Section 7.1 Integration by parts

Integration by parts comes from the product rule.

$$\frac{d}{dx} f(x)g(x) = f(x)g'(x) + g(x)f'(x)$$

$$\int \frac{d}{dx} f(x)g(x)\,dx = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx$$

$$f(x)g(x)\,dx = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx$$

rearrange terms

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx}$$

Letting $u = f(x)$, $v = g(x)$

The formula for integration by parts is

$$\boxed{\int u\,dv = uv - \int v\,du}$$

<u>Ex</u> $I = \int x\cos x\,dx$

<u>Solution</u> $u = x$ $\quad dv = \cos x\,dx$

$du = dx$ $\quad v = \sin x$

$$I = x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + C$$

<u>Check</u> $\frac{d}{dx}\left[x\sin x + \cos x + C\right]$

$= \sin x + x\cos x - \sin + 0$

$= x\cos x$

Ex $I = \int \ln x \, dx = \int (\ln x)(1)\, dx$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}\, dx \qquad v = x$

$I = x\ln x - \int (\frac{1}{x}) x\, dx = x\ln x - \int dx = x\ln x - x + C$

Ex $I = \int x e^x \, dx$

$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

$I = xe^x - \int e^x dx = xe^x - e^x + C$

Ex $I = \int x^2 e^x \, dx$

$u = x^2 \qquad dv = e^x dx$

$du = 2x\, dx \qquad v = e^x$

$I = x^2 e^x - 2\int x e^x dx$

Now use Int by parts a second time.

This was done above

$I = x^2 e^x - 2\left[xe^x - e^x + C_1\right]$

$I = x^2 e^x - 2xe^x + 2e^x + C$, where $C = -2C_1$

Ex Find $\int_0^1 \tan^{-1} x\,dx$

Solution first find $I = \int \tan^{-1} x\,dx$

$u = \tan^{-1} x \qquad dv = dx$

$du = \frac{dx}{1+x^2} \qquad v = x$

So $I = x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx$

Now find $\int \frac{x}{1+x^2}\,dx$

Let $t = 1+x^2$

$dt = 2x\,dx$

$\int \frac{x}{1+x^2}dx = \frac{1}{2}\int \frac{2x\,dx}{1+x^2}$

$= \frac{1}{2}\int \frac{dt}{t} = \ln|t| = \frac{1}{2}\ln(1+x^2)$

So $\int \tan^{-1} \cancel{x\,dx} = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$

So $\int_0^1 \tan^{-1} x\,dx = (x\tan^{-1} x)\Big|_{x=0}^{x=1} - \frac{1}{2}\ln(1+x^2)\Big|_{x=0}^{x=1}$

$= \left[1\left(\frac{\pi}{4}\right) - 0(0)\right] \cancel{+} \frac{1}{2}\left[\ln(1+1) - \ln(1+0)\right]$

$= \left(\frac{\pi}{4} - 0\right) \cancel{+} \frac{1}{2}\left[(\ln 2) - 0\right]$

$= \frac{\pi}{4} - \frac{\ln 2}{2}$

Ex Find $I_1 = \int e^x \sin x\,dx$

$u = \sin x \qquad dv = e^x dx$
$du = \cos x\,dx \qquad v = e^x$

So $I_1 = e^x \sin x - \int e^x \cos x\,dx$

Now find $I_2 = \int e^x \cos x\,dx$

$u = \cos x \qquad dv = e^x dx$
$du = -\sin x\,dx \qquad v = e^x$

$I_2 = e^x \cos x + \int e^x \sin x\,dx$

So $I_1 = \int e^x \sin x\,dx = e^x \sin x - \left(e^x \cos x + \int e^x \sin x\,dx\right)$

$$\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$$
$$+\int e^x \sin x\,dx \qquad\qquad +\int e^x \sin x\,dx$$

$$2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x + C_1$$

So $\int e^x \sin x\,dx = \frac{1}{2}\left[e^x \sin x - e^x \cos x + C_1\right]$

Check $\frac{d}{dx}\left[\frac{1}{2}(e^x \sin x - e^x \cos x)\right]$

$$= \frac{1}{2}\left[\cancel{e^x \cos x} + e^x \sin x - \cancel{e^x \cos x} + e^x \sin x + 0\right]$$

$$= \frac{1}{2}\left[e^x \sin x + e^x \sin x\right]$$

$$= \frac{1}{2}\left[2e^x \sin x\right]$$

$$= e^x \sin x$$