11.9 Representations of Functions as power series

If you find this section difficult, that is o.k., it is one of the more difficult sections in the course

Some functions, like $f(x) = e^{-x^2}$ do not have antiderivatives. Hence you cannot integrate them by using the fundamental theorem of calculus.

If you expand the function out in a power series, you can integrate the function by integrating the power series expansion term by term.

First Let's consider how to expand a function in a power series

Let $f(x) = \frac{1}{1-x}$. This looks a lot like the sum of a geometric series: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $-1<r<1$

So, if we let $a=1$ and $r=x$, then a power series for $\frac{1}{1-x}$, centered at 0 is.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\cdots+, \quad |x|<1$$

Note This power series represents $f(x)=\frac{1}{1-x}$ only on the interval $(-1,1)$. But $f(x)$ is defined for all $x$, $x\neq 1$

If you wanted the p.s. for $x=-1$, one way to do it is.

$$\frac{1}{1-x} = \frac{1}{2-(x+1)} = \frac{\frac{1}{2}}{1-\left[\frac{x+1}{2}\right]} = \frac{a}{1-r}$$ (from the sum of a geometric series)

In this case, $a=\frac{1}{2}$, $r=\frac{x+1}{2}$. So

So for $|x+1|<2$, we have

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)\left(\frac{x+1}{2}\right)^n$$

$$= \frac{1}{2}\left[1 + \frac{(x+1)}{2} + \frac{(x+1)^2}{4} + \frac{(x+1)^3}{8} + \cdots\right]; \quad |x+1|<2$$

which converges on the interval $(-3,1)$

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Ex Find a power series for $f(x) = \frac{4}{x+2}$, centered at 0

Solution Writing $f(x)$ in the form $\frac{a}{1-r}$ yields

$$\frac{4}{x+2} = \frac{2}{1-\left(\frac{-x}{2}\right)} = \frac{a}{1-r}$$

so $a=2$, $r=\frac{-x}{2}$. So a power series for $f(x)$ is given by

$$\frac{4}{x+2} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} 2\left(\frac{-x}{2}\right)^n = 2\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\cdots\right)$$

This power series converges when $\left|\frac{-x}{2}\right|<1$

so the interval of convergence is $(-2,2)$

Note we could have obtained the above by long division

$$\begin{array}{r l}
 & 2 - x + \frac{1}{2}x^2 - \frac{1}{4}x^3 + \cdots \\
2+x \,\big)\!\! & 4 \\
 & \underline{4+2x} \\
 & \quad -2x \\
 & \quad \underline{-2x - x^2} \\
 & \qquad x^2 \\
 & \qquad \underline{x^2 + \frac{1}{2}x^3} \\
 & \qquad\quad -\frac{1}{2}x^3 \\
 & \qquad\quad -\frac{1}{2}x^3 - \frac{1}{4}x^4 \cdots
\end{array}$$

Ex. Find a power series for $f(x) = \frac{1}{x}$, centered at 1

Solution: Writing $f(x)$ in the form $\frac{a}{1-r}$, yields

$$\frac{1}{x} = \frac{1}{1-(-x+1)} = \frac{a}{1-r}, \quad \text{so } a = 1,\ r = 1-x = -(x-1)$$

which implies

$$\frac{1}{x} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \left[-(x-1)\right]^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$$

This power series converges when $|x-1| < 1$. So the interval of convergence is $(0,2)$

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Differentiation and Integration of Power Series

Thm: If the power series $\sum c_n (x-a)^n$ has radius of convergence $R>0$ then the function $f$ defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is differentiable and continuous on the interval $(a-R, a+R)$ and

i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

ii) $\int f(x)\,dx = C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radius of convergence of the power series i) and ii) are both R.

i) Is called term by term differentiation

ii) called term by term integration.

A general guideline for working with power series is things work term by term. The guideline also work in the following thm

<u>Thm</u> Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$, then

1) $f(xk) = \sum_{n=0} a_n k^n x^n$, note $k$ is also raised to higher power

2) $f(x^N) = \sum_{n=0}^{\infty} a_n x^{nN}$

3) $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$

In the thm, the interval of convergence for the resulting series. For Example in 3) the interval of convergence is the <u>intersection</u> of the interval of convergence of the original two series

Ex $\underbrace{\sum_{n=0} x^n + \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n}_{(-1,1)\, \cap\, (-2,2)} = \sum_{n=0}^{\infty} \left[x^n + \left(\frac{x}{2}\right)^n\right] = \underbrace{\sum_{n=0}^{\infty} \left(1 + \frac{1}{2^n}\right) x^n}_{(-1,1)}$

