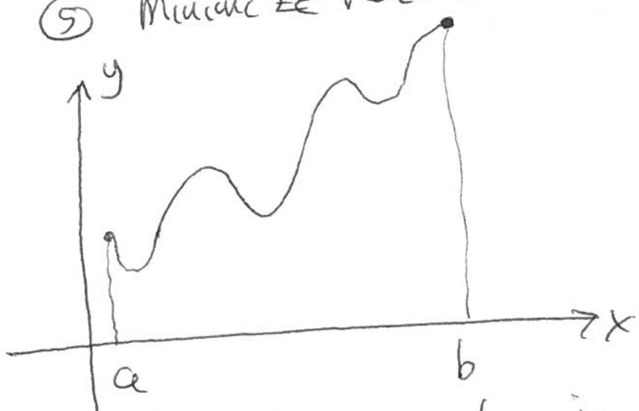
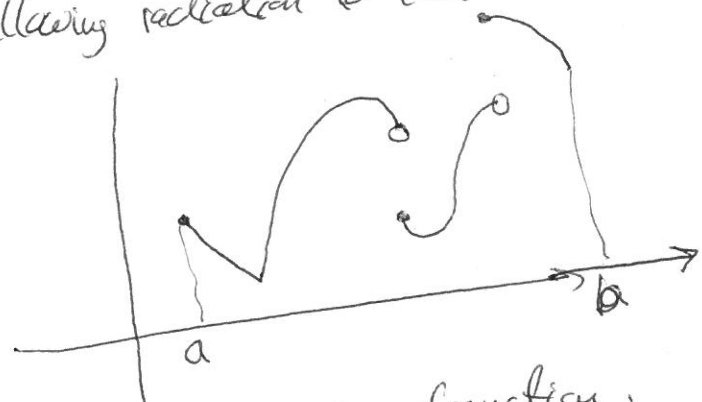
021 Section 4.1

Section 4.1 Maximums and minimums of functions of one variable

Often, one wants to optimize, (maximize or minimize) a quantity

1. Minimize costs
2. Maximize profits
3. Maximize fluid flow
4. Minimize fluid flow
5. Minimize radiation while allowing radiation to kill a tumor.





Def Let c be a number in the domain D of a function.

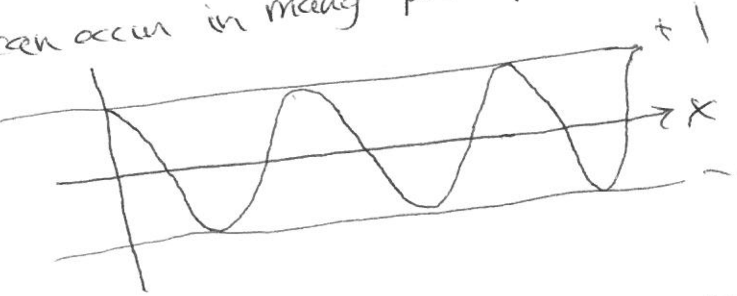
Then $f(c)$ is an

- Absolute, (global) maximum of f on D iff $f(c) \geq f(x)$ for all x in D
- Absolute, (global), minimum " " " " " $f(c) \leq f(x)$ " " " "

Note A function can have 0 or 1 absolute maximums

" " " " " " " minimums.

However, these values can occur in many places.



Ex $y=f(x)=\cos x$ on $[0,6\pi]$

The absolute max is $y=+1$

This occurs at

$x=0, 2\pi, 4\pi, 6\pi$

The absolute minimum is $y=-1$, occurring when $x=\pi, 3\pi, 5\pi$

Ex Let $g(t)=27$ for all t

The absolute max is 27, this occurs at every value of t

" " min " 27 " " " " "

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~~Outline for Test 2, Tues Oct 22, 2019 Bring a Calculator~~

