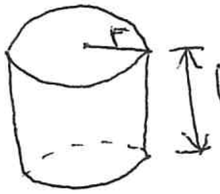
Section 8.2 Area of a surface of revolution

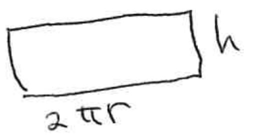
Quiz Thursday Feb 6th

1. Find an antiderivative using partial fractions (7.4)
   hint - synthetic division will be handy
2. Finding an arclength (8.1)
3. An improper integral of type I (7.8)
4. " " " " " II (7.8)

Start with, for a circular cylinder radius r, height h



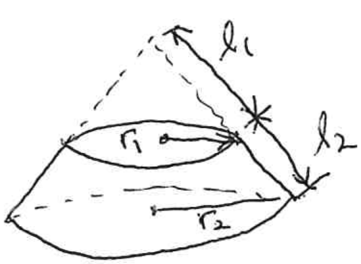
The lateral (side) area is



$$A = 2\pi r h$$

For a circular cone, radius r, slant height l

$$A = \frac{1}{2} l^2 \left(\frac{2\pi r}{l} \right) = \pi r l$$



$$A = \pi r_2 (l_1 + l_2) - \pi r_1 l_1$$
$$= \pi \left[(r_2 - r_1) l_1 + r_2 l_2 \right]$$

By similar triangles: $\frac{l_1}{r_1} = \frac{l_1 + l_2}{r_2}$

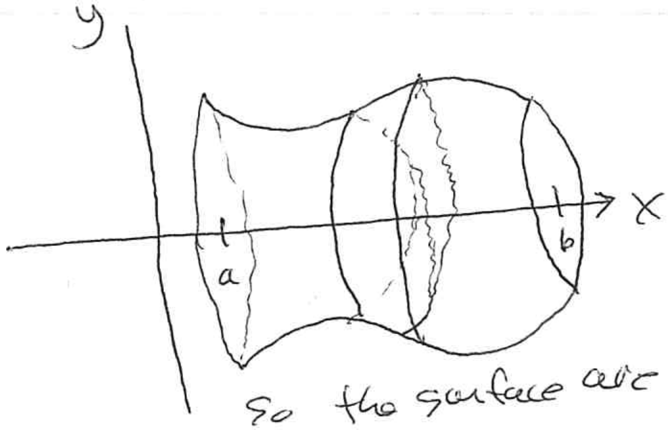
so $r_2 l_1 = r_1 l_1 + r_1 l_2$

or $(r_2 - r_1) l_1 = r_1 l_2$

$$A = \pi (r_1 l_2 + r_2 l_2)$$

Set $r = \frac{1}{2}(r_1 + r_2)$

$$A = 2\pi r l_2$$



The length of one band is the slant height: $l = |P_{i-1}P_i|$

average radius is $r = \frac{1}{2}(y_{i-1} + y_i)$

So the surface area is $\approx 2\pi \left[\frac{y_{i-1}+y_i}{2}\right] |P_{i-1}P_i|$

and $|P_{i-1}P_i| = \sqrt{1+(f'(x_i^*))^2}\ \Delta x$ arclength

If Δx is small: $y_i \approx f(x_i) \approx f(x_i^*)$, so

likewise $y_{i-1} = f(x_{i-1}) \approx f(x_i^*)$

$$2\pi\left[\frac{y_{i-1}+y_i}{2}\right]|P_{i-1}P_i| \approx 2\pi f(x_i^*)\sqrt{1+(f'(x_i^*))^2}\ \Delta x$$

So, the total surface area is

$$\approx \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+(f'(x_i^*))^2}\ \Delta x$$

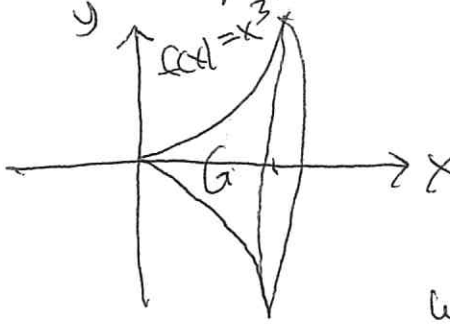
So, the exact surface area is

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+[f'(x_i^*)]^2}\ \Delta x$$

i.e. $$\boxed{S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\, dx}$$

$$S = \int_a^b (\text{circumference})(\text{arclength})$$

Ex Find the area of the surface formed by revolving the graph of $f(x) = x^3$ on the interval $[0,1]$ about the x-axis.



