Section 2.6 Limits at infinity; Horizontal Asymptotes

We are concerned with the "long term" behavior of the function $f(x)$

i.e. $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

A function have ~~have~~ any number of vertical asymptotes (the number has to be a non-neg integer)

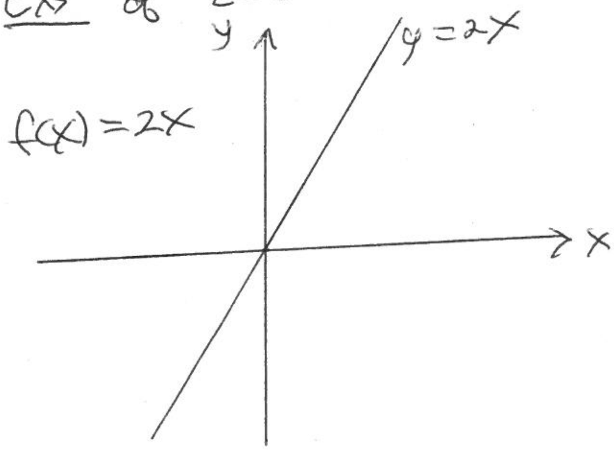
Here is an example of a function with 4 vertical asymptotes

$$g(t) = \frac{1}{(t-3)(t+4)^2(t-1)t}$$

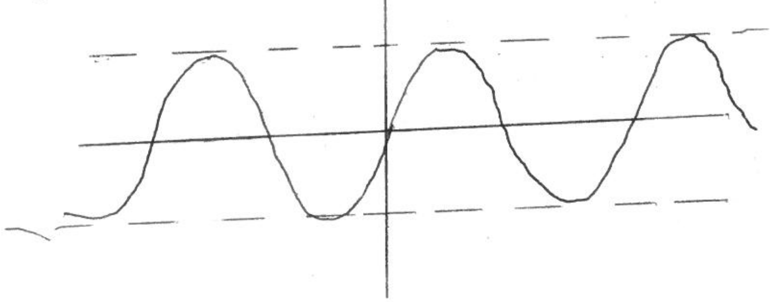
The situation for ~~vertical~~ horizontal asymptotes is simpler

A function $f(x)$ can have 0, 1 or 2 horizontal asymptotes.

Exs of zero horizontal asymptotes

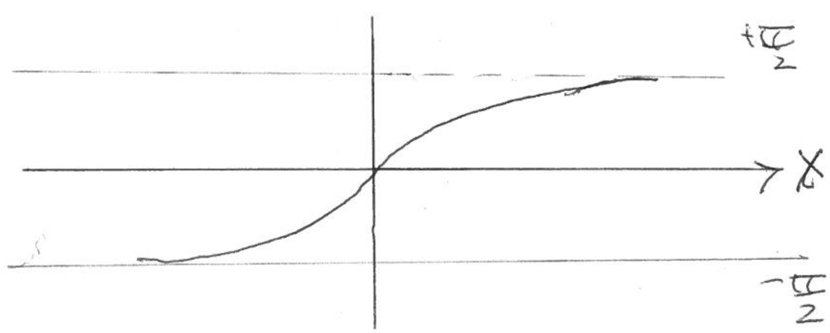


$g(t) = \sin t$



Ex of two horizontal asymptotes

$y = f(x) = \tan^{-1}(x)$

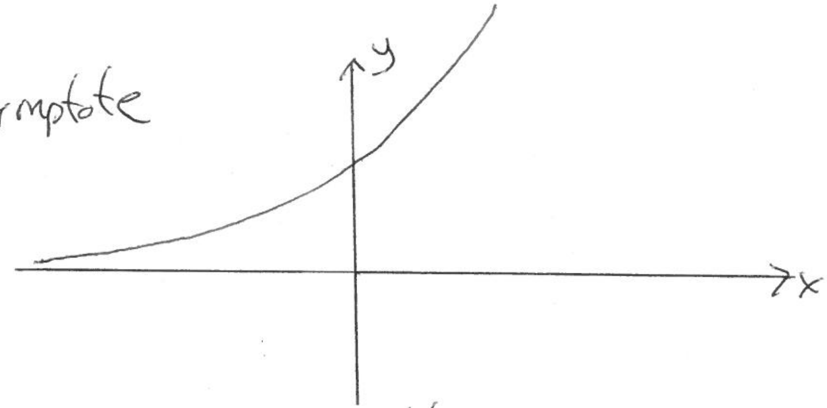


Note $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$, so $y = \frac{\pi}{2}$ is a horizontal asymptote

~~lim~~ $\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$, so $y = -\frac{\pi}{2}$ " " " "

<u>Ex</u> of 1 horizontal asymptote

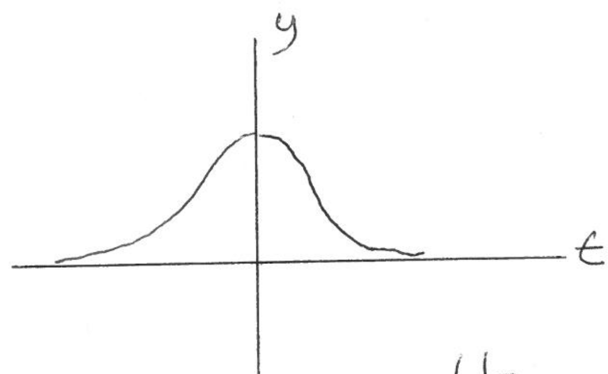
$y = f(x) = e^x$



$\lim_{x \to -\infty} f(x) = 0$

So $y = 0$ is a horizontal asymptote

<u>Ex</u> $y = g(t) = e^{-t^2}$

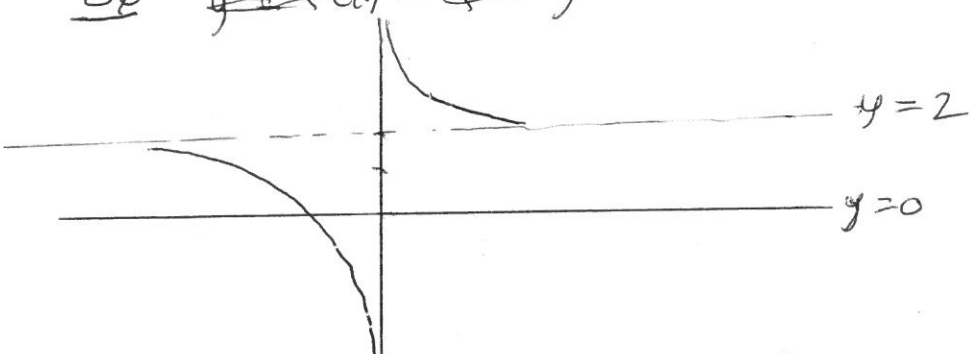


$\lim_{t \to \infty} g(t) = 0$

$\lim_{t \to -\infty} g(t) = 0$

So $y = 0$ is the only horizontal asymptote

<u>Ex</u> ~~$y = f(x) = e^x$~~ $y = f(x) = 2 + \frac{1}{x}$



$\lim_{x \to \infty} f(x) = 2$, $\lim_{x \to -\infty} f(x) = 2$

So, $y = 2$ is the only horizontal asymptote

Intuitive Def of a limit at infinity

Let f be a function defined on some interval $[a, \infty)$ then $\lim_{x \to \infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarly close to L as x becomes sufficiently large

