SC1007
Data Structures and Algorithms

Graph

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Overview

- Graph Terminology
- Graph Representation
  - Adjacency Matrix
  - Adjacency List
- Traversal of Graphs
  - Breadth-first Search
  - Depth-first Search

Graph Terminology

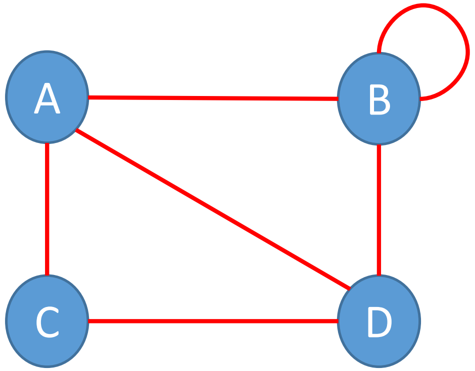
- A graph $G = (V,\ E)$ consists of two finite sets:
  - A set V of vertices/ nodes
  - |V| is the number of vertices

  - A set E of edges/arcs/links that connect the vertices
  - $E = \{(x, y) | x, y \in V\}$
  - $| E |$ is the number of edges ranged from 0 to $\frac{|V|(|V|-1)}{2}$

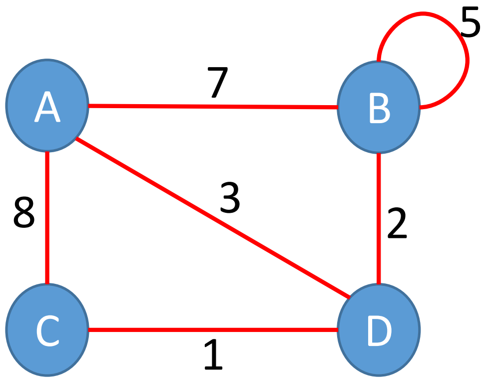
  - Degree of a vertex is the number of edges incident to it

  - A tree is a special graph with no cycle

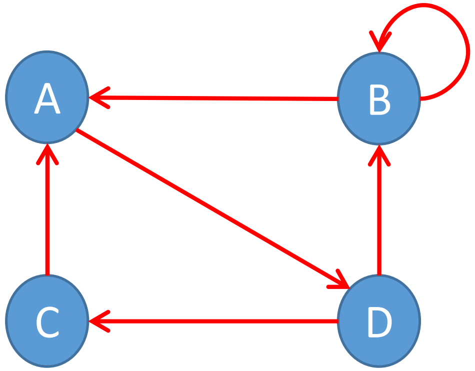
Graph Terminology



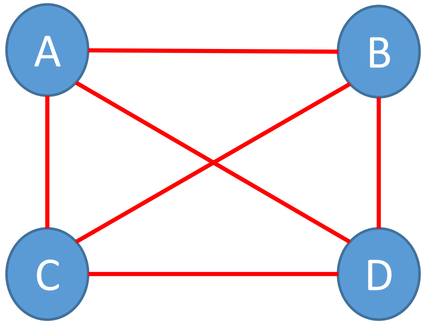
Undirected Graph



Weighted Graph



Directed Graph



Complete Graph

Graph Terminology

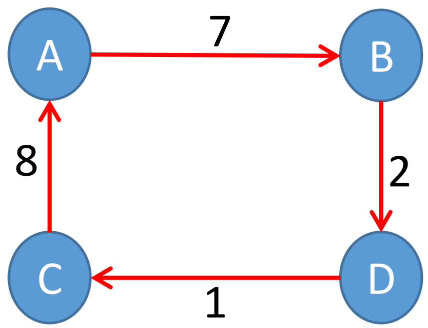
- If e = (*x*, *y*) is an edge in an undirected graph, then *e* is **incident** with *x* and *y*; *x* is **adjacent** to *y* and vice versa.

- If E is unordered, then G is undirected; otherwise, G is a directed graph.

- If e = (x, y) is an edge in a directed graph, then y can be reached from x through one edge, so target y is adjacent to source x (but it doesn't mean x is adjacent to y).

Graph Terminology

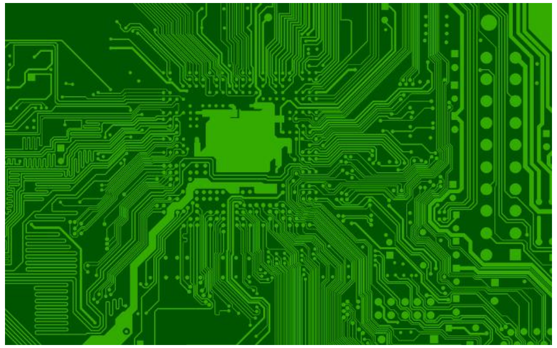
- A path is a sequence of distinct vertices, each adjacent to the predecessor (except for the first vertex). $|V| = |E|+1$
  - ABDC
- A cycle is a path containing at least three vertices such that the last vertex on the path is the same as the first. $|V| = |E|$
  - ABDCA