~~① 10 pts total, 4 parts – Finding derivatives.~~

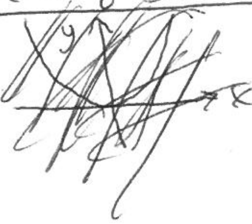
~~② 3 pts log different~~

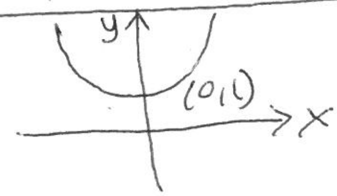
~~③ 4 pts half life problem.~~ ~~① Mi~~

~~④ 4 pts related rates~~

~~⑤ 3 pts Approximation~~

~~⑥ 4 pts total 2 parts – derivatives of hyperbolics, inverse hyperbolics.~~

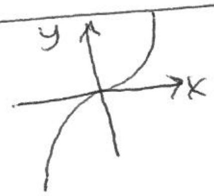




EX $f(x)=x^2+1$ on $(-\infty,\infty)$

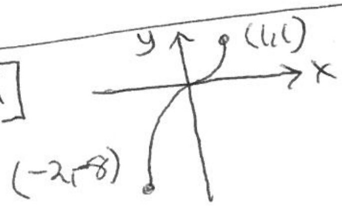
No global max

global min of 1 at $(0,1)$



EX $f(x)=x^3$ on $(-\infty,\infty)$

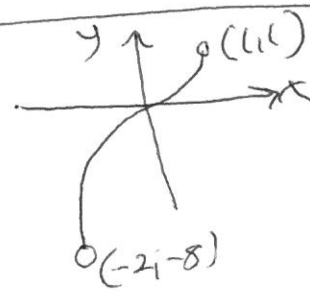
No global maxs, no global mins



EX $f(x)=x^3$ on $[-2,1]$

Global max, $y=1$ at $(x,y)=(1,1)$

Global min $y=-8$ at $(x,y)=(-2,-8)$



EX $f(x)=x^3$ on $(-2,1)$

No global max.

Note The pt $(1,1)$ is not on the graph of the function

because $x=1$ is not in the domain.

Note There is no real number "next to" 1

Likewise no minimum

$f(x)$ is however bounded

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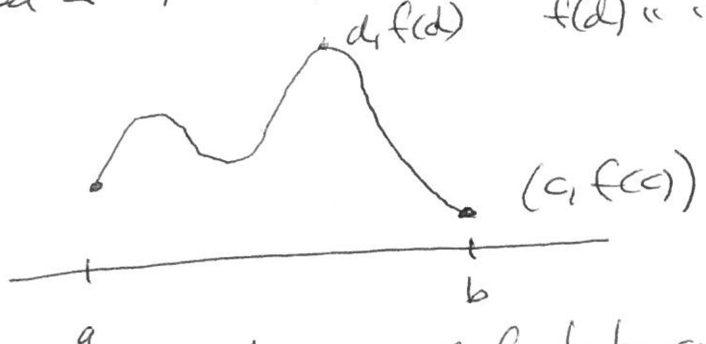
<u>Thm</u> The extreme value theorem

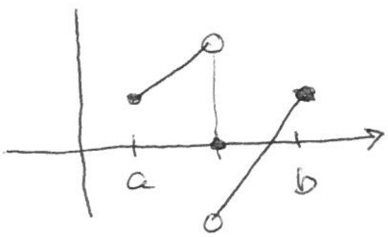
If f is <u>continuous</u> on a <u>closed</u> and <u>bounded</u> interval, $I = [a,b]$

Then $\exists\ c, d \in I$, s.t. $f(c)$ is a global min

$f(d)$ " " " max



<u>Ex</u> to show that we need f to be continuous

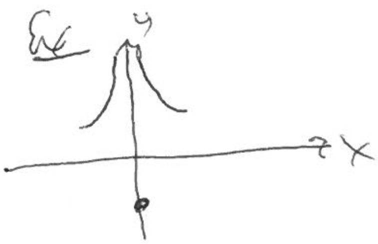


$[a,b]$ is closed and bounded

f is not continuous,

f does not have a global max or a global min

<u>Ex</u>



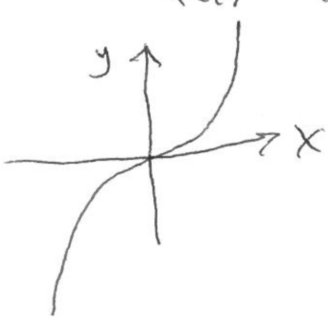
$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ -3 & \text{if } x = 0 \end{cases} \quad \text{on } [-1,1]$$

Again, no extrema.

