022 Appendix G Complex Numbers   Friday, June 21, 2019

Complex numbers – want to solve the equation

$$x^2+1=0$$
$$x^2=-1$$
$$x=\sqrt{-1}$$, can't be a ~~real~~ real number.

Define $i=\sqrt{-1}$

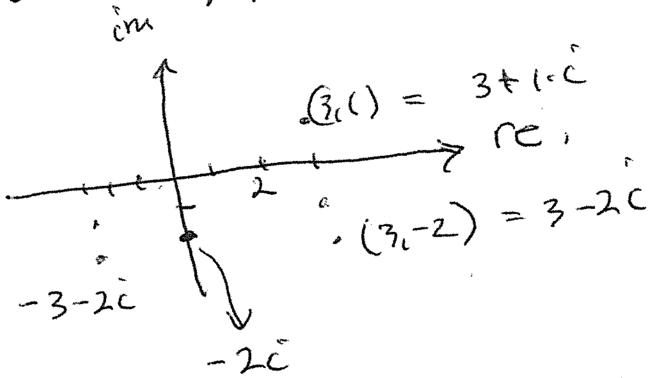
The set of complex numbers is the set of numbers in the form $a+bi$, where a and b are real numbers.

a is called the real part

b " " " imaginary part

Note if $b=0$, we have $a+bi=a+0i=a$, the real numbers.

We can graph the ~~rea~~ complex numbers in the plane



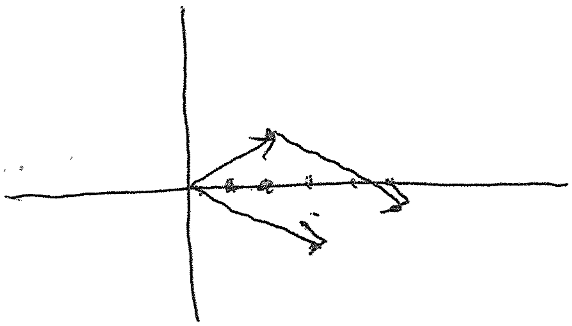
The x-axis is the real axis

The y-axis " " imag axis

For $a+bi$ and $c+di$, they are equal iff

$a=c$ <u>and</u> $b=d$.

We add $a+bi$ ~~and~~ $c+di$ by $(a+bi)+(c+di)=(a+c)+(b+d)i$



~~Add 2+i~~
Find $(2+i)+(3-2i)$
P_1 P_2
~~$(2+3)+(i-2i) = 5$~~

$(2+i)+(3+2i)$
$= (2+3)+(1-2)i$
$= 5-i$

So adding vectors is like adding adding complex numbers in the plane.

and $(a+bi)-(c+di) = (a-c)+(b-d)i$
subtracting complex numbers is like subtracting vectors in the plane.

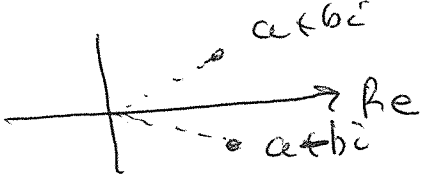
Multiplication of Complex Numbers

$(a+bi)(c+di)$
$= ac + adi + bci + bdi^2$
$= ac + (ad+bc)i + bdi^2$, recall $i=\sqrt{-1}$ so $i^2=-1$
$= ac + (ad+bc)i - bd$
$= ac - bd + (ad+bc)i$

Ex $(3-2i)(-4+6i) = -12+(18+8)i - 12i^2$
$= -12 + \cancel{12}) + (2 \cancel{6}i$
$26i \quad = 26i$

For division of two complex numbers, we need the idea of the conjugate (complex conjugate)

if $z = a+bi$, the conjugate is $\bar{z} = a-bi$



Note $(a+bi)(a-bi)$
$= a^2 - abi + abi - b^2 i = a^2 + b^2$

We can now divide two complex numbers and get a complex number.

$$\frac{a+bi}{c-di} = \frac{a+bi}{c-di}\left(\frac{c+di}{c+di}\right) = \frac{(ac-bd)+(ad+bc)i}{c^2+d^2}$$

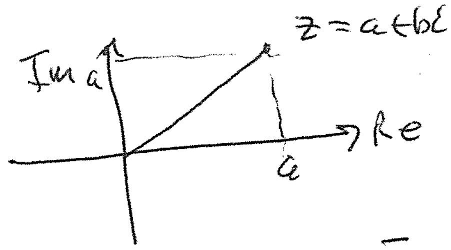
$$= \frac{ac-bd}{c^2+d^2} + \frac{ad+bc}{c^2+d^2} i$$

Ex $$\frac{-1+3i}{2+5i} = \left(\frac{-1+3i}{2+5i}\right)\left(\frac{2-5i}{2-5i}\right) = \frac{13+11i}{2^2+5^2} = \frac{13}{29} + \frac{11}{29} i$$

Properties of the conjugate

$\overline{z+w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z}\bar{w}$, $\overline{z^n} = \bar{z}^n$