Ex Find a power series, centered at 0, for $f(x) = \frac{3x-1}{x^2-1}$

Using partial fractions: $\frac{3x-1}{x^2-1} = \frac{2}{x+1} + \frac{1}{x-1}$

Noting that $\frac{2}{x+1} = \frac{2}{1-(-x)} = \sum_{n=0}^{\infty} 2(-1)^n x^n$, $|x| < 1$

and $\frac{1}{x-1} = \frac{-1}{1-x} = -\sum_{n=0}^{\infty} x^n$, $|x| < 1$

So, $\frac{3x-1}{x^2-1} = \sum_{n=0}^{\infty} \left[2(-1)^n - 1\right] x^n = 1 - 3x + x^2 - 3x^3 + x^4 - \cdots$

The interval of convergence is $(-1,1)$

<u>EX</u> Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$

Find the interval of convergence of a) $\int f(x)dx$, b) $f(x)$ c) $f'(x)$

<u>Solution</u> $f'(x) = \sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + x^3 + \cdots$

and $\int f(x)dx = C + \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}$

$= C + \frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} + \cdots$

By using the ratio test, you can check that the radius of convergence for a), b) and c) is $R=1$. We now check endpoints

For a) $\sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}$ if $x=-1$, we have $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)}$, converges by alt series test

if $x=1$, $\sum_{n=1}^{\infty} \frac{1^{n+1}}{n(n+1)}$, converges by p-test.

So Interval of convergence is $[-1,1]$

For b) $\sum_{n=1}^{\infty} \frac{x^n}{n}$, if $x=-1$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, converges by alter series test

if $x=1$, $\sum_{n=1}^{\infty} \frac{1^n}{n} = \sum \frac{1}{n}$, diverges, harmonic series.

so interval of convergence is $[-1,1)$

For c) $\sum_{n=1}^{\infty} x^{n-1}$ if $x=-1$, $\sum_{n=1}^{\infty} (-1)^{n-1} = 1-1+1-1+\cdots$ diverges

if $x=1$, $\sum_{n=1}^{\infty} 1^{n-1} = 1+1+1+1+\cdots$ diverges

So, radius of convergence is $(-1,1)$

EX Recall that $\frac{1}{1-x} = 1+x+x^2+x^3 = \sum_{n=0}^{\infty} x^n, \ |x|<1$

Let $f(x) = \frac{1}{1-x}$. Write $f(x)$ as a power series by differentiating each term

$$\frac{1}{1-x} = 1+x+x^2+x^3+\cdots = \sum_{n=0}^{\infty} x^n$$

Differentiate both sides

$$\frac{1}{(1-x)^2} = 1+2x+3x^2+\cdots = \sum_{n=1}^{\infty} n x^{n-1}$$

We can tidy this up by replacing $n$ by $n+1$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n$$

The ~~interval~~ radius of convergence is once again $R=1$

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Finding a power series for $f(x) = \ln x$, centered at 1

Solution From the example on page ③ of todays notes

$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$. Now integrate both sides.

$$\ln x = \int \frac{1}{x}\,dx + C = C + \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}$$

$$\ln x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} = \frac{x-1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

Now the interval of convergence of $\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{n+1}$ is $(0,2)$

so ~~the~~ for $x=0$, $\frac{1}{x}$ is undefined so exclude $x=0$

for $x=2$, $\sum (-1)^n \frac{(x-1)^n}{n+1} = \sum \frac{(-1)^n \cdot 1^n}{n+1}$, ~~diverges~~ converges

so interval of convergence is $(0,2]$

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Ex Find a p.s. for $g(x) = \arctan x$, centered at 0

Solution $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$. So use the series

$$f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

Substituting $x^2$ for $x$ yields

$$f(x^2) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Now, integrate both sides

$$\arctan x = \int \frac{1}{1+x^2}\,dx + C = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

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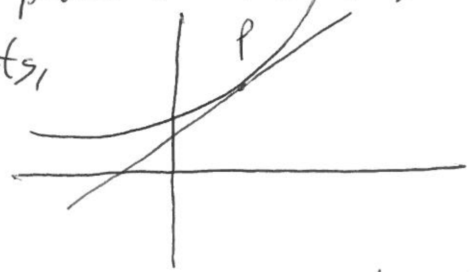
Sec 11.10 Taylor And Maclaurin Series

We saw in calculus 1, that the best straight line approximation to a curve at a point is the tangent line, provided that the tangent line exists,

