Limit Comparsion test

Let $\sum a_n$, $\sum b_n$ have only positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is a positive finite number,

Then either both series converge or both series diverge

Pf Let m, M be s.t. $0 < m < c < M$

since $\frac{a_n}{b_n} \to c$, there is a N s.t.

$$m < \frac{a_n}{b_n} < M \quad \text{when } n > N$$

If $\sum b_n$ converges then $\sum M b_n$ converges

This implies $\sum a_n$ converges by the comparsion test

If $\sum b_n$ diverges, then $\sum m b_n$ diverges

So, by the comparsion thm, $\sum a_n$ diverges

Ex $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$, as $n \to \infty$, the looks like

$\sum \frac{\sqrt{n}}{n^2} = \sum \frac{1}{n^{3/2}}$. This is a p-series with

$p = \frac{3}{2} > 1$, so it converges.

Now $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n^2+1}}{1/n^{3/2}} = \lim_{n\to\infty} \left(\frac{\sqrt{n}}{n^2+1}\right)\left(\frac{n^{3/2}}{1}\right)$

$= \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1$. and 1 in in $(0,\infty)$

So, by the limit comparsion test, $\sum \frac{\sqrt{n}}{n^2+1}$ converges

Ex $\sum_{n=1}^{\infty} \frac{n2^n}{4n^3+1}$. Compare with $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

Consider $\lim_{x\to\infty} \frac{2^x}{x^2} \overset{H.}{=} \lim_{x\to\infty} \frac{(\ln 2)2^x}{2x} \overset{H.}{=} \frac{(\ln 2)^2 2^x}{2} \to \infty$

so $a_n \not\to 0$, so $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ diverges

Now $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n2^n}{4n^3+1}}{\frac{2^n}{n^2}}$

$= \lim_{n\to\infty} \frac{n\cdot 2^n \cdot n^2}{(4n^3+1)2^n} = \lim_{n\to\infty} \frac{1}{4+\frac{1}{n^3}} = \frac{1}{4}$

now $0 < \frac{1}{4} < \infty$

So since $\sum \frac{2^n}{n^2}$ diverges, the limit comparison test

gives that $\sum_{n=1}^{\infty} \frac{n2^n}{4n^3+1}$ diverges

Section 11.5 Alternating Series

An alternating series is a series with the terms alternating from + to −.

In closed form, an alternating series usually has a factor of −1.

If we have a series with all positive terms, b_n. $\sum b_n$.

We can form an alternating series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} b_n$$

Ex The harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

yields the alternating harmonic series.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$$

The alternating series test

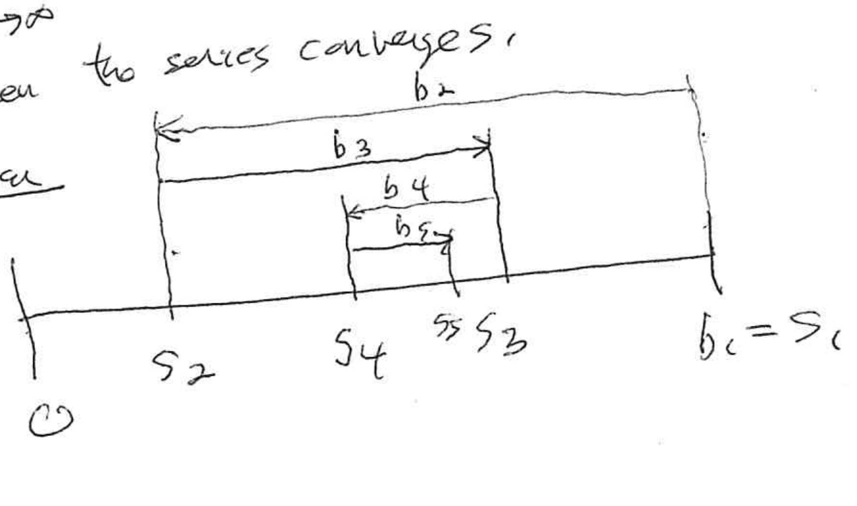
If $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4$ is s.t.

i) $b_{n+1} \le b_n$ for sufficiently large n

ii) $\lim_{n\to\infty} b_n = 0$

Then the series converges.

Motivation



<u>Proof</u> of the alternating series test.

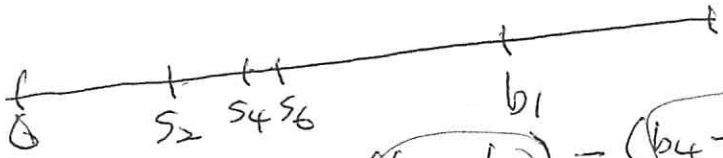
Every partial sum is either an even partial sum or an odd partial sum.