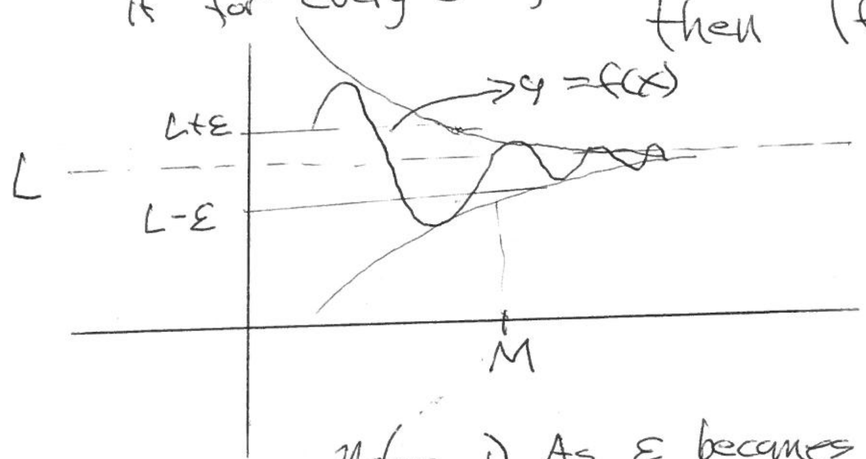
(The def of $\lim_{x \to -\infty} f(x) = L$ is similar)

More rigorously

$f(x)$ has a horizontal asymptote at $y = L$

i.e. $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$

if for every $\varepsilon > 0$, there is a M s.t. if $x > M$ then $|f(x) - L| < \varepsilon$



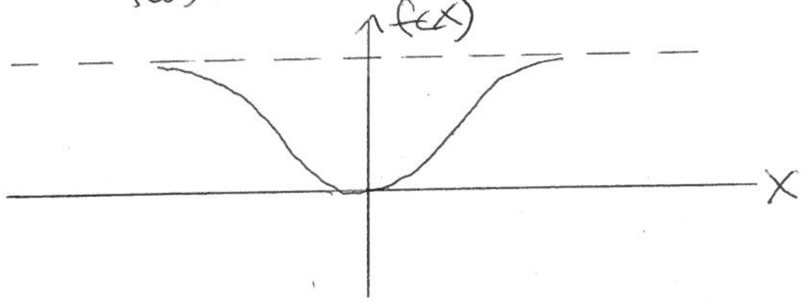
Notes 1) As ε becomes smaller, M becomes larger

2) A function can cross a horizontal asymptote (but not a vertical asymptote)

<u>Ex</u> Let $f(x) = \dfrac{3x^2}{x^2+1}$, note $f(x) = f(-x)$

so the graph is symmetric wrt the y-axis, i.e. f is an even function.

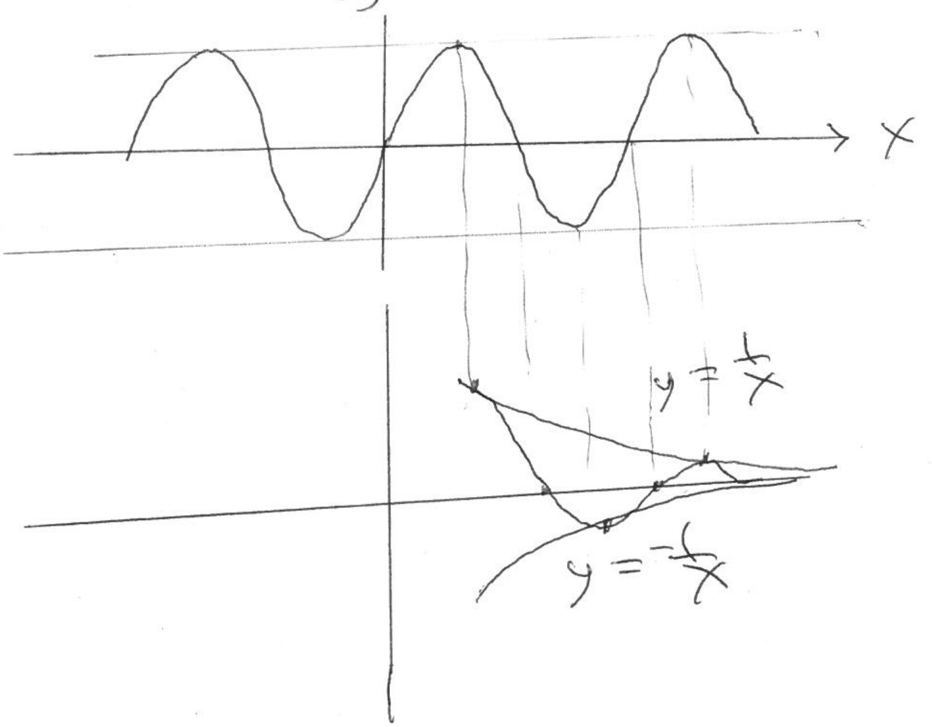
x	0	± 1	± 10	± 100
$f(x)$	0	1.5	2.97	2.997



Ex of a function crossing a horizontal asymptote

$y = f(x) = \frac{1}{x}\sin x$ on $[\frac{\pi}{2}, \infty)$

<u>Note</u> $-1 \le \sin x \le 1$, so we are multi the values of $\frac{1}{x}$ by values between -1 and $+1$



Thm If $r \geq 0$, then $\lim_{x\to\infty} \frac{1}{x^r} = 0$

$r=2, \ \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \ldots$

Application Find $\lim_{x\to\infty} \frac{4x^3-7}{-7x^3+8}$

Note 3 is the highest power of x that occurs

$$\lim_{x\to\infty} \frac{4x^3-7}{-7x^3+8} \frac{\left(\frac{1}{x^3}\right)}{\left(\frac{1}{x^3}\right)}$$

$$= \lim_{x\to\infty} \frac{\frac{4\cancel{x^3}}{\cancel{x^3}} - \frac{7}{x^3}}{\frac{-7\cancel{x^3}}{\cancel{x^3}} + \frac{8}{x^3}} = \lim_{x\to\infty} \frac{4 - \frac{7}{x^3}}{-7+\frac{8}{x^3}}$$

$$= \frac{4 - \lim_{x\to\infty} \frac{7}{x^3}}{-7 + \lim_{x\to\infty}\frac{8}{x^3}} = \frac{4 - 7\lim_{x\to\infty}\frac{1}{x^3}}{-7+8\lim_{x\to\infty}\frac{1}{x^3}} = \frac{4-7(0)}{-7+8(0)} = \frac{-4}{7}$$

In general, we have

$$\lim_{x\to\pm\infty} \frac{ax^m + \text{lower terms}}{bx^n + \text{lower terms}} = L$$

and $L = 0$ if $n > m$

$L = \frac{a}{b}$ if $n = m$

$L = +\infty$ or $L = -\infty$ if $m > n$

Ex $\lim_{x\to+\infty} \frac{\sqrt{3x^4+6}}{5x^2-2}$,

$\sqrt{x^4} = x^2$

The highest power of x is 2.

$$= \lim_{x\to\infty} \frac{\sqrt{3x^4+6}}{5x^2-2}\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right) = \lim_{x\to\infty} \frac{\sqrt{\frac{3x^4}{x^4}-\frac{6}{x^4}}}{\frac{5x^2}{x^2}-\frac{2}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{3+\frac{6}{x^4}}}{5-\frac{2}{x^2}} = \frac{\sqrt{3}}{5}$$

021 Sec 2.6

Ex $\lim_{x \to -\infty} \frac{\sqrt{2x^2+1}}{3x-5}$, highest power of x is 1

Note if $x = -6$, $|x| = 6 = -(-6) = -x$

So, if $x < 0$, $\sqrt{x^2} = |x| = -x$

So $\frac{1}{x} \sqrt{2x+1} = \frac{-1}{\sqrt{x^2}} \sqrt{2x^2+1}$

$= -\sqrt{\frac{2x^2}{x^2} + \frac{1}{x^2}} = -\sqrt{2 + \frac{1}{x^2}}$

So $\lim_{x \to -\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x \to -\infty} \frac{-\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = -\frac{\sqrt{2}}{3}$

