Math 021

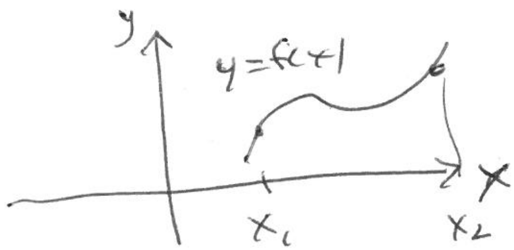
Introductory Remarks

Calculus is one of the great intellectual achievements of all time. It is also one of the most important.

Calculus has its roots with two ancients problems

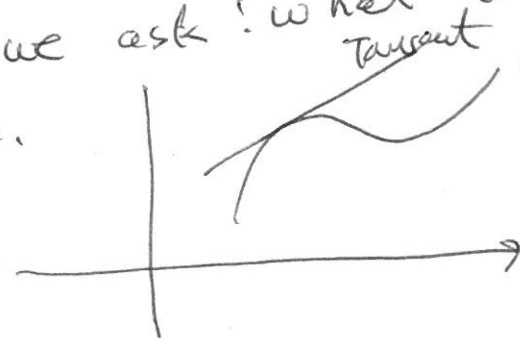
1) The tangent problem.

We know how to find the slope of a straight line. slope $= \dfrac{\text{change in } y}{\text{change in } x}$



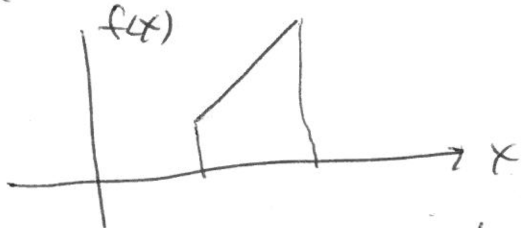
$$\text{slope} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

In calculus we ask: what is the slope of a tangent line.



The Area Problem

We know how to find the area under a straight line



Calculus will allow us to find the area under a curve like $y = x^2$, between $x=a$ and $x=b$

~~What~~ What allows us to solve both problems is the idea of a limit.

The area problem goes back to the ancient Greeks say, to about 300 B.C.E.

The tangent problem goes back to about 1500 A.D.

These two problems, area and tangent, are two sides of the same coin.

We will be concerned with the tangent problem
in chapters 2, 3, 4 and the area problem in
chapters 5 and 6.

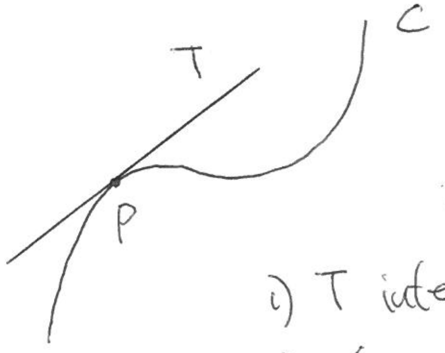
Calculus makes math easier, not easy but easier.
We will solve problem in just a few minutes that
the greatest minds of ancient could not come close to
solving. They did not have calculus, we do.

Having said that, Calculus is hard.
It was developed over the course of a couple of
hundred years by some of the brightest people who
ever lived. Don't expect it to come easy,
Expect to be frustrated at times.

I know that when I learned calculus
there were times that I was so frustrated
that I snapped my pencil in half.

But, work steady and consistently, you will
learn it.

021 Sec 2.1 The tangent problem

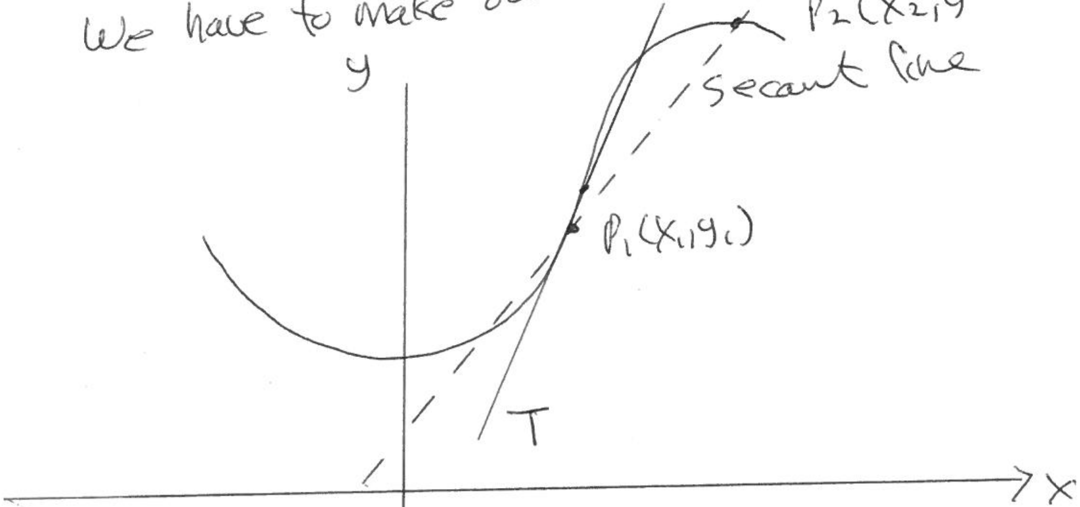


T is a tangent line to the curve C at a pt P if

i) T intersects C at P

2) (in most, but not all, cases) T stays on the same side of C for pts on T near P

We have an intuitive idea as to what a tangent line is,
We have to make our intuition rigorous.

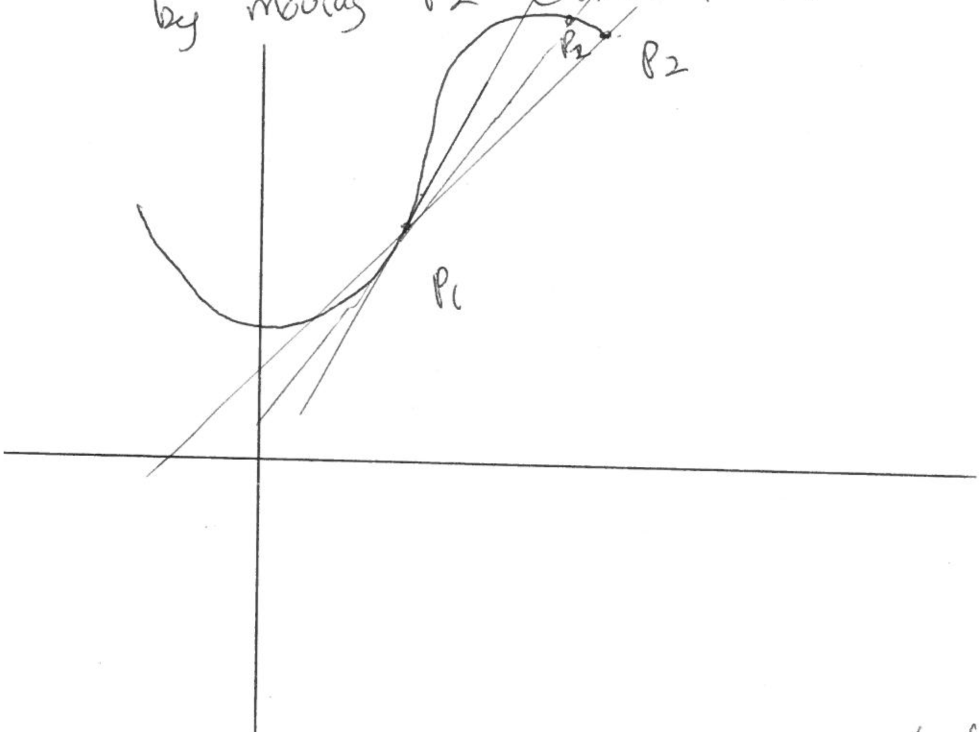


We will approximate the tangent line at P_1 by the secant line joining P_1 and P_2

~~Hence~~ So, the slope of a tangent line will be approximated by the slope of the secant line, which is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

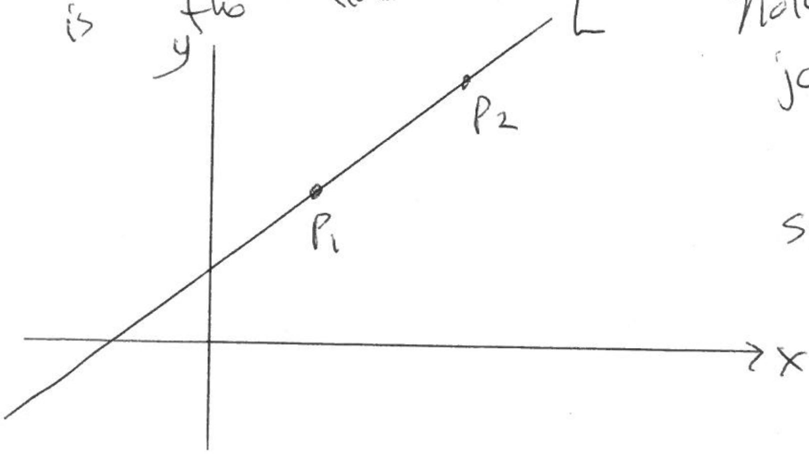
We get better approximations of the tangent line by moving P_2 closer to P_1



In finding a tangent line, a calculus concept, we look at a limit (calculus concept) of secant lines (a precal concept)

The following example is so easy that it can actually be tricky

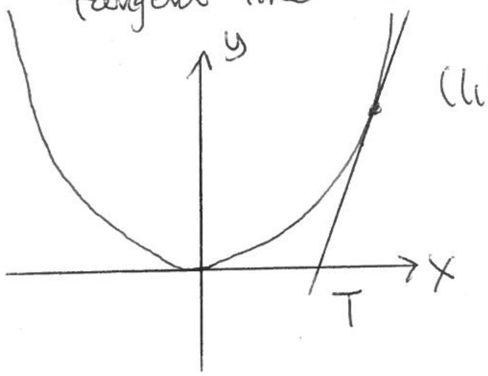
For a straight line L, a tangent line T, to L, is the line itself

