-7}{3x-2} + 1} = \frac{2}{3} - \frac{3}{\frac{-9}{3x-2}} = \frac{2}{3} + \frac{1}{3} \cdot (3x-2)$

$= \frac{2}{3} + \frac{3x-2}{3} = \frac{3x}{3} = x$

$f_4(x) = f_1(f_3(x)) = \frac{2}{3} - \frac{3}{3 \cdot f_3(x) + 1} = \frac{2}{3} - \frac{3}{3x+1}$

$f_4(x) = f_1(x)$

$1000 \% 4 = 0$ So $f_{1000}(x) = f_4(x) = f_1(x)$

Thus $f_{1001}(x) = f_2(x) = -\dfrac{3x-7}{9x-6}$

Then $x-3 = -\dfrac{3x-7}{9x-6}$

$9x^2 - 27x - 6x + 18 = -3x - 7$

$9x^2 - 30x + 25 = 0$

$(3x-5)^2 = 0$

$x = \frac{5}{3} \Rightarrow 8$

14. $\sum_{k=1}^{1995} \frac{1}{f(k)}$

$1^4 = 1$

$2^4 = 16$

$3^4 = 81$

$4^4 = 256$

$5^4 = 625$

$6^4 = 1296$

$7^4 = 2401$

if $n - \frac{1}{2} < \sqrt[4]{k} < n + \frac{1}{2}$

$n = 1$ $\sqrt[4]{k} < 1.5$

$k < 2.25^2 = 5.0625$

$\sqrt[4]{k} < 2.5$

$k < 6.25^2 = 39.025$

$\sqrt[4]{k} < 3.5$

$k < 150.0625$

$1 - 5 = 1 \Rightarrow 5$

$6 - 39 = 2 \Rightarrow 34 \cdot \frac{1}{2} = 17$

$40 - 150 = 3 \Rightarrow 111 \cdot \frac{1}{3} = 37$

$151 - 410 = 4 \Rightarrow 260 \cdot \frac{1}{4} = 65$

$411 - 915 = 5 \Rightarrow 505 \cdot \frac{1}{5} = 101$

$916 - 1785 = 6 \Rightarrow 870 \cdot \frac{1}{6} = 145$

$1786 - 1995 = 7 \Rightarrow 210 \cdot \frac{1}{7} = 30$

$5 + 17 + 37 + 65 + 101 + 145 + 30$

$22 + 37 + 166 + 175$

$59 + 341 = 400$

15. $x_0 = 0$

$|x_k| = |x_{k-1} + 3|$

$x_k^2 = x_{k-1}^2 + 6x_{k-1} + 9$

$x_k^2 - x_{k-1}^2 = 6x_{k-1} + 9$

$$\sum_{k=1}^{2007} x_k^2 - x_{k-1}^2 = \sum_{k=1}^{2007} 6x_{k-1} + 9$$

$$x_1^2 - x_0^2 + x_2^2 - x_1^2 + \cdots + x_{2007}^2 - x_{2006}^2 = x_{2007}^2 - x_0^2 = x_{2007}^2 = \sum_{k=1}^{2007} 6x_{k-1} + 9$$

$$= 6(x_0 + x_1 + \cdots + x_{2006}) + 2007 \cdot 9$$

$$x_{2007}^2 - 2007 \cdot 9 = 6(x_1 + \cdots x_{2006})$$

$$x_1 + \cdots + x_{2006} = \frac{1}{6}(x_{2007}^2 - 2007 \cdot 9)$$

$x_1 + \cdots + x_{2006}$ minimized when x_{2007}^2 close to $2007 \cdot 9$

Thus, $2007 \cdot 9 = 18063$

$100 = 10000$

$140 = 19600$

$130 = 16900$

$135 = 18225$

So $\dfrac{18225 - 18063}{6} = \dfrac{162}{6} = 27$