Hermite Wavelet Collocation Method for Solving Three Dimensional Partial Differential Equation

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Abstract

In this paper, we solved a three-dimensional partial differential equation using a Hermite wavelet. The Hermite wavelet collocation technique has been applied to solve three-dimensional Helmholtz and Poisson equations. We have meticulously developed an error analysis of the proposed scheme, ensuring the reliability of our method. To show the efficiency and accuracy of the developed method, we have solved two examples of the Helmholtz equation and two examples of the Poisson equation. Also, we have compared the maximum absolute errors with the methods present in the literature, further reinforcing the robustness of our approach.

Keywords: Hermite wavelet; Helmholtz equation; Poisson equation; Collocation method.
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1. Introduction

The three-dimensional partial differential equation (3D-PDE) arises in the mathematical modelling of various real-life applications of engineering and sciences. The Helmholtz equation and Poisson equation (P.E.) occur in the mathematical modelling of computational physics, such as simulating the behaviour of electromagnetic waves; theoretical chemical science for predicting the behaviour of chemical reactions; radiation for modelling the spread of radioactive materials; seismology, for understanding the propagation of seismic waves; optics, for designing optical systems; and quantum mechanics, for describing the behaviour of quantum particles. Further applications can be seen in heat conduction, ideal fluid flow, elasticity, electrostatics, fluid mechanics and torsion problems. The well-known Schrodinger's equation is an extension of Helmholtz equation (H.E.). The solution of P.E. is the potential field caused by a given electrical charge. Further, the H.E. and P.E. are examples of elliptic PDE. The second order 3D-PDE in three independent variables ω_1, ω_2 and ω_3 is given by

$$\begin{aligned} A(\omega_1,\omega_2,\omega_3)\frac{\partial^2 u}{\partial \omega_1{}^2}+B(\omega_1,\omega_2,\omega_3)\frac{\partial^2 u}{\partial \omega_2{}^2}+C(\omega_1,\omega_2,\omega_3)\frac{\partial^2 u}{\partial \omega_3{}^2} \\ = f(\omega_1,\omega_2,\omega_3,u_{\omega_1},u_{\omega_2},u_{\omega_3},u_{\omega_1,\omega_2},u_{\omega_2,\omega_3},u_{\omega_3,\omega_1}) \end{aligned} \tag{1.1}$$

on the bounded region $\Omega=\{(\omega_1,\omega_2,\omega_3): a_1<\omega_1<a_2, b_1<\omega_2<b_2, c_1<\omega_3<c_2\}$ with boundary conditions

$$\begin{aligned} u(0,\omega_2,\omega_3) &= f_1(\omega_2,\omega_3), \\ u(\omega_1,0,\omega_3) &= f_2(\omega_3,\omega_1), \\ u(\omega_1,\omega_2,0) &= f_3(\omega_1,\omega_2), \end{aligned} \tag{1.2}$$

and

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In the equation 1.1, if $A = B = C = 1$ and f is zero, then the equation 1.1 becomes a 3D Laplace equation. Also, if f is continuous function of ω_1, ω_2 and ω_3, then it becomes 3D-P.E.. The Poisson equation is a partial differential equation describing scalar fields' behaviour in space. It is named after the French mathematician Simon Denis Poisson, who introduced the equation in the early 19th century. u is the scalar field and f is the source term. The Poisson equation is often used in physics and engineering to model various phenomena, including electrostatics, gravitational fields, fluid mechanics, and heat transfer. It is a fundamental equation in many fields of study and has numerous applications in theoretical and applied sciences. The 3D-P.E. is given by

$$A(\omega_1,\omega_2,\omega_3)\frac{\partial^2 u}{\partial \omega_1{}^2} + B(\omega_1,\omega_2,\omega_3)\frac{\partial^2 u}{\partial \omega_2{}^2} + C(\omega_1,\omega_2,\omega_3)\frac{\partial^2 u}{\partial \omega_3{}^2} = f(\omega_1,\omega_2,\omega_3) \tag{1.3}$$