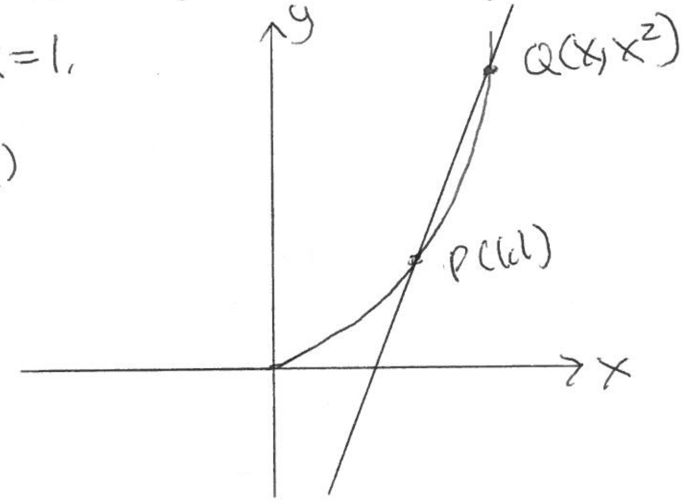


Note any secant line joining P_1 and P_2 is the line L itself, so, the limit of the secant lines is L itself,

021 Sec 2.1

Standard Ex $y = f(x) = x^2$; Find an equation of the tangent line when $x = 1$.





$$m_{PQ} = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1}$$

Note $x \neq 1$. If $x=1$ ~~x=0~~ we have $\frac{0}{0}$, we can't divide by 0, This is o.k., We are only interested in what happens as Q approaches P. i.e. as $x \to 1$. For now, we don't care what happens when $x = 1$

$$m_{PQ} = \frac{(x-1)(x+1)}{x-1} = x+1$$

So m_{PQ} when $x=1$ is $1+1=2$

Hence at the pt (1,1) the tangent line has a slope $m=2$

For an equation of the tangent line is

$$y = mx + b$$

$$y = 2x + b$$

$$1 = 2(1) + b \Rightarrow b = -1$$

$$y = 2x - 1$$

If we set up tables, we get evidence but not proof

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

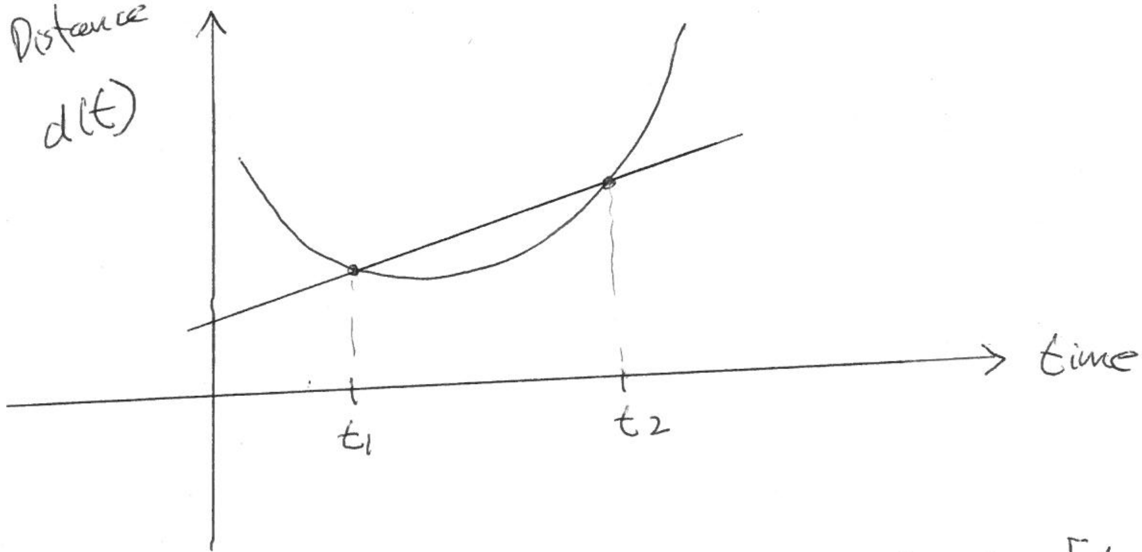
This is evidence, but not proof that the slope of the line tangent to the graph of $f(x) = x^2$ at $x = 1$ is 2

If we had more extensive tables we could get x closer to 1, but no matter how close we got to 2, we could always get closer.

The velocity problem

Recall that Distance = (velocity)(time)

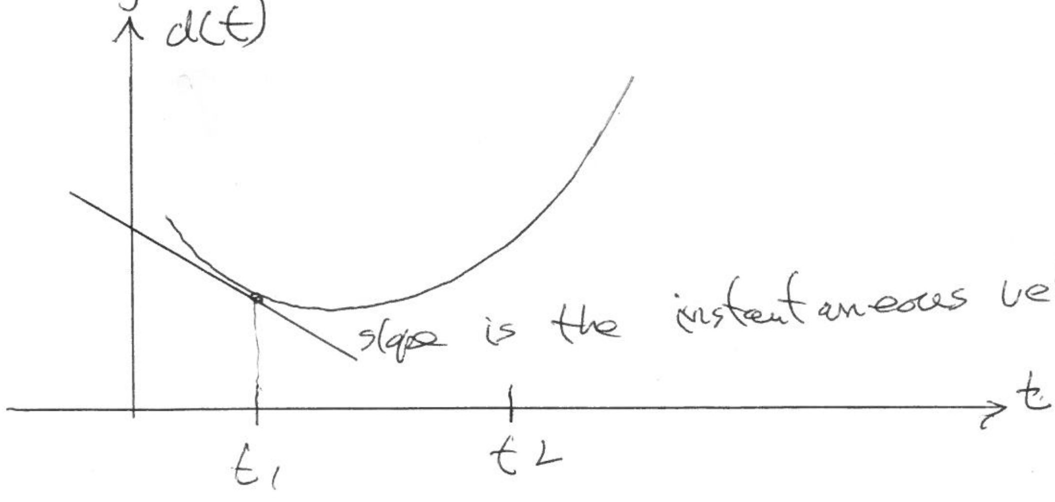
so velocity = $\frac{\text{Distance}}{\text{time}}$



The average velocity over the time interval $[t_1, t_2]$ is $\frac{\text{rise}}{\text{run}} = \frac{d(t_2) - d(t_1)}{t_2 - t_1}$, = slope of the secant line

Instantaneous velocity is the limit of the average velocity as t_2 approaches t_1. Note instantaneous is sometimes just called velocity

Hence the instantaneous velocity is the slope of the tangent line to the curve at $t = t_1$



<u>Application</u> Neglecting air-resistance, the distance that an object falls in t seconds is

$$s(t) = 4.9t^2$$

~~meters~~ seconds   meters

<u>EX</u> An object is dropped from a high altitude. Ignoring air resistance, what is the velocity at 5 seconds?

Partial Solution

Time Interval	Av Velocity m/s
$5 \le t \le 6$	53.9
$5 \le t \le 5.1$	49.49
$5 \le t \le 5.05$	49.049
$5 \le t \le 5.001$	49.0049

Evidence that the velocity is about 49 meters/second

Let's do a proof

$$m_{pg} = \frac{4.9(a+h)^2 - 4.9a^2}{(a+h)-a}$$

The value of a is fixed, in our case $a=5$

h is the change in t

$$m_{pg} = \frac{4.9(a^2+2ah+h^2)-4.9a^2}{h}$$

$$= \frac{4.9a^2+9.8ah+4.9h^2-4.9a^2}{h}$$

$$= \frac{9.8ah+4.9h^2}{h} = \frac{h(9.8a+4.9h)}{h}$$

$$= 9.8a + 4.9h$$

As $h \to 0$, $4.9h \to 0$

So $m_{pg} = 9.8a$. In our case $a=5$

$m = (9.8)5 = 49$ m/sec

At $t=6$ seconds the velocity is

$m = (9.8)6 = 58.8$ m/sec

<u>Another Ex</u> Let $f(x) = \begin{cases} \frac{x^3-1}{x-1} & \text{if } x \neq 1 \\ 7 & \text{if } x = 1 \end{cases}$

What does $f(x)$ approach as x approaches 1

i.e. find $\lim_{x \to 1} f(x)$

We gather evidence with tables

x	$f(x)$
0.75	2.313
0.9	2.710
0.99	2.970
0.999	2.997

x	$f(x)$
1.25	3.813
1.1	3.310
1.01	3.030
1.001	3.0003

looks like $\lim_{x \to 1} f(x) \approx 3$

Now, let's do a proof

set $g(x) = \frac{x^3-1}{x-1} = \frac{(x-1)(x^2+x+1)}{x-1}$

Since we are only concerned with what happens as $x \to 1$, and not what happens when $x=1$, we can cancel the $x-1$'s and not be dividing by zero

$g(x) = x^2+x+1$

$g(1) = 1^2+1+1 = 3$

so $\lim_{x \to 1} f(x) = 3.$

Note that this value is independent of the value of $f(1)$

Now, lets consider $\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1$ for $x \neq 1$

Define $f(x) = x+1$

$g(x) = \frac{x^2-1}{x-1}$

We have that f and g are different functions

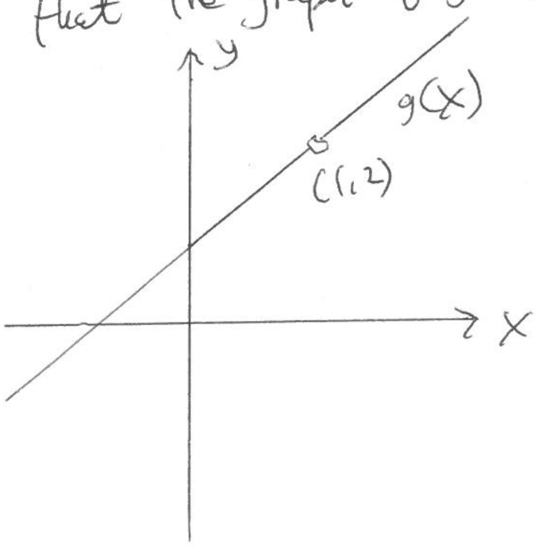
Since the domains are different.

