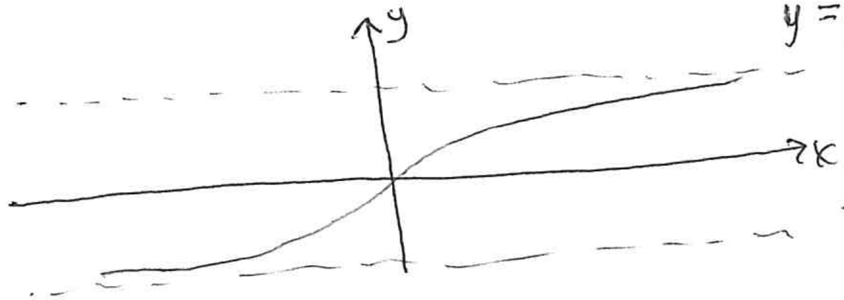
Important Example

Find $I = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

$$I = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

now $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x$



$$I = \lim_{t\to-\infty} \tan^{-1}x\Big]_{x=t}^{x=0} + \lim_{v\to+\infty} \tan^{-1}x\Big]_{x=0}^{x=v}$$

$$I = \left(\cancel{\tan^{-1}0} - \lim_{t\to-\infty}\tan^{-1}t\right) + \left(\lim_{v\to\infty}\tan^{-1}v - \cancel{\tan^{-1}0}\right)$$

$$= -\lim_{t\to-\infty}\tan^{-1}t + \lim_{v\to\infty}\tan^{-1}v$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi$$

For what values of p, $p>0$ does $\int_1^\infty \frac{1}{x^p}dx$ converge?

<u>Solution</u> We need $\int_1^\infty x^{-p}dx$ to be a finite number.

Compute $\lim_{t\to\infty}\int_1^t x^{-p}dx = \lim_{t\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right]_{x=1}^{x=t}$, needs to be finite

$$= \lim_{t\to\infty} \frac{t^{-p+1}}{-p+1} + \frac{1}{-p+1}$$

$$= \frac{1}{-p+1}\left[t^{1-p} + 1\right] = \frac{1}{p-1}\lim_{t\to\infty}\left[\frac{1}{t^{p-1}} - 1\right]$$

Now, if $p>1$, then $p-1>0$,

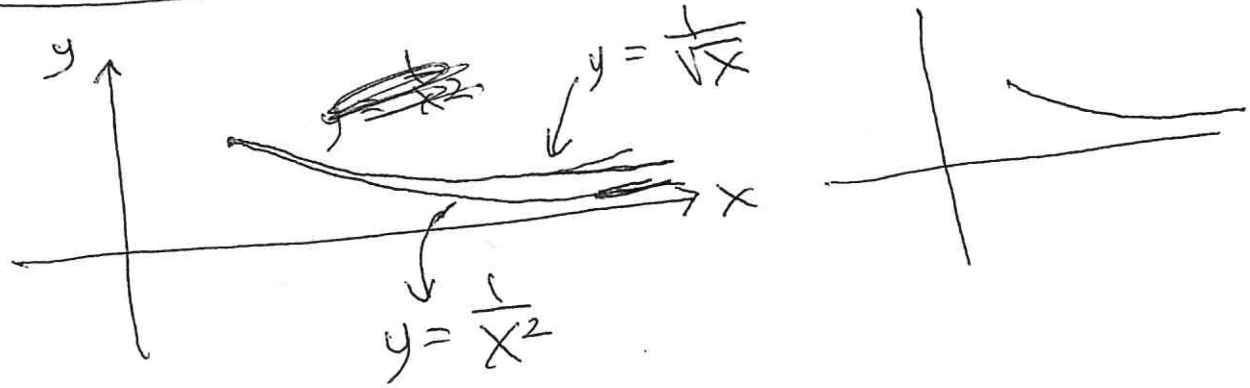
so, as $t\to\infty$, $t^{p-1}\to\infty$ and $\frac{1}{t^{p-1}}\to 0$