<u>Ex</u> to show we need $[a,b]$ to be bounded

$f(x) = x^3$ on $(-\infty, \infty)$

closed, unbounded

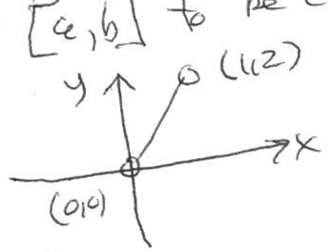


f continuous,

No global extrema

<u>Ex</u> to show we need $[a,b]$ to be closed

$y = f(x)$ on $(0,1)$



No extrema

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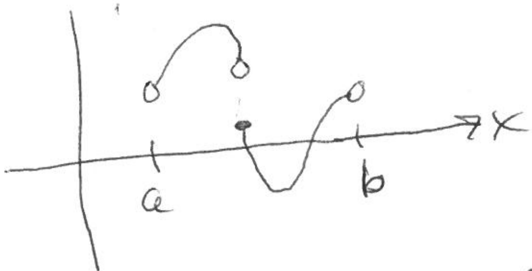
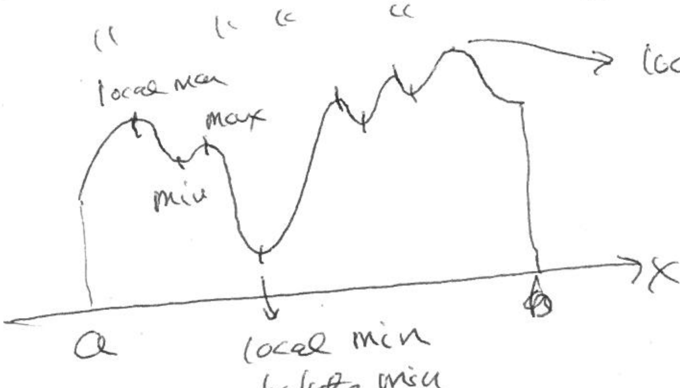
<u>Note</u> Our theorem only guarantees when extrema must exist

<u>EX</u> of a discontinuous function on an open interval that has extrema



<u>Def</u> $f(x)$ has a relative, (local) max at $x=c$, if $f(x) \le f(c)$, $\forall x$ near c

" " " " min at $x=c$, if $f(x) \ge f(c)$, $\forall x$ near c



local min, $\exists\, \varepsilon > 0$ s.t.
if $x \in (c-\varepsilon, c+\varepsilon)$
then $f(x) \ge f(c)$

<u>Note</u> An absolute max is also a relative max
(likewise for min)

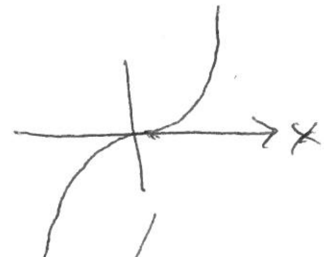
But a rel max need not be a global max
(likewise for min)

<u>Note</u> A local minimum can be greater than a local maximum

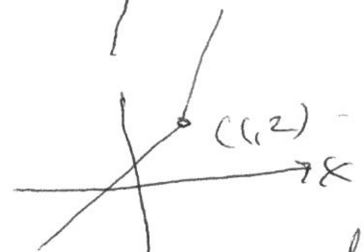
021 Sec 4.1

Fermat's Thm: If $f(x)$ has a relative extrema at $x=c$, then $f'(c)=0$ or $f'(c)$ does not exist

Caution The converse is not true

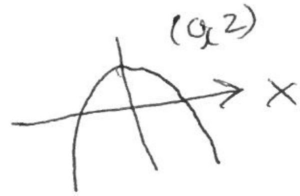


Ex 1 $f(x)=x^3$
$f'(x)=3x^2$, $f'(x)=0$, when $x=0$
But $(0,0)$ is neither a local min nor a local max