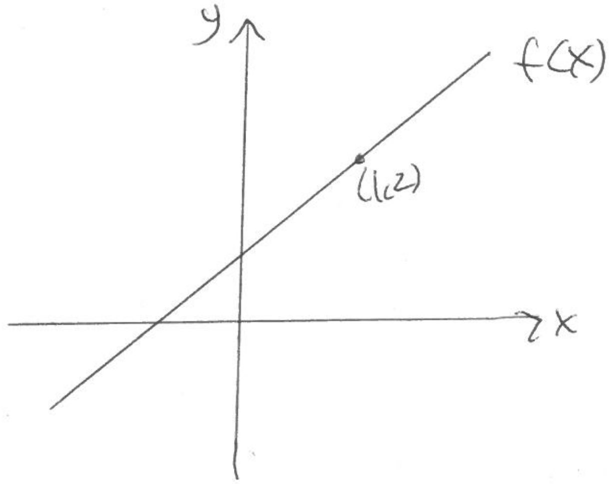
Dom f = all real numbers

Dom g = all real numbers, except $x=1$

Since the domains are different, the functions are different

The graph of $g(x)$ is the graph of $f(x)$ except that the graph of $g(x)$ has a hole (undefined) at $x=1$





Section 2.2 The Limit of a function

The idea of a limit is the most important idea in calculus. One can think of calculus as algebra and geometry combined with a limiting process

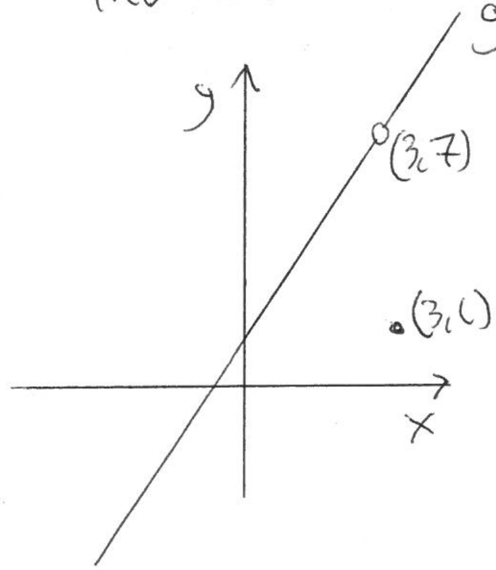
We saw the idea of limit at the end of section 2.1

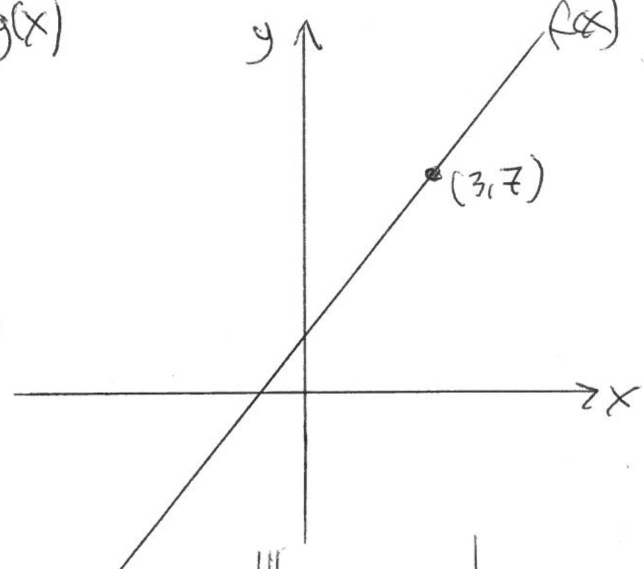
To get started lets define ~~3~~ 2 functions

$f(x) = 2x+1$ for all x

$$g(x) = \begin{cases} 2x+1 & \text{if } x \neq 3 \\ 1 & \text{if } x = 3 \end{cases}$$

This will be similar to the last page of notes for sec 1.1





x	f(x)
2	5
2.5	6
2.9	6.8
2.99	6.99

x	f(x)
4	9
3.5	8
3.1	7.2
3.01	7.02

x	g(x)
2	5
2.5	6
2.9	6.8
2.99	6.99

x	g(x)
4	9
3.5	8
3.1	7.2
3.01	7.02

We have that $\lim_{x \to 3} f(x) = 7$ and $\lim_{x \to 3} g(x) = 7$

We now give an intuitive def of a Limit

Let $f(x)$ be defined on some open interval containing the value a, except possibly at a itself.

(we don't care if $f(a)$ exists, and if $f(a)$ does exist, we don't care what it is)

We write $\lim_{x \to a} f(x) = L$

[Read it as, "the limit of $f(x)$ as x approaches a, equals L

or

"as x approaches a, the limit of $f(x)$ equals L]

If as x becomes closer and closer to $x=a$
$f(x)$ " " " " L

We can rephrase it, in a slightly more rigorous way

as, " as x becomes arbitrarily close to a
$f(x)$ " " " L"

I emphasize again, for $\lim_{x \to a} f(x) = L$

we don't care at all about what happens to $f(x)$ when x Equals a,

A shorthand notation for $\lim_{x \to a} f(x) = L$

is $f(x) \to L$ as $x \to a$

Estimating a Limit Numerically

Let $f(x) = \dfrac{x}{\sqrt{x+1} - 1}$ for $x \neq 0$

Using tables, make an educated guess for $\lim_{x \to 0} f(x)$

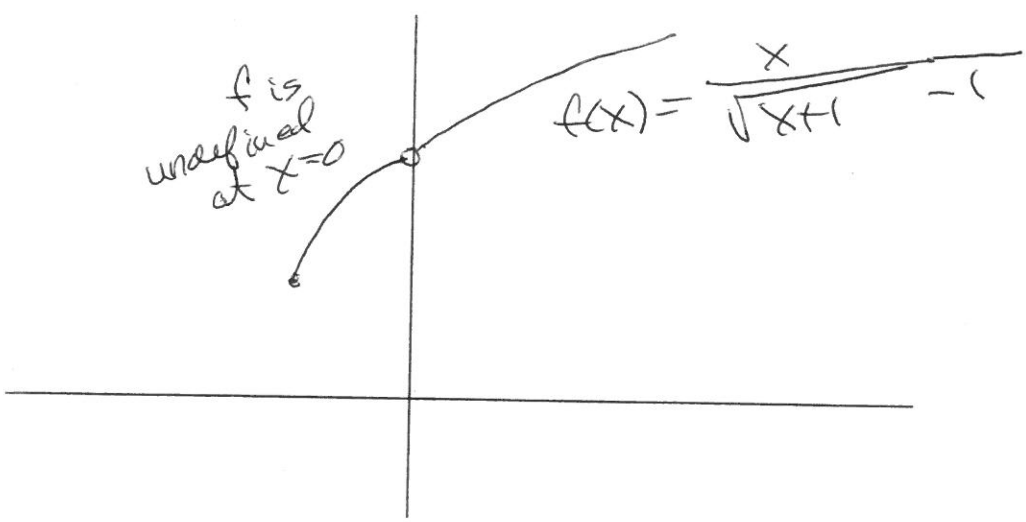
	$x \to 0$ from the left			$x \to 0$ from the right		
x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$f(x)$	1.99499	1.99950	1.99995	2.00005	2.00050	2.004955

$f(x) \longrightarrow 2$ $\longleftarrow f(x) \to 2$

I wrote $f(x) \to 2$, but at this point, for all we know $\lim_{x \to 0} f(x)$ could be anything.

Well, $\lim_{x \to 0} f(x) = 2$. We will prove this later

Here is a rough graph



021 Sec 2.2

Our book on page 85, Ex 2 is very good

$$\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}, \quad \text{Let } f(t) = \frac{\sqrt{t^2+9}-3}{t^2}$$

If you make a table

t	$\frac{\sqrt{t^2+9}-3}{t^2}$
± 1.0	0.162277...
± 0.5	0.165525...
± 0.1	0.166620...
± 0.05	0.166655...
± 0.01	0.166666...

looks like $\lim_{t \to 0} f(t) = \frac{1}{6}$

If we look closer

t	$f(t)$
± 0.001	0.166667
± 0.0001	0.166670
± 0.00001	0.167000
± 0.000001	0.00000

It now looks like $\lim_{t \to 0} f(t) = 0$

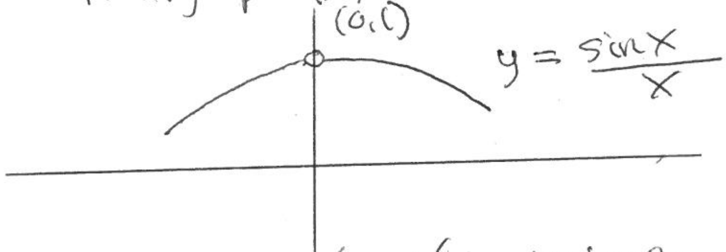
We will prove, in the next section, that

indeed $\lim_{t \to 0} f(t) = \frac{1}{6}$

Ex Guess $\lim_{x\to 0} \frac{\sin x}{x}$. Note, substituting $x=0$, we get $\frac{\sin(0)}{0} = \frac{0}{0}$

which is undefined.