Graph Terminology

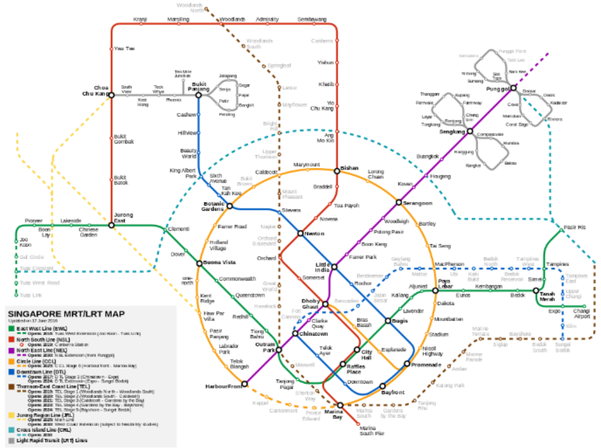
- An undirected graph is connected if there is a path from any vertex to any other vertex.
- A directed graph is strongly connected if there is a path from any vertex to any other vertex.
- A graph is cyclic if it contains one or more cycles; otherwise it is acyclic.
- A complete graph on n vertices is a simple undirected graph that contains exactly one edge between each pair of distinct vertices.
  - $|\mathrm{E}| = \frac{|V|(|V|-1)}{2}$

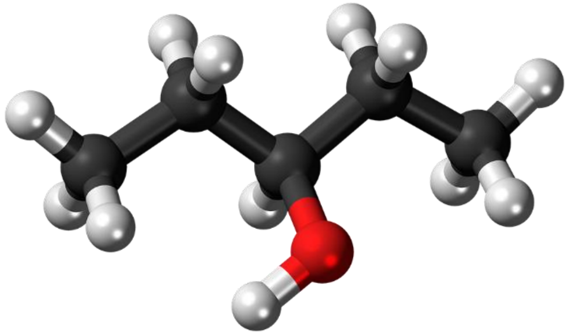
Graph Applications



Electrical circuits

- V = {electrical devices}
- E = {linkage between devices}





Maps

- V = {stations}
- E = {underground route}

Organic Chemistry

- V = {atoms}
- E = {bonds between atoms}

Computer Networks

V = {computers}

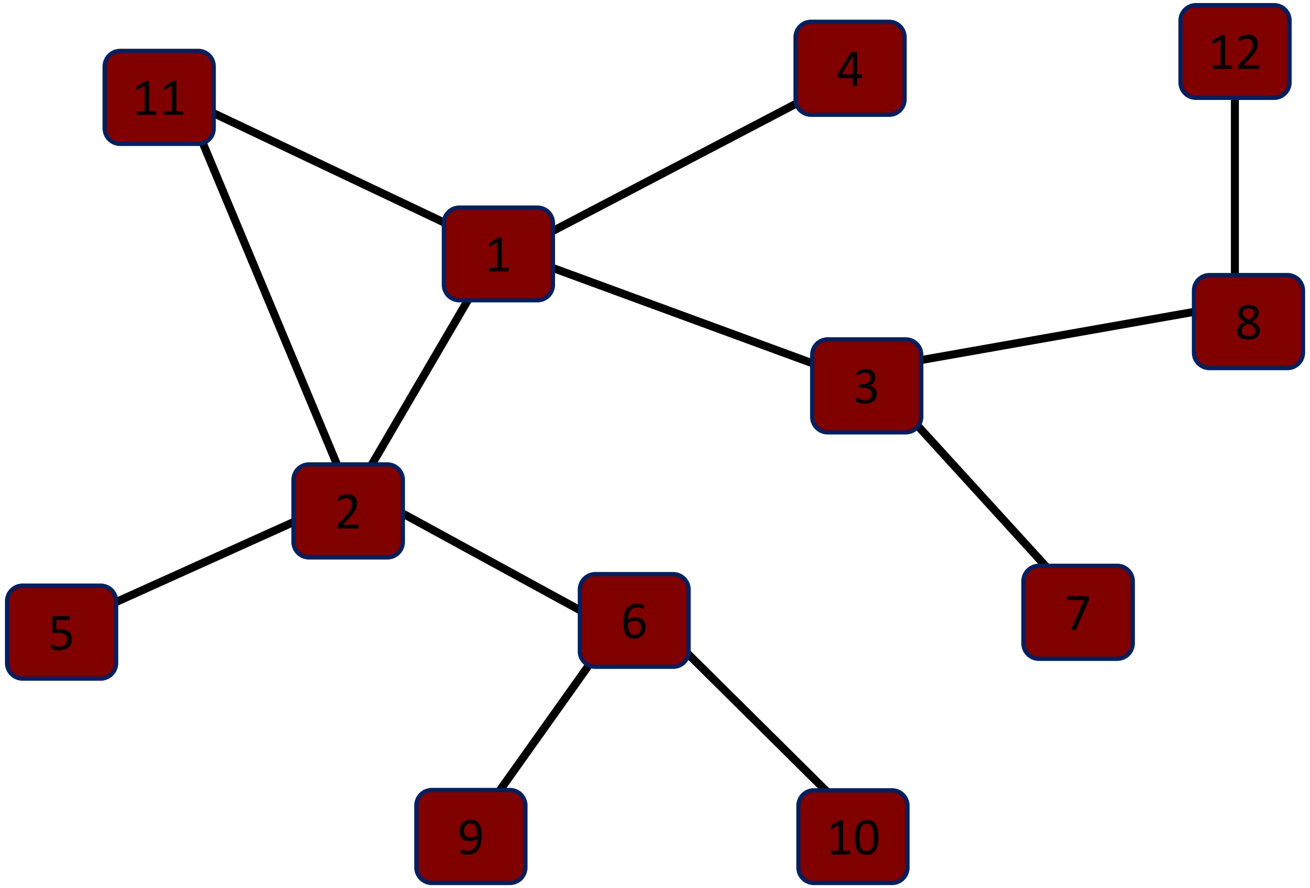
E = {connections between computers}

- Aforl. (2014). A map of Singapore's Mass Rapid Transit (MRT) and Light Rail Transit (LRT) systems [Image]. Retrieved from https://commons.wikimedia.org/wiki/File:Singapore_MRT_and_LRT_System_Map.svg
- File: Electric circuit [Image]. (2013). Retrieved from https://pixabay.com/en/board-chip-circuit-electric-158973
- Chemistry-atoms [Image]. (2015). Retrieved from https://pixabay.com/en/pentanol-molecule-chemistry-atoms-867210/