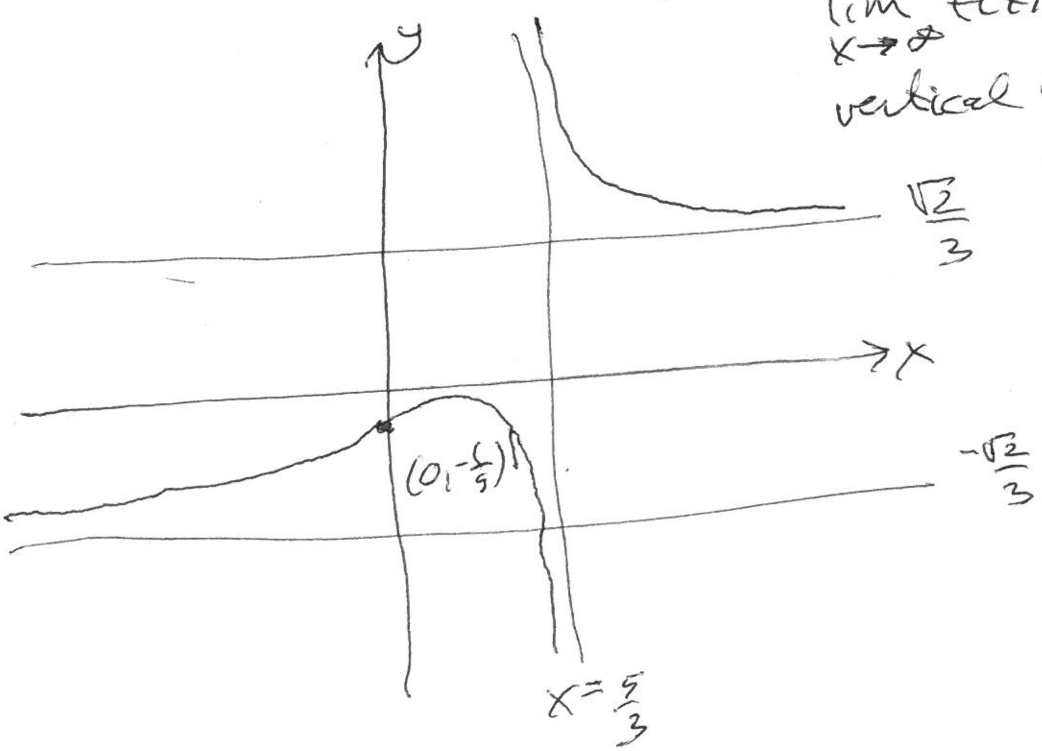
Now find $\lim_{x \to \infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x \to \infty} \frac{\sqrt{2x^2+1}}{3x-5} \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}$

$= \lim_{x \to \infty} \frac{\sqrt{\frac{2x^2}{x^2} + \frac{1}{x^2}}}{\frac{3x}{x} - \frac{5}{x}} = \lim_{x \to \infty} \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = \frac{\sqrt{2}}{3}$

Graph $y = f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$, $\left.\begin{array}{l} \lim_{x \to -\infty} f(x) = -\frac{\sqrt{2}}{3} \\ \lim_{x \to \infty} f(x) = \frac{\sqrt{2}}{3} \end{array}\right\}$ horizontal asymptotes

vertical asymptote $x = \frac{5}{3}$

$f(0) = -\frac{1}{5}$



2.6 Find $\lim_{x \to -\infty} \left(\sqrt{64x^2+9x} + 8x\right)$

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$$= \lim_{x \to -\infty} \left(\sqrt{64x^2+9x} + 8x\right)\left(\frac{\sqrt{64x^2+9x} - 8x}{\sqrt{64x^2+9x} - 8x}\right)$$

$$= \lim_{x \to -\infty} \frac{\cancel{64x^2} + 9x - \cancel{64x^2}}{\sqrt{64x^2+9x} - 8x}$$

$$= \lim_{x \to -\infty} \frac{9x}{\sqrt{64x^2+9x} - 8x}$$

The highest power of x that occurs is $x^1 = x$

But as $x \to -\infty$, we use $\sqrt{x^2} = |x| = -x$. This affects the denominator

So $$\lim_{x \to -\infty} \frac{\frac{9x}{x}}{\left(\sqrt{64x^2+9x} - 8x\right)\left(\frac{1}{x}\right)}$$

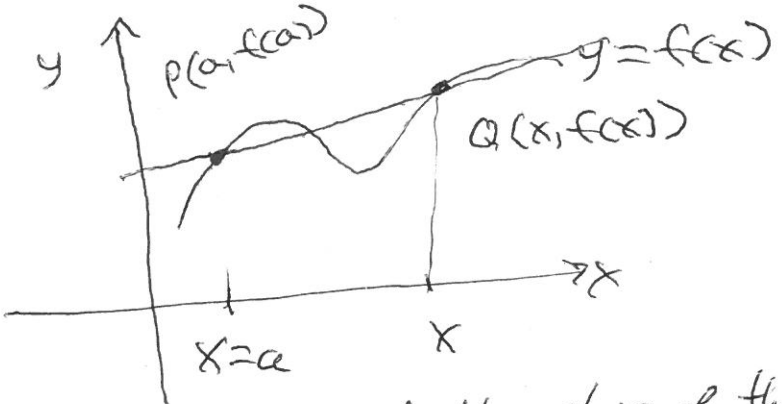
$$= \lim_{x \to -\infty} \frac{9}{\frac{\sqrt{64x^2+9x}}{-\sqrt{x^2}} - 8} = \lim_{x \to -\infty} \frac{9}{-\sqrt{\frac{64x^2+9x}{x^2}} - 8}$$

$$= \lim_{x \to -\infty} \left[\frac{9}{-\sqrt{64+\frac{9}{x}} - 8}\right] = \lim_{x \to -\infty} \left[\frac{9}{-\sqrt{64} - 8}\right]$$

$$= \frac{9}{-8-8} = \frac{-9}{16}$$

021 Sec 2.7

Section 2.7 Derivatives and rates of change



Recall that the slope of the secant line joining P and Q is

$$m_{PQ} = \frac{f(x) - f(a)}{x - a} = \frac{\Delta y}{\Delta x}$$

Def The tangent line to the curve $y = f(x)$ at the point P is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$ provided that this limit exists.

Def The derivative of $f(x)$ at P is

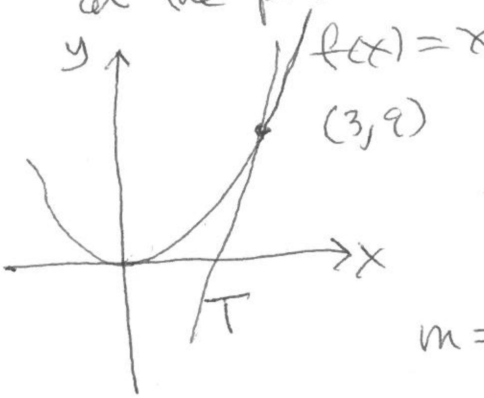
$$f'(x)\Big|_P = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Hence, the derivative of $f(x)$ at P is the limit of the slopes of the secant lines.

i.e. the derivative at P is the slope of the tangent line at P.