If the equation 1.1 is of the form

Then equation **??** is called 3D-H.E. The Helmholtz equation is a partial differential equation that appears in many areas of physics and engineering. It is named after Hermann von Helmholtz, a German physicist who first derived the equation in 1853; u is the unknown function, and k is a constant known as the wave number. The equation describes the propagation of waves in a medium and has applications in acoustics, electromagnetics, fluid dynamics, and other fields. Solutions to the Helmholtz equation are typically sought using the separation of variables or numerical methods. In many applications, the solutions are subject to boundary or initial conditions, which must be incorporated into the solution process. The Helmholtz equation has essential applications in studying wave phenomena, including the behaviour of sound, electromagnetic, and other types of waves.

Bialecki and Wang [3] presented a revised nodal cubic spline collocation technique to improve accuracy for elliptic equations. To overcome the limitations of previous methods, their approach refines nodal placement and interpolation techniques using cubic splines to provide higher-order convergence.

Hadjidimos et al. In 1993 [**?**], spline collocation methods for multiple dimensions were extended with an iterative approach by Deuflhard et al. For example, the line cubic spline collocation methods are presented to solve the elliptic PDEs in more dimensions and they bring an efficient way of solving it. Houstis et al. (1988) [**?**]. Convergence of Cubic Spline Collocation methods. These results validate the $O(h^4)$ convergence rates of our methods and lend credibility to their being highly accurate tools for approximating solutions to elliptic PDEs.

2. Hermite Wavelet

Hermite wavelets are a family of wavelet functions derived from the Hermite polynomials, a set of orthogonal polynomials used in mathematical physics. The Hermite wavelets are used in signal processing and image analysis, particularly in analysing signals and images exhibiting certain features, such as edges and corners. The Hermite wavelets are defined as the product of a scaling function and a wavelet function derived from the Hermite polynomials. The scaling function is a low-pass filter that decomposes signals and images into different frequency components. In contrast, the wavelet function is a high-pass filter that captures details and edges in the signal or image. The Hermite wavelets have valuable properties, such as orthogonality and compact support, which make them well-suited for signal and image analysis. They have been used in various applications, including image denoising, edge detection, and feature extraction.

3. Method For Solution

In this section, we shall solve both 3D-H.E. and 3D-P.E., by assuming highest derivative From here, we can find the unknown constants C_{ijk} with the help of initial and boundary conditions (1.2), (**??**) and put these values of unknown constants C_{ijk} in equation (**??**) for required solution of 3D-PE.

(a)(b)

Figure 1: Slice planes of the errors of problem 1 for $N = 10$ and $N = 12$

4. Convergence Analysis

In this section, we have established the convergence of the proposed method.

The solution of equation (1.1) can be written in analytical form as

5. Simulations and Results

In this section, two examples of the Helmholtz equation and two examples of the Poisson equations have been solved by hermite wavelet collocation method. To show the accuracy, applicability and robustness of the proposed method maximum absolute errors (MAE) have been tabulated and volumetric data of the calculated MAE along slice planes have been plotted.

Example 1. Consider the following 3D-P.E.

$$\frac{\partial^2 u}{\partial \omega_1{}^2} + \frac{\partial^2 u}{\partial \omega_2{}^2} + \frac{\partial^2 u}{\partial \omega_3{}^2} = f(\omega_1, \omega_2, \omega_3),$$

on

$$\Omega = \{(\omega_1, \omega_2, \omega_3) : 0 < \omega_1 < 1, 0 < \omega_2 < 1, 0 < \omega_3 < 1\},$$

with boundary conditions

$$f_1(\omega_2, \omega_3) = f_2(\omega_3, \omega_1) = f_3(\omega_1, \omega_2) = 0,$$

and

$$g_1(\omega_2, \omega_3) = g_2(\omega_3, \omega_1) = g_3(\omega_1, \omega_2) = 0,$$

where

$$f(\omega_1, \omega_2, \omega_3) = sin(\pi\omega_1)sin(\pi\omega_2)sin(\pi\omega_3)$$

and the exact solution is

$$u(\omega_1, \omega_2, \omega_3) = \frac{-1}{3\pi^2} sin(\pi\omega_1)sin(\pi\omega_2)sin(\pi\omega_3).$$