Graph Representation

- Adjacency Matrix

- Adjacency List



Adjacency Matrix

- Use a matrix (2-D array) with size $|V| \times |V|$

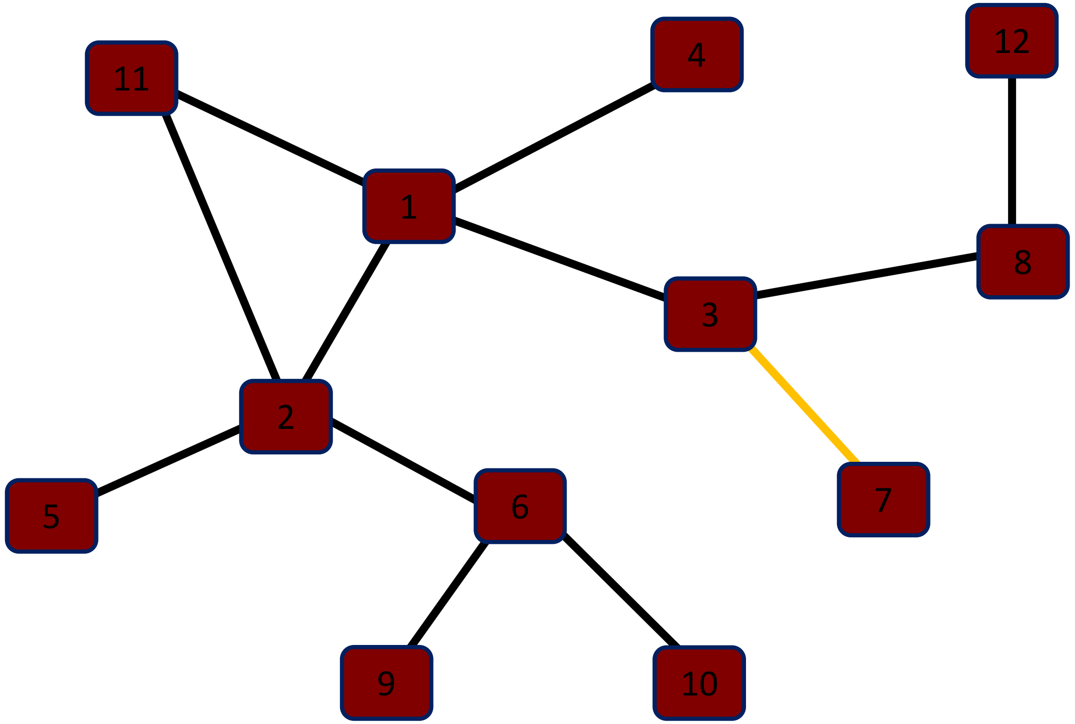
```
typedef struct _graph{
        int vSize;
        int eSize;
        int **AdjM;
}Graph;
```

- $(u, v) \in E$ implies AdjM[u][v] = 1; Otherwise AdjM[u][v] = 0.
- If a graph is undirected, then AdjM is symmetric
  - AdjM[u][v] = AdjM[v][u]
- If a graph is directed, then AdjM[u][v] = 1 iff $(u, v) \in E$ but it does not imply $(v, u) \in E$ and AdjM[v][u] = 1.

Adjacency Matrix

```c
typedef struct _graph{
        int vSize;
        int eSize;
        int **AdjM;
}Graph;
```

- access time for AdjM[u][v] is constant
- when graph is sparsely connected, most of the entries in AdjM are zeros



	1	2	3	4	5	6	7	8	9	10	11	12
1	0	1	1	1	0	0	0	0	0	0	1	0
2	1	0	0	0	1	1	0	0	0	0	1	0
3	1	0	0	0	0	0	1	1	0	0	0	0
4	1	0	0	0	0	0	0	0	0	0	0	0
5	0	1	0	0	0	0	0	0	0	0	0	0
6	0	1	0	0	0	0	0	0	1	1	0	0
7	0	0	1	0	0	0	0	0	0	0	0	0
8	0	0	1	0	0	0	0	0	0	0	0	1
9	0	0	0	0	0	1	0	0	0	0	0	0
10	0	0	0	0	0	1	0	0	0	0	0	0
11	1	1	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	1	0	0	0	0

