Ex Sketch the curve given by

$$x = 3\cos\theta, \quad y = 4\sin\theta \qquad 0 \le \theta \le 2\pi$$

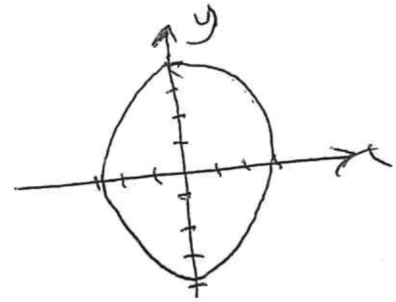
by elimination $\theta$ and finding the corresponding rectangular equation.

Solution: $\cos\theta = \frac{x}{3}$, $\sin\theta = \frac{y}{4}$

Use $\cos^2\theta + \sin^2\theta = 1$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$$

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$$

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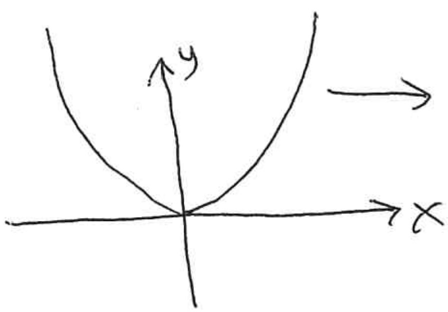
Ex $\left.\begin{array}{l} x = e^{t} \\ y = e^{2t} \end{array}\right\}$ Solution. $x = e^t$

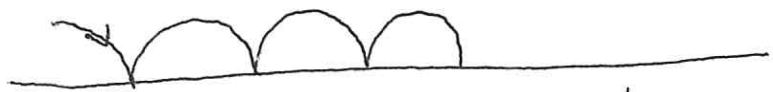
$y = (e^t)^2$, $y = x^2$




$\longrightarrow$ a parabola

022 Sec 10.1 Friday, Feb 14, 2020 ②

The cycloid

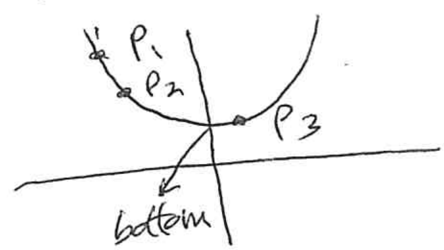


Roll a wheel along a flat straight line
Map out the path of a particle on the edge of the wheel
The equations (parametric) of a cycloid are

$x = r(\theta - \sin\theta)$
$y = r(1 - \cos\theta)$

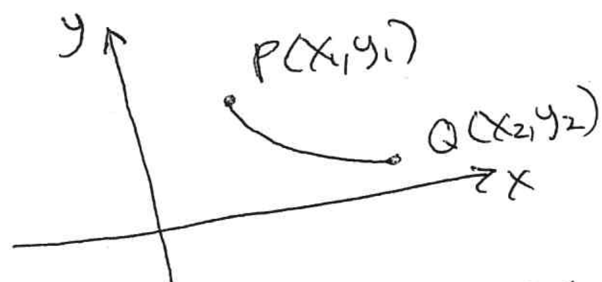
The cycloid gives the solution to two classic problems in mechanics

1) The Tautochrone prome
same time




A particle is acted upon
by gravity and only gravity ~~What should the curve~~
The ~~particle moves along a curve,~~ ~~the distance in the same~~
~~be so that the~~ particle moves the ... What should the shape
~~period~~ Think of it this way. What should the shape
of the curve be so that two particles will reach
the bottom of the curve at the same time, if they
start moving at the same time. Answer part of an inverted
cycloid.

The $\underbrace{\text{Brachisto}}_{\text{shortest}}\underbrace{\text{chrone}}_{\text{time}}$ Problem




The only force acting on the
particle is gravity.
What is the path from P to Q of shortest time.
Answer part of an inverted cycloid.

Calculus of curves given parametrically

<u>Tangent</u> If f and g are differentiable functions and we want to find the tangent line at a point on the curve given parametrically $x = x(t)$, $y = y(t)$. The chain rule gives

$$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt}$$

If $\frac{dx}{dt} \neq 0$, then $\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$ (1)

We can use (1) to find the slope $\frac{dy}{dx}$ of the tangent line to a parametric curve without having to eliminate the parameter $t$.

