Probable Topics for test scheduled for Thurs, March 19th.

1) Limit of sequences 11.1
2) Series - including geometric series 11.2
3) Integral test to bound the sum of a series 11.3
4) Comparison Test 11.4
5) Alternating Series 11.5
6) Absolute Convergence, ratio test 11.6
7) Tangents to a plane curve.

Section 11.6 Absolute Convergence - ratio and root test.

Def A series $\sum a_n$ is said to be absolutely convergent if $\sum |a_n|$ converge

Ex Any convergent series with only positive terms or only negative terms.

Ex $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^3}$ is <u>absolutely convergent</u> because $\sum_{i=1}^{\infty} \frac{1}{n^3}$ converges. since $\sum \frac{1}{n^3}$ is a p-series, $p = 3 > 1$.

Def If $\sum a_n$ converges but $\sum |a_n|$ diverges, the original series $\sum a_n$ is said to be <u>conditionally convergent</u>

Ex The alternating harmonic series: $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges

But the harmonic series $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

So, the alternating harmonic series is conditionally convergent

<u>Thm</u> If Σa_n is absolutely convergent then it is convergent.

<u>Ex</u> $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$. This series will have negative terms but not every term will be negative.

<u>Note</u> For all n, $-1 \le \sin(n) \le 1$

so $\displaystyle \frac{-1}{n^2} \le \frac{\sin(n)}{n^2} \le \frac{1}{n^2}$ for all n.

Now $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, p-series, $p = 2 > 1$.

So $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges, So by the comparison thm, $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges

<u>Ex</u> $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n}$, $\quad -\frac{1}{2} + \frac{2}{4} - \frac{6}{8} + \frac{24}{16}$

$\displaystyle\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n!}{2^n} = \infty$, so $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{2^n}$ diverges

$$\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2}$$

Ex $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$. This is an alternating series.

Note $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ diverges, p-series $p = \frac{1}{3} < 1$

So $\sum \frac{(-1)^n}{\sqrt[3]{n}}$ is not absolutely convergent

in the alternating series

But $b_n \geq 0$, $b_{n+1} < b_n$, $\lim_{n \to \infty} b_n = 0$

So the original series converges by the alter series test,

So $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ is conditional convergent.

Non Ex $\sum_{n=1}^{\infty} \frac{(-1)^{\frac{n(n+1)}{2}}}{3^n} = -\frac{1}{3} - \frac{1}{9} + \frac{1}{27} + \frac{1}{81} - - + +$

This is not an alternating series, so don't use the alternating series test.

But $\sum_{n=1}^{\infty} \left| \frac{(-1)^{\frac{n(n+1)}{2}}}{3^n} \right| = \sum_{n=1}^{\infty} \frac{1}{3^n}$, which is a geometric series with $r = \frac{1}{3} < 1$, so it converges

so our orginal series is absolutely convergent

The ratio test

1) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L<1$, then $\sum a_n$ is absolutely convergent hence convergent.

2) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L>1$, or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$

Then $\sum a_n$ diverges

3) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, no conclusion can be drawn.

i.e. $\sum a_n$ might converge

$\sum a_n$ absolutely convergent

$\sum a_n$ conditionally convergent

$\sum a_n$ diverge

Ex $\sum_{n=1}^{\infty} \frac{n!}{2^n}$; $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)!}{2^{n+1}}}{\frac{n!}{2^n}}\right|$

$= \lim_{n\to\infty} \left|\frac{(n+1)!\,2^n}{2^{n+1}\,n!}\right| = \lim_{n\to\infty} \frac{n+1}{2} = \infty > 1$

So $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ diverges

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 2^{n+1}}{3^n}$: $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2 2^{(n+1)+1}}{3^{n+1}} \cdot \frac{3^n}{n^2 2^{n+1}}\right|$

$$= \lim_{n\to\infty}\left|\frac{2}{3}\,\frac{(n+1)^2}{n^2}\right| = \frac{2}{3}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 = \frac{2}{3}\cdot 1 = \frac{2}{3} < 1$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ is absolutely convergent

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Ex $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{(n+1)!}\,\frac{n!}{n^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{(n+1)n^n}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)^n}{n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left[1+\frac{1}{n}\right]^n = e > 1$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges

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Non Ex $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+1}}{(n+1)+1}\cdot\frac{n+1}{\sqrt{n}}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+1}\,(n+1)}{\sqrt{n}\,(n+2)}\right| = 1$$

$$\cancel{\lim_{n\to\infty}}\ \lim_{n\to\infty}\left|\frac{\sqrt{n+1}}{\sqrt{n}}\right| \lim_{n\to\infty}\frac{(n+1)}{(n+2)} = 1\cdot 1 = 1$$

So, the ratio test does not work.

~~We now use the alternating series test~~

Note i) $\dfrac{\sqrt{n}}{n+1} \to 0$ as $n\to\infty$, so $a_n \to 0$

ii) $\dfrac{\sqrt{n}}{n+1} > \dfrac{\sqrt{n+1}}{n+2}$ i.e. $a_n > a_{n+1}$

iii) The terms alternate in sign

So $\displaystyle\sum (-1)^n \frac{\sqrt{n}}{n+1}$ converges.

but use a comparison with $\sum \frac{1}{\sqrt{n}}$ to show $\displaystyle\sum (-1)^n \frac{\sqrt{n}}{n+1}$ does not converge absolutely.