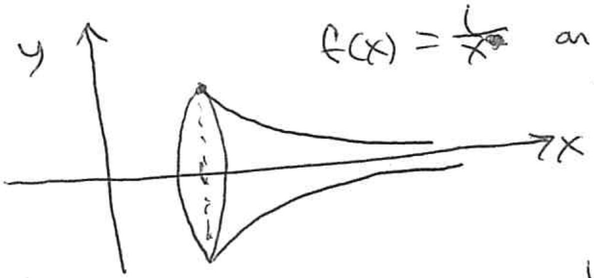
Gabriel's Horn

$f(x) = \cancel{\frac{1}{x^2}}$ on $x \in [1, \infty)$

Clarification $f(x) = \frac{1}{x}$



Volume $V = \lim_{b \to \infty} \pi \int_1^b \frac{1}{x^2}\, dx$

$= \pi \lim_{b \to \infty} \frac{-1}{x}\Big|_1^b = \pi \left[\lim_{b \to \infty} -\frac{1}{b} - \left(\frac{-1}{1}\right) \right] = \pi$

S.A. $f(x) = x^{-1}$ $f'(x) = -x^{-2} = \frac{-1}{x^2}$, $\left(f'(x)\right)^2 = \frac{1}{x^4}$

S.A. $2\pi \lim_{a \to \infty} \int_1^a \frac{1}{x} \sqrt{1 + \frac{1}{x^4}}\, dx$

Now, we use a comparison thm.

for $x > 1$ $\quad 1 < \sqrt{1 + \frac{1}{x^4}}$

so $\int_1^a \frac{1}{x} \sqrt{1 + \frac{1}{x^4}}\, dx > \int_1^a \frac{1}{x} \sqrt{1}\, dx \geq \int_1^a \frac{1}{x}\, dx$

we know $\int_1^\infty \frac{1}{x}\, dx$ diverges

So $\lim_{a \to \infty} \int_1^\infty \frac{1}{x} \sqrt{1 + \frac{1}{x^4}}\, dx$ diverges.

So, Gabriel's Horn has finite volume $(= \pi)$

" " " infinite surface area.

So we can fill it with π gallons of paint and it seems that there is not enough paint to coat the inside.

Find area of the surface generated by rotating the curve $y = e^x$ $0 \le x \le 1$ about the x-axis.

$y = e^x$ $\qquad \frac{dy}{dx} = e^x$

$$S = \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_0^1 e^x \sqrt{1+e^{2x}}\, dx$$

$$= 2\pi \int_1^e \sqrt{1+u^2}\, du \qquad (u = e^x)$$

$$= 2\pi \int_{\frac{\pi}{4}}^{\alpha} \sec^3\theta\, d\theta \qquad (u = \tan\theta,\ \alpha = \tan^{-1} e)$$

$$= 2\pi \cdot \frac{1}{2}\Big[\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta|\Big]_{\frac{\pi}{4}}^{\alpha}$$

$$= \pi \Big[\sec\alpha \tan\alpha + \ln|\sec\theta + \tan\theta|\Big]_{\frac{\pi}{4}}^{\alpha}$$

$$= \pi\Big[\sec\alpha \tan\alpha + \ln(\sec\alpha + \tan\alpha) - \sqrt{2} - \ln(\sqrt{2}+1)\Big]$$

Now $\tan\alpha = e$, so $\sec^2\alpha = 1 + \tan^2\alpha = 1 + e^2$

So $S = \pi\Big[e\sqrt{1+e^2} + \ln(e + \sqrt{1+e^2}) - \sqrt{2} - \ln(\sqrt{2}+1)\Big]$

section 8.3 Applications to physics

Mass – is a measure of a body's resistance to changes in straight line motion

Mass is independent of the gravitational forces involved.

Recall $F = ma$

system of measurement	Measure of Mass	Measure of Force
U.S.	slug	Pound $= (\text{slug})(\text{ft}/\text{sec})^2$
International	Kilogram	Newton $= (\text{Kilogram})(\text{m}/\text{sec})^2$
C.G.S.	Gram	Dyne $= (\text{gram})(\text{cm}/\text{sec})^2$

EX Find the mass in slugs, of an object whose weight at sea level is 1 pound.