Ex 2 $f(x) = \begin{cases} 1+x & \text{if } x<1 \\ 2x & \text{if } x \geq 1 \end{cases}$

at $(x,y)=(1,2)$, we have a corner so $f'(x)$ d.n.e. at $x=1$
but no extrema at $(x,y)=(1,2)$



Exs of Fermat's Thm

Ex 1 $f(x) = -x^2+2$
$f'(x)=-2x$
$f'(x)=0$ when $x=0$, we have a local max when $x=2$

Ex 2 $g(t)=|t|$ 

we have a local min at $t=0$

$g'(t)$ d.n.e. when $t=0$,

Def Let $f(x)$ be defined on a set I, $x=c$ is a critical number for $f(x)$ if $f'(c)=0$ or $f'(c)$ d.n.e.

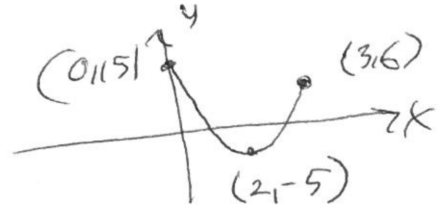
To find extrema for $f(x)$ defined on $I=[a,b]$
1) Find the critical ~~values~~ numbers for $f(x)$ that occur in I
2) Compute f at the critical ~~vals~~ numbers in 1)
3) Compute $f(a)$ and $f(b)$

The largest value of f that occurs in 2) or 3) is the global max
The smallest " " " " " " " " " " global minimum

021 Sec 4.1

EX Find the global extrema for
$f(x)=2x^3-3x^2-12x+15$ on $[0,3]$

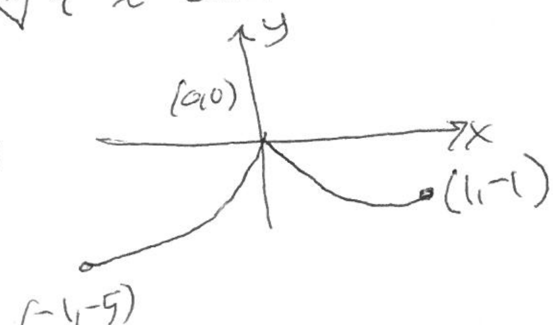
Solution $f'(x)=6x^2-6x-12=6(x-2)(x+1)$
$f'(x)=0$ when $x=-1$, $x=2$,
Now $x=-1$ is not in $[0,3]$, so we exclude $x=-1$,
$f(0)=15 \rightarrow$ global max
$f(2)=-5 \rightarrow$ global min
$f(3)=6$



The graph is not part of a parabola,
but is part of a cubic.

EX Find the extrema of $f(x)=2x-3x^{2/3}$ on $[-1,3]$
Solution $f'(x)=2-\dfrac{2}{x^{\frac{1}{3}}}=2\left(\dfrac{x^{\frac{1}{3}}-1}{x^{1/3}}\right)$
Now $f'(x)=0$ when $x^{\frac{1}{3}}-1=0$ or $x^{1/3}=1$, cube $x=1$
$f'(x)$ dne. when $x=0$
We now compute: $f(-1)=-5$
$f(0)=0$
$f(1)=-1$
$f(3)=6-3\sqrt[3]{9}\approx-0.24$

Global max of 0 at $(0,0)$
Global min of -5 at $(-1,-5)$



021 section 4.1

~~Test Tomorrow - take your exams, bring a calculator~~

Ex Find the extrema for

$y = f(x) = 2\sin x - \cos(2x)$ on the interval $[0, 2\pi]$

Solution $f'(x) = 2\cos x + 2\sin(2x)$

We now use $\sin(2x) = 2\cos x \sin x$

$f'(x) = 2\cos x + 4\cos x \sin x$

$2\cos x + 4\cos x \sin x = 0$

$2\cos x(1 + 2\sin x) = 0$

$\cos x(1 + 2\sin x) = 0$

$\cos x = 0$ in $[0, 2\pi]$ when $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$

and $1 + 2\sin x = 0$ when $\sin x = -\frac{1}{2}$, $x = \frac{7\pi}{6}$, $x = \frac{11\pi}{6}$