so $\int_1^\infty \frac{1}{x^p}dx = \frac{1}{p-1}$, if $p>1$, so converges

But if $p<1$, then $p-1<0$

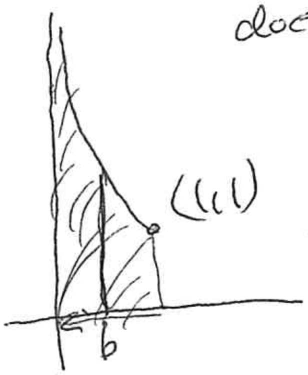
and $\frac{1}{t^{p-1}} = t^{1-p} \to \infty$ as $t\to\infty$

To sum up $\int_1^\infty \frac{dx}{x^p}$ $\begin{cases} \text{converges to } \frac{1}{p-1} & \text{if } p>1 \\ \text{diverges} & \text{if } 0<p\le 1 \end{cases}$



We now ask, for which values of p, $p > 0$

does $I = \int_0^1 \frac{1}{x^p}\,dx$ converge



Solution: $I = \int_0^1 x^{-p}\,dx = \lim_{b\to 0^+} \int_b^1 x^{-p}\,dx$

$$= \lim_{b\to 0^+} \left. \frac{x^{-p+1}}{-p+1} \right]_b^1$$

$$I = \lim_{b\to 0^+} \left[\frac{1}{1-p} - \frac{b^{1-p}}{1-p}\right] = \frac{1}{1-p} \lim_{b\to 0^+} \left[1 - b^{1-p}\right]$$

$$= \frac{1}{1-p}\left[1 - \lim_{b\to 0^+} b^{1-p}\right]$$

Now if $p > 1$, $b^{1-p} \to \infty$ as $b \to 0$

So $\frac{1}{1-p}\left[1 - b^{1-p}\right] \to \infty$, I diverges

If $0 < p \leq 1$, $\lim_{b\to 0^+} b^{1-p} \to 0$ as $b \to 0$

so, $\frac{1}{1-p}\left[1 - b^{1-p}\right] \to \frac{1}{1-p}\left[1 - 0\right] = \frac{1}{1-p}$

Conclusion

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{1-p} & \text{if } 0 < p < 1 \\ \infty & \text{diverges if } p \geq 1 \end{cases}$$

EX Find $I = \int_{-1}^{2} \frac{dx}{x^3}$

note $f(x) = \frac{1}{x^3}$ has a discontinuity, (actually a vertical asymptote) at $x=0$

So $I = \int_{-1}^{0} \frac{dx}{x^3} + \int_{0}^{2} \frac{dx}{x^3}$

First find $\int_{0}^{2} \frac{dx}{x^3} = \lim_{b \to 0^+} \int_{b}^{2} x^{-3}\,dx = \lim_{b \to 0^+} \frac{x^{-2}}{-2} \Big|_{x=b}^{x=2}$

$= \frac{2^{-2}}{2} + \frac{1}{2} \lim_{b \to 0^+} \frac{1}{b^2} = \infty.$

So $\int_{-1}^{2} \frac{dx}{x^3}\,dx$ diverges

no need to find $\int_{-1}^{0} \frac{dx}{x^3}$

If we did not see that $f(x) = \frac{1}{x^3}$ is discontinuous in $[-1,2]$
we could easily get a wrong answer.

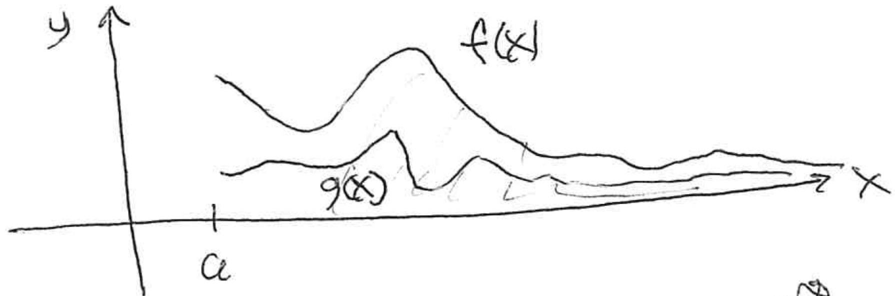
$$\int_{-1}^{2} \frac{dx}{dx} = \int_{-1}^{2} x^{-3}\,dx = \frac{-1}{2x^2} \Big|_{x=-1}^{x=2} = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}$$

this is an incorrect answer.

