If f has a power series expansion centered at $x=a$,
then the power series is the Taylor Series of f centered at $x=a$

and $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

The special case $a=0$, is so important that it gets its own name - The Maclaurin series.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

If we truncate the Taylor series for f about a, after $n+1$ ~~xxx~~ terms, (i.e. after the n'th power) we get the n'th-degree Taylor polynomial of f at a.
The n'th-degree MacLaurin polynomial is defined the same way.

There are functions whose Taylor series does not converge and functions $f(x)$ whose Taylor series converges but does not converge to $f(x)$. We will not be concerned with those functions.

How can we tell when a function $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$
in " " " " " " equals its Taylor series

<u>If</u> $f(x)$ equals its Taylor series
and $T_n(x)$ is the n'th - degree Taylor polynomial of f at a.

Then $f(x) = \lim_{n\to\infty} T_n(x)$

Let $R_n(x)$ be the remainder of the taylor series.

I.E. $R_n(x) = f(x) - T_n(x)$

so that $f(x) = T_n(x) + R_n(x)$

Think of $R_n(x)$ as the error when approximating $f(x)$ by $T_n(x)$

Ex Let $f(x) = e^x$. Consider the MacLaurin series

$$e^x = \underbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}}_{T_3(x)} + \underbrace{\frac{x^4}{4!} + \frac{x^5}{5!} + \cdots}_{R_3(x)}$$

If we show that $\lim_{n\to\infty} R_n(x) = 0$, then

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty}\left[f(x) - R_n(x)\right] = f(x) - \lim_{n\to\infty} R_n(x) = f(x)$$

So $\lim_{n\to\infty} R_n(x) = 0$

In use the above we often use

[9] Taylor's Inequality: If $|f^{(n+1)}(x)| < M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor Series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a| \leq d$$

The above is the books version. I prefer an equivalent but stated somewhat differently

<u>Thm</u> If f can be differentiated $n+1$ times on an interval I containing the number x_0, and if M is an upper bound for $|f^{(n+1)}(x)|$ on I. ie. $|f^{(n+1)}(x)| \leq M$ for all x in I, then

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-x_0|^{n+1} \quad \text{for all } x \text{ in } I.$$

I will now give you an application. It is rather involved. Don't stress over it. It is here if you want it.

<u>Ex</u> Use an n'th degree MacLaurin polynomial for e^x to approximate e to five decimal places.

<u>Solution</u> all derivatives of e^x equal e^x

i.e. if $f(x) = e^x$ $f^{(i)}(x) = e^x$ for all (i), gotta luv e^x ☺

Now, The n'th MacLaurin polynomial for e^x is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

$$\text{so } e = e^1 \approx \sum_{k=0}^{n} \frac{1^k}{k!} = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

But what should n equal to get 5 digit accuracy

What is the 1st n s.t. $|R_n(1)| \leq 0.000005$

<u>Note</u> 5 zeros because we want to allow for rounding.

We use the remainder Thm with $f(x) = e^x$, $x=1$, $x_0 = 0$
and I the interval $[0,1]$

so $|R_n(1)| \le \frac{M}{(n+1)!}$

where M is an upper bound on the values of $f^{(n+1)}(x) = e^x$
for x in $[0,1]$. Now f is an increasing function, so its
max on the interval $[0,1]$ is at $x=1$. Let $M = e^1 = e$

so $|R_n(1)| \le \frac{e}{(n+1)!}$

well, this is not so great. Our estimate involves e, which is
what we are trying to find.
But we know that $e<3$. So use 3 in place of e.
(or any number that you know is bigger than e)

So we want: $\frac{3}{(n+1)!} \le 0.000005$

or $(n+1)! \ge 600{,}000$

Now $9! = 362{,}880$
$10! = 3{,}628{,}800$.

so use $n = 9$.

$e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{9!} \approx 2.71828$

(a computer check gives $e \approx 2.718281846$)

so pretty good

The Examples of finding Taylor series for functions are all pretty standard: $f(x) = \sin x$, $f(x) = \cos x$,

$f(x) = (1+x)^k$, k any real number

These examples are in pretty much every calculus book. Rather than copy them here, please read pg 763 - 765 in the textbook.

Email, or call, me if you have any questions.