$f(0) = -1$

$f(\frac{\pi}{2}) = 3$

$f(\frac{7\pi}{6}) = -\frac{3}{2}$

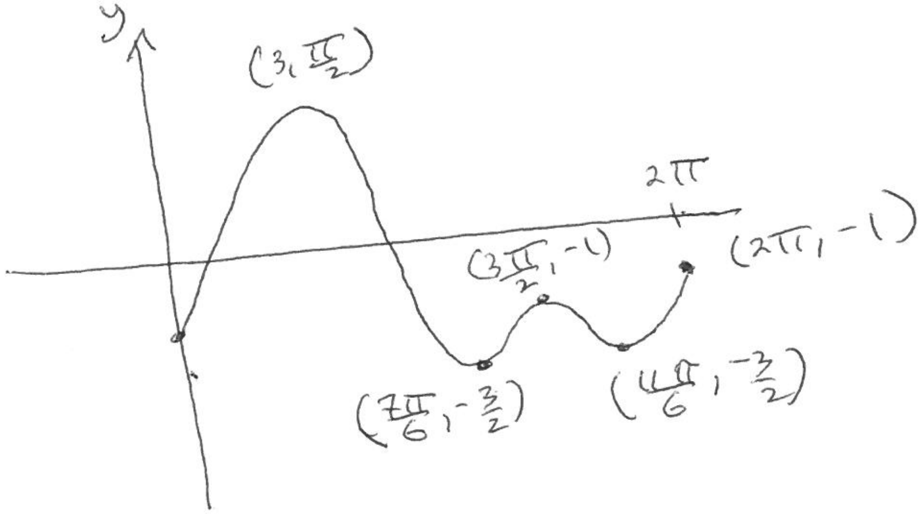
$f(\frac{3\pi}{2}) = -1$

$f(\frac{11\pi}{6}) = -\frac{3}{2}$

$f(2\pi) = -1$

The absolute minimum is $y = -\frac{3}{2}$ at $x = \frac{7\pi}{6}$, $x = \frac{11\pi}{6}$

The absolute maximum is $y = 3$ at $x = \frac{\pi}{2}$



021 Sec 4.1

EX Find the critical numbers and local extrema for $f(x) = 5x^{2/3} - x^{5/3}$

$f'(x) = \frac{10}{3}x^{-\frac{1}{3}} - \frac{5}{3}x^{2/3}$

$f'(x) = \frac{5}{3}\left(2x^{-\frac{1}{3}} - x^{2/3}\right)$

$= \frac{5}{3}x^{-\frac{1}{3}}(2-x)$

$x^{-\frac{1}{3}}x^{K} = x^{2/3}$

so $-\frac{1}{3} + K = \frac{2}{3}$

$K = 1$

so, $f'(x)$ d.n.e. at $x=0$

$f'(x) = 0$ at $x=2$

critical numbers are $x=0, x=2$

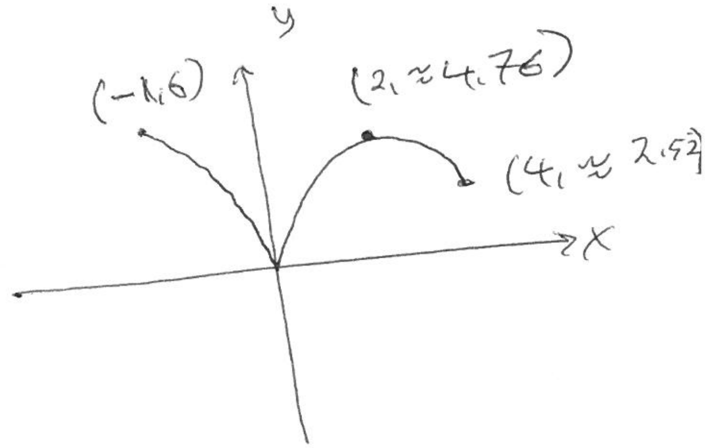
Find Global extrema for $f(x) = y = 5x^{2/3} - x^{5/3}$ on $[-1,4]$