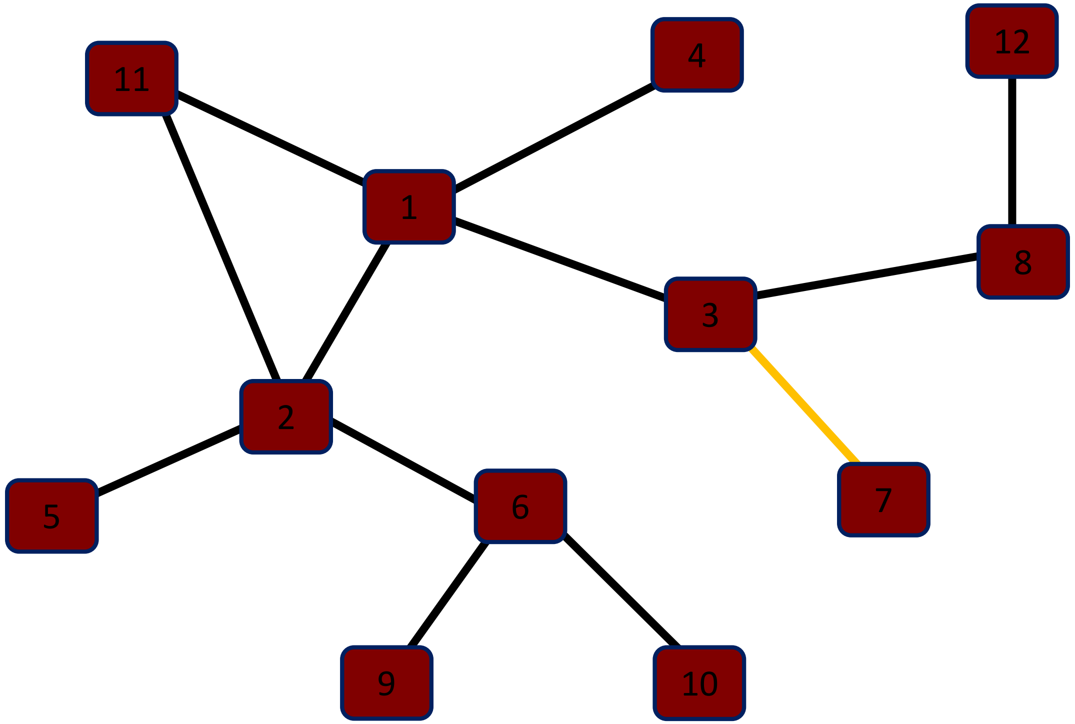
Adjacency List

```
struct _listnode
{
        int id; //or weight
        struct _listnode *next;
};
typedef struct _listnode ListNode;
typedef struct _graph{
       int vSize;
       int eSize;
       ListNode **AdjL;
}Graph;
```

- Use an array to represent the vertices
- For each vertex, use a linked list to represent the connections to other vertices
- Access time for AdjM[u][v] is linear
- Space complexity is lower, $O(|V|+|E|)$

Adjacency List

- Array size is |V|.
- Total number of nodes in link lists is 2|E|



1	$\to$ 2 $\to$ 3 $\to$ 4 $\to$ 11
2	$\to$ 11 $\to$ 1 $\to$ 5 $\to$ 6
3	$\to$ 1 $\to$ 8 $\to$ 7
4	$\to$ 1
5	$\to$ 2
6	$\to$ 10 $\to$ 9 $\to$ 2
7	$\to$ 3
8	$\to$ 12 $\to$ 3
9	$\to$ 6
10	$\to$ 6
11	$\to$ 2 $\to$ 1
12	$\to$ 8

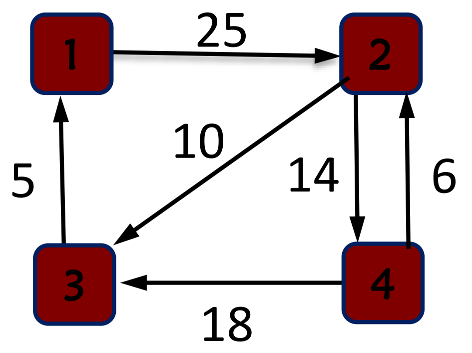
Represent Weighted Graphs

- In the array of adjacency lists, the weight can be stored as a data field in each list node
- In the adjacency matrices, the weight can be stored
  - The element at the *u*-th row and the v-th column can be defined as:

$$AdjM[u][v] = \begin{cases} W(u,v) & \text{if } (u,v) \in E \\ c & \text{otherwise} \end{cases}$$

- Constant c can be defined as 0 (weight as capacity) or some very large number ∞ (weight as cost)

Represent Weighted Graphs



	1	2	3	4
1	0	25	0	0
2	0	0	10	14
3	5	0	0	0
4	0	6	18	0

1	→ (2, 25)
2	→ (3, 10) → (4, 14)
3	→ (1, 5)
4	→ (2, 6) → (3, 18)

Summary

- Concepts and terminologies of graph, such as
  - A graph consists of a set of vertices and a set of edges
  - Directed vs. undirected graphs
  - The definitions of path and cycle, etc.
- Two data structures used to represent graphs:
  - Adjacency matrix
  - Array of adjacency lists
  - Their advantages and disadvantages for different applications

Traversal of Graphs

- To traverse a graph means to visit the vertices of the graph in some systematic order.

