3.7 Rates of Change in the Sciences ①

A particle moves back and forth in a straight line, (called rectilinear motion)

Describe the motion of the particle if position at time t is:

$$s = f(t) = t^3 - 9t^2 + 15t + 10 \text{ ft.}$$

Solution

v = velocity $= f'(t) = 3t^2 - 18t + 15 = 3(t-1)(t-5)$ ft/sec

The particle is at rest when $v = 0$, so $t = 1, 5$ seconds

The particle is moving in a positive direction when $v(t) > 0$

so $0 \le t < 1$ or $t > 5$

The particle is moving in a negative direction when $v(t) < 0$

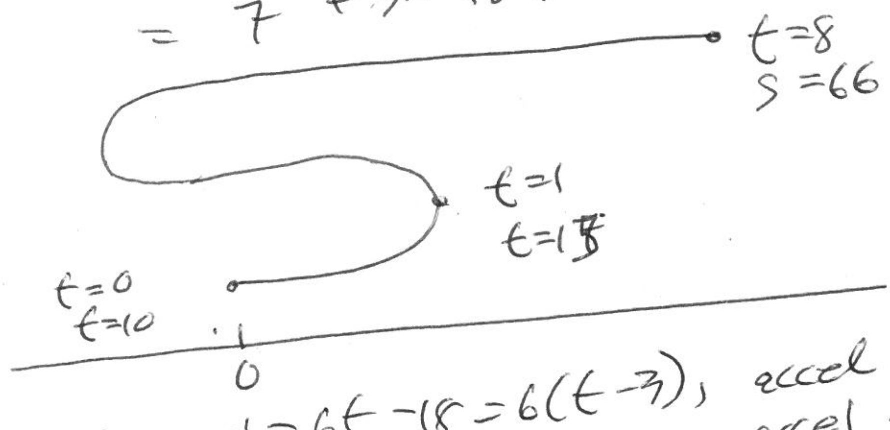
i.e. $1 < t < 5$

The total distance traveled in the first 8 seconds

is $|f(1) - f(0)| + |f(5) - f(1)| + |f(8) - f(5)|$

$= |17 - 10| + |-15 - 17| + |f(8) - f(5)|$

$= 7 + 32 + 81 = 120$ ft



Accel $= v' = 6t - 18 = 6(t-3)$, accel > 0, if $t > 3$

accel < 0 if $t < 3$

3.7 1.5

$0 < t < 1$ moving forward $v > 0$

slowing down $a < 0$

$1 < t < 3$ moving backwards $v < 0$

speeding up $a < 0$, but the direction is neg

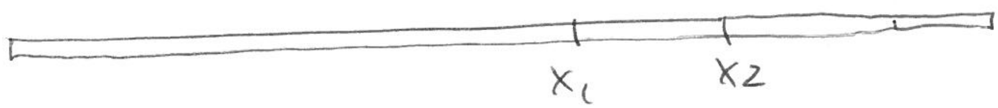
$3 < t < 5$ moving backwards $v < 0$

slowing down, $a > 0$

$t > 5$, moving forward $v > 0$

speeding up $a > 0$

Density: Suppose that you have a rod
whose mass between two pts on the rod x_1 and x_2



is $\Delta m = f(x_2) - f(x_1)$

The average density is $\dfrac{\Delta m}{\Delta x} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

As $x_2 \to x_1$ (i.e. $\Delta x \to 0$) we get the
<u>linear density</u> ρ

$$\rho = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} = \frac{dm}{dx}$$

Ex Suppose that the mass of the part of a metal rod that lies between its left end and a pt x meters to the right is $3x^2$ kg. Find the d(linear) density of the rod at 2 m.

density $= \rho(x) = 3x^2$

$\rho'(x) = 6x$

$\rho'(2) = \cancel{6} \cdot 2 = 12$ kg/m

3.7 ③

Compressibility in thermodynamics

If a substance is kept at a constant temperature, then its volume V depends on the pressure P

We consider the rate of change of volume wrt pressure i.e. $\frac{dV}{dP}$

As P increases, V decreases, so $\frac{dV}{dP} < 0$.

The compressibility is defined by introducing a minus sign and dividing by V, so

isothermal compressibility $= \beta = -\frac{1}{V}\frac{dV}{dP}$

β measures how fast per unit volume, the volume of a substance decreases as the pressure on it increases at constant temperature.

