A polynomial being reducible, factorable depends on what numbers are allowed

x^2-3 is irreducible if you only allow integers

but x^2-3 is not irreducible if you allow all real numbers

$x^2-3=(x-\sqrt{3})(x+\sqrt{3})$

Likewise x^2+1 is irreducible if you only allow real numbers

but x^2+1 is reducible if you allow complex numbers

$x^2+1=(x-\sqrt{-1})(x+\sqrt{-1})$

Ex $I=\int \frac{2x+4}{x^3-2x^2}\,dx = \int \frac{2x+4}{x^2(x-2)}\,dx$

$$\frac{2x+4}{x^2(x-2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}=\frac{Ax(x-2)+B(x-2)+Cx^2}{x^2(x-2)}$$

So $2x+4=Ax(x-2)+B(x-2)+Cx^2$

We now multiply out the RHS and collect terms

$$2x+4=(A+C)x^2+(-2A+B)x-2B$$

$$\left.\begin{aligned} A+C&=0\\ -2A+B&=2\\ -2B&=4\end{aligned}\right\}\quad \begin{aligned}B&=-2\\ A&=-2\\ C&=2\end{aligned}$$

So, $\frac{2x+4}{x^2(x-2)}=-\frac{2}{x}-\frac{2}{x^2}+\frac{2}{x-2}$

So $I=\int\frac{2x+4}{x^2(x-2)}\,dx=-2\int\frac{dx}{x}-2\int\frac{dx}{x^2}+2\int\frac{dx}{x-2}$

$I=-2\ln|x|+\frac{2}{x}+2\ln|x-2|+C$

$I=2\ln\left|\frac{x-2}{x}\right|+\frac{2}{x}+C$

Ex $I = \int \frac{x^2+x-2}{(3x-1)(x^2+1)}\,dx$

$$\frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx+C}{x^2+1}$$

So $x^2+x-2 = A(x^2+1) + (Bx+C)(3x-1)$

$x^2+x-2 = (A+3B)x^2 + (-B+3C)x + (A-C)$

So
$A + 3B = 1$ (1)
$-B+3C = 1$ (2)
$A - C = -2$ (3)

From (3) $A = C-2$, substitute into (1)
(1) $C-2+3B = 1$
$C+3B = 3$
$C = 3-3B$, substitute into (2)
(2) $-B+3C = -B+3(3-3B) = 1$
$= -B+9-9B$, $B = \frac{-8}{-10} = \frac{4}{5}$
$-10B = -8$,

We can now find $A = -\frac{7}{5}$, $B = \frac{4}{5}$, $C = \frac{3}{5}$

So $\frac{x^2+x-2}{(3x-1)(x^2+1)} = \left(-\frac{7}{5}\right)\frac{1}{3x-1} + \left(\frac{4}{5}\right)\frac{x}{x^2+1} + \left(\frac{3}{5}\right)\frac{1}{x^2+1}$

So $I = -\frac{7}{5}\int \frac{1}{3x-1}\,dx + \frac{4}{5}\int \frac{x}{x^2+1}\,dx + \frac{3}{5}\int \frac{dx}{x^2+1}$

use $u = 3x-1$, $du = 3x\,dx$ $\qquad$ $u = x^2+1$, $du = 2x\,dx$

$$I = -\frac{7}{15}\ln|3x-1| + \frac{2}{5}\ln|x^2+1| + \frac{3}{5}\tan^{-1}x + C$$

EX $I = \int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx$

Since the degree of the numerator is greater than the degree of the denominator we have to divide

$$
\begin{array}{r l}
 & 3x^2 \quad +1 \quad r\ 1 \\
x^2+x-2 \,) & 3x^4+3x^3-5x^2+x-1 \\
 & -(3x^4+3x^3-6x^2) \\
 & x^2+x-1 \\
 & -(x^2+x-2) \\
 & 1
\end{array}
$$

so, $\frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2} = 3x^2+1+\frac{1}{x^2+x-2}$

so $I = \int \left(3x^2+1+\frac{1}{x^2+x-2}\right)dx$

$= x^3+x+\int \frac{1}{x^2+x-2}\,dx$

now $\frac{1}{x^2+x-2} = \frac{1}{(x+2)(x-1)} = \frac{A}{x+2}+\frac{B}{x-1} = \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$

so, $1 = A(x-1)+B(x+2)$

if $x=1$, $1=3B$ so $B=\frac{1}{3}$

if $x=-2$, $1=-3A$, so $A=-\frac{1}{3}$

so $I = x^3+x+\int\left(-\frac{1}{3}\right)\frac{1}{x+2}\,dx + \int\left(\frac{1}{3}\right)\frac{1}{x-1}\,dx$

$I = x^3+x-\frac{1}{3}\ln|x+2|+\frac{1}{3}\ln|x-1|+C$

$I = x^3+x+\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|+C$

Ex $\int \frac{8x^3+13x}{(x^2+2)^2}\,dx$

$$\frac{8x^3+13x}{(x^2+2)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2}$$

$$8x^3+13x = (Ax+B)(x^2+2) + Cx+D$$

$$8x^3+13x = Ax^3 + Bx^2 + (2A+C)x + (2B+D)$$

$8 = A$

$B = 0$

$(2A+C) = 13, \Rightarrow C = -3$

$(2B+D) = 0 \Rightarrow D = 0$

$$I = \int \left(\frac{8x}{x^2+2} + \frac{-3x}{(x^2+2)^2}\right)dx$$

$$= 4\ln(x^2+2) + \frac{3}{2(x^2+2)} + C$$