I like how they find the Maclaurin series for $\cos x$. They first show that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Now we know that $\frac{d}{dx} \sin x = \cos x$

so The MacLaurin series for $\cos x$ = Maclaurin series for $\frac{d}{dx} \sin x$

so, the MacLaurin series for ~~sin x~~

$$\cos x = \frac{d}{dx}(\sin x)$$

$$= \cancel{\frac{d}{dx}(\cos}$$

$$\frac{d}{dx}(\sin x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

The binomial series was developed by Newton
(O.K, he was not a very nice person, but what a genius)

The book develops on pg 766 the Maclaurin Series
for $f(x) = (1+x)^k$. This particular series is the
MacLaurin Series:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

The binomial series converges for $|x|<1$.
The convergence at $x=-1$ or $x=+1$, depends on the value of k

Suppose $k=2$, then you already know that

$$(1+x)^2 = 1+2x+x^2,$$

(the binomial expansion)

The binomial series for $k=2$ is

$$1+2x+\frac{2(2-1)}{2!}x^2 + \frac{2(2-1)(2-2)}{3!}x^3 + \frac{2(2-1)(2-2)(2-3)}{3!}x^3 +$$

note $2-2=0$, so $=0$

so all terms for x^3 or higher contain a zero, so

$$(1+x)^2 = 1+2x+x^2,$$

The binomial series is a vast expansion of binomial expansion.

The binomial theorem

The book shows how to use the binomial theorem to expand out $\frac{1}{\sqrt{4-x}}$ in a Maclaurin Series.

Let me do a different example

022 Sec 11.10

Find a power series for $f(x) = \sqrt[3]{1+x}$

<u>Solution</u> Using the binomial series

$$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots$$

with $k = \frac{1}{3}$, we write

$$\cancel{(1+x)^{\frac{1}{3}} = 1 + (\tfrac{1}{3})x + \frac{\frac{1}{3}(1-\frac{1}{3})x^2}{2!} + \tfrac{1}{3}(1-\tfrac{1}{}}$$

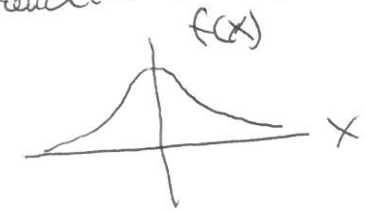
$$(1+x)^{\frac{1}{3}} = 1 + (\tfrac{1}{3})x + \frac{\frac{1}{3}(\frac{1}{3}-1)x^2}{2!} + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)x^3}{3!}$$

$$(1+x)^{\frac{1}{3}} = 1 + \frac{x}{3} + \frac{2x^2}{3^2 2!} + \frac{2 \cdot 5 x^3}{3^3 3!} - \frac{2 \cdot 5 \cdot 8 x^4}{3^4 4!},$$ which

converges for $-1 \le x \le 1$.

Approximating a definite integral by using a power series

The function $f(x) = e^{-x^2}$ is perhaps the most important function in probability and statistics. It is a bell-shaped curve



(Because of the Central Limit Theorem all sampling distributions tend towards a bell shape curve

This was an aside if you know some statistics)

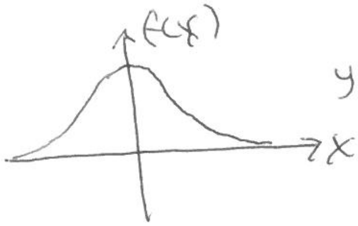
However, there is no function $F(x)$ s.t.

$$F'(x) = e^{-x^2}$$

So you can not use the Fundamental Theorem of Calculus to find areas under the curve.

But find areas under a bell-shape curve is <u>very</u> important.

Here is what you do.



$y = e^{-x^2}$

We use a power series to approximate

$\int_0^1 e^{-x^2} dx$ with an error of less than 0.01

<u>Solution</u> We need the power series for e^{-x^2}.

We need the power series for e^x

To do this, simply take the power series for e^x and replace x by $-x^2$

So $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$

So $\int_0^1 e^{-x^2} dx = \left[x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \cdots \right]_0^1$

we just integrated term by term.

using the first four terms

$\left[x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} \right]_0^1 = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} = 0.74$

By the alternating series test, the above estimate of at most $|a_{n+1}| = \left| \frac{1}{9 \cdot 4!} \right| = \left| \frac{1}{216} \right| \approx 0.005$

So, to an accuracy of at most 0.005

~~$\int_0 e^{x^2}$~~ $\int_0^1 e^{-x^2} dx \approx 0.74$

We can multiply and divide power series.

see pages 769-771

A Venn Diagram for you