021 Sec 2.7

Ex Let $f(x) = x^2$. Find an equation of the tangent line at the point $(3, 9)$



Let m be the slope of the tangent line.

$$m = \lim_{x \to 3} \frac{f(x) - 3^2}{x - 3} = \lim_{x \to 3} \frac{x^2 - 3^2}{x - 3}$$

$$m = \lim_{x \to 3} \frac{(x-3)(x+3)}{(x-3)} = \lim_{x \to 3} (x+3) = 3 + 3 = 6$$

So slope $m = 6$, a point is $(3, 9)$

$$y = mx + b$$
$$y = 6x + b$$
$$9 = 6(3) + b$$
$$b = -9$$
$$y = 6x - 9$$

So ~~an~~ equation of T is: $y = 6x - 9$

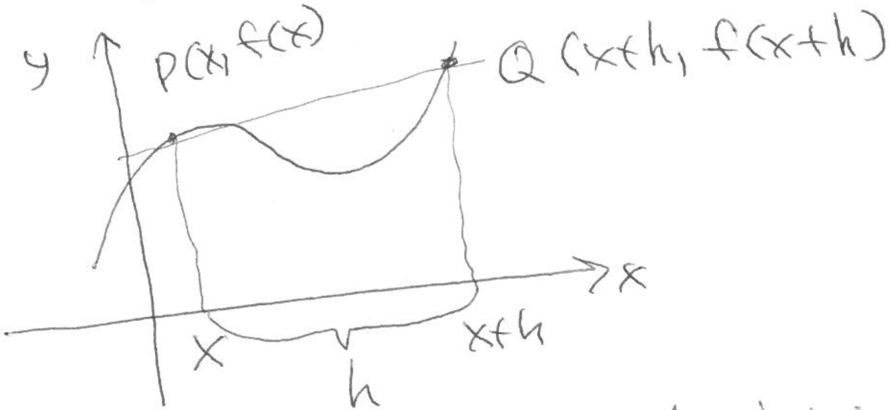
Def The derivative, $f'(a)$ is the instantaneous rate of change of $y = f(x)$ when $x = a$.

Aside The average rate of change of $f(x)$ on the interval $[x_1, x_2]$ is

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

021 Sec 2.7

An equivalent form for the derivative:



The slope of the secant line joining P and Q is

$$m_{PQ} = \frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$$

The derivative; (the slope of the tangent line
instantaneous rate of change)

is $$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

021 Sec 2.7

Ex From the definition of the derivative, find the derivative of,
$f(x) = 4x^2 - 3x + 7$ at the pt $(x, f(x))$

Solution $m = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$m = \lim_{h \to 0} \frac{[4(x+h)^2 - 3(x+h) + 7] - [4x^2 - 3x + 7]}{h}$$

$$= \lim_{h \to 0} \frac{(4x^2 + 8xh + 4h^2 - 3x - 3h + 7) - (4x^2 - 3x + 7)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{4x^2} + 8xh + 4h^2 - \cancel{3x} - 3h + \cancel{7} - \cancel{4x^2} + \cancel{3x} - \cancel{7}}{h}$$

$$= \lim_{h \to 0} \frac{8xh + 4h^2 - 3h}{h} = \lim_{h \to 0} \frac{\cancel{h}(8x + 4h - 3)}{\cancel{h}}$$

$$= \lim_{h \to 0} (8x + 4h - 3) = 8x - 3, \quad \boxed{f'(x) = 8x - 3}$$

Now find an equation of the tangent line when

$x = 0$, $f'(x) = m = 8(0) - 3 = -3$

pt $(0, f(0)) = (0, 7)$

$y = mx + b \Rightarrow$ $7 = 0x + b$, $b = 7$

$y = -3x + 7$

When $x = 1$: $m = 8(1) - 3 = 5$, $f(1) = 4 - 3 + 7 = 8$

so slope $m = 5$, pt is $(1, 8)$

$y = mx + b$ is $8 = 5(1) + b \Rightarrow b = 3$

$y = 5x + 3$

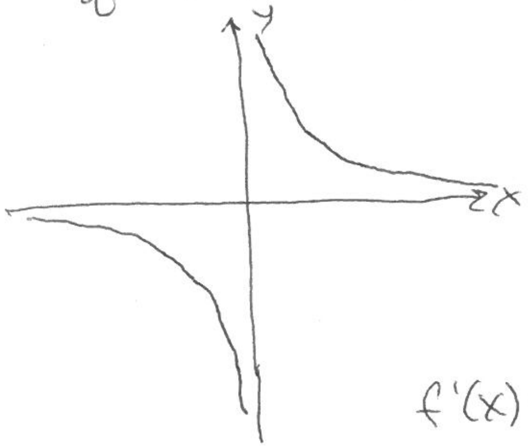
The derivative of a function $f(x)$ is another function $f'(x)$ st $f'(a)$ gives the slope of the tangent line to $f(x)$ when $x = a$.

Change the value of a, you usually change the value of the derivative

Different tangent lines, usually different slopes

More Examples of finding derivatives from the definition.

Ex Find the slope of the tangent line, i.e. find the derivative of $f(x)$ if $y = f(x) = \frac{1}{x}$, $x \neq 0$



$$f'(x) = m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

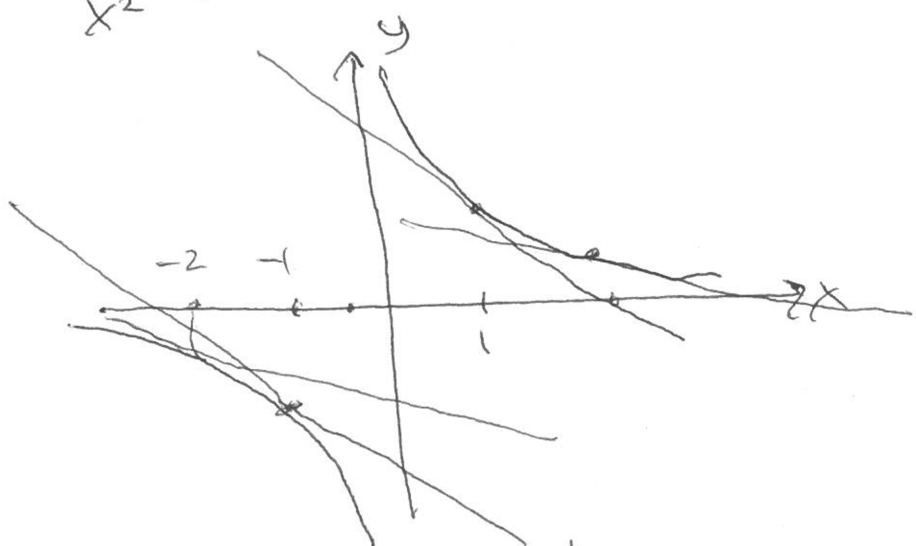
$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}, \text{ LCD is } (x+h)x$$

$$f'(x) = \lim_{h \to 0} \frac{\frac{x-(x+h)}{(x+h)x}}{h} = f'(x) = \lim_{h \to 0} \frac{\frac{-h}{(x+h)x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{-\cancel{h}}{(x+h)x}}{\cancel{h}} \left(\frac{\frac{1}{\cancel{h}}}{\frac{1}{\cancel{h}}} \right) = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

Aside $f'(x) = -\frac{1}{x^2}$ is an even functn so $f'(x) = f'(-x)$

x	$f'(x)$
± 1	-1
± 2	$-\frac{1}{4}$
± 3	$-\frac{1}{9}$

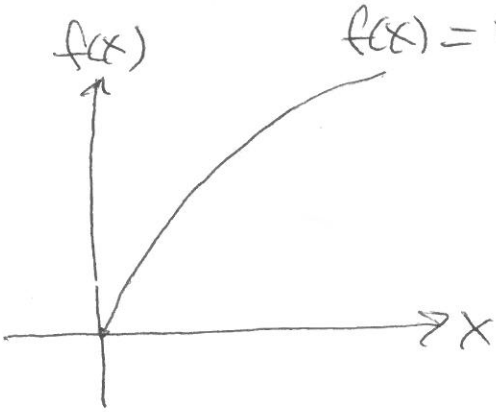


Fun fact: For $f(x) = \frac{1}{x}$, the tangent lines for $x = a$ and $x = -a$ are parallel