For acceleration due to gravity, use $32\ \text{ft}/\text{sec}^2$

$$\cancel{F} = ma \Rightarrow m = \frac{F}{a}$$

$$\text{mass} = \frac{1\ \text{lbs}}{32\ \text{ft}/\text{sec}^2} = 0.03125 \frac{\text{pounds}}{\text{foot}/\text{sec}^2}$$

$$= 0.03125 \text{ slugs}$$

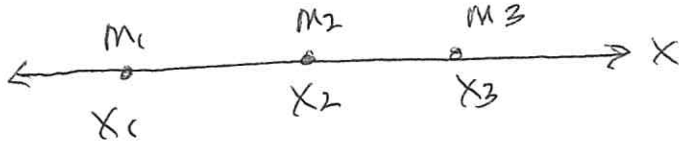
We can now define the moment of a force

In this setting, the center of mass is called the center of gravity.

Suppose that a system of point masses, $m_1, m_2, \ldots, m_n$ are located at $x_1, x_2, \ldots, x_n$

Since $F = ma$, the total force is

$$F = m_1 a + m_2 a + \cdots + m_n a = Ma \text{ with } M = \sum_{i=1}^{n} m_i$$



The moment, torque about the origin is

$$T_0 = (m_1 a)x_1 + (m_2 a)x_2 + \cdots + (m_n a)x_n = M_0 a$$

The center of gravity is

$$\frac{T_0}{F} = \frac{M_0 a}{m a} = \frac{M_0}{m} = \bar{x}$$

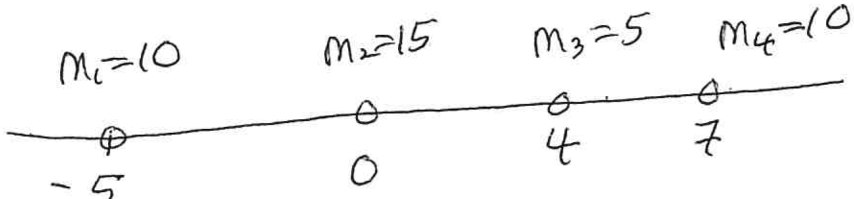
Note: The center of gravity and the center of mass have the same location.

$$\text{So, } \sum_{i=1}^{n} m_i(x_i - \bar{x}) = \sum_{i=1}^{n} m_i x_i - \sum_{i=1}^{n} x_i \bar{x} = 0$$

$$\text{So } \bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\text{moment of the system about the origin}}{\text{total mass of the system}}$$

if $\bar{x} = 0$ i.e. $\sum_{i=1}^{n} m_i x_i = 0$, the system is said to be in equilibrium

EX



~~The moment about the origin is~~

$$\cancel{m = 10+15+5+10 = 40}$$

The moment about the origin is:

$$M_0 = \sum_{i=1}^{4} m_i x_i = 10(-5) + 15(0) + 5(4) + 10(7) = 40$$

The sum of the masses is: $m = 10+15+5+10 = 40$

So, the center of gravity is $\bar{x} = \frac{M_0}{m} = \frac{40}{40} = 1$

Center of mass in a 2-dim system

We have a system of masses m_i, in the xy plane at (x_i, y_i) resp

We define 2 moments, one wrt x-axis
one " " y-axis

<u>Def</u> Let point masses $m_1, m_2, \dots, m_n$ be located at $(x_1, y_1), (x_2, y_2), \dots; (x_n, y_n)$

① The moment about the y-axis is

$$M_y = \sum_{i=1}^{n} m_i x_i$$ - note x_i is the distance of the mass m_i to the y-axis

② The moment about the x-axis is

$$M_x = \sum_{i=1}^{n} m_i y_i$$

Letting $m = \sum_{i=1}^{n} m_i$ be the total mass of the system

we have that the center of mass is $(\bar{x}, \bar{y})$

is. $$\bar{x} = \frac{M_y}{m}$$

$$\bar{y} = \frac{M_x}{M}$$