022 Sec 11.10 Friday, June 14th 2019 ①

Quiz on Monday, June 17

① Develop a Maclaurin series.

② Find the interval of convergence for a power series

Section 11.11 Applications of Taylor Series

Approximate the function $f(x) = x^{1/3}$ by a Taylor Polynomial of degree 2 at $a = 8$.

$f(x) = x^{1/3}$ $\quad f(8) = 2$

$f'(x) = \frac{1}{3}x^{-2/3}$ $\quad f'(8) = \frac{1}{12}$

$f''(x) = \frac{10}{27}x^{-\frac{8}{3}}$

$$P_2(x) = f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2$$

$$P_2(x) = 2 + \frac{1}{12}(x-8) + \frac{1}{288}(x-8)^2$$

How good is this approximation for $7 \leq x \leq 9$

Answer ~~T~~Taylor Series is not an alternating series, so we can't use the estimate on alternating series.

We do use Taylor's Inequality with $n=2$, $a=8$

$$|R_2(x)| \leq \frac{M}{3!}|x-8|^3, \text{ where } |f'''(x)| \leq M$$

with $x \geq 7$, we have $x^{8/3} \geq 7^{8/3}$, so

$$f'''(x) = \frac{10}{27} \cdot \frac{1}{x^{8/3}} \leq \frac{10}{27} \cdot \frac{1}{7^{8/3}} < 0.0021$$

So, let $M = 0.0021$. With $7 \leq x \leq 9$,

$-1 \leq