The modulus, (absolute value) $|z|$ of $z = a+bi$, is the distance between the origin $(0,0)$ and the pt $(a+bi)$



$$|z| = \sqrt{a^2+b^2}$$

Compute $\quad z\bar{z} = (a+bi)(a-bi) = a^2 + abi - abi - b^2i^2 = a^2+b^2$

$$z\bar{z} = |z|^2$$

So, in general: $\quad \dfrac{z}{w} = \dfrac{z\bar{w}}{w\bar{w}} = \dfrac{z\bar{w}}{|w|^2}$

Since $i^2 = -1$, we have that $i = \sqrt{-1}$

and also $(-i)^2 = (-1\cdot i)^2 = (-1)^2 i^2 = 1\cdot i^2 = -1$

So $-i$ is also a square root of -1

We say that $+i$ is the principle square root of -1.

In general for $c > 0$,

$$\sqrt{-c} = \sqrt{c}\, i$$

Ex Find the roots of the equation: $3x^2+4x+10 = 0$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(3)10}}{2\cdot 3} = \frac{-4 \pm \sqrt{16-120}}{6}$$

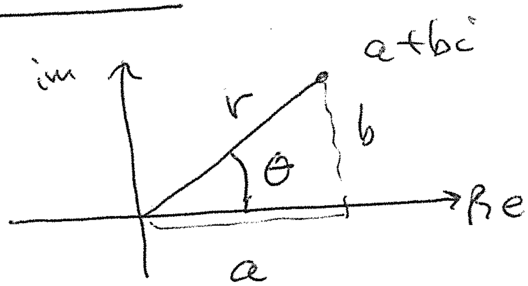
$104 \mid 2$
$52 \mid 2$
26

$$x = \frac{-4 \pm \sqrt{-104}}{6} = \frac{-4 \pm 2\sqrt{-26}}{3} = \frac{-2}{3} \pm \frac{\sqrt{-26}}{3}$$

$$x = \frac{-2}{3} + \frac{\sqrt{26}}{3} i, \quad x = -\frac{2}{3} - \frac{\sqrt{26}}{3} i$$

Note these roots are complex conjugates of each other

Polar Form



Since any complex number $z = a+bi$

can be viewed as a point in the plane (a,b)

we can view that pt (a,b) in polar coordinates,

$$z = a+bi = (r\cos\theta) + (r\sin\theta)i$$

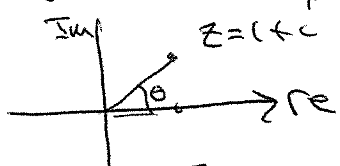
$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z| = \sqrt{a^2+b^2}$ and $\tan\theta = \frac{b}{a}$

The Angle θ is called the argument of z and we write $\theta = \arg(z)$

<u>note</u> $\arg(z)$ is not unique. to $\arg(z)$ add any multiple of 2π, and get a different but valid $\arg(z)$,

<u>Ex</u> $z = 1+i$

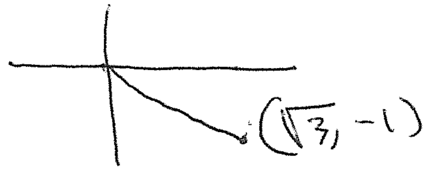


$$r = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\tan\theta = \frac{b}{a} = \frac{1}{1} = 1, \text{ so } \theta = \frac{\pi}{4}$$

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Ex $z = \sqrt{3} - i$



$$r = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = 2$$

$\tan\theta = \frac{-1}{\sqrt{3}}$, w is in the fourth quadrant, so set $\theta = -\frac{\pi}{6}$

$$w = 2\left[\cos\left(\frac{-\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

Mult and Division of Complex numbers in Polar Form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$z_1 z_2 = r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

with some trig identities

$$z_1 z_2 = r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$$

So, to multiply two complex numbers in polar form, multiply the moduli and add the arguments,

In a similar way,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right], z_2 \neq 0$$

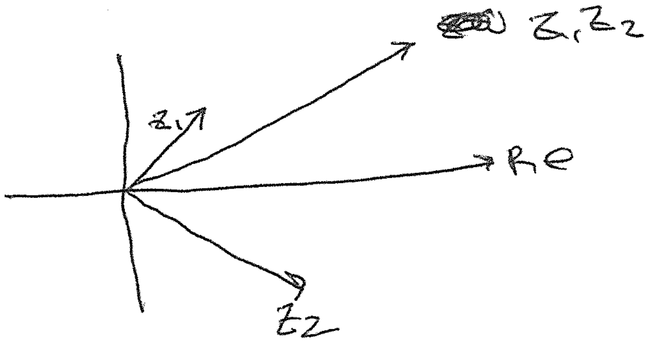
so to divide two complex numbers in polar form, divide the moduli and subtract the arguments,

Ex $((1+i)(\sqrt{3} - i)$ in polar form,

$$\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) 2\left(\cos\left(\frac{-\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$= 2\sqrt{2}\left[\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\right]$$

$$= 2\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$



$$\frac{z_1}{z_2} = \frac{\sqrt{2}}{2}\left[\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right)\right]$$