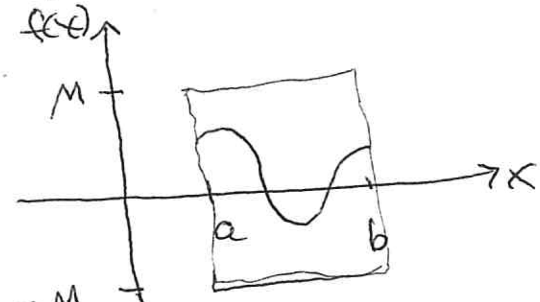
Section 7.8 Improper Integrals

Up to now, we have only dealt with

$\int_a^b f(x)\,dx$, where a and b are real numbers and $f(x)$ is bounded on $[a,b]$

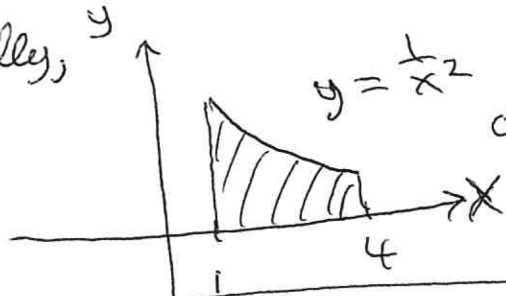
i.e. $\exists$ M s.t. $-M \le f(x) \le M$ $\forall x$ in $[a,b]$

there is / real number / for all



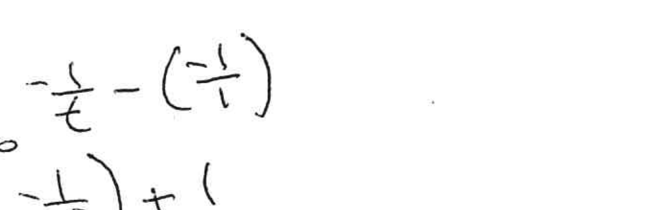
Ex $\int_1^4 \frac{1}{x^2}dx = \int_1^4 x^{-2}dx = \frac{x^{-1}}{-1}\Big]_{x=1}^{x=4} = \frac{-1}{x}\Big|_1^4 = -\frac{1}{4} - (-1) = \frac{-1}{4} - 1 = \frac{3}{4}$

Geometrically,



$y = \frac{1}{x^2}$ area is $\frac{3}{4}$

Ask what is $\int_1^\infty \frac{1}{x^2}dx$



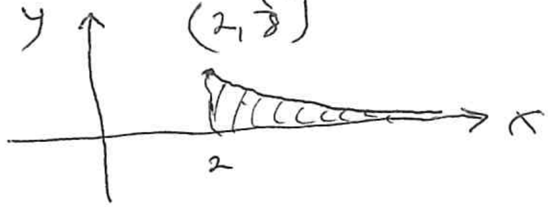
Def $\int_1^\infty \frac{1}{x^2}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}dx$

$= \lim_{t\to\infty} \frac{-1}{x}\Big|_{x=1}^{x=t} = \lim_{t\to\infty} \frac{-1}{t} - \left(\frac{-1}{1}\right)$

$= \left(\lim_{t\to\infty} \frac{-1}{t}\right) + 1$

$= 0 + 1 = 1$

Ex $\displaystyle\int_2^\infty \frac{1}{x^3}\,dx$



Note $\displaystyle\int_2^\infty \frac{1}{x^3}\,dx$

$$= \lim_{t\to\infty} \int_2^t \frac{1}{x^3}\,dx$$

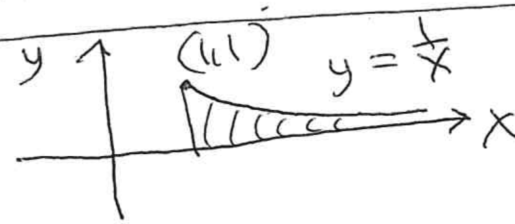
now $\displaystyle\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-2}}{-2} = \frac{-1}{2x^2}$

$$\cancel{= \lim_{t\to\infty} \frac{1}{t^3}} = \lim_{t\to\infty} \left.\frac{-1}{2x^2}\right|_{x=2}^{x=t}$$

$$= \lim_{t\to\infty} \frac{-1}{2t^2} - \left(\frac{-1}{2\cdot 2^2}\right)$$

$$= 0 + \frac{1}{8} = \frac{1}{8}$$

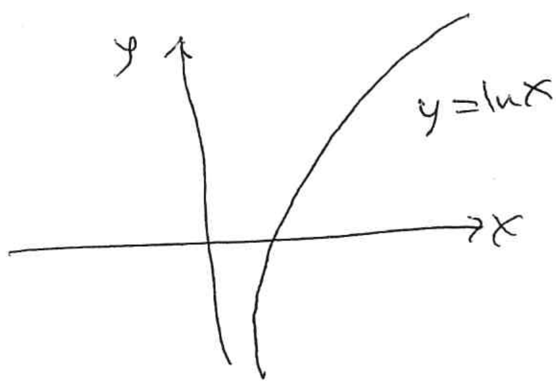
Ex $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

