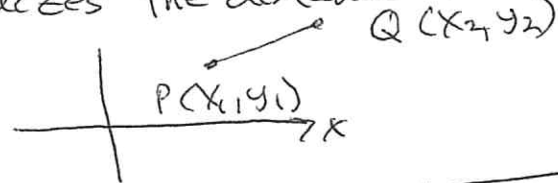
Sec 8.1 Arclength

Arclength is one of those ideas, like area or tangent, for which we have an intuitive feel. We need to make it precise and rigorous

Arclength generalizes the distance formula.

Distance formula :

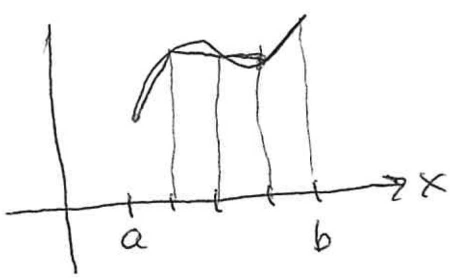



The distance between $P$ and $Q$ is $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

Suppose we have a curve defined on $x \in [a,b]$

When we partition $[a,b]$ into $n$ subintervals of equal width

The partition pts will induce a partition of the curve




We approximate the arclength of a subarc by the length of the straight line segment joining the endpts

Let $|P_{i-1}-P_i|$ be the ~~length~~ distance between $P_{i-1}$ and $P_i$

An approximation of the length of the curve $C$ is

$$\sum_{i=1}^{n} |P_{i-1}-P_i| = \sum_{i=1}^{n} |P_{i-1}P_i|$$

We get better approximations by increasing the number of partition pts. We define the length of the curve to be:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}-P_i|$$

We now develop a formula to compute arclength.
We will only consider curves whose defining function has a continuous derivative. Such function are called smooth

Start $|P_{i-1} - P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$

$$= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

We can apply the mean value theorem to f on the interval $[x_{i-1}, x_i]$ to find $x_i^*$ in the interval $[x_{i-1}, x_i]$

s.t. : $f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$

$$\Delta y_i = f'(x_i^*) \Delta x$$

So $|P_{i-1} P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$

$$= \sqrt{(\Delta x_i)^2 + [f'(x_i^*)\Delta x_i]^2}$$

$$= \sqrt{(\Delta x_i)^2 + [f'(x_i^*)^2](\Delta x_i)^2}$$

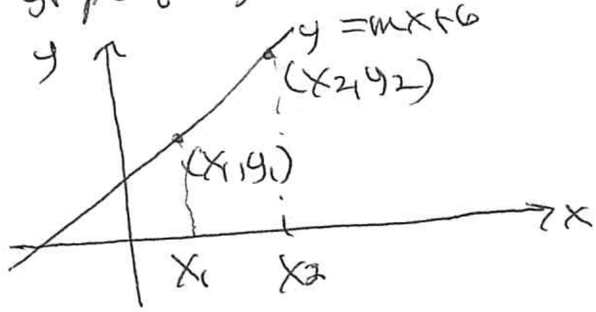
$$= \sqrt{1 + [f'(x_i^*)^2]}\ \Delta x_i$$

So $L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1} P_i| = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\ \Delta x$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\ dx$$

$$L=\int_a^b \sqrt{1+(f'(x))^2}\,dx$$

<u>EX</u> Find the arclength from $(x_1, y_1)$ to $(x_2, y_2)$ for the graph of $y=f(x)=mx+b$