Solution

$$S.A. = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx$$

with $f(x) = x^3$

$f'(x) = 3x^2$, $(f'(x))^2 = (3x^2)^2 = 9x^4$

So $S.A. = 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4}\, dx$

let $u = 1 + 9x^4$

$du = 36x^3\, dx$

$$S.A. = \frac{2\pi}{36} \int_{x=0}^{x=1} (1 + 9x^4)^{\frac{1}{2}} 36x^3\, dx$$

$$= \frac{\pi}{18} \int_{u=1}^{u=10} u^{\frac{1}{2}}\, du = \frac{\pi}{18} u^{3/2}\left(\frac{2}{3}\right) \Big|_{u=1}^{u=10}$$

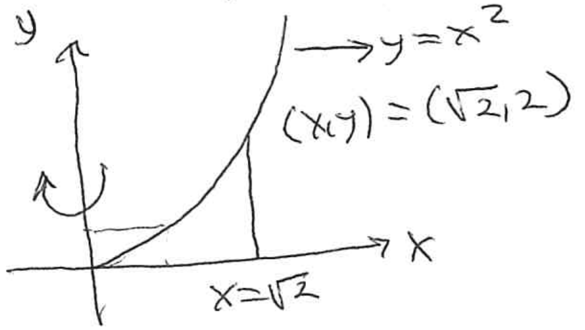
$$= \frac{\pi}{27}\left(10^{3/2} - 1^{3/2}\right)$$

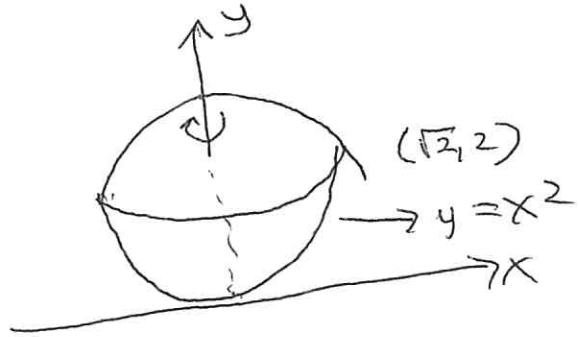
$$= \frac{\pi}{27}\left(10^{3/2} - 1\right)$$

$$\approx 3.563$$

Note the radius was the distance from the x-axis, so $r = f(x)$

Ex Find the area of the surface formed by revolving
The graph of $f(x) = x^2$ on x in $[0, \sqrt{2}]$ about the y-axis





$$S = 2\pi \int_a^b r(x) \sqrt{1 + (f'(x))^2}\, dx$$

$f(x) = x^2$, $f'(x) = 2x$, $(f'(x))^2 = (2x)^2 = 4x^2$

Now the radius is now given by the distance from the y-axis, i.e. radius is x

$$S = 2\pi \int_{x=0}^{x=\sqrt{2}} x\sqrt{1 + 4x^2}\, dx$$

Let $u = 1 + 4x^2$
$du = 8x\, dx$

$$S = \frac{2\pi}{8} \int_0^{\sqrt{2}} 8x\sqrt{1 + 4x^2}\, dx$$

$$= \frac{\pi}{4} \int_{u=1}^{u=9} u^{\frac{1}{2}}\, du = \frac{\pi}{4} u^{3/2} \frac{2}{3} \Big|_{u=1}^{u=9}$$

$$= \frac{\pi}{6} \left[u^{3/2}\right]_1^9 = \frac{\pi}{6} \left[(9^{\frac{1}{2}})^3 - 1^{3/2}\right]$$

$$= \frac{\pi}{6} \left[3^3 - 1\right] = \frac{\pi}{6} \left[27 - 1\right]$$

$$= \frac{\pi}{6} (26) = \frac{13\pi}{3} \approx 13.614$$