021 Sec2.7

From the def, find the derivative of

$f(x)=\sqrt{x}=x^{\frac{1}{2}}, \quad x>0$



$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$=\lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$$

$$=\lim_{h\to 0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{\sqrt{x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}=\frac{1}{2}x^{-\frac{1}{2}}$$

$$(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})$$

$$=(\sqrt{x+h})^2\cancel{+\sqrt{x}\sqrt{x+h}-\sqrt{x}\sqrt{x+h}}-(\sqrt{x})^2$$

$$=x+h-x$$

Q21 Sec 2.7

From the definition of a derivative, find the derivative of $g(t) = \frac{1}{\sqrt{t}}$

$$g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}$$

$$m = g'(t) = \lim_{h \to 0} \frac{\frac{1}{\sqrt{t+h}} - \frac{1}{\sqrt{t}}}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{1}{\sqrt{t+h}} - \frac{1}{\sqrt{t}}\right)\left(\frac{1}{h}\right)}{\cancel{h}\left(\frac{1}{\cancel{h}}\right)}$$

$$= \lim_{h \to 0} \left(\frac{1}{\sqrt{t+h}} - \frac{1}{\sqrt{t}}\right)\left(\frac{1}{h}\right)$$

$$= \lim_{h \to 0} \frac{\sqrt{t} - \sqrt{t+h}}{\sqrt{t+h}\sqrt{t}\, h}$$

$$= \lim_{h \to 0} \frac{\sqrt{t} - \sqrt{t+h}}{\sqrt{t+h}\sqrt{t}\, h}\left(\frac{\sqrt{t} + \sqrt{t+h}}{\sqrt{t} + \sqrt{t+h}}\right)$$

$$= \lim_{h \to 0} \frac{\cancel{t} - (\cancel{t}+h)}{\sqrt{t+h}\sqrt{t}\, h(\sqrt{t} + \sqrt{t+h})}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\sqrt{t+h}\sqrt{t}\,\cancel{h}(\sqrt{t} + \sqrt{t+h})}$$

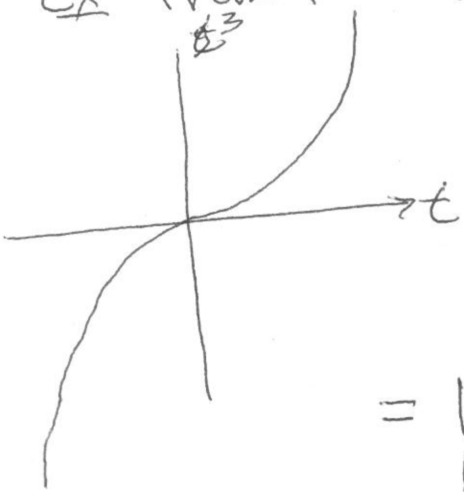
$$= \lim_{h \to 0} \frac{-1}{\sqrt{t+h}\sqrt{t}(\sqrt{t} + \sqrt{t+h})}$$

$$= \frac{-1}{\sqrt{t}\sqrt{t}(\sqrt{t} + \sqrt{t})} = \frac{-1}{t\,2\sqrt{t}}$$

$$= \frac{-1}{2t^{3/2}}$$

021 Sec 2.7

<u>Ex</u> From the def, find the derivative of $g(t) = t^3$



$$g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}$$

$$= \lim_{h \to 0} \frac{(t+h)^3 - t^3}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{t^3} + 3t^2h + 3th^2 + h^3) - \cancel{t^3}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(3t^2 + 3th + h^2)}{\cancel{h}}$$

$$= \lim_{h \to 0} (3t^2 + 3th + h^2) = 3t^2$$

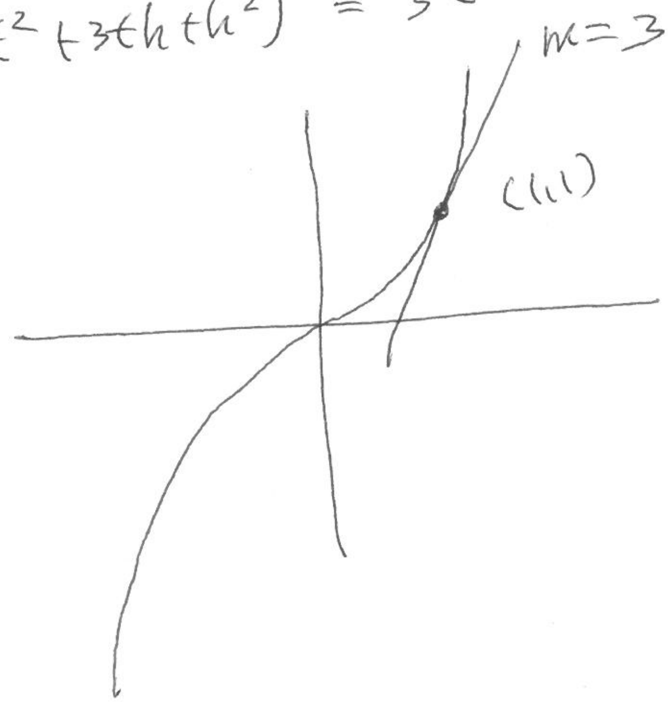
So $g'(1) = 3$

$g'(0) = 0$

Tangent line at $x=0$ has slope 0,

The tangent line at $x=0$ is the x-axis,

it cuts through the curve



021 sec 2.7

Notation for a first derivative

TFAE - (the following are equivalent)

Let $y = f(x)$

$$f'(x), \ \frac{dy}{dx}, \ y', \ \frac{df}{dx}, \ \dot{y}$$

$$Df(x), \quad D_x f(x)$$

<u>Def</u> A function f is differentiable at $x=a$
iff $f'(a)$ is a real number

Also, $f(x)$ is differentiable on an <u>open</u> interval I
iff f is differentiable for all x in I.

O21 Sec 2.7/2.8

~~Quiz - Tuesday, Sept 17, 2019~~

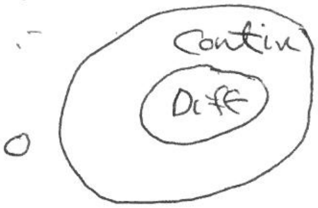
~~① Finding a derivative from the definition~~

~~② Two-parts, finding limits at $+\infty, -\infty$~~

~~③ Finding a limit - involves algebra~~

H.W. Read sec 2.7, ~~especially example~~ 6 and 7, page 146-147

<u>Thm</u> - If $f(x)$ is differ at $x=a$, then $f(x)$ is continuous at $x=a$



The converse is not true. We now give an example of a function which is continu at $x=0$ but not differ at $x=0$

$y=f(x)=|x|$

Recall, that a derivative is a limit and a limit exists iff both one-sided limits exist and are equal

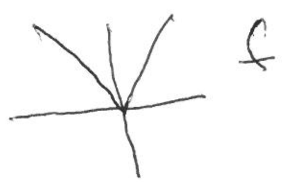
$$f'(x)=\lim_{h\to 0}\frac{|x+h|-|x|}{h} = \lim_{h\to 0}\frac{|h|}{|h|} = \lim_{h\to 0}\frac{|h|}{h}$$

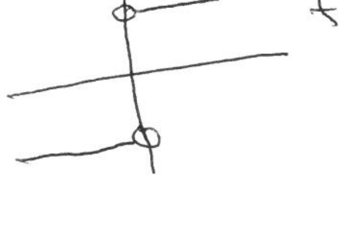
Recall that if $h>0$, $|h|=h$, so $\lim_{h\to 0}\frac{|h|}{h}=\frac{h}{h}=1$

So $\lim_{h\to 0^+} f(x)=+1$

Recall if $h<0$, $|h|=-h$, so $\lim_{h\to 0}\frac{|h|}{h}=\frac{-h}{h}=1$

so $\lim_{h\to 0^-} f(x)=-1$. So the one-sided limits exist but are unequal so, the limit, which is the derivative, dne.