Plotting points, see page 86 gives



Note If a limit exists, it is unique.

A much more interesting, and harder ex, is Find $\lim_{x\to 0} \sin \frac{\pi}{x}$

Solution Let $f(x) = \sin\left(\frac{\pi}{x}\right)$. Note ~~sin sin~~ since, sin(anything) is $-1 \le \sin(\text{anything}) \le 1$

We find some values

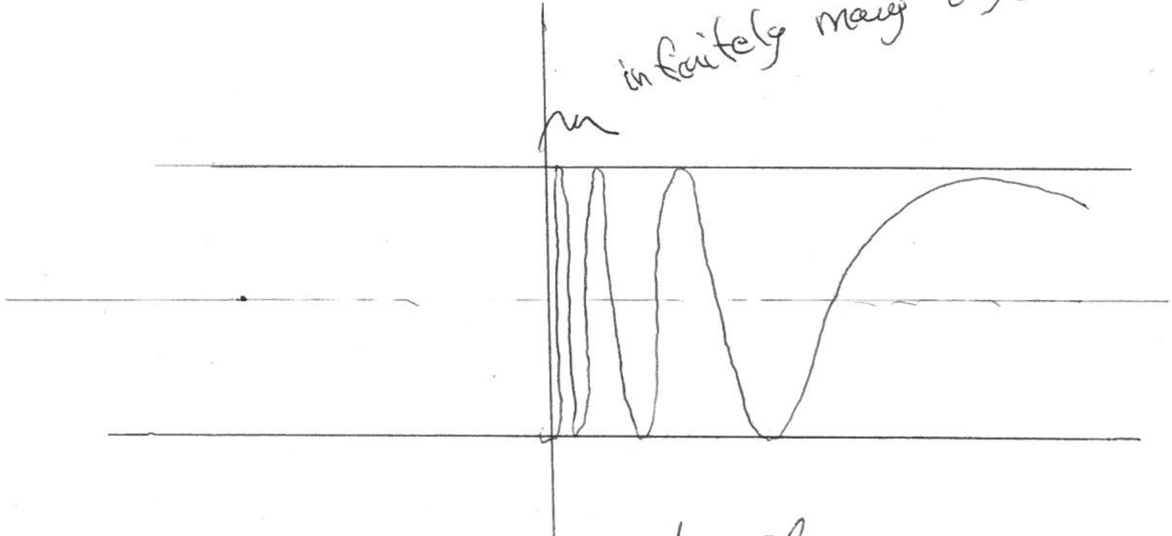
$f(1) = \sin \pi = 0$

$f(\frac{1}{2}) = \sin 2\pi = 0$

$f(\frac{1}{3}) = \sin(3\pi) = 0$

$f\left(\frac{1}{n}\right) = \sin n\pi = 0$ for all $n \in \mathbb{Z}$, n an integer

So, we have a sequence of values of x approaching 0 such that at all these values $\sin(\frac{\pi}{x}) = 0$



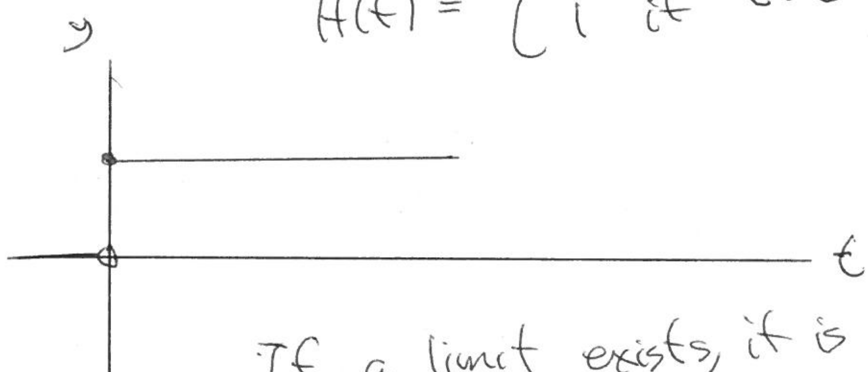
So $\lim \sin(\frac{\pi}{x})$ does not exist.

One-sided Limits

Let's graph the current flowing through a closed circuit with a switch. Initially, the switch is open so there is 0 current flowing through the circuit. You closed the switch and pretty much instantly the current jumps up to some fixed value. Let's say the value is 1.

The function that models that circuit is called the Heaviside function H

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



If a limit exists, it is unique

As $t \to 0$ from the left, the value of $H(t)$ is always equal to 0

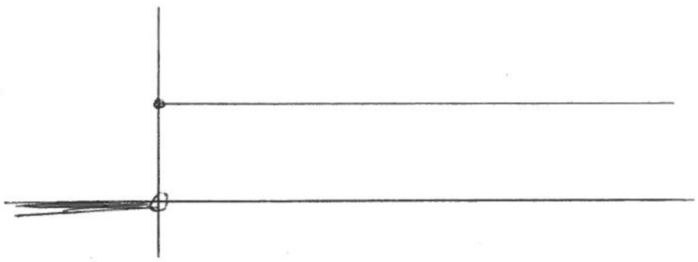
If $t \to 0$ " " right, " " " " " " " to 1

at the same time.

A function cannot approach 0 and 1 at the same time.

So $\lim_{t \to 0} H(t)$ does not exist.

But the ~~only~~ one sided limits do exist



$$\lim_{t \to 0^-} H(t) = 0 \quad \text{and} \quad \lim_{t \to 0^+} H(t) = 1$$

Note $\lim_{t \to 0^-} H(t)$

this "$-$" is not an exponent, but a direction.

in this case as t approaches 0 from the left

likewise $\lim_{t \to 0^+} H(t)$

means as t approach 0 from the right

Def $\lim_{x \to a^-} f(x) = L$ is the left hand (or left sided) limit

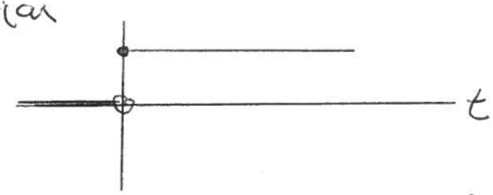
of $f(x)$ as x approaches a from the left

(equivalently, from below)

The connection between one sided limits and limits

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = L$ <u>and</u> $\lim_{x \to a^-} f(x) = L$

The Heaviside Function



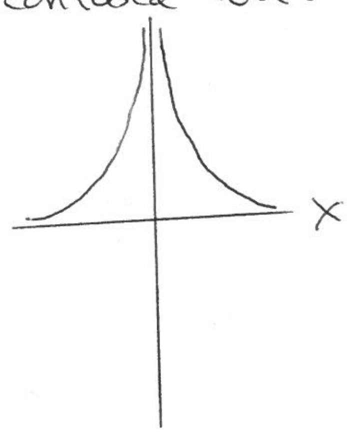
is an example where the one sided limits exist but the limit does not exist.

Review Ex 7 page 88

Infinity Limits (not to be confused with limits at infinity)

<u>Standard Example</u>

graph $f(x) = \frac{1}{x^2}$



x	$\frac{1}{x^2}$
± 1	1
$\pm \frac{1}{2}$	4
$\pm \frac{1}{5}$	25
$\pm \frac{1}{10}$	100

As $x \to 0$, $f(x) \to +\infty$. We write $\lim_{x \to 0} \frac{1}{x^2} = +\infty$

Intuitive Def of an infinite limit. Let f be of a function defined on an open interval containing a. Note f need not be defined at a itself. Then $\lim_{x \to a} f(x) = \infty$

means that as $x \to 0$, $f(x)$ becomes, (and remains) larger than any ~~fixe~~ real number.

i.e. as x becomes arbitrary close to 0

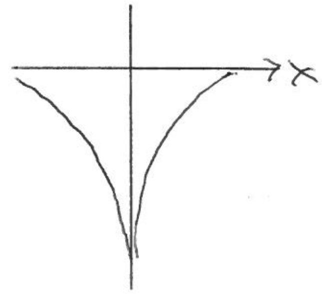
large

We have a similar def for $\lim_{x \to a} f(x) = -\infty$

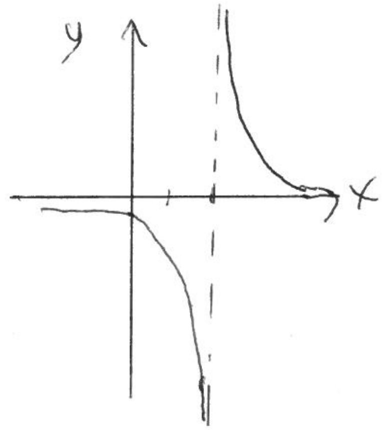
We can also have one sided infinite limits

Here are examples

1) $f(x) = \frac{-1}{x^2}$ $\lim_{x \to 0} f(x) = -\infty$



2) $g(x) = \frac{1}{(x-2)}$



$\lim_{x \to 2^-} g(x) = -\infty$

$\lim_{x \to 2^+} g(x) = +\infty$

We can now define a vertical asymptote

Def The vertical line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one, (possibly more) of the following statements is true