Comparison Theorem

Let $f(x)$, $g(x)$ be continuous functions on $[a,\infty)$

Let $f(x) \geq g(x) \geq 0$ on $[a,\infty)$



a) If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ also converges

b) If $\int_a^\infty g(x)\,dx$ ~~converges~~ diverges, then $\int_a^\infty f(x)\,dx$ diverges

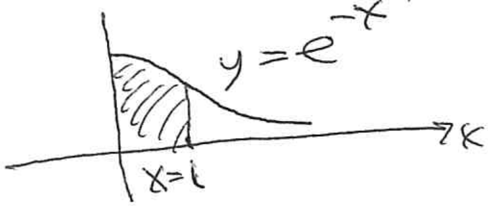
Note This thm is an extension of the comparison from calc I. If $f(x)$, $g(x)$ are continuous on closed interval $[a,b]$ ~~hence~~ $f(x) \geq g(x)$, $\forall x$ in $[a,b]$

then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

Aside If $\int_a^\infty f(x)\,dx$ diverges, no conclusion about $\int_a^\infty g(x)\,dx$

If $\int_a^\infty g(x)\,dx$ converges, no conclusion about $\int_a^\infty f(x)\,dx$

EX Show that $\int_0^\infty e^{-x^2}dx$ converges



First note that $f(x)=e^{-x^2}$ does not have an antiderivative

i.e. there does not exist a function F, s.t. $F'=f$

So, we have to be clever.

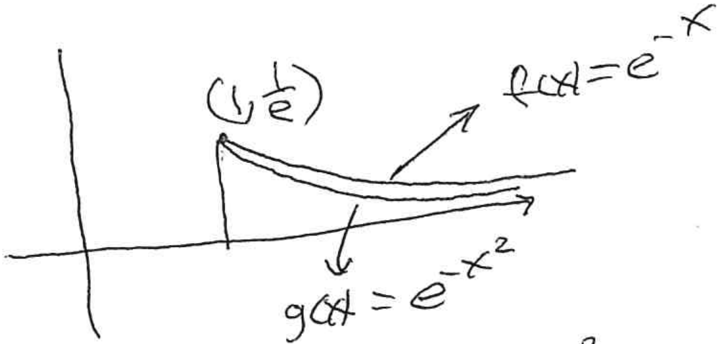
We ~~compare~~ first note

$$\int_0^\infty e^{-x^2}dx = \underbrace{\int_0^1 e^{-x^2}dx}_{\text{finite number}} + \int_1^\infty e^{-x^2}dx$$

Want to show that $\int_1^\infty e^{-x^2}dx$ converges

Compare $\int_1^\infty e^{-x^2}dx$ with $\int_1^\infty e^{-x}dx$

for $x\geq 1$, $x^2\geq x$, so $-x^2\leq -x$, so $e^{-x^2}\leq e^{-x}$



So $\int_1^\infty e^{-x}dx \geq \int_1^\infty e^{-x^2}dx$

We now show that $\int_1^\infty e^{-x}dx$ converges

$$\int_1^\infty e^{-x}\,dx = -e^{-x}\Big]_1^\infty$$

$$= -\lim_{b\to\infty} e^{-b} - (-e^{-1})$$

$$= 0 + \frac{1}{e}$$

$$\int_1^\infty e^{-x}\,dx = \frac{1}{e}$$

Hence $\int_1^\infty e^{-x^2}\,dx$ converges.

and $0 < \int_1^\infty e^{-x^2}\,dx < \frac{1}{e}$