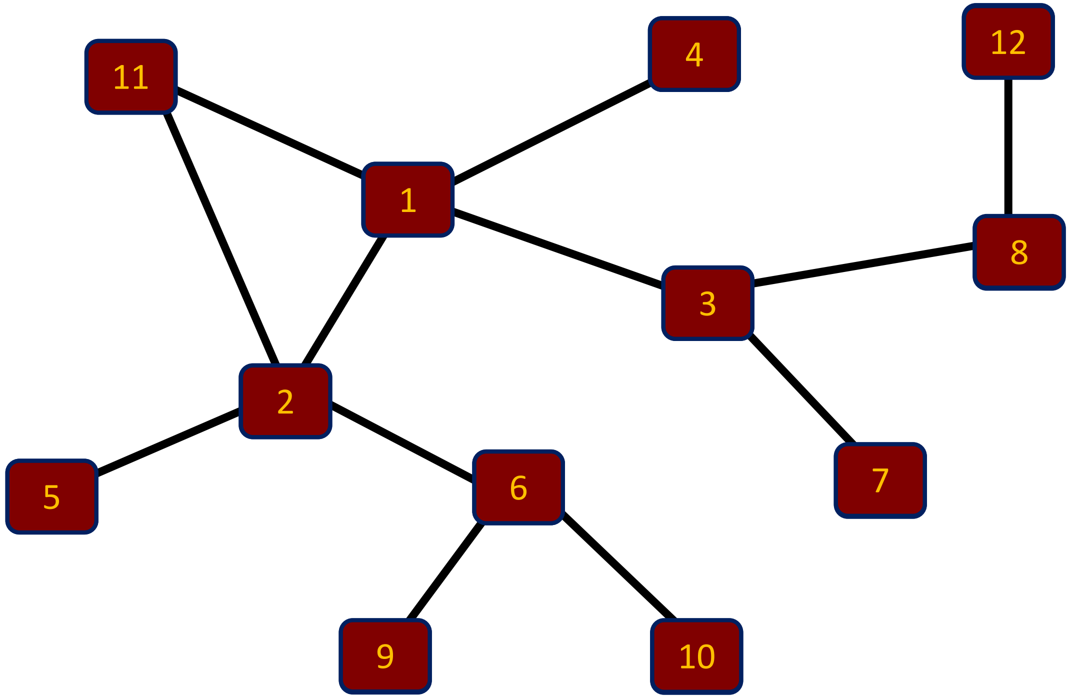
- In some applications, we may need to do some processing at every vertex of a graph.

- To visit each vertex and edge exactly once, we can apply:
  - Breadth-first Search

  - Depth-first Search

Breadth First Search (BFS)

- Work similar to level-order traversal of the trees
- BFS systematically explores the edges directly connected to a vertex before visiting vertices further away.

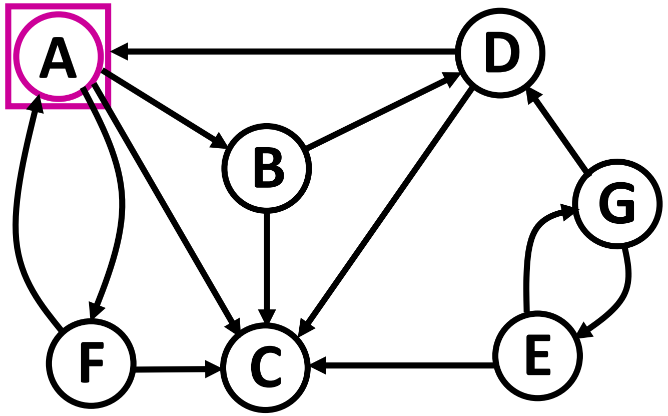


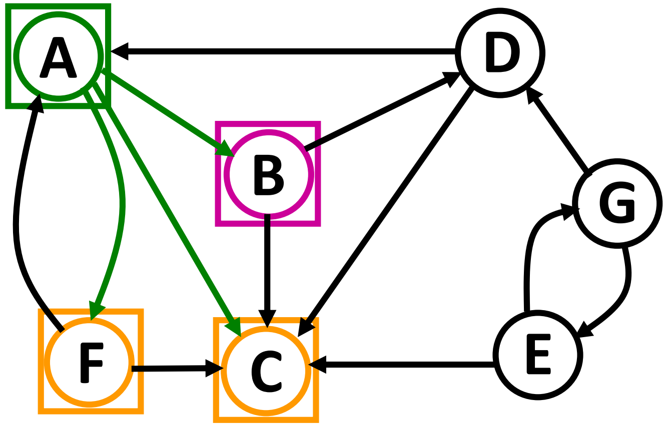
Breadth First Search (BFS)

```
typedef struct _linkedlist{
     ListNode *head;
     int size;
} LinkedList;
```

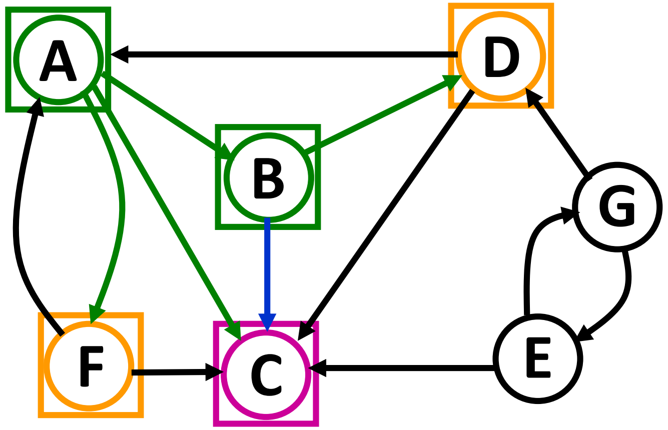
```
typedef ListNode QueueNode;
typedef struct _queue{
   int size;
   ListNode *head;
   ListNode *tail;
} Queue;
```