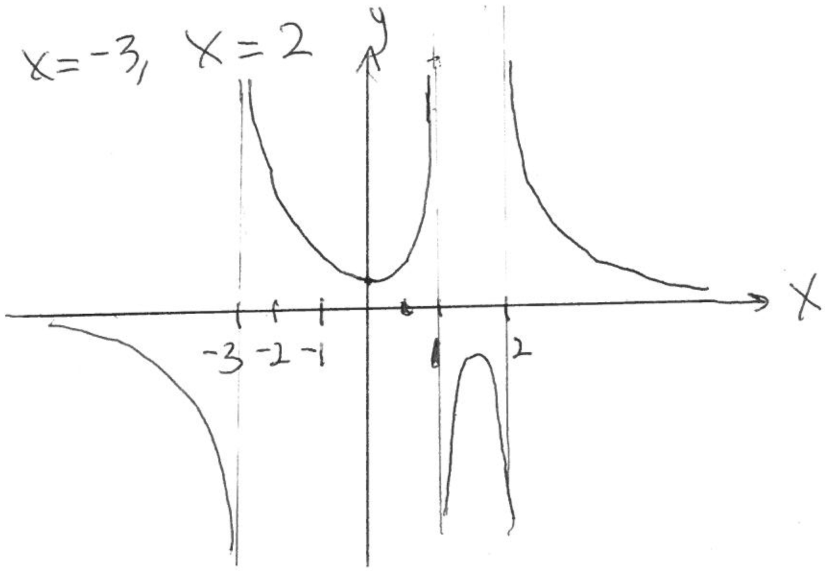
$\lim_{x \to a} f(x) = \infty$, $\lim_{x \to a^-} f(x) = \infty$ $\lim_{x \to a^+} f(x) = \infty$

$\lim_{x \to a} f(x) = -\infty$, $\lim_{x \to a^-} f(x) = -\infty$ $\lim_{x \to a^+} f(x) = -\infty$

It is easy to construct a function with vertical asymptotes at a fixed set of values in the domain.

Ex Find a function $f(x)$ that has vertical asymptotes at

$x=1, \ x=-3, \ x=2$



$$f(x) = \frac{1}{(x-1)(x+3)(x-2)}$$

Place to look ~~where~~ for a vertical asymptotes is where you would be dividing by 0.

Calculating Limits using the limit Laws

We will see that, for the most part, limits behave nicely, and our intuition is pretty much a good guide

This is not a very exciting section, but it is necessary

Let c be a constant and $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ both exist,

Then Then

1) $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2) $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3) $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$

4) $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

5) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0$

6) $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$, $n\in\mathbb{Z}^+$; set of positive integers

7) $\lim_{x\to a} c = c$

8) $\lim_{x\to a} x = a$

9) $\lim_{x\to a} x^n = a^n$, $n\in\mathbb{Z}^+$

10) $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, $n\in\mathbb{Z}^+$, if n is even we assume $a>0$

11) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, $n\in\mathbb{Z}^+$, if n is even, assume $\lim_{x\to a} f(x) > 0$

Ex $\lim_{x\to 2}(3x^4-7x^2+2x-1)$

$= \lim_{x\to 2} 3x^4 - \lim_{x\to 2} 7x^2 + \lim_{x\to 2} 2x - \lim_{x\to 2} 1$ (repeated use of 1) and 2))

$= 3\lim_{x\to 2} x^4 - 7\lim_{x\to 2} x^2 + 2\lim_{x\to 2} x - \lim_{x\to 2} 1$ (repeated use of 3)

$= 3\left(\lim_{x\to 2} x\right)^4 - 7\left(\lim_{x\to 2} x\right)^2 + 2\lim_{x\to 2} x - \lim_{x\to 2} 1$ (repeated use of 6)

use 8) use 8) use 8) use 7)

$= 3(2^4) - 7(2^2) + 2(2) - 1$

$= 48 - 28 + 4 - 1$

$= 23$

Note if $f(x) = (3x^4-7x^2+2x-1)$

then $f(2) = 3\cdot 2^2 - 7\cdot 2^2 + 2\cdot 2 - 1 = 23$

Ex $\lim_{x\to -4} \dfrac{x^2-4x}{x+3} \overset{\text{use 5)}}{=} \dfrac{\lim_{x\to -4}(x^2-4x)}{\lim_{x\to -4}(x+3)}$

$= \dfrac{\lim_{x\to -4}(x^2) - \lim_{x\to -4}(4x)}{\lim_{x\to -4} x + \lim_{x\to -4} 3}$

$= \dfrac{\lim_{x\to -4} x^2 - 4\lim_{x\to -4} x}{-4+3} = \dfrac{(-4)^2 - 4(-4)}{-1} = \dfrac{16+16}{-1} = -32$

note if $g(x) = \dfrac{x^2-4x}{x+3}$

then $g(-4) = \dfrac{(-4)^2-4(-4)}{-4+3} = \dfrac{16+16}{-1} = -32$

Now, to find the limit of a polynomial or a rational function, (where we are not dividing by 0)

$$\lim_{x \to a} f(x) = f(a)$$

Exs Read the ~~exs~~ Ex 3) on pg 98

Ex $\lim_{x \to 3} \frac{x^2-9}{x-3}$, Now if we set $g(x) = \frac{x^2-9}{x-3}$, $g(3) = \frac{0}{0}$ undefined. so direct substitution. because $g(3)$ is

i.e. 3 is not in the domain of g.

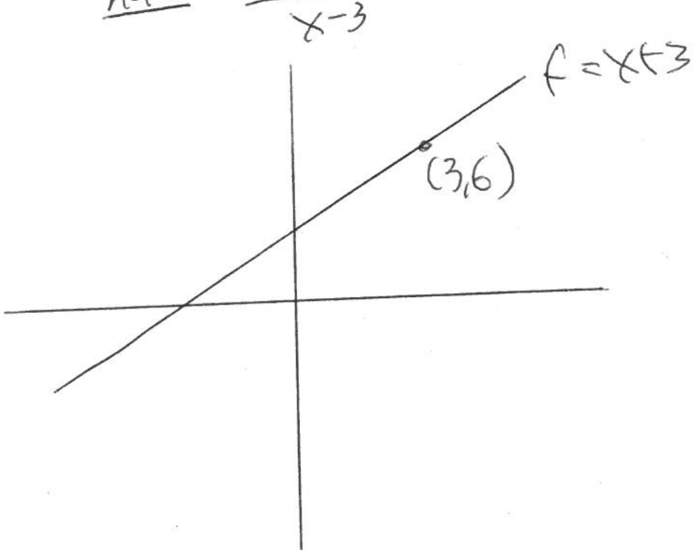
correct solution: $\lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3}$

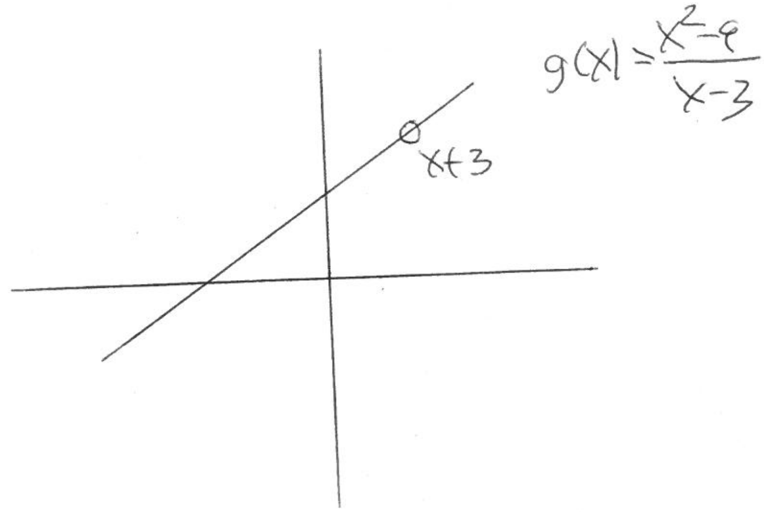
Now, we only care about what happens as $x \to 3$, we don't care what happens at $x = 3$

we can cance the $(x-3)$'s

$$= \lim_{x \to 3} (x+3) = 3+3 = 6$$

Note $\frac{x^2-9}{x-3} = x+3$ for all $x \neq 3$





021 Sec 2.3

EX find $\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$

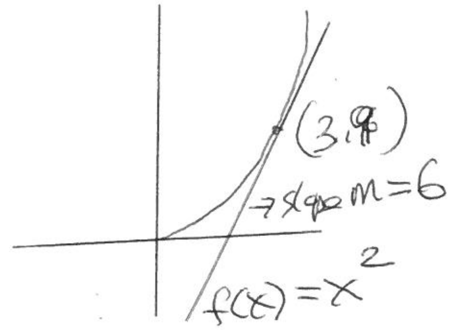
secant line has slope $\frac{\Delta y}{\Delta x} = \frac{(x+h)^2 - x^2}{(x+h) - x}$

Note $\lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$

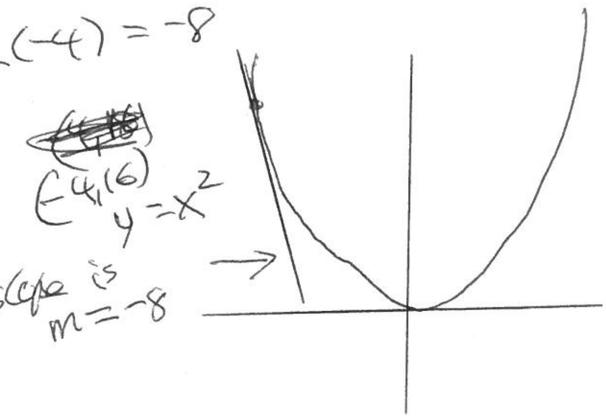
$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$

$= \lim_{h \to 0} (2x+h) = 2x + 0 = 2x$

So if $x = 3$, this limit is $2(3) = 6$



if $x = -4$, this limit is $2(-4) = -8$



Back to a problem that we considered with tables in sec 2.2

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$$

$$= \lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}\left(\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}\right)$$

$$= \lim_{t\to 0} \frac{(\cancel{t^2}+\cancel{9})-\cancel{9}}{\cancel{t^2}(\sqrt{t^2+9}+3)}$$

$$= \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3}$$

$$= \lim_{t\to 0} \frac{1}{\sqrt{0^2+9}+3}$$

$$= \lim_{t\to 0} \frac{1}{\sqrt{9}+3}$$

$$= \lim_{t\to 0} \frac{1}{3+3} = \frac{1}{6}$$

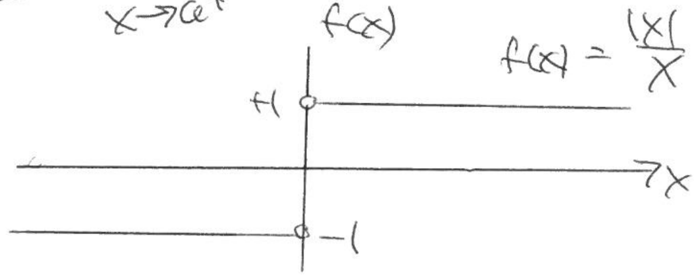
<u>So</u> $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \frac{1}{6}$

The limit laws hold for one sided limits as well.