We will show that the subsequence of even partials sums converge up to S. The subsequence of odd partial sums converge down to S

So the entire sequence will converge to S,

In general: $S_{2n} = S_{2n-2} + (b_{2n-1} - b_{2n}) \geq S_{2n-2}$

so $0 \leq S_2 \leq S_4 \leq S_6 \leq \ldots\ldots \leq S_{2n}$



Also, $S_{2n} = b_1 - \underbrace{(b_2 - b_3)}_{\geq 0} - \underbrace{(b_4 - b_5)}_{\geq 0} - \cdots - \underbrace{(b_{2n-2} - b_{2n-1})}_{\geq 0} - b_{2n}$

So $S_{2n} \leq b_1$, for all n

So, the sequence of partial sums $\{S_{2n}\}$ is increasing and bounded above.

So, by the Monotone convergence thm, it converges

Say $\lim_{n\to\infty} S_{2n} = S$.

For the odd partial sums: $\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} (S_{2n} + b_{2n+1})$

$= \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} b_{2n+1} = S + 0 = S$

So, the even partial sums converge to S

the odd " " " " S

so the series ~~conver~~ converge to S

<u>Ex</u> The alternating harmonic series

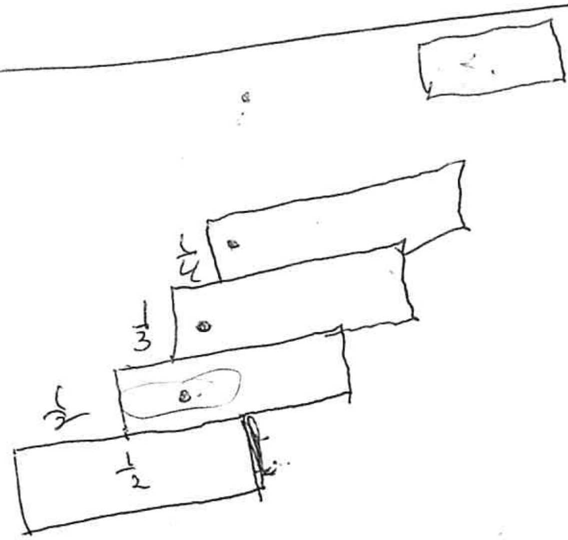
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

has $b_{n+1} < b_n$, since $\frac{1}{n+1} < \frac{1}{n}$

and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

So the series converges

Turns out $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$



<u>Non Ex</u> $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 5n}{3n+7}$

The terms do alternate in sign

but $\displaystyle\lim_{n\to\infty} |a_n| = \frac{5}{3} \neq 0$, so

the alternating series test does not apply

<u>Note</u> $\displaystyle\lim_{n\to\infty} S_{2n} = +\infty$ $\displaystyle\lim_{n\to\infty} S_{2n-1} = -\infty$

<u>Ex</u> $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$, clearly the series alternates

so $b_n = \dfrac{n^2}{n^3+1} > 0, \ \forall n$

and $\displaystyle\lim_{n\to\infty} b_n = 0$

To check to see if the sequence is decreasing

Let $f(x) = \dfrac{x^2}{x^3+1}$

so $f'(x) = \dfrac{(x^3+1)2x - x^2(3x^2)}{(x^3+1)^2} = \dfrac{2x - x^4}{(x^3+1)^2}$

so $f'(x) < 0$ for large n.

so f is decreasing, ~~so $\sum$~~

so $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ converges

022 Sec 11.5 Thurs, March 5, 2020

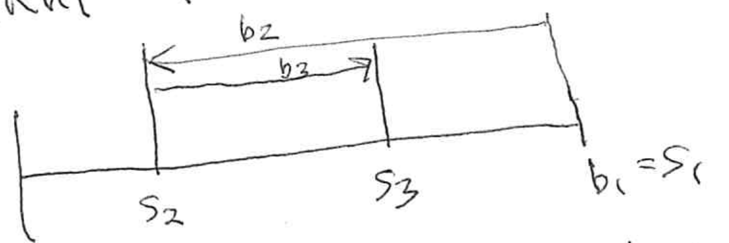
Estimating error with an alternating.

Let $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ with

i) $b_n > 0$, ii) $b_{n+1} \le b_n$, iii) $\lim_{n\to\infty} b_n = 0$

Then $|R_n| = |S - S_n| \le b_{n+1}$

Idea



Ex $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{1}{n!}\right) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$

Note $\frac{1}{(n+1)!} < \frac{1}{n!}$ and $\lim_{n\to\infty} \frac{1}{n!} = 0$

so, the ~~conver~~ converges.

Estimate $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{1}{n!}\right)$ with the first six terms.

Solution $S_6 = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} = \frac{91}{144}$

$S_6 \approx 0.63194$

error $|S - S_6| = |R_6| \le a_7 = \frac{1}{5040} = \frac{1}{7!}$

≈ 0.0002

so $.63194 - 0.0002 \le S \le 0.63194 + 0.0002$

$0.63174 \le S \le 0.63214$