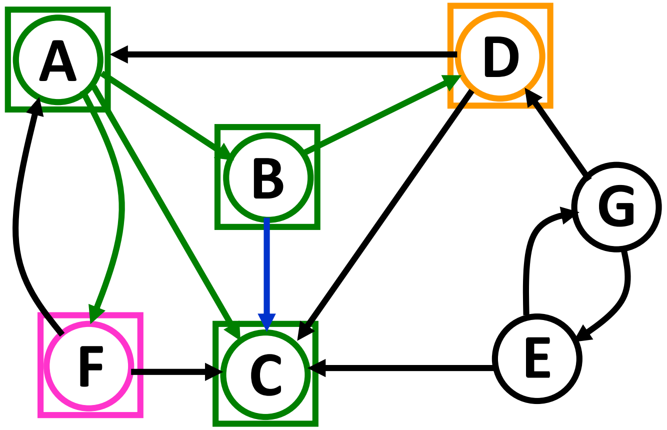
- A queue is used to monitor which vertices to visit the next
- Action taken during visiting v_i depends on specific applications

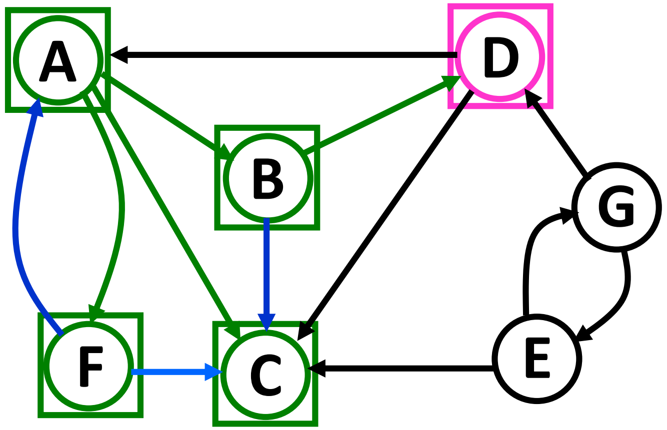


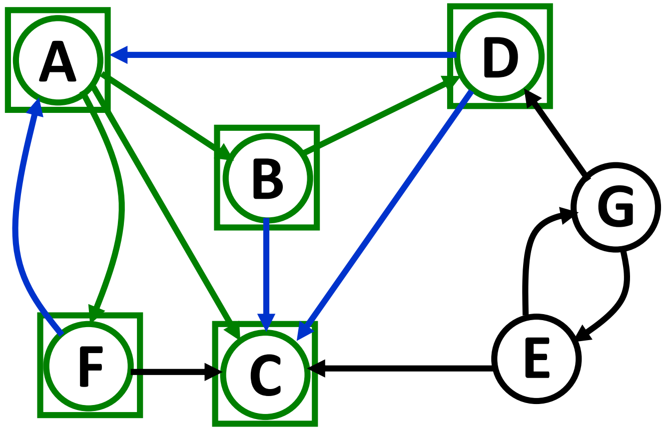


Breadth First Search (BFS)









BFS Algorithm

function BFS(Graph G, Vertex v)
    create a Queue, Q
    enqueue v into Q
    mark v as visited
    while Q is not empty **do**
        dequeue a vertex denoted as w
        for each unvisited vertex u adjacent to w **do**
            mark u as visited
            enqueue u into Q
        end for
    end while
end function

Breadth First Search (BFS)

- If a vertex has several unmarked neighbours, it would be equally correct to visit them in any order.

- If the shortest path from s to any vertex v is defined as the path with the minimum number of edges, then BFS finds the shortest paths from s to all vertices reachable from s.

- The tree built by BFS is called the **breadth first spanning tree** (when graph G is connected).

Applications of BFS

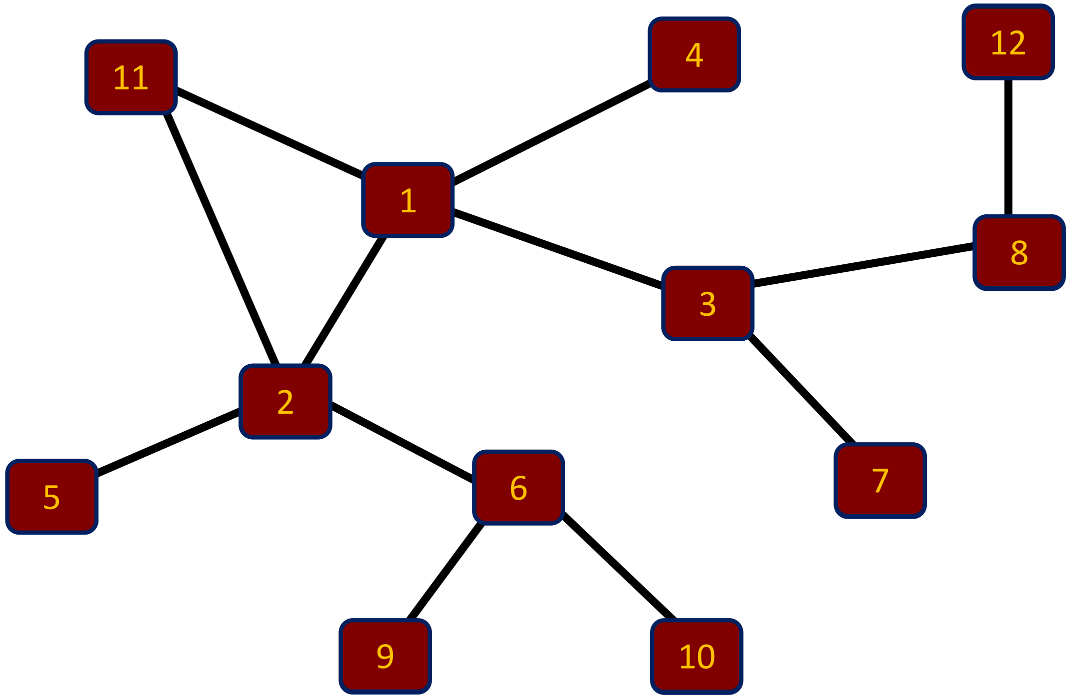
- Finding all connected components in a graph
- Finding all vertices within one connected component
- Finding the shortest path between two vertices