We we also use the fact that

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$

and $\lim_{x \to a^+} f(x) = L$



Ex Determine $\lim_{x \to 0} \frac{|x|}{x}$

now $|x| = x$ if $x \geq 0$
$-x$ if $x < 0$

So $\frac{|x|}{x} = \frac{x}{x} = 1$ if $x \geq 0$

$\frac{|x|}{x} = \frac{-x}{x} = -1$ if $x < 0$

So $\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} (-1) = -1$

$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} (+1) = +1$

So, $\lim_{x \to 0^-} \frac{|x|}{x} \neq \lim_{x \to 0^+} \frac{|x|}{x}$

So $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Ex Let $g(t) = \begin{cases} t^2+1 & \text{for } t \le 2 \\ -2t+9 & t > 2 \end{cases}$

Find $\lim_{t\to 2} g(t)$

Solution $\lim_{t\to 2^-} g(t) = \lim_{t\to 2^-} (t^2+1) = 2^2+1 = 5$

$\lim_{t\to 2^+} g(t) = \lim_{t\to 2^+} (-2t+9) = (-2)(2)+9 = 5$

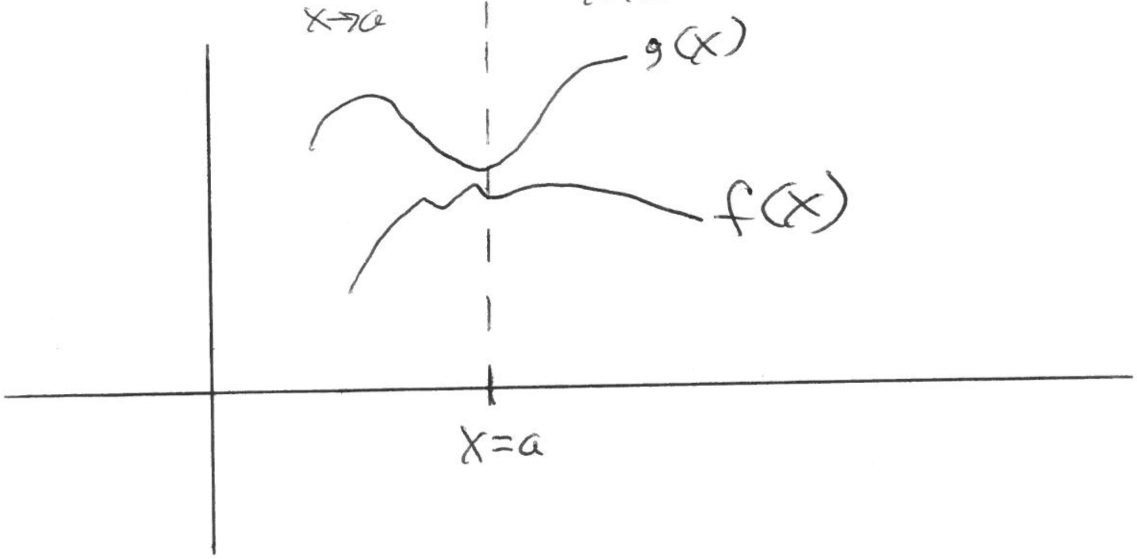
Since LHL = RHL

left hand limit = right hand limit

The $\lim_{t\to 2} g(t) = 5$

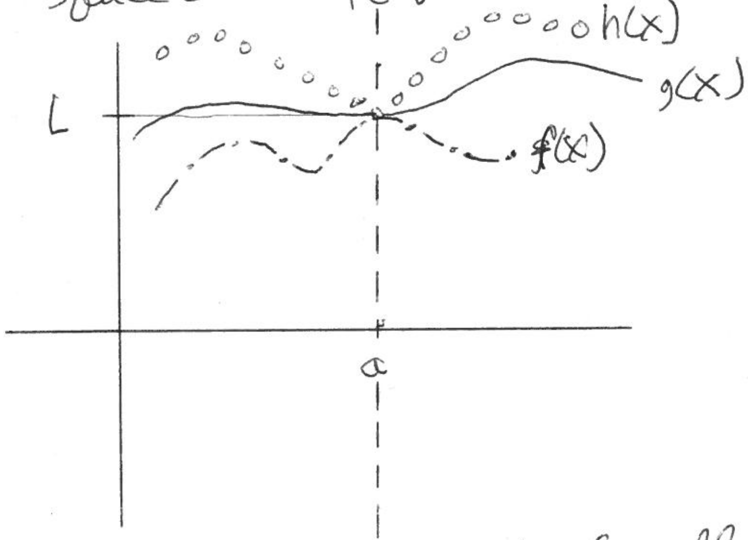
Thm If $f(x) \le g(x)$ for all x near $x=a$, except possibly at a itself, then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$



021 Sec 2.3

The Squeeze Thm, (often called the Pinching thm)



If $f(x) \le g(x) \le h(x)$ for all x near a, (except possibly at a itself) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

Then $\lim_{x \to a} g(x) = L$

We will need the squeeze theorem when we derive the derivatives of the trig functions

Ex Find $\lim_{x \to 0} x^4 \sin\frac{1}{x}$

First since $\lim_{x \to 0} \sin(\frac{1}{x})$ does not exist

we cannot say that

$$\lim_{x \to 0} x^4 \sin\frac{1}{x} = \lim_{x \to 0} x^4 \lim_{x \to 0} \sin\frac{1}{x}$$

Instead, we will use the squeeze thm,

We will use the following fact

$$-1 \le \sin(\text{anything}) \le +1$$

so so $-x^4 \le \sin(\text{anything}) \le x^4$

so $-x^4 \le \sin\frac{1}{x} \le x^4$

Now $\lim_{x \to 0} (-x^4) = 0$

$\lim_{x \to 0} (x^4) = 0$

so by the squeeze thm $\lim_{x \to 0} \sin(\frac{1}{x}) = 0$

so

 The precise def of a limit

The idea of a limit was used very succesfully by Isaac Newton and his successors. (see bar on page 109) but as mathematics became more rigorous, the limits of the intuitive became obvious.

What does, "arbitrarty close" really mean.

or, "closer and closer"

I will now throw the definition of a limit at you. Don't panic. I expect that you will have trouble with it. That's o.k. Very few people understand this definition the first time that they see it. I certainly did not.

Def Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the limit of f(x) as x approaches a equals L.

if for every number $\varepsilon > 0$, then there is a number $\delta > 0$

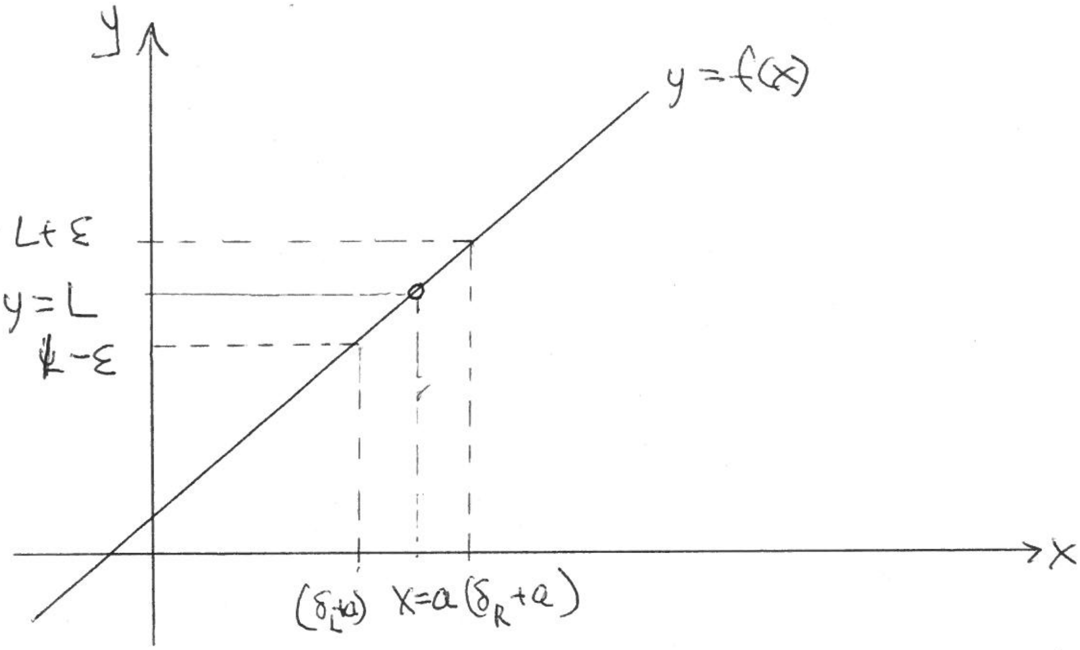
s.t. if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$

we write $\lim_{x \to a} f(x) = L$.