<u>Note</u> If the curve has a horizontal tangent, then $\frac{dy}{dt} = 0$

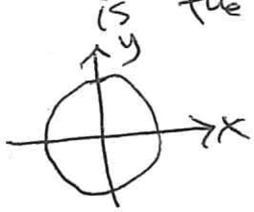
" " " " " vertical " , " $\frac{dx}{dt} = 0$.

To find the second derivative: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

Ex Find $\frac{dy}{dx}$ for the curve given by $x=\cos t$, $y=\sin t$

Solution $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\sin t}{\frac{d}{dt}\cos t} = \frac{\cos t}{-\sin t} = -\cot(t)$

Note The graph of the curve given by: $x=\cos t$, $y=\sin t$ is the unit circle, centered at the origin,




$x^2+y^2=1$. In calc I, you found $\frac{dy}{dx}$ using implicit differentiation

$$\frac{dy}{dx} = \frac{-x}{y} = \frac{-\cos t}{\sin t} = -\cot(t)$$

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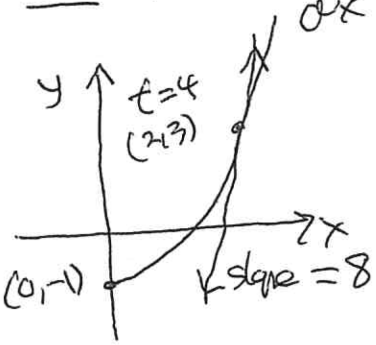
Ex Find $\frac{dy}{dx}$ for the curve given by: $x=\sqrt{t}=t^{\frac{1}{2}}$, $y=\frac{1}{4}(t^2-4)$, $t\geq 0$




Solution $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(\frac{t}{2})t}{\frac{1}{2}t^{-\frac{1}{2}}} = t^{3/2}$

So at $(x,y) = (2,3)$, i.e. when $t=4$

$$\frac{dy}{dx} = 4^{3/2} = 8$$

To test for concavity $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt} = \frac{\frac{d}{dt}(t^{3/2})}{\frac{dx}{dt}} = \frac{\frac{3}{2}t^{\frac{1}{2}}}{\frac{1}{2}t^{-1/2}}$

$\frac{d^2y}{dx^2} = 3t$. So at $(2,3)$, $t=3\cdot4=12>0$

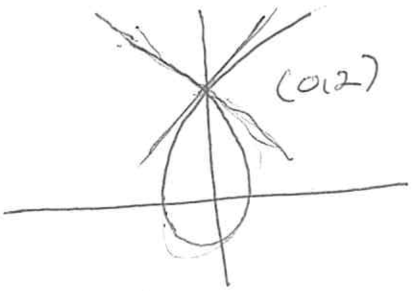
~~So at (3,4), t=~~

So the curve is concave up

at $t=4$, $(x,y)=(2,3)$

Ex A curve with two tangents at a point

Prolate cycloid




$$x = 2t - \pi \sin t$$
$$y = 2 - \pi \cos t$$

Find equations of both tangent lines.

Solution

For $(x,y) = (0,2)$, we have $t = -\frac{\pi}{2}$ or $t = +\frac{\pi}{2}$

And $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\pi \sin t}{2 - \pi\cos t}$

We have $\frac{dy}{dx} = -\frac{\pi}{2}$ when $t = -\frac{\pi}{2}$

$\frac{dy}{dx} = \frac{\pi}{2}$ when $t = \frac{\pi}{2}$

We have our slopes, our pt is $(0,2)$

When $t = -\frac{\pi}{2}$, ~~the~~ an equation of the tangent line is

$y - 2 = -\frac{\pi}{2}x$, $y = -\frac{\pi}{2}x + 2$

When $t = +\frac{\pi}{2}$, an equation of the tangent line is

$y - 2 = \frac{\pi}{2}x$, or $y = \frac{\pi}{2}x + 2$