The calculated MAE using hermite wavelet collocation method (HWCM) for different values of K, M and N and MAE calculated by the method given in [**?**] have been compared and tabulated in the Table 1. Also, graph of the slice planes have been plotted in the Figure 1

Table 1: Calculated maximum absolute errors using HWCM and Haar wavelet collocation method [**?**] for different values of K, M and N for example 1

M	K	N	Error by HWCM	Error by Haar
1	1	1	7.90×10^{-03}	—-
2	1	2	8.9227×10^{-04}	8.9227×10^{-04}
4	1	4	3.7439×10^{-05}	6.4125×10^{-04}
4	2	8	1.3825×10^{-05}	2.0157×10^{-04}

(a)(b)

Figure 2: Slice planes of the errors of problem 2 for $N = 10$ and $N = 12$

Example 2. Consider the following 3D-P.E.

$$\frac{\partial^2 u}{\partial {\omega_1}^2} + \frac{\partial^2 u}{\partial {\omega_2}^2} + \frac{\partial^2 u}{\partial {\omega_3}^2} = f(\omega_1, \omega_2, \omega_3),$$

on

$$\Omega = \{(\omega_1, \omega_2, \omega_3) : 0 < \omega_1 < 1, 0 < \omega_2 < 1, 0 < \omega_3 < 1\},$$

with boundary conditions

$$f_1(\omega_2, \omega_3) = f_2(\omega_3, \omega_1) = f_3(\omega_1, \omega_2) = 0,$$

and

$$g_1(\omega_2, \omega_3) = g_2(\omega_3, \omega_1) = g_3(\omega_1, \omega_2) = 0,$$

where

$$\begin{aligned} f(\omega_1, \omega_2, \omega_3) &= 6\omega_1\omega_2\omega_3[{\omega_1}^2{\omega_2}^2(1-\omega_1)(1-\omega_2)(1-2\omega_3) + {\omega_2}^2 \\ {\omega_3}^2&(1-\omega_2)(1-\omega_3)(1-2\omega_1) + {\omega_3}^2{\omega_1}^2(1-\omega_1)(1-\omega_3)(1-2\omega_2)] \end{aligned} \tag{5.1}$$

and the exact solution is

$$u(\omega_1, \omega_2, \omega_3) = {\omega_1}^3{\omega_2}^3{\omega_3}^3(1-\omega_1)(1-\omega_2)(1-\omega_3).$$

The calculated MAE using hermite wavelet collocation method (HWCM) for different values of K, M and N and MAE calculated by the method given in [**?**] have been compared and tabulated in the Table 2. Also, graph of the slice planes have been plotted in the Figure 2

Table 2: Calculated maximum absolute errors using hermite wavelet collocation method (HWCM) and Haar wavelet collocation method [**?**] for different values of K, M and N for example 2.

M	K	N	Error by HWCM	Error by Haar
1	1	1	2.4414×10^{-04}	—-
2	1	2	8.9892×10^{-05}	8.9892×10^{-05}
4	1	4	1.5230×10^{-011}	4.5356×10^{-05}
4	2	8	7.0485×10^{-012}	1.1495×10^{-05}

Conclusion

In summary, here we proposed the Hermite wavelets with good properties to solve 3D Poisson and Helmholtz equations, which can be extended directly to higher dimensional problems. Using the Hermite wavelet collocation approach, we also solved these equations and performed a detailed error analysis to verify the effectiveness of our solution. Our method produces maximum absolute errors that are as good as or better than existing benchmarks for a range of numerical examples, including Helmholtz and Poisson equations. The goal of this study is to present a powerful computational tool that combines high efficiency and accuracy, which, in turn, could be an appropriate candidate for three-dimensional PDE problems in future.

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