<u>Solution</u> $f'(x)=m=\dfrac{y_2-y_1}{x_2-x_1}$

so $L=\int_{x_1}^{x_2}\sqrt{1+(f'(x))^2}\,dx$

$$L=\int_{x_1}^{x_2}\sqrt{1+\left(\frac{y_2-y_1}{x_2-x_1}\right)^2}\,dx$$

$$=\int_{x_1}^{x_2}\sqrt{\frac{(x_2-x_1)^2+(y_2-y_1)^2}{(x_2-x_1)^2}}\,dx$$

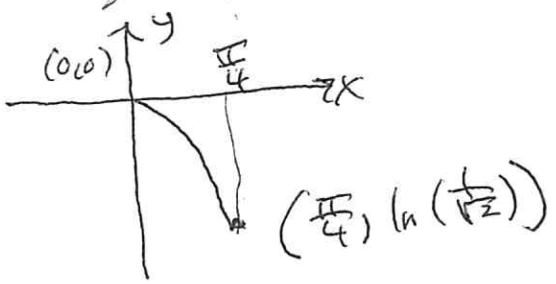
$$=\left(\sqrt{\frac{(x_2-x_1)^2+(y_2-y_1)^2}{(x_2-x_1)^2}}\;x\right)\Bigg|_{x=x_1}^{x=x_2}$$

$$=\sqrt{\frac{(x_2-x_1)^2+(y_2-y_1)^2}{\cancel{(x_2-x_1)^2}}}\;\cancel{(x_2-x_1)}$$

$$=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

This is the standard distance formula

EX Find the length of the curve of the graph of $y = \ln(\cos x)$ from $x=0$ to $x=\frac{\pi}{4}$




Solution We first find $\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$

So $L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$= \int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2 x}\, dx$

$= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 x}\, dx$

$= \int_0^{\frac{\pi}{4}} \sec x\, dx$

$= \left(\ln|\sec x + \tan x|\right)\Big|_{x=0}^{x=\frac{\pi}{4}}$

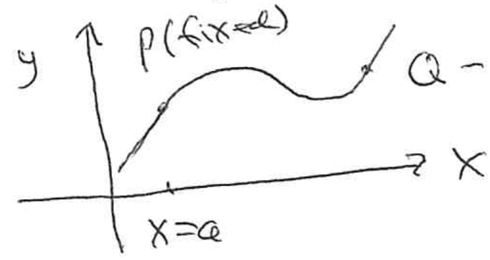
$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0|$

$= \ln|\sqrt{2}+1| - \ln|1+0|$

$= \ln|\sqrt{2}+1| - \ln(1)$, note $\ln 1 = 0$

$= \ln|\sqrt{2}+1|$

The arclength function




Q - moves along the curve

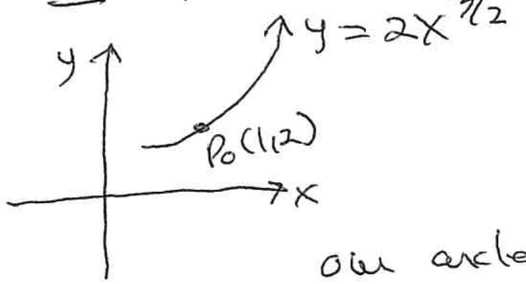
We seek a function that gives the length of a curve starting at some fixed pt P on that curve to a varying pt Q on the curve

If a smooth curve C has the equation $y = f(x)$, $a \le x \le b$. Let $S(x)$ be the distance along C from an initial, fixed pt $P(a, f(a))$ to the pt $Q(x, f(x))$ called the arclength function.

So, S is a function in x, called the arclength function

and $$S(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

EX Find the arclength function for $y = 2x^{3/2}$, starting at $P_0(1,2)$ $a=1$




Solution: $y' = 3x^{\frac{1}{2}}$

so $(y')^2 = 9x$

So $1 + (y')^2 = 1 + 9x$

So our arclength function is $S(x) = \int_1^x \sqrt{1+9t}\, dt$

use $u = 1 + 9t$
$du = 9dt$

$$S(x) = \frac{1}{9} \int_1^x \sqrt{1+9t}\; 9dt$$

$$= \frac{2}{27} (1+9t)^{3/2} \Big]_{t=1}^{t=x}$$

$$S(x) = \frac{2}{27} \left[ \left( (1+9x)^{3/2} - 10\sqrt{10} \right) \right]$$

So the length of the curve from $x=1$ to $x=2$

is $$S(2) = \frac{2}{27} \left[ (1+9 \cdot 2)^{3/2} - 10\sqrt{10} \right]$$

$$S(2) = \frac{2}{27} \left[ 19^{3/2} - 10^{3/2} \right]$$