Time Complexity of BFS

- Each edge is processed once in the while loop for a total cost of $O(|E|)$
- Each vertex is queued and dequeued once for a total cost of $O(|V|)$
- The worst-case time complexity for BFS is
  - $\Theta(|V| + |E|)$ if graph is represented by adjacency lists
  - $\Theta(|V|^2)$ if graph is represented by an adjacency matrix
    - each vertex takes $\Theta(|V|)$ to scan for its neighbours

Depth First Search (DFS)

- Work similar to preorder traversal of the trees
- DFS systematically explores along a path from vertex v as deeply into the graph as possible before backing up.



Depth First Search (DFS)

```
struct _listnode
{
      int item;
      struct _listnode *next;
} ListNode;
```

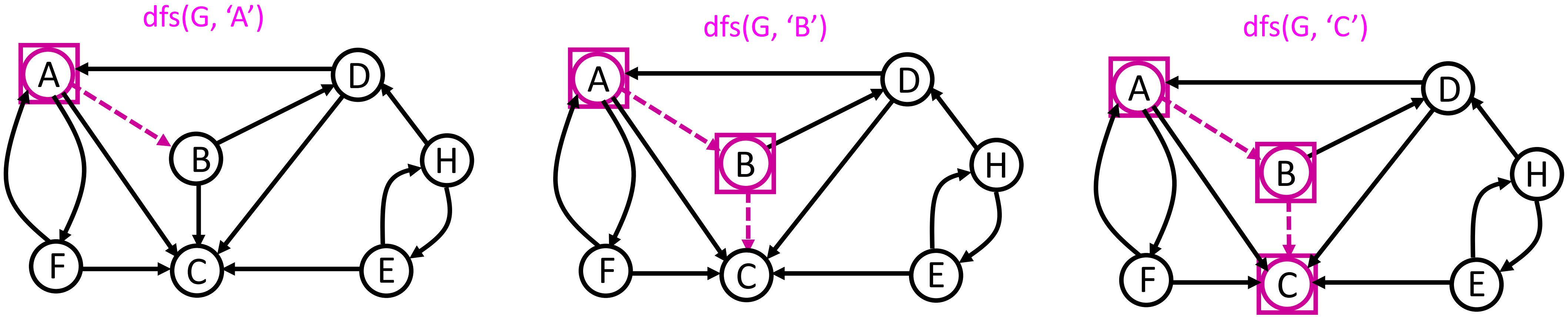
```
typedef struct _linkedlist{
      ListNode *head;
      int size;
} LinkedList;
```

```
typedef ListNode StackNode;

typedef LinkedList Stack;
```

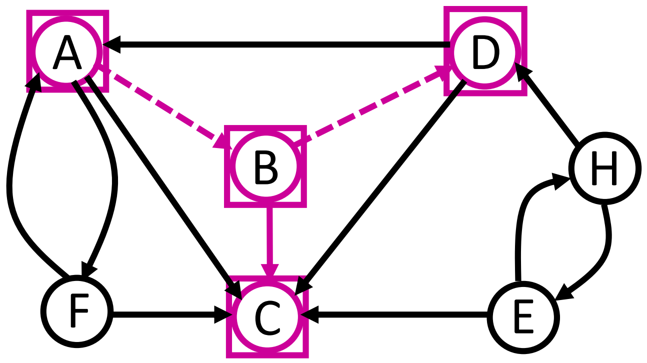
- A stack is used to monitor which vertices to visit the next

- Action taken during visiting v_i depends on specific applications

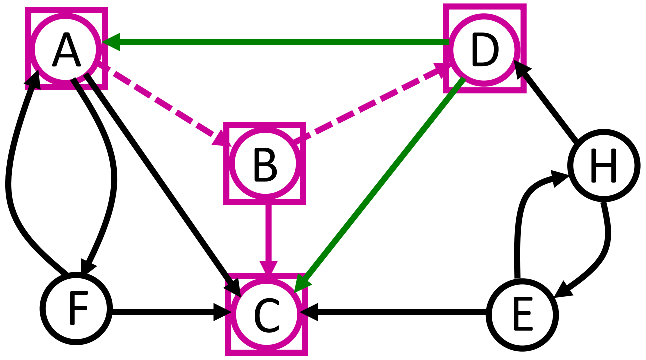


Depth First Search (DFS)

