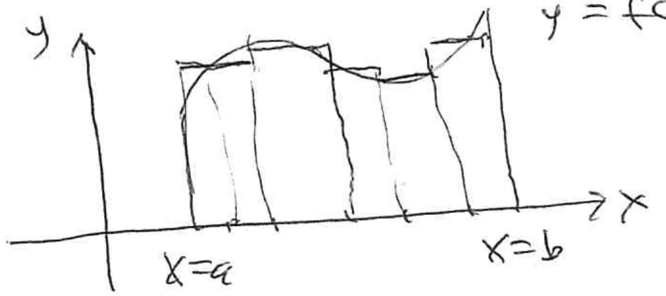
Section 7.7 Approximate Integration.

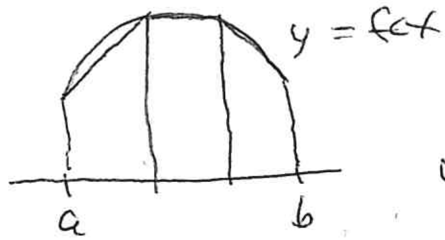


$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right)\right]$$

The trapezoidal Rule

Let f be continuous on $[a,b]$. The trapezoidal rule for approximating $\int_a^b f(x)\,dx$ is given by

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right]$$



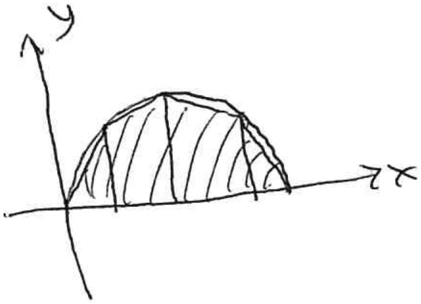
Ex Approximate $\int_0^{\pi} \sin x\,dx$ using $n=4$ trapezoids $\frac{b-a}{2n} = \frac{\pi-0}{2(4)} = \frac{\pi}{8}$

$$\int_0^{\pi} \sin x\,dx = \frac{\pi}{8}\left[\sin 0 + 2\sin\frac{\pi}{4} + 2\sin\frac{\pi}{2} + 2\sin\frac{3\pi}{4} + \sin\pi\right]$$

$$= \frac{\pi}{8}\left[0 + 2\left(\frac{1}{\sqrt{2}}\right) + 2(1) + 2\left(\frac{1}{\sqrt{2}}\right) + 0\right]$$

$$= \frac{\pi}{8}\left[0+\sqrt{2}+2+\sqrt{2}+0\right] = \frac{\pi(1+\sqrt{2})}{4}$$

$$\approx 1.896$$



Note The exact area is

$$\int_0^{\pi} \sin x\,dx = -\cos x \Big|_{x=0}^{x=\pi}$$

$$= -\cos\pi - (-\cos 0)$$

$$= -(-1) - (-1) = 1+1 = 2$$

Error Analysis

If f has a continuous second derivative on $[a,b]$ then the error E in approximating

$\int_a^b f(x)dx$ by the trapezoidal rule is

$$E \le \frac{(b-a)^3}{12n^2}\left[\max|f''(x)|\right], \text{ for } a \le x \le b$$

<u>Aside</u> The Trapezoidal Rule can be written

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n}\left[\frac{f(x_i)-f(x_{i-1})}{2}\right]\Delta x, \quad \Delta x = \frac{b-a}{n}$$

Simpson's Rule - we need the number of subintervals to be even.

We approximate the curve of our function $f(x)$ by a parabola

Let f be continuous on $[a,b]$. then,

$$\int_a^b f(x)dx \approx \frac{b-a}{3n}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+4f(x_{n-1}) + f(x_n)\right]$$

1, 4, 2, 4, 2, ... , 4, 2, 4, 1

<u>Ex</u> Approximate $\int_0^\pi \sin x\,dx$ with $n=4$, using Simpson's Rule

$$\int_0^\pi \sin x\,dx = \frac{\pi-0}{3\cdot 4}\left[\sin(0)+4\sin\left(\frac{\pi}{4}\right)+2\sin\frac{\pi}{2}+4\sin\frac{3\pi}{4}+\sin\pi\right]$$

$$= \frac{\pi}{12}\left[0, 4\left(\frac{1}{\sqrt{2}}\right)+2\cdot 1+4\left(\frac{1}{\sqrt{2}}\right)+0\right]$$

$$= \frac{\pi}{12}\left[4\sqrt{2}+2\right] \approx 2.005$$

<u>Note</u>: Using simpson's rule we got a much better estimate than ~~the~~ when we used the trapezoidal method

Bounds for the error using Simpson's Rule

If f has a continuous fourth derivative on the open interval (a,b), then the error in computing $\int_a^b f(x)dx$ using Simpson's Rule is

$$E \leq \frac{(b-a)^5}{180n^4}\left[\max\left|f^{(4)}(x)\right|\right], \text{ for } x \text{ in } [a,b]$$