2.7

2) Find an equation of the tangent line to the curve at the given point. $y = 3x^2 - 12x + 1$ at $(5, 16)$

Solution Most of the work will be in finding the slope which is given by the derivative. I prefer to work with the general formula. Let m be the slope

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[3(x+h)^2 - 12(x+h) + 1] - [3x^2 - 12x + 1]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + \cancel{h}^2 - \cancel{12x} - 12h \cancel{+1} - \cancel{3x^2} + \cancel{12x} \cancel{-1}}{h}$$

$$= \lim_{h \to 0} \frac{6xh + h^2 - 12h}{h} = \lim_{h \to 0} \frac{\cancel{h}(6x + h - 12)}{\cancel{h}} = \lim_{h \to 0} (6x + h - 12)$$

$m = 6x - 12$. So when $x = 5$, $m_5 = 6(5) - 12 = 18$

So the equation of the tangent line is

$$y = mx + b$$

$$y = 18x + b.$$

To find b, use the fact that the pt $(5, 16)$ is on the line.

$$y = 18(5) + b$$

$$16 = 80 + b$$

$$b = -74$$

So, the equation of the tangent line at $(5, 16)$ is

$$y = 18x - 74$$

2.7 #3) Find an equation of the tangent line to the curve

$y = \frac{x+2}{x-4}$ at the pt $(3, -5)$

Solution: slope $m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\frac{(x+h)+2}{(x+h)-4} - \frac{x+2}{x-4}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{[(x+h)+2](x-4) - [(x+2)][(x+h)-4]}{[(x+h)-4][x-4]}}{h}$$

$$= \lim_{h \to 0} \frac{(x^2 - \cancel{4x} + xh - 4h + \cancel{2x} - 8) - [x^2 + xh - \cancel{4x} + \cancel{2x} + 2h - 8]}{[x^2 - \cancel{4x} + xh - 4h - \cancel{4x} + 16]} \left(\frac{1}{h}\right)$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} - \cancel{2x} + \cancel{xh} - 4h - \cancel{8} - \cancel{x^2} + \cancel{2x} - \cancel{xh} - 2h + \cancel{8}}{h(x^2 - 8x + xh - 4h + 16)}$$

ignore

$$\cancel{= \lim_{h \to 0} \frac{-6(h)}{h(x^2 - 8x + x^2 - 4h + \ldots)}}$$

ignore

$$\cancel{= \lim_{h \to 0} xh - 4h}$$

$$= \lim_{h \to 0} \frac{-4h - 2h}{h(x^2 - 8x + xh - 4h + 16)}$$

$$= \lim_{h \to 0} \frac{-6\cancel{h}}{\cancel{h}(x^2 - 8x + xh - 4h + 16)}$$

$$= \lim_{h \to 0} \frac{-6}{x^2 - 8x - \cancel{4h} + 16} = \frac{-6}{x^2 - 8x + 16}$$

$m = \frac{-6}{(x-4)^2}$, at $x = 3$, $m = \frac{-6}{(3-4)^2} = \frac{-6}{(-1)^2}$

$m = -6$, so $y = -6x + b$. at $(3, -5)$

$-5 = -6(3) + b$

$-5 = -18 + b \Rightarrow b = 13$

so an equation of the tangent line is

$y = -6x + 13$

or $y = 13 - 6x$

2.7 #5 $y = \sqrt{4-77x}$ at $(-1, 9)$

$$m = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{4-77(x+h)} - \sqrt{4-77x}}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sqrt{4-77(x+h)} - \sqrt{4-77x}}{h}\right)\left(\frac{\sqrt{4-77(x+h)} + \sqrt{4-77x}}{\sqrt{4-77(x+h)} + \sqrt{4-77x}}\right)$$

$$= \lim_{h\to 0} \frac{(4-77(x+h)) - (4-77x)}{h(\sqrt{4-77(x+h)} + \sqrt{4-77x})}$$

$$= \lim_{h\to 0} \frac{\cancel{4} \cancel{-77x} -77h \cancel{-4} \cancel{+77x}}{h(\sqrt{4-77(x+h)} + \sqrt{4-77x})}$$

$$= \lim_{h\to 0} \frac{-77\cancel{h}}{\cancel{h}(\sqrt{4-77(x+h)} + \sqrt{4-77x})}$$

$$= \lim_{h\to 0} \frac{-77}{\sqrt{4-77(x+h)} + \sqrt{4-77x}} = \frac{-77}{\sqrt{4-77x} + \sqrt{4-77x}}$$

$$= \frac{-77}{2\sqrt{4-77x}}$$ when $x=-1$, $m = \frac{-77}{2\sqrt{4-77(-1)}}$

$$m = \frac{-77}{2\sqrt{81}} = \frac{-77}{2\cdot 9} = \frac{-77}{18}$$

so $y = mx+b$ is $y = -\frac{77}{18}x + b$

at the pt $(-1, 9)$

$$9 = \frac{-77}{18}(-1) + b \Rightarrow b = 9 - \frac{77}{18}$$

$$b = \frac{85}{18}$$

$$y = \frac{85}{18} - \frac{77x}{18}$$

7) A cliff diver plunges from a height of 49 ft above the water surface. The distance the diver falls in t seconds is given by the function

$$d(t) = 16t^2 \text{ ft}$$

a) How ~~many~~ many seconds will it take for the diver to hit the water?

The diver must fall 49 ft in t seconds

so solve $49 = 16t^2$

$\frac{49}{16} = t^2 \Rightarrow t = \sqrt{\frac{49}{16}} = \frac{7}{4} = 1.75$ sec

b) with what velocity (in ft/sec) does the diver hit the water

velocity is the derivative of position

so find $\frac{d}{dt} 16t^2 = \lim_{h \to 0} \frac{16(t+h)^2 - 16t^2}{h}$

$$= \lim_{h \to 0} \frac{\cancel{16t^2} + 32th + 16h^2 - \cancel{16t^2}}{h}$$

$$= \lim_{h \to 0} \frac{32th + 16h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(32t + 16h)}{\cancel{h}}$$

$$= \lim_{h \to 0} (32t + 16h) = 32t$$

So velocity is $v(t) = 32t$.

now find $v(1.75) = 32(1.75) = 56$ ft/sec

2.7 #9.

This much like #7, but the rock is thrown upward where the ~~diver~~ diver just dropped.

In any case, here the position function is

$$H(t) = 12t - 1.86t^2$$

velocity is $H'(t) = \lim\limits_{t \to 0} \dfrac{[12(t+h) - 1.86(t+h)^2] - [12t - 1.86t^2]}{h}$

$$H'(t) = \lim_{h \to 0} \frac{\cancel{12t} + 12h - \cancel{1.86t^2} - 3.72th - 1.86h^2 - \cancel{12t} + \cancel{1.86t^2}}{h}$$

$$= \lim_{h \to 0} \frac{12h - 3.72th - 1.86h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(12 - 3.72t - 1.86h)}{\cancel{h}}$$

$$= \lim_{h \to 0} (12 - 3.72t - 1.86h) = 12 - 3.72t$$

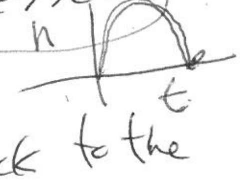
$$v(t) = 12 - 3.72t$$

a) $12 - 3.72(2) = 4.56$ m/sec

b) $12 - 3.72a$ m/s

c) $12t - 1.86t^2 = 0$

$t(12 - 1.86t) = 0$, $t = 0$, $t = \dfrac{12}{1.86} \approx 6.45$ sec



$t = 0$ when the rock is initially thrown

$t \approx 6.4516$ sec. when it comes back to the surface

d) At what velocity will it strike the surface

$v(6.4516) = 12 - 3.72(6.4516) = -24$ m/sec

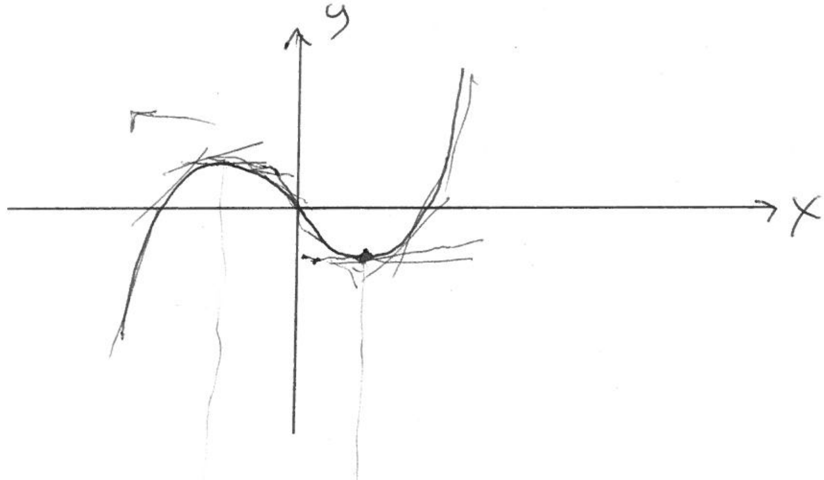
the negative sign is because the rock is going down

