022 Complex numbers Monday, June 24, 2019

Quiz Tuesday, June 25, 2019

① Find an equation of a plane containing 3 given pts.

② Express a complex fraction in the form $a+bi$

③ Express a complex number in polar form,

Polar form of a complex number $z=a+bi$

is $z=r(\cos\theta + i\sin\theta)$, θ is the acute angle from the positive x-axis

Recall if $z_1 = r_1(\cos\theta_1 + i\sin\theta_2)$

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1+\theta_2) + \sin(\theta_1+\theta_2)]$$

for $z = r(\cos\theta + i\sin\theta)$

$$z^2 = r^2(\cos(2\theta) + i\sin(2\theta))$$

$$z^3 = r^3(\cos(3\theta) + i\sin(3\theta))$$

De Moivre's Thm

Let $z = r(\cos\theta + i\sin\theta)$, $n \in \mathbb{Z}^+$

then $z^n = [r(\cos\theta + i\sin\theta)]^n$

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

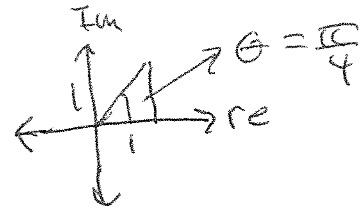
If $r>1$, z^n lies further away from the origin as n increases

i.e. if $r>1$, $r^n > r^{n-1}$

If $r<1$, z^n lies closer to the origin as n increases

for $0<r<1$, $r^n < r^{n-1}$

Ex Find $z=(3+3i)^4$



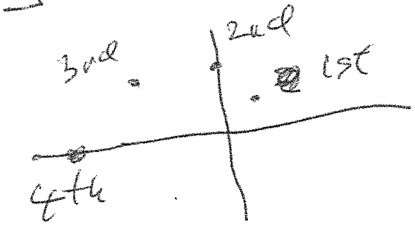
rewrite $z=\sqrt{3}(1+i)^4$

$r=\sqrt{3}$, $\theta=\frac{\pi}{4}$

So $Z^4=4\cdot 3^4\left[\cos\left(4\frac{\pi}{4}\right)+i\sin\left(\frac{4\pi}{4}\right)\right]$

$=4\cdot 81\left[\cos\pi + i\sin\pi\right]$

$=4\cdot 81\left[-1+i(0)\right]=-4(81)$



$$\begin{array}{r}81\\ \times\ 4\\ \hline 324\end{array}$$

We now use De Moivre's Thm to find the nth root of a complex number. ~~Def~~ an nth root of the complex number z is a complex number w s.t.

$$w^n=z$$

Letting $w=s(\cos\phi+i\sin\phi)$

$z=r(\cos\theta+i\sin\phi)$

we get $s^n(\cos n\phi+i\sin n\phi)=r(\cos\theta+i\sin\theta)$

and $\sin n\phi=\sin\theta$

so we need $s^n=r$ and $\cos n\phi=\cos\theta$

$s=r^{\frac{1}{n}}$

so $n\phi=\theta+2k\pi$

$\phi=\frac{\theta+2k\pi}{n}$

So $w=r^{\frac{1}{n}}\left[\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$

<u>Def</u> Roots of a complex number

Let $z = r(\cos\theta + i\sin\theta)$ and $n \in \mathbb{Z}^+$

Then z has the distinct roots

$$w_k = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

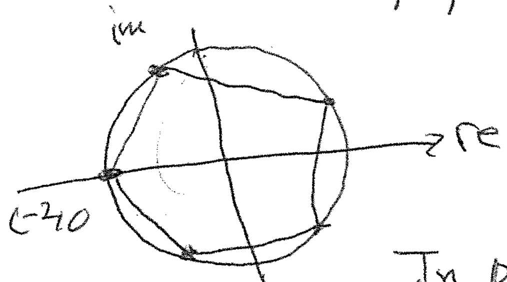
$k = 0, 1, 2, \dots, n-1$

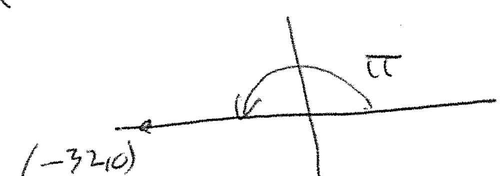
<u>Note</u> All the roots are spaced evenly on the same circle

<u>Ex</u> Find all solutions to $z^5 + 32 = 0$ i.e. solve $z^5 = -32$

<u>Note</u> You can get one real solutions, $z = -2$, without complex variables. To find the other 4 solutions we need complex variable.

We can graph the solutions without finding





In polar form: $-32 = 32\left[\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)\right]$

k an integer.

Letting $z = r(\cos\theta + i\sin\theta)$

We have $z^5 = r^5\{\cos(5\theta) + i\sin(5\theta)\}$

$z^5 = 32\{\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)\}$

So $r^5 = 32$ so $r = 2$

$5\theta = \pi + 2k\pi$

$\theta = \dfrac{\pi + 2k\pi}{5}$, with $k = 0, 1, 2, 3, 4$

so $z = 2\left[\cos\left(\dfrac{\pi + 2k\pi}{5}\right) + i\sin\left(\dfrac{\pi + 2k\pi}{5}\right)\right]$

$k = 0, 1, 2, 3, 4$

If $k = 0$, $z = z_1 = 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$

$z_2 = 2\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$

$z_3 = 2\left(\cos\frac{5\pi}{5} + i\sin\frac{5\pi}{5}\right)$

$= 2((\cos\pi) + i\sin\pi)$

$= 2(-1 + 0i) = -2$

$z_4 = 2\left(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right)$

$z_5 = 2\left(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}\right)$

Ex Find the roots of $(-1+i)^{\frac{1}{3}}$

Start with $-1+i = \sqrt{2}\left\{\cos\frac{3\pi}{4} + 2k\pi\right\} + i\sin\left(\frac{3\pi}{4} + 2k\pi\right)$

So $(-1+i)^{\frac{1}{3}} = 2^{\frac{1}{6}}\left\{\cos\left(\dfrac{3\pi/4 + 2k\pi}{3}\right) + i\sin\left(\dfrac{3\pi/4 + 2k\pi}{3}\right)\right\}$

so with $k=0$, $z_1 = 2^{\frac{1}{6}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$k=1$, $z_2 = 2^{\frac{1}{6}}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$

$k=2$, $z_3 = 2^{\frac{1}{6}}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$

Complex Exponentials

want to define $e^{z} = e^{x+iy}$

Recall that if x is a real number.

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Recall x by $z = a+bi$

$$e^{z} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

This complex functions follows the rules of exponents

i.e. $e^{z_1+z_2} = e^{z_1}e^{z_2}$

Let $z = iy$, $y \in \mathbb{R}$, use $i^2 = -1$, $i^3 = i^2(i) = -i$, $i^4 = 1$, $i^5 = i$

So $e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \cdots$

$$= 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!}$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)$$

Euler's Formula

$$\boxed{e^{iy} = \cos y + i \sin y}$$

So $e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$

$$e^{\pi i} = e^0 e^{\pi i} = e^0(\cos\pi + i\sin\pi) =$$
$$= 1(\cos\pi + i\sin\pi)$$

$$e^{\pi i} = -1 + i(0)$$
$$e^{\pi i} = -1$$

$$\boxed{e^{\pi i} + 1 = 0}$$