$f(-1) = 6$

$f(0) = 0$

$f(2) = 5 \cdot 2^{2/3} - 2^{5/3} \approx 4.76$

$f(4) = 5 \cdot 4^{2/3} - 4^{5/3} \approx 2.52$



Absolute max is
$y = 6$ when $x = -1$

Absolute min is
$y = 0$ when $x = 0$

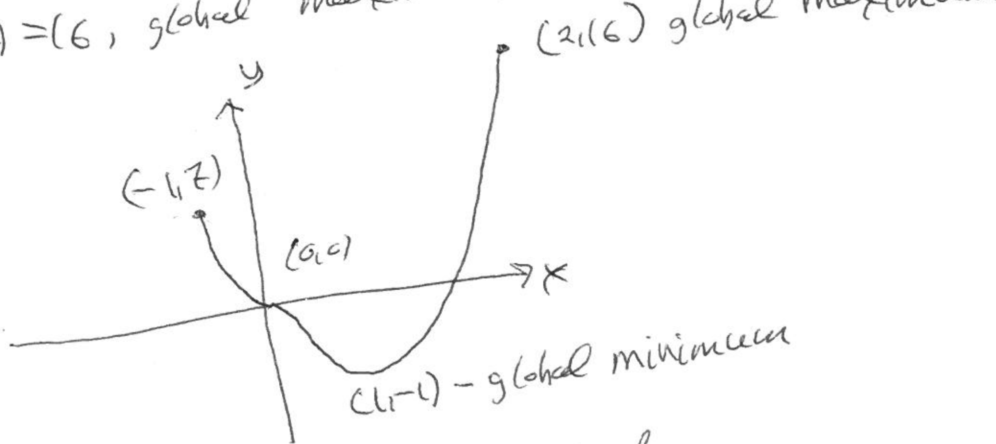
021 Sec 4.1

Ex Find the extrema of $f(x) = 3x^4 - 4x^3$ on the interval $[-1, 2]$

Solution $f'(x) = 12x^3 - 12x^2$
$= 12x^2(x-1)$
$f'(x) = 0$ at $x = 0, 1$

critical number are $x=0, x=1$

$f(-1) = 7$
$f(0) = 0$
$f(1) = -1$, global minimum
$f(2) = 16$, global maximum

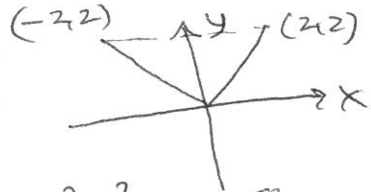


Note (0,0) is a critical pt
but not a max nor a min.

021 Sec 4.2

Section 4.2 The mean-value theorem (MVT)

In general, with f defined on [a,b] and f continuous on [a,b]
there is not a pt c in (a,b) with $f'(c) = \frac{f(b)-f(a)}{b-a}$



Ex. $f(x) = |x|$ on $[-2,2]$

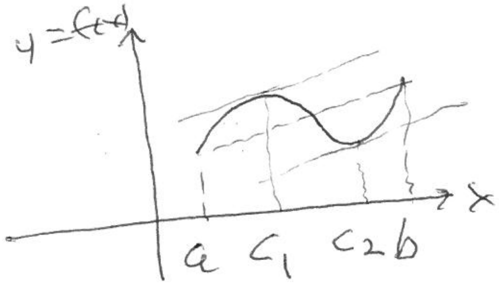
Note here $\frac{f(b)-f(a)}{b-a} = \frac{2-2}{2-(-2)} = \frac{0}{4} = 0$

but $f'(x) = -1$ on $(-2,0)$
$f'(x) = +1$ on $(0,2)$
$f'(x)$ d.n.e. at $x = -2, 2, 0$
$f'(x)$ never equals 0.
so $f'(x)$ never equals 0.

So, in our example, there is not a pt c in $[-2,2]$ where the instantaneous rate of change equals the average rate of change.

However if $f(x)$ is differentiable on the open interval (a,b) and continuous on the closed interval $[a,b]$, there is at least one number c in (a,b) s.t.

$f'(c) = \frac{f(b)-f(a)}{b-a}$, the slope of the secant line joining the endpoints.