021 Sec 2.8

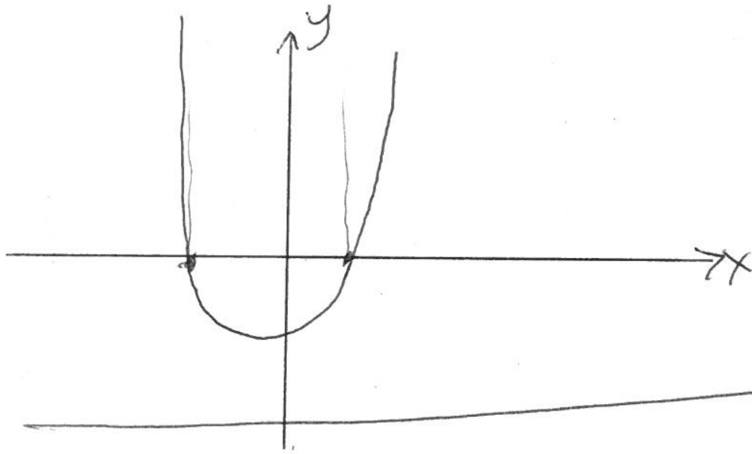
$y = f(x)$ Alternate notations for the derivative ①

$f'(x)$, $\frac{dy}{dx}$, $\frac{df}{\partial x}$, $\dot{y}$ – all represent the first derivative

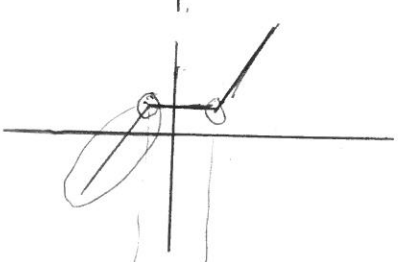
Graphing $f'(x)$ given the graph of $f(x)$



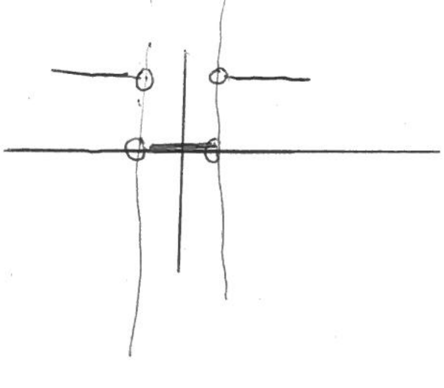
$y = f(x)$



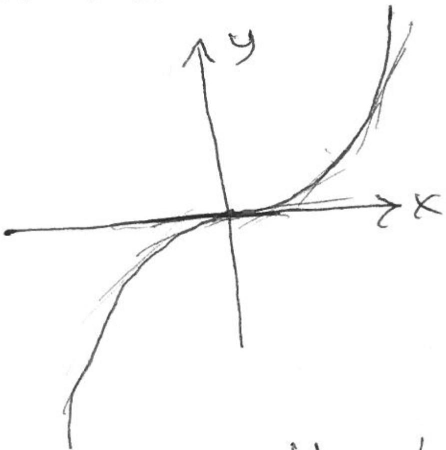
$f'(x)$



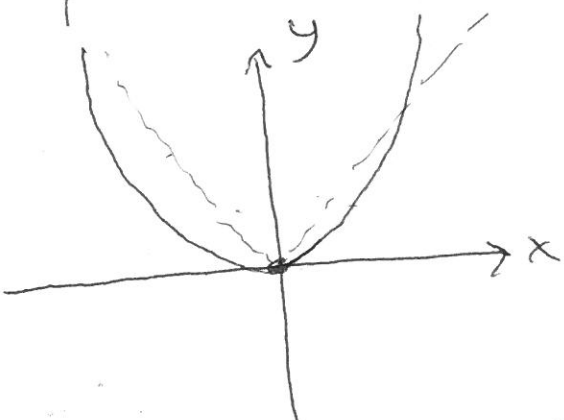
$y = g(t)$



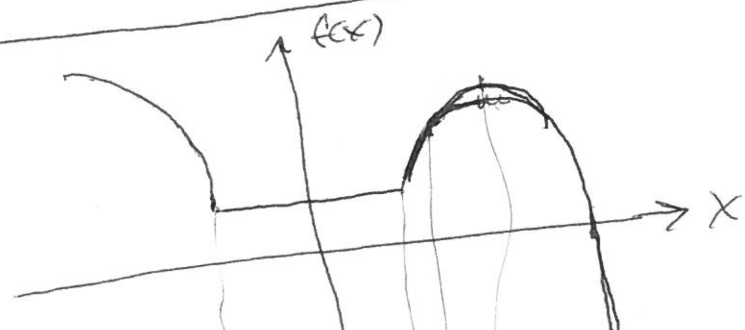
$g'(t)$



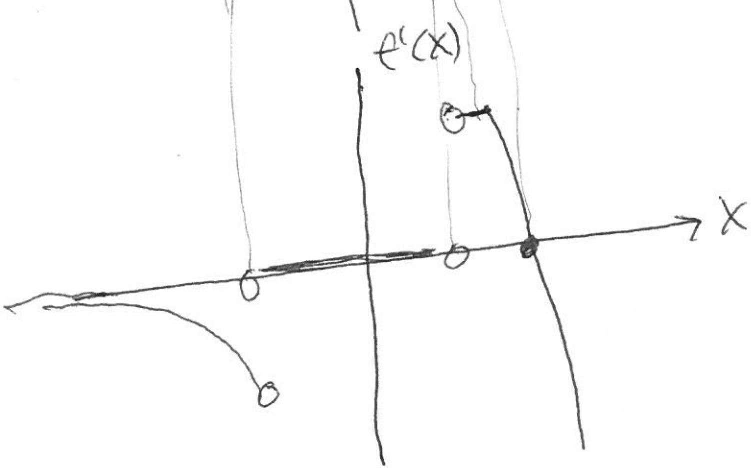
$y = f(x)$



$y = f'(x)$



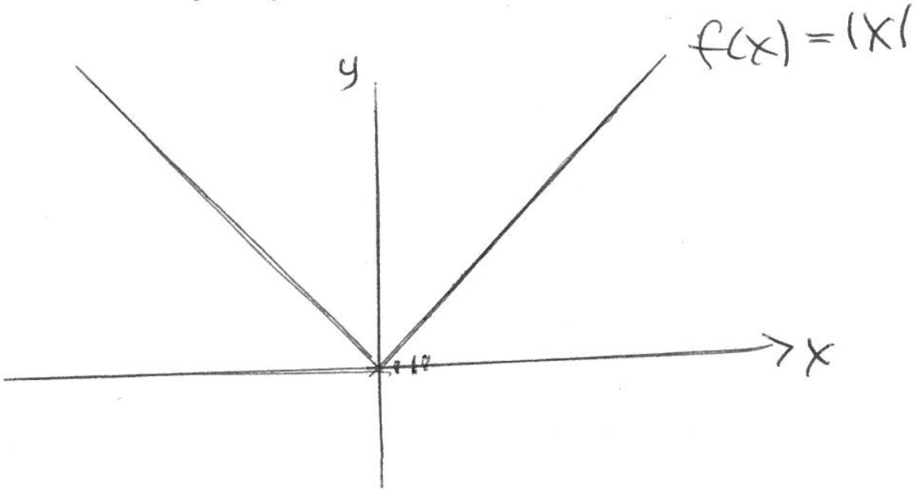
$f(x)$



$f'(x)$

Sec 2.8

Where is $y = f(x) = |x|$ differentiable



We are asking when does $\lim\limits_{h \to 0} \frac{|x+h| - |x|}{h}$ exist

If $x > 0$, $|x| = x$

so $\lim\limits_{h \to 0} \frac{|x+h| - |x|}{h} = \lim\limits_{h \to 0} \frac{(x+h) - x}{h} = \frac{h}{h} = 1$