Ex Let V (in cubic meters) of air at 25°C is related to pressure P (in kilopascals) by $V = \frac{5.3}{P}$

Find the rate of change of V wrt P when $P = 100$ kPa.

The rate of change

is $\left.\frac{dV}{dP}\right|_{P=100} = \left.\frac{-5.3}{P^2}\right|_{P=100}$

$= \frac{-5.3}{10,000} = -0.00053 \, m^2 / kPa.$

So, the compressibility at that pressure is

$\beta = \left.-\frac{1}{V}\frac{dV}{dP}\right|_{P=100} = \frac{+0.00053}{\frac{5.3}{100}} = 0.01$

$= 0.01 \, (m^3/kPa)/m^3$

3.7

Economics

Let $C(x)$ be the total costs to produce x units.

If production is increased from x_1 to x_2.

The additional cost is $\Delta C = C(x_2) - C(x_1)$

and the average rate of change of the cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}$$

As $\Delta x \to 0$, the marginal cost

is: marginal cost $= \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$

The marginal cost is an approximation of the additional cost of producing one more item (the n+1 item).

The marginal cost can change with n.

Ex Suppose that the ~~marginal~~ cost is

$$C(x) = 84 + 0.16x - 0.0006x^2 + 0.000003x^3$$

Then ~~$C'(x) = 16x$~~ $C'(x) = 0.16 - 0.0012x + 0.000009x^2$

$C'(100) \approx 0.13$. The is an approximation of producing the 101st item

Section 3.8 Exponential Growth and Decay

When the rate of change is proportional to the amount present

$$f'(t) = k\,f(t)$$

$\downarrow$

k is the constant of proportionality

__Aside__ $f'(t) = k f(t)$ is an example of a differential equation

The function that solves the above differential equation is

$$y(t) = A_0 e^{kt}$$

__check__ $y'(t) = k A_0 e^{kt} = k\left[A_0 e^{kt}\right] = k\,y(t)$

__Note__ if $t=0$

$$y(0) = A_0 e^{k\cdot 0} = A_0 e^{0} = A_0 \cdot 1 = A_0$$

so A_0 is the initial amount

i.e. $A_0 = y(0)$

__note__ The only solutions of $\frac{dy}{dx} = ky$

are of the form $y(t) = y(0)e^{kt}$

where k is the constant of proportionality

021 Sec 3.8

Ex Suppose that a bacteria sample is sc.

At $t=2$ hours, the population is 3.1 grams

At $t=10$ " " " " 5.8 grams

The population growth follows an exponential model.

How large is the population at $t=12$ hours?

Solution We are given that $A(t) = A(0)e^{kt} = A_0 e^{kt}$

We know that $A(10) = 5.8 = A_0 e^{10k}$

$A(2) = 3.1 = A_0 e^{2k}$

So, $\frac{5.8}{3.1} = \frac{A_0 e^{10k}}{A_0 e^{2k}} = \frac{e^{10k}}{e^{2k}} = e^{10k-2k} = e^{8k}$

$1.871 \approx e^{8k}$

$\ln(1.871) \approx \ln e^{8k} = 8k$

$0.6264 \approx 8k \Rightarrow k \approx \frac{0.6264}{8} \approx 0.07831$

so $A(t) = A_0 e^{0.07831t}$

To find the value of A_0, use either $A(2)=3.1$ or $A(10)=5.8$

$A(2) = 3.1 = A_0 e^{(0.07831)(2)} = A_0 e^{0.1566}$

so $A_0 = \frac{3.1}{e^{0.1566}} \approx \frac{3.1}{1.16953} \approx 2.6506$

so $A(t) = 2.6506e^{(0.07831t)}$

so $A(12) = 2.6506e^{(0.07831)12} \approx 6.7359$ grams

When will the population reach 12 grams

Solve $12 = 2.6506e^{0.07831t}$ for t

$4.527 \approx e^{0.07831t}$

$\ln(4.527) \approx 0.07831t$

$1.51 \approx 0.07831t$

$t \approx \frac{1.51}{0.07831} \approx 19.33$ hours

$t \approx 19$ hours and 20 minutes

021 Section 3.8

Background on C-14 dating

A team from U. of Chicago led by William Libby found a method to estimate the age of carbonaceous fossilized matter.