Mean Value Thm for derivatives:
Let $f(x)$ be continuous on the ~~open~~ closed interval $[a,b]$
$f(x)$ " diff " " open " (a,b)
Then there is a c, s.t. $a < c < b$ and
$f'(c) = \frac{f(b)-f(a)}{b-a}$
instantaneous r.o.c. = average rate of change

021 Section 4.2

Ex of the Mean Value Theorem:

Let $f(x)=x^{1/3}$ on $[0,2]$. Note f is continuous on $[0,2]$
f is differen on $(0,2)$

Find the value of c for which the MVT applies.

Solution $\dfrac{f(b)-f(a)}{b-a} = \dfrac{2^{1/3}-0^{1/3}}{2-0} = \dfrac{2^{1/3}}{2} = 2^{-2/3}$

Also $f'(x) = \frac{1}{3}x^{-2/3}$

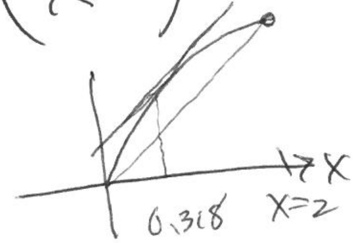
Set $2^{-2/3} = \frac{1}{3}x^{-\frac{2}{3}}$, solve for x

$3\cdot 2^{-2/3} = x^{-2/3}$

$\left(3\cdot 2^{-2/3}\right)^{\left(-\frac{3}{2}\right)} = \cancel{\left(x^{-2/3}\right)\left(-\frac{3}{2}\right)} = \left(x^{-\frac{2}{3}}\right)^{\left(-\frac{3}{2}\right)}$

$3^{-3/2}\cdot 2 = x$

$c = x = 2\cdot 3^{-3/2} \approx 0.318$



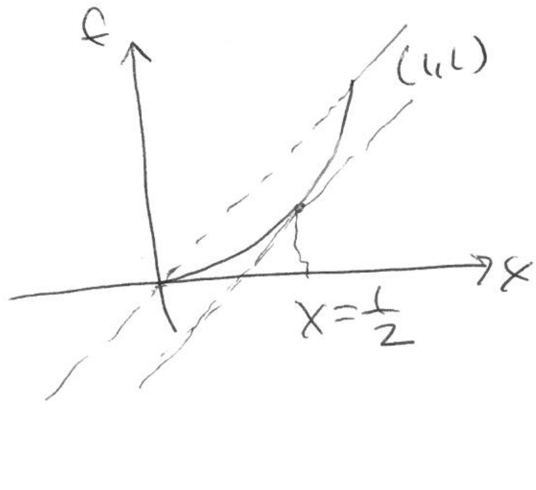
Let $f(x)=x^2$ on $[0,1]$; Find the value of c for which the MVT applies.

Solution $\dfrac{f(1)-f(0)}{1-0} = \dfrac{1^2-0^2}{1} = 1$

$f'(x) = 2x$

set $2x = 1$

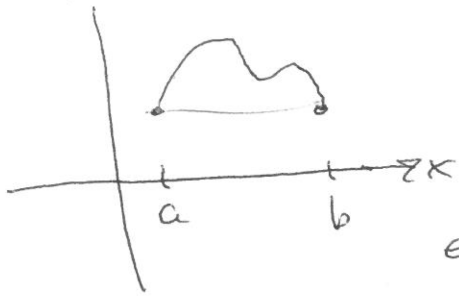
$c = x = \frac{1}{2}$



021 Sec 4.2

A special case of MVT is Rolle's Thm

__Rolle's Thm__ Let f be continuous on $[a,b]$ and f be differentiable on (a,b). Also let $f(a)=f(b)$ then $\exists c$ in (a,b) s.t. $f'(c)=0$



Note the average rate of change of f on $[a,b]=0$ and the slope of the joining the endpoints is 0 $\frac{f(b)-f(a)}{b-a}=\frac{0}{b-a}=0$