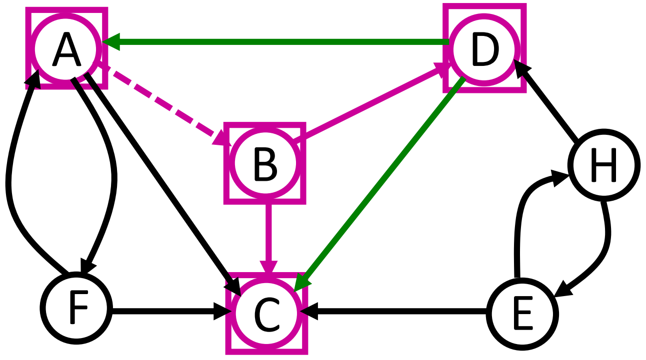
Back to dfs(G, ‘B’) then dfs(G, ‘D’)



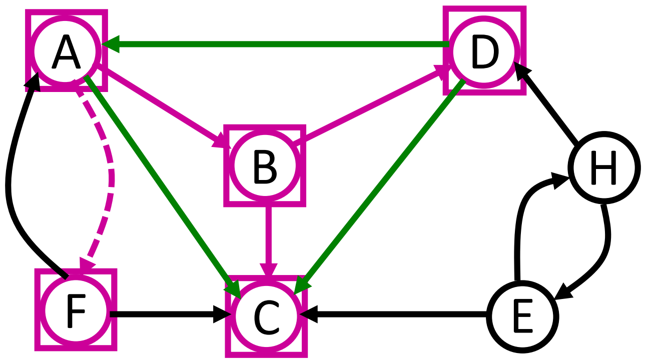
dfs(G, ‘D’)



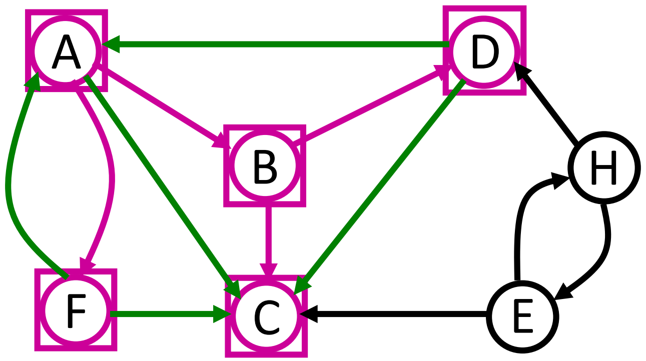
Back to dfs(G, ‘B’)



Back to dfs(G, ‘A’) then dfs(G, ‘F’)



Back to dfs(G, ‘A’)



This directed graph is **not** strongly connected

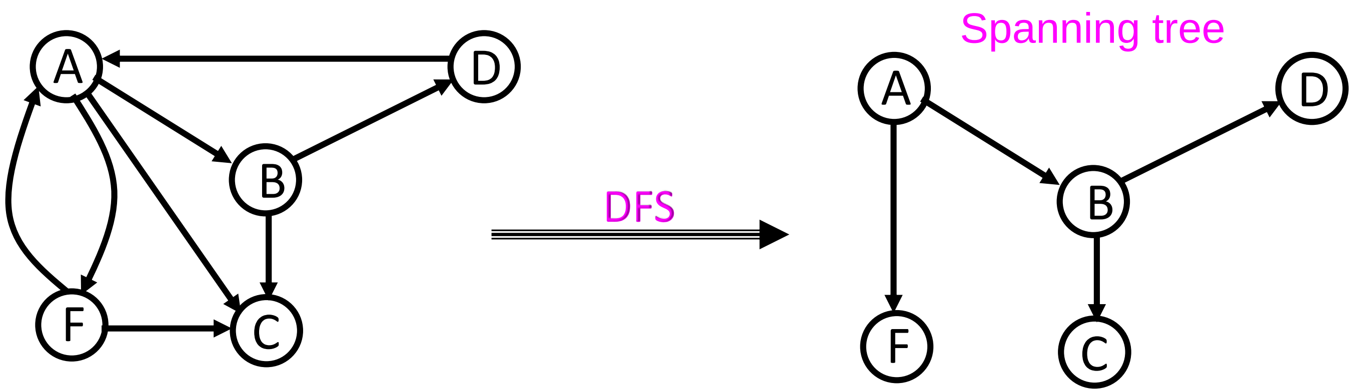
DFS Algorithm

```
function DFS(Graph G, Vertex v)
    create a Stack, S
    push v into S
    mark v as visited
    while S is not empty do
        peek the stack and denote the vertex as w
        if no unvisited vertices are adjacent to w then
            pop a vertex from S
        else
            push an unvisited vertex u adjacent to w
            mark u as visited
        end if
    end while
end function
```

Depth First Search (DFS)

- If a vertex has several neighbours it would be equally correct to go through them in any order.

- If the graph is strongly connected, the tree T, constructed by the DFS algorithm is a spanning tree, i.e., a set of |V|-1 edges that connect all vertices of the graph. T is called the **depth first search tree**.



Applications of DFS

- Topological Sorting
- Finding connected components
- Finding articulation points (cut vertices) of the graph
- Finding strongly connected components
- Solving puzzles

Time Complexity of DFS

- The DFS algorithm visits each node exactly once; every edge is traversed once in forward direction (exploring) and once in backward direction (backtracking).

- Using adjacency-lists, time complexity of DFS is $O(|V| + |E|)$.

Summary

- Two elementary algorithms for graph traversal
  - Breadth-first search (BFS): Use queue
  - Depth-first search (DFS): Use stack
- Time complexity of BFS or DFS:
  - Using adjacency lists: $O(|V| + |E|)$
  - Using adjacency matrix: $O(|V|^2)$