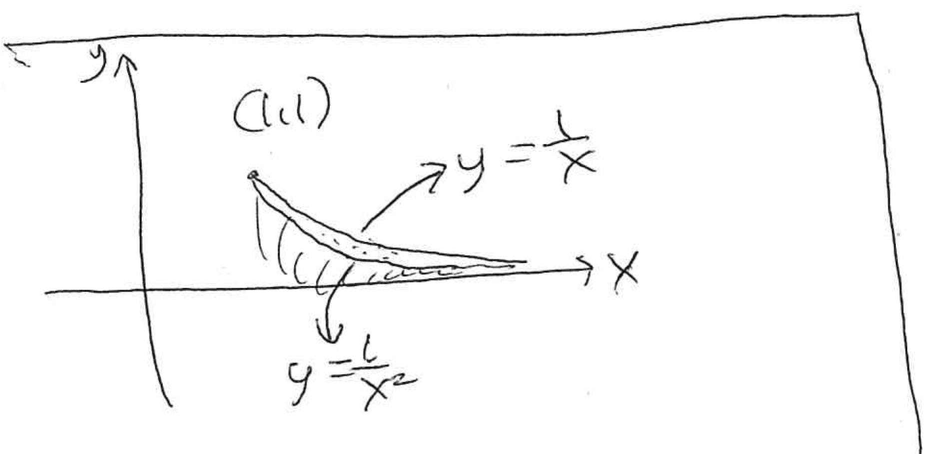


$$= \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \ln(x)\Big|_{x=1}^{x=t} = \cancel{\lim_{t\to\infty} \ln(x}$$

$$= \lim_{t\to\infty} \ln(t) - \ln 1 = \lim_{t\to\infty} \ln(t) - 0 = \lim_{t\to\infty} \ln(t) = +\infty$$





Def Improper Integrals of Type I,

a) If $\int_a^t f(x)dx$ exists for every real number $t \geq a$

then $\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$

provided that this limit is a finite number

b). If $\int_t^b f(x)dx$ exists for all $t \leq b$, then

$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$

provided that this limit is a real number.

Note If the improper integral is a real number, we say that the improper integral converges to that real number

We say that an improper integral that does not converge, diverges.

Indefinite integral

$\int f(x)dx$ is a function $F(x)$ s.t. $F'(x) = f(x)$

Definite integral

$\int_a^b f(x)dx$ is the signed area between $y = f(x)$ and the x-axis on the interval $a \leq x \leq b$

Improper Integral given above will be a limit of definite integrals

Another improper integral of type 1

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

The LHS exists iff both improper integrals on the RHS exists i.e. are finite

<u>Caution</u> In general $\int_{-\infty}^{\infty} f(x)\,dx \neq \lim_{t\to\infty} \int_{-t}^{t} f(x)\,dx$

<u>Non Ex</u> $\int_{-\infty}^{\infty} \frac{1}{x}\,dx \neq \lim_{t\to\infty} \int_{-t}^{t} \frac{1}{x}\,dx$

$$= \lim_{t\to\infty} \ln|x| \Big]_{x=-t}^{x=t}$$

$$= \lim_{t\to\infty} \ln|t| - \lim_{t\to -\infty} \ln|-t|$$

Note $\ln|t| - \ln|-t| = 0$

$$= \lim_{t\to 0} 0 = 0$$

But we just saw that $\int_{1}^{\infty} \frac{1}{x}\,dx = +\infty$

<u>Non Ex</u> $f(x) = \frac{1}{x^2}$

If we just considered

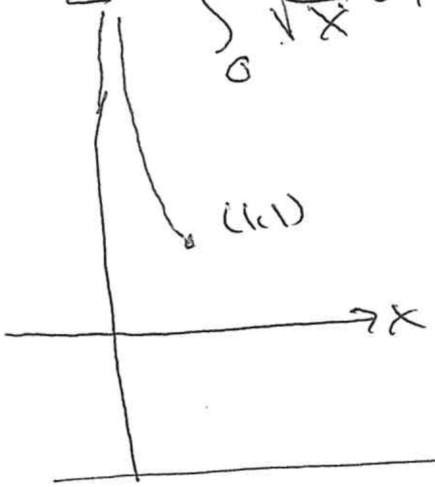
$$\int_{-\infty}^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_{-t}^{t} \frac{1}{x^2}\,dx$$

we miss the vertical asymptote at $x=0$

Improper Integral of Type 2

Basically when we have a vertical asymptote

Ex $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 x^{-\frac{1}{2}}\,dx = \lim_{b \to 0^+} \int_b^1 x^{-\frac{1}{2}}\,dx$



$= \lim_{b \to 0^+} 2x^{\frac{1}{2}} \Big|_{x=b}^{x=1}$

$= 2\left(\lim_{b \to 0^+} \sqrt{1} - \lim_{b \to 0} \sqrt{b}\right) = 2(\sqrt{1} - 0) = 2$

Ex $\int_0^1 \frac{1}{x^2}\,dx = \lim_{b \to 0^+} \int_b^1 x^{-2}\,dx$, now $\int x^{-2}\,dx = -\frac{1}{x}$

$= \lim_{b \to 0^+} -\frac{1}{x} \Big|_{x=b}^{x=1} = -\frac{1}{1} - \lim_{b \to 0^+}\left(-\frac{1}{x}\right)$

$= -1 + \lim_{b \to 0^+}\left(\frac{1}{x}\right)$

$= -1 + \infty = \infty$

so $\int_0^1 \frac{1}{x^2}\,dx$ diverges

Def of Improper Integral of Type 2

a) If f is contin on $[a, b)$ and is discontinuous at b, then

$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$

b) If f is contin on $(a, b]$, and is discontinu at a, then

$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$