so, $f'(x) = 1$, if $x > 0$

If $x < 0$, $|x+h| = -(x+h)$

so $\lim\limits_{h \to 0} \frac{|x+h| - |x|}{h} = \lim\limits_{h \to 0} \frac{-(x+h) - (-x)}{h} = \frac{-h}{h} = -1$

so $f'(x) = -1$ if $x < 0$

If $x = 0$. $\lim\limits_{h \to 0^+} \frac{|x+h| - |x|}{h} = +1$

$\lim\limits_{h \to 0^-} \frac{|x+h| - |x|}{h} = -1$

since the one sided limits are not equal, we have $\lim\limits_{h \to 0} \frac{|x+h| - |x|}{h}$ does not exist

so, $|x|$ is not differentiable at $x = 0$

021 Sec 2.8

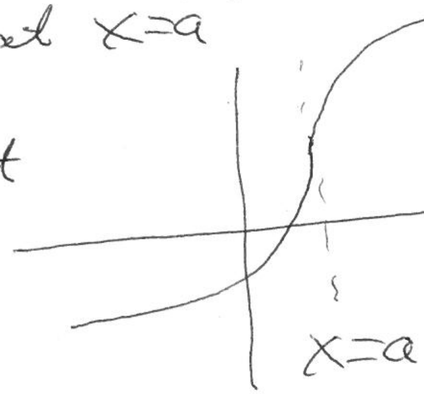
section 2.8 The derivative as a function

How can a function not be differentiable at $x=a$

1) f has a vertical tangent at $x=a$

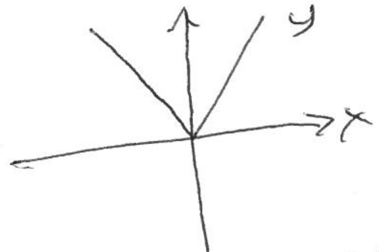
good

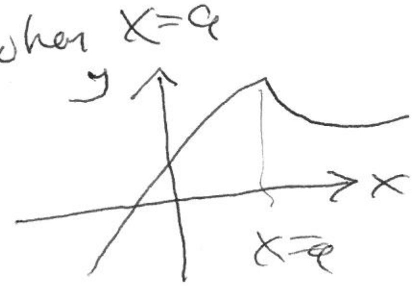
We have a tangent line at $x=8$, but the tangent line is vertical, so the slope of the tangent is undefined



2) f has corner (cusps) when $x=a$

bad



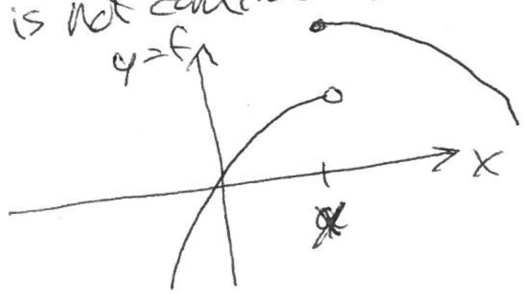


Here the one sided derivaties exists but are not equal

But the function is continuous at $x=a$

3) f is not continuous at $x=a$

ugly



In this case when we look at

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

on one side we get a vertical line

Also, at the function is not contin.

<u>Thm</u> If f is differ at $x=a$, then f is contin at $x=a$.

A little logic

Let P be the statement: f(x) is differ at $x=a$

" Q " " " : f(x) is continu at $x=a$

Our theorem says $P \Rightarrow Q$

Another way of saying the same thing is not $Q \Rightarrow$ not P

If not Q then not P,

i.e. If f is not continuous at $x=a$
then f is not diff at $x=a$

this is the contrapositive of $P \Rightarrow Q$

A statement and its contrapositive are equivalent.

The converse to $P \Rightarrow Q$ is $Q \Rightarrow P$

A statement and its converse are not equivalent:

In our case: the converse is

If f is not differen at a, then f is not continu at a.

The inverse of a statement is equivalent to the converse. In fact the inverse is the contrapositive of the converse. not $P \Rightarrow$ not Q.

In our case: the inverse.

If is not differen at a, then f is not continu at a

$P \Rightarrow Q$	$Q \Rightarrow P$
not $Q \Rightarrow$ not P	not $P \Rightarrow$ not Q

021 Sec 2.8

We will need $\frac{d}{dx} x^n = nx^{n-1}$

Ex $\frac{d}{dx} x^5 = 5x^{5-1} = 5x^4$

Consider: $f(x) = 3x^4 - 7x + 3$ $\qquad g(x) = 12x^3 - 7$

$f'(x) = 12x^3 - 7$ $\qquad g'(x) = 36x^2$

So $[f'(x)]' = [g(x)]' = g'(x) = 36x^2$

We write $f''(x) = \frac{d^2 f}{dx^2} = 36x^2$

This is the second derivative of f wrt x

$f(x) = 3x^4 - 7x + 3$

$f'(x) = 12x^3 - 7$

$f''(x) = 36x^2$

$f'''(x) = 72x = \frac{d^3 f}{dx^3}$

$f^{iv}(x) = 72 = \frac{d^4 f}{dx^4}$

$f^{v}(x) = 0$ $\qquad f^{(vi)}(x) = 0, \; f^{(vii)}(x) = 0,$

In this case $\qquad f^{(n)}(x) = 0, \; \forall n \geq 5$

Trick Ex $\quad g(x) = 38.7x^{45} + 16.2x^{40}$

find $g^{(50)}(x) = 0$. Note, everytime you take a derivative, you reduce the exponents by 1, so in this problem you get to 0 and stay there.

Ex $f(x) = x^{\frac{1}{2}}$, $\quad f'(x) = \frac{1}{2}x^{1-\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}}$

$f''(x) = -\frac{1}{4}x^{-3/2}$

Note no derivative is 0, in this case

021 Section 2.8

<u>Ex</u> Given that

$$\frac{d}{dx}\sin X = \cos X \quad \text{and} \quad \frac{d}{dx}\cos X = -\sin X$$

find $f^{(2019)}(X)$ if $f(X) = \sin X$

<u>Solution</u>:

$$f(X) = f^{(0)}(X) = \sin X$$
$$f'(X) = \cos X$$
$$f''(X) = -\sin X$$
$$f'''(X) = -\cos X$$
$$f^{iv}(X) = \sin X$$

The ~~derivatives~~ derivatives have a cycle of length 4,

$$\begin{array}{r} 504 \text{ r. } 3 \\ 4 \overline{)\,2019} \end{array}$$

The cycle repeat itself every four derivatives, and have that $f^{(2019)}(X) = f'''(X) = -\cos X$

021 Section 2.8

Application Let $f(t)$ be the position of a particle at time t.
Then $f'(t)$ is the rate of change of the position of the particle at time t.
i.e. $f'(t) = \text{velocity}$

So $f''(t) = [f'(t)]' = $ rate of change of velocity wrt time.
i.e. $f''(t) = \text{acceleration}$.

$f'''(t) = $ rate of change of acceleration wrt time
$f'''(t) = $ the jerk

Ex Let $f(t) = t^3$ be the position of a particle at time t.
so $f'(t) = 3t^2$ is " velocity
$f''(t) = 6t$ is " acceleration
$f'''(t) = 6$ is the jerk