The radioisotopy C-14 is produced in the atmosphere when cosmic rays interact with nitrogen 14

The ratio $\frac{\text{C-14 (radioactive)}}{\text{C-14 (stable)}}$ seems to be constant

for all living material up to the time of death.

This ratio is also constant in the atmosphere.

When the organism dies, the proportion of C-14 declines

The half-life of C-14 is 5730 years.

This can be used to determine the age of clothes, bones etc, up to about 65,000 years.

For t > 65,000 years the method is still theoretically sound but small errors in measurements are compounded too many times to have reliable results

021 Section 3.8

Let $m(t)$ be the mass remaining at time t from an initial mass m_0, i.e. $m(0) = m_0$

The relative decay rate

$$-\frac{1}{m}\frac{dm}{dt} \text{ is known to be constant}$$

$$\text{so } -\frac{1}{m}\frac{dm}{dt} = km, \quad k<0$$

so $m(t) = m_0 e^{kt}$, this is an exponential model

$k<0$, means that we have decay

Ex A fossilized bone contains 0.1% of its original amount of C-14. Determine the age of the bone.

Use $A(t) = A_0 e^{kt}$

$$\frac{1}{2}\cancel{A_0} = \cancel{A_0} e^{5730k}$$

$$\frac{1}{2} = e^{5730k}, \text{ take ln of both sides}$$

$$\ln\left(\frac{1}{2}\right) = \ln 1 - \ln 2 = -\ln 2 = 5730k$$

$$k = \frac{-\ln 2}{5730} \approx -0.00012097$$

$$\text{So } A(t) = A_0 e^{-0.00012097t}$$

$$0.001\cancel{A_0} = \cancel{A_0} e^{-0.00012097t}$$

$$\ln 0.001 = -0.00012097t$$

$$t = \frac{\ln 0.001}{-0.00012097}$$

$$t \approx 57{,}103 \text{ years}$$

$$0.001 = e^{-0.00012097t}$$

$$\ln 0.001 = \ln(e^{-0.00012097t})$$

$$\ln 0.001 = -0.00012097t$$

021 Sec 3.8

Newton's Law of Cooling (Heating)

The rate of change of the temperature T of an object wrt time t is proportional to the difference of the temperature of the object and the surrounding temperature.

$$\frac{dT}{dt} = k(T - T_s)$$

This is not quite an ~~exponential~~ exponential model ($\frac{dT}{dt} = kT$)

To transform it into an exponential model

We use a change of variables.

$$y(t) = T(t) - T_s \rightarrow \text{constant}$$

$$\text{so } y'(t) = T'(t) - 0$$

$$y'(t) = T'(t)$$

Ex A water bottle at room temperature (72°F) is placed in a fridge at 44°F. After $\frac{1}{2}$ hour, the water temp is 61°F. What is the temp of the water after <u>another</u> half hour.

solution $\frac{dT}{dt} = k(T - 44)$.

Let $y(t) = T(t) - 44$

so $y(0) = T(0) - 44 = 72 - 44 = 28$

We now have a true exponential model.

$$\frac{dy}{dt} = ky \text{ with } y(0) = 28$$

$$y(t) = y_0 e^{kt} = 28e^{kt}, \text{ find } k.$$

We have $T(30) = 61$, so $y(30) = 61 - 44 = 17$

so $28e^{30k} = 17$, solve for k

$$k = \frac{\ln\left(\frac{17}{28}\right)}{30} \approx -0.01663$$

021 Sec 3.8

so $y(t) = 28e^{-0.01663t}$

Almost there Recall $y(t) = T(t) - 44$

so $T(t) = 44 + y(t)$

$T(t) = 44 + 28e^{-0.01663t}$

After another half hour, $t = 60$

$T(60) \approx 54.3°F$

When does the water temp reach $50°F$

Solution solve $50 = 44 + 28e^{-0.01663t}$

$\frac{6}{28} = e^{-0.01663t}$

$\ln\left(\frac{6}{28}\right) = -0.01663t$

$t = \frac{\ln\left(\frac{6}{28}\right)}{-0.01663} \approx 92.6$ minutes

1:32:36

When will the water temp reach $40°F$ - never.