Note 1) ε is the lower case Greek letter epilison

2) δ " " " " " delta.

3) ε and δ are usually small positive numbers.



Basically, this rigorous defines kinda works like this.

We are given a function $y = f(x)$ and a point a.

I tell you that I want an interval I on the x-axis containing the value $x=a$. I want to be sure that all the values of $f(x)$, for x in I, $f(x)$ is within ε units of L. i.e. $f(x) \in$ of the interval $(L-\varepsilon, L+\varepsilon)$

an element

So, You have to find a small positive number δ

s.t. if x is in the interval $(a-\delta, a+\delta)$

s.t. = such that

then $f(x)$ is in the interval $(L-\varepsilon, L+\varepsilon)$

Ex Using the rigorous def of a limit, show that

$$\lim_{x \to 2} (3x-5) = 1$$

Solution We must show that given any (small) positive number ε, we can find a positive number δ s.t.

$$\underbrace{|(3x-5) - 1|}_{f(x) - L} < \varepsilon \quad \text{if} \quad 0 < |x - \underbrace{2}_{a}| < \delta$$

Two things to do

1) Discover a value of δ for which this statement holds.
2) Then, prove that the choice of δ works

For the discovery

Rewrite $|(3x-5)-1| < \varepsilon$ if $0 < |x-2| < \delta$

$|3x-6| < \varepsilon$ if $0 < |x-2| < \delta$

factor out a 3

$3|x-2| < \varepsilon$ if $0 < |x-2| < \delta$

$|x-2| < \frac{\varepsilon}{3}$ if $0 < |x-2| < \delta$

So let $\delta = \frac{\varepsilon}{3}$

Now, we show that our choice $\delta = \frac{\varepsilon}{3}$ works

i.e. $|(3x-5)-1| < \varepsilon$ if $0 < |x-2| < \frac{\varepsilon}{3}$

we kinda just did this

$|3x-6| < \varepsilon$ if $0 < |x-2| < \frac{\varepsilon}{3}$

$3|x-2| < \varepsilon$ if $0 < |x-2| < \frac{\varepsilon}{3}$

$|x-2| < \frac{\varepsilon}{3}$ if $0 < |x-2| < \frac{\varepsilon}{3}$

In general, for $y = mx + b$

Let ε be given, then choose $\delta = \frac{\varepsilon}{m}$.

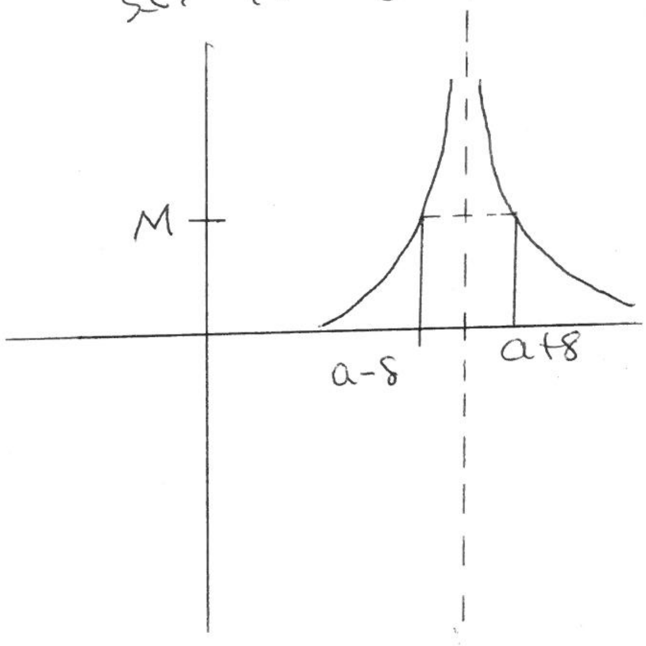
Infinite limits

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

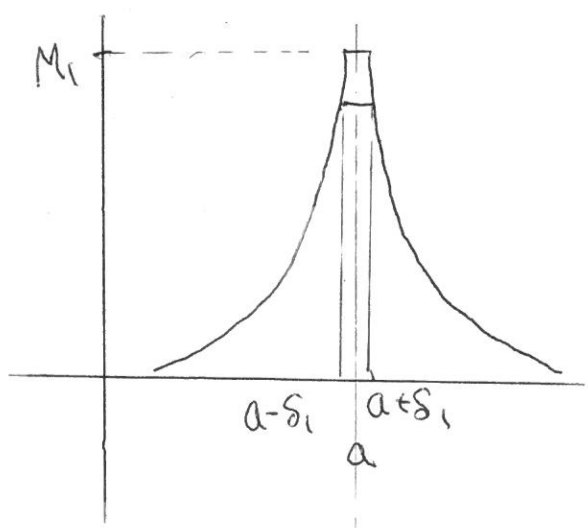
if for every large positive number M, there is a positive number δ

s.t. if $0 < |x - a| < \delta$ then $f(x) > M$



change M, say M_1 is bigger than your first choose of M.

Then a new δ, δ_1, will be required with $0 < \delta_1 < \delta$



Ex Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$

Let M be a given positive number (large)

We seek a number δ s.t. if $0 < |x| < \delta$ then $\frac{1}{x^2} > M$

Now $\frac{1}{x^2} > M$ iff $x^2 < \frac{1}{M}$ iff $\sqrt{x^2} < \sqrt{\frac{1}{M}}$ iff $|x| < \frac{1}{\sqrt{M}}$

So, if $\delta = \frac{1}{\sqrt{M}}$ and $0 < |x| < \delta = \frac{1}{\sqrt{M}}$, then $\frac{1}{x^2} > M$.

~~This [illegible]~~ This shows $\frac{1}{x^2} \to \infty$ as $x \to 0$.

Section 2.5 Continuity

Roughly speaking, a function is continuous if its graph can be drawn without lifting the pen from the paper.

I stressed that in finding ~~a limit~~ $\lim_{x \to a} f(x)$, we don't care what happens to $f(x)$ at a. We only care what happens to $f(x)$ near a.

With continuity, we very much care what happens to $f(x)$ at $x = a$.

O.K. here is the def

Def A function f is <u>continuous</u> at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

There is much embedded in the above def. We have

1) $\lim_{x \to a} f(x)$ exists.

This means that

$$\left.\begin{array}{l} \lim_{x \to a^+} f(x) \text{ exists} \\ \lim_{x \to a^-} f(x) \text{ exists} \end{array}\right\} \text{ and they are equal}$$

2) $f(a)$ exists

3) 1) equals 2)

$$\lim_{x \to a} f(x) = f(a)$$

If $f(x)$ is not continuous at $x=a$

i.e. $\lim\limits_{x \to a} f(x) \neq f(a)$

then we say $f(x)$ is discontinuous at $x=a$

or a is a point of discontinuity for f

The place to look for discontinuities is often when you would be dividing by zero.

There is an early Clint Eastwood movie called, "The Good, the Bad, the Ugly". I like to refer to the 3 types of discontinuities as, the good, the bad, the ugly

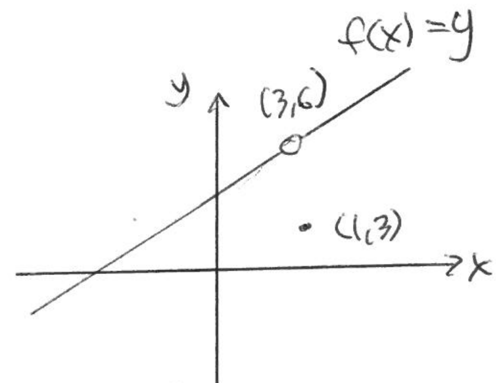
Types of Discontinuities

Type 1 Removable Discontinuities (The good) called a removable discontinuity

When a) $\lim\limits_{x \to a} f(x)$ exists

b) $f(a)$ exists

c) but $\lim\limits_{x \to a} f(x) \neq f(a)$

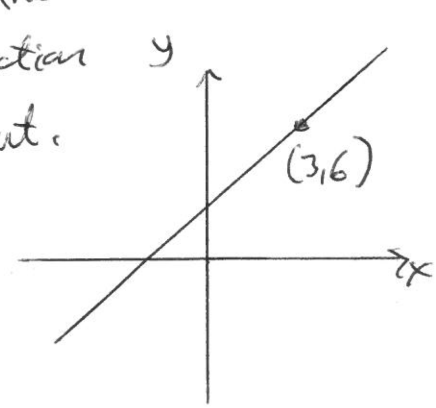


Ex $f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 1 & \text{if } x = 3 \end{cases}$

Note $\dfrac{x^2-9}{x-3} = \dfrac{(x-3)(x+3)}{x-3} = x+3$ if $x \neq 3$

what makes this a, "good" discontinuity is that we transform $f(x)$ into a continuous function by changing the value of $f(x)$ at one point.

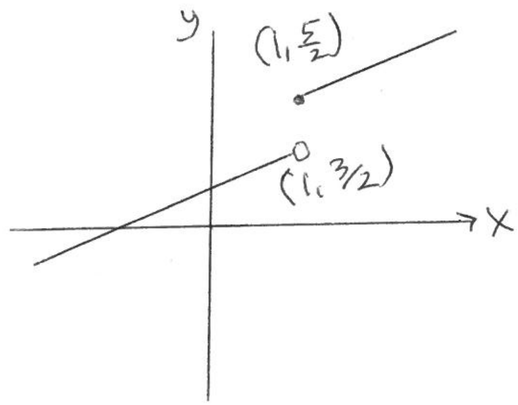
In this case, make $f(3)=6$



Type 2 Discontinuity of Type 2 (the bad)

When $\lim_{x \to a} f(x)$ does not exist

but $\lim_{x \to a^-} f(x)$ exists and $\lim_{x \to a^+} f(x)$ exists



Ex $g(x) = \begin{cases} \frac{x}{2} + 1 & \text{if } x < 1 \\ \frac{x}{2} + 2 & \text{if } x \geq 1 \end{cases}$

What makes this type of discontinuity bad is that to transform g into a continuous we have to redefine g on an interval, (it might be a very short interval, but still an open interval.