__EX__ Let $f(x)=5-\frac{4}{x}$ on $[1,4]$

First, we show that the MVT applies

1) The only pt of discontinuity for $f(x)$ is $x=0$, but 0 is not in $[1,4]$

so f is continuous on $[1,4]$

2) $f'(x)=\frac{4}{x^2}$, which exists on $(1,4)$

so, we can apply the MVT

The average rate of change is $\frac{f(4)-f(1)}{4-1}=\frac{(5-\frac{4}{4})-(5-\frac{4}{1})}{4-1}=1$

So, we seek a value of c, s.t. $f'(c)=1$

__Solution__ $f'(x)=\frac{4}{x^2}$, want $\frac{4}{x^2}=1$

$x^2=4$, $x=2$, $x=-2$

Since $x=-2$ is not in our domain $[1,4]$

we only want $x=2$

021 Sec 4.2

Application Two fixed radar stations on a straight highway are 5 miles apart

When a car passes R_1, it is traveling at 55 miles per hour.
4 minutes later the car passes R_2 at 50 mph.
The speed limit is 65 mph. Was the driver speeding?
Let $s(t)$ be the distance traveled in miles at time t.

Solution Let $s(t)$ be the distance traveled in miles at time t.

$s(0)=0$, $s(\frac{1}{15})=5$, 4 min $= \frac{1}{15}$ hr.

So the average velocity of the car, over the 5 mile stretch

is $av = \dfrac{s(\frac{1}{15})-s(0)}{\frac{1}{15}-0} = \dfrac{5}{\frac{1}{15}} = 75$ mph.

The MVT applies, so at some pt, the driver was going 75 mph

Note The velocity at the endpts don't enter into the argument at all

021 Sec 4.2

Thm If $f'(x)=0$ for all x in (a,b) then f is constant on (a,b)

Pf. Let x_1, x_2 be any two points with $a<x_1<x_2<b$

Want to show $f(x_1)=f(x_2)$

f is differ on (a,b) so f is contin on $[x_1,x_2]$

f is diff on (x_1,x_2)

So, the MVT applies

So, $\exists\, c$ with $x_1<c<x_2$ with

$$f'(c)=\frac{f(x_2)-f(x_1)}{x_2-x_1}$$ never 0

$$(x_2-x_1)f'(c)=f(x_2)-f(x_1)$$

We assumed $f'(c)=0$. and that $x_2>x_1$, so $x_2-x_1>0$

so LHS $=0$. So RHS $=0$ so $f(x_2)-f(x_1)=0$

$f(x_2)=f(x_1)$

Since the choice of x_1 and x_2 are arbitrary, we are done.

Corollary If $f'(x)=g'(x)$ for all x in (a,b)

then $f(x)=g(x)+K$, where K is some constant.

Pf Construct a new function.

$$F(x)=f(x)-g(x)$$

so $$F'(x)=f'(x)-g'(x)$$

we assumed $f'=g'$.

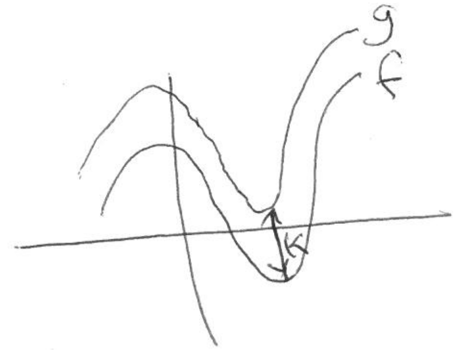
So $F'(x)=0 \quad \forall x$

By our theorem: $F(x)=K$, K some constant.

so $F(x)=K$

$F(x)=f(x)-g(x)$

so $f(x)-g(x)=K$



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Application of the corollary

Show that: $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ for all x in their common domain.

Solution Let $f(x) = \tan^{-1}x + \cot^{-1}x$

So $f'(x) = \frac{1}{1+x^2} + \frac{-1}{1+x^2}$

$f'(x) = 0, \forall x$

So f is constant for all x

To find the value of C, we compute $f(x)$ for any x

$f(0) = \tan^{-1}(0) + \cot^{-1}(0)$
$= 0 + \frac{\pi}{2} = \frac{\pi}{2}$

$f(1) = \tan^{-1}(1) + \cot^{-1}(1)$
$= \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$