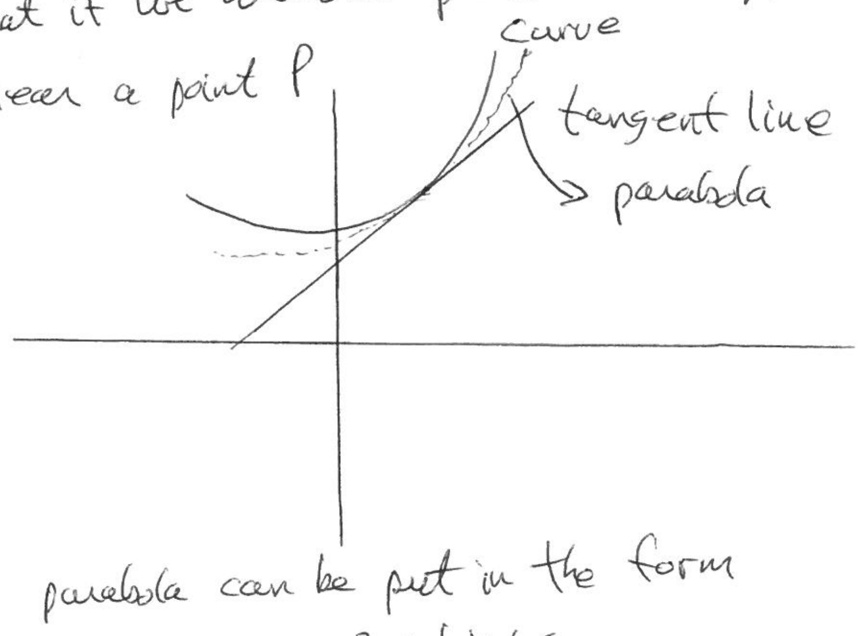



Recall that the slope of the tangent line is the value of the derivative

Now, a tangent line is a line and can be put in the form

$$y = mx + b$$

But what if we allowed parabola's to approximate our curve near a point P




A parabola can be put in the form

$$y = ax^2 + bx + c$$

Question: How do we find the parabola that "best" approximates a curve $y = f(x)$ near a pt P.

How do we find the best cubic

$$y = ax^3 + bx^2 + cx + d$$

best approximates $y = f(x)$ near a pt P

Our goal, which we will ~~achieve~~, achieve, is to solve these problems.

Let's find the n'th order polynomial $P$, that best approximates $f(x)$ near $x=0$

Let $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n$

<u>Key point</u> Any power series is determined by its ~~polynomials~~ coefficients. So $p(x)$ is determined by $a_0, a_1, a_2, \ldots, a_n$.

We have to find the values of $a_0, a_1, \ldots, a_n$.

<u>Key point</u> By, "best", we will mean

$p(0) = f(0), \quad p'(0) = f'(0), \quad p''(0) = f''(0), \ldots, p^{(n)}(0) = f^{(n)}(0)$

Now $p(0) = f(0) \Rightarrow \boxed{a_0 = f(0).}$ Just plug in 0 for $x$ in $P(x)$

$P(0) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n$

For $a_1$, consider that

$P'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + (n-1)a_nx^{n-2} + na_nx^{n-1}$

$p'(0) = f'(0) \Rightarrow \boxed{f'(0) = a_1}$

For $a_2$: $p''(x) = 2a_2 + 2\cdot3a_3x + \cdots + (n-2)(n-1)a_nx^{n-3} + (n-1)na_nx^{n-2}$

and $p''(0) = f''(0) \Rightarrow 2a_2 = f''(0)$ so $a_2 = \frac{f''(0)}{2} = \frac{f''(0)}{2!}$

One more

For $a_3$: $p'''(x) = 2\cdot3a_3 + \cdots + (n-3)(n-2)(n-1)a_nx^{n-4} + (n-2)(n-1)na_nx^{n-3}$

$p'''(0) = f'''(0) \Rightarrow 2\cdot3a_3 = f'''(0)$

so $a_3 = \frac{f'''(0)}{2\cdot3} = \frac{f'''(0)}{3!}$

Continuing on in this fashion

$$a_i = \cancel{\frac{f^{(i)}(0)}{i!}} = \frac{f^{(i)}(0)}{i!}$$

$$\text{so } f(x) \approx \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!} x^i$$

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Notes The book uses "c" where I used "a".

If $n=2$, $f(x) \approx \underbrace{f(0) + \frac{f'(0)}{1!}x}_{\text{tangent line}} + \frac{f''(0)}{2!}x^2$

This is the best quadratic approximation of $f$ near $x=0$

If $n=3$, $f(x) \approx f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)x^3}{3!}$

This is the best cubic approximation of $f$ near $x=0$.

Notes The book does approximations of $f$ near $x=a$.
I decided to do the simpler case of $x=0$ first.
The ideas are the same.