Type 2 discontinuities are often called jump discontinuties

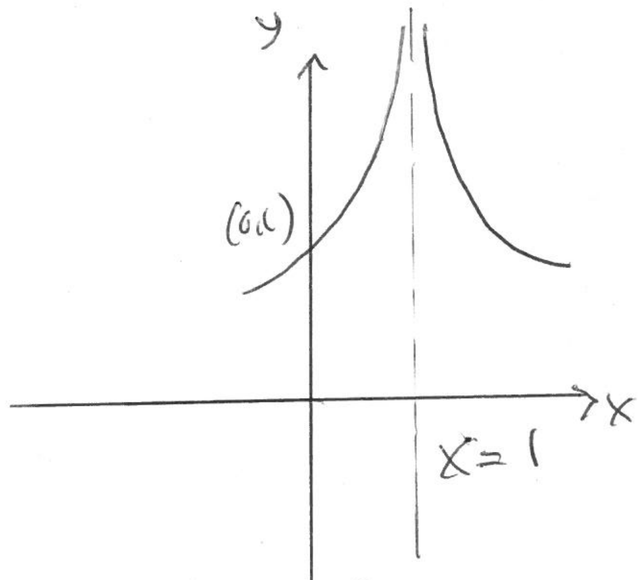
Note The original function didn't have an asymptote, the revised function doesn't have an asymptote

Sec 2.5

The ugly - When $\lim_{x \to a} f(x)$ does not exist, because the one-sided limits don't exist

(called essential discontinuities

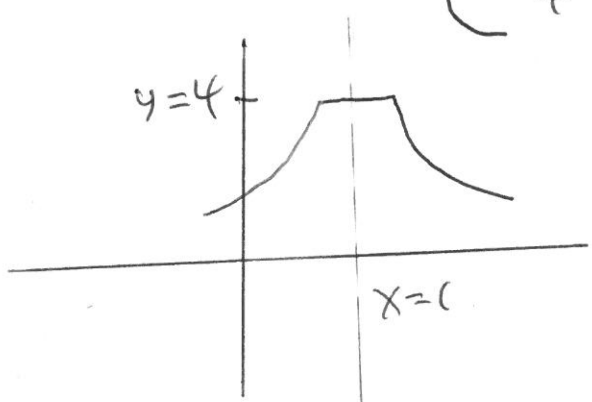
<u>Ex</u> $y = f(x) = \frac{1}{(x-1)^2}$



Note we have a vertical asymptote at $x=1$

We can create a new function, call it g(x) from f(x) by truncation

$$y = g(x) = \begin{cases} \frac{1}{(x-1)^2}, & \text{for } x < \frac{1}{2},\ x > \frac{3}{2} \\ 4 & \text{for } \frac{1}{2} < x < \frac{3}{2} \end{cases}$$



<u>Note</u> we have changed the function in an essential way.
f(x), (the original function) is unbounded with a vertical asymptote

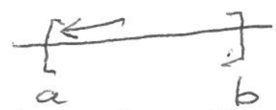
The altered function, g(x) is unbounded and does not have any vertical asymptotes

<u>Def</u> A function f is continuous on an open interval (a,b) if it is continuous in every point in that open interval.

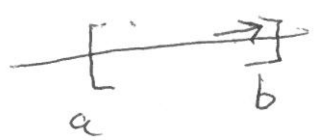
<u>Def</u> A function g is continuous on the closed interval, $[a,b]$ iff

1) g is continuous on the open interval (a,b)
2) g is continuous from the right, (from above) at $x=a$

   i.e. $\lim_{x\to a^+} g(x) = g(a)$.



3) g is continuous from the left, (from below) at $x=b$



   i.e. $\lim_{x\to b^-} g(x) = g(b)$

<u>Ex</u> Show that $f(x) = 1-\sqrt{1-x^2}$ is continuous on the interval $[-1,1]$

<u>Solution</u> Using the limits for $f(x)$ on $(-1,1)$

we see that $\cancel{\lim_{x\to a} = \lim_{x}}$

$$\lim_{x\to a} f(x) = \lim_{x\to a}\left(1-\sqrt{1-a^2}\right) = f(a) \checkmark$$

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so f is continuous on the open interval, $(-1,1)$

Now check endpoints

$$\lim_{x\to -1^+} f(x) = 1-\sqrt{1-(-1)^2} = 1 = f(-1) \checkmark$$

$$\lim_{x\to 1^-} f(x) = 1-\sqrt{1-1^2} = 1 = f(1)$$

So, we have continuity at the endpoints.

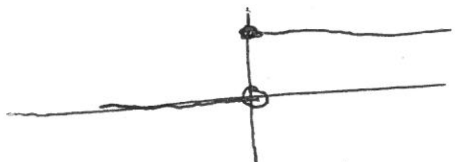
<u>Thm</u> If f and g are both continuous at $x=a$ and c is a constant, then the following are continuous at $x=a$

1) $f+g$ 2) $f-g$ 3) cf 4) $f \cdot g$ 5) $\frac{f}{g}$ for $g(a) \neq 0$

The following classes of functions are continuous on their domains

1) Polynomials 2) Rational function 3) Exponentials
4) Logs 3) Trig functions 6) Inverse Trig function

<u>Note</u> Heaviside is <u>not</u> continuous at $x=0$, but is defined at 0



Easy to construct a function that is discontin at a set of points
$7, -2, 4, 13$ $\qquad f(x) = \dfrac{1}{(x-13)(x+2)(x-4)(x-7)^2}$

Continuity and Composition
Let f be continuous at b and let $\lim_{x\to a} g(x) = b$
then $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$

<u>Ex</u> Find $\lim_{x\to\frac{\pi}{4}} \sqrt[3]{\sin x}$

<u>Solution</u> Here $\lim_{x\to\frac{\pi}{4}} \sqrt[3]{\sin x} = \lim_{x\to\frac{\pi}{4}} (\sin x)^{\frac{1}{3}}$

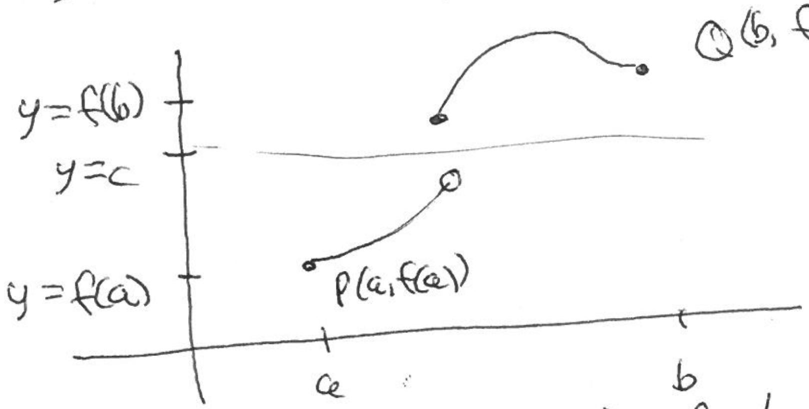
In this case, $g(x) = \sin x$, $f(x) = x^{\frac{1}{3}}$

So $\lim_{x\to\frac{\pi}{4}} (\sin x)^{\frac{1}{3}} = \left(\lim_{x\to\frac{\pi}{4}} \sin x\right)^{\frac{1}{3}} = \left(\sin\frac{\pi}{4}\right)^{\frac{1}{3}}$

$= \left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{3}} = \left(2^{-\frac{1}{2}}\right)^{\frac{1}{3}} = 2^{-\frac{1}{6}}$

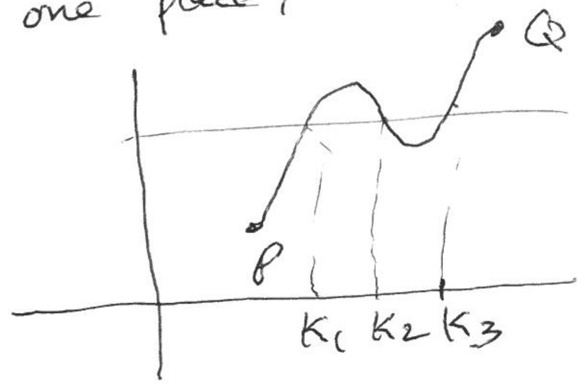
021 Sec 2.5

Sec 2.5 The Intermediate Value Theorem



If we don't require f to be a continuous function, it is possible to draw a graph from P(a, f(a)) to Q(b, f(b)) without crossing the horizontal line $y=C$

If we require f to be a continuous function, then f must cross the horizontal line $y=c$ in at least one place.



The intermediate value theorem.

Let $f(x)$ be continuous on the closed interval $[a,b]$

Let c be between $f(a)$ and $f(b)$

Then there is a real number K with $a<K<b$

s.t. $f(K)=c$

Application of the theorem

Use the intermediate value theorem to show that $f(x) = x^3 - 34$ has a real root.

Solution $f(x)$ is a polynomial, so continuous everywhere

A root means $f(K) = 0$

Now $f(0) = -34 < 0$, $f(4) = 30 > 0$

and $-34 < 0 < 30$

So, by the intermediate value theorem

$\exists K$ with $0 < K < 4$ and $f(K) = 0$

This is an example of an existance proof.
It shows that K must exist, but doesn't
show the value of K.