$\to t$

$t = 44$ is a horizontal asymptote

If we tried to solve it mathematically, we would have

$40 = 44 + 28e^{-0.01663t}$

$-\frac{4}{28} = e^{-0.01663t}$

$-$ $+$

but note for any real number x, $e^x > 0$

021 Sec 3.8

The number e as a limit

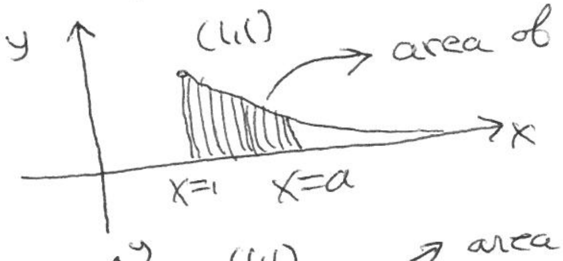
Two ways to define and compute e

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x, \quad e = \lim_{x\to\infty}(1+x)^{\frac{1}{x}}$$

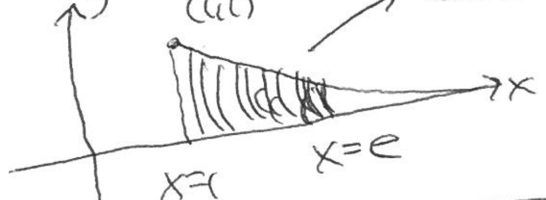
$$e \approx 2.718\cdots$$

Graph $f(x) = \frac{1}{x}$ on $[1,\infty)$

area of shaded region is $\ln a$



area of shaded region is 1 is when $x = e$



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e is irrational, so e can not be expressed in the form $e = \frac{a}{b}$, a, b are integers $b \neq 0$

e is not an algebraic number. This means that e is not the root of a polynomial with integer coeff

A number that is not algebraic is said to be transcendental

Note $\sqrt{2}$ is irrational but is algebraic since $\sqrt{2}$ isolves the equation $x^2 - 2 = 0$

Note e, π are both transcendental.

Euler Identity

$$\boxed{e^{\pi i} + 1 = 0}$$

021 Sec 3.8

Continuosly Compounding Interest

You make one deposit into a savings account and leave it alone

The amount in the account will depend on:

must agree $\left\{ \begin{array}{l} t \text{ - time} \\ r \text{ - interest rate, written as a decimal} \end{array} \right.$ so 5% = 0.05 per unit time

n - number of compounding period per unit time

A_0 - the initial amount

$A(r, t, A_0, n) = A_0\left(1+\frac{r}{n}\right)^{nt}$, nt is the total number of compounding periods.

Ex Let t be fixed at 3 years

r " " " 6%/year

$A_0 = \$1000$

n will vary

Simple interest - (you do not get interest on the interest)

$S = A_0 + A_0 \cdot r \cdot t$

$S = 1000 + 1000(.06)3 = \$1180$

Compound interest: you do get interest on the interest

$A(n) = 1000\left(1+\frac{.06}{n}\right)^{3n}$

$n=1$, the amount is \$1191.02 up \$11.02 from simple interest

$n=2$, " " " \$1194.05 up another \$3.03

$n=4$ " " " \$1195.62 " " \$1.57

$n=12$ " " " \$1196.68 " " \$1.06

$n=365$ " " " \$1197.20 " " \$0.52

As n increases, $A(n)$ increases

but the increases seem to taper off

021

The best that you can do is when interest is compounded at every moment. This is continuous interest

$$A(t) = \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt} = A_0 \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}$$

$$= A_0 \lim_{n\to\infty}\left[\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right]^{rt}$$

Now let $m=\frac{n}{r}$, $A(t) = A_0\left[\lim_{n\to\infty}\left(1+\frac{1}{m}\right)^{m}\right]^{rt}$

so $A(t) = A_0 e^{rt}$

In our case $A(3) = 1000\, e^{(.06)3} = 1000\, e^{0.18}$

$A(3) \approx \$\,1197.21$

We now show that: $A(t) = A_0 e^{rt}$ is an exponential model

<u>Pf</u> $A'(t) = rA_0e^{rt} = r\left[A_0e^{rt}\right] = rA(t)$

so $A'(t) = rA(t)$

So, we do have an exponential model with the rate of change $A'(t)$, is proportional to the amount present. The constant of proportionality is the interest rate per unit time written as a decimal.

<u>Often</u> in place of A_0, one uses the letter P, P is for principle. We have

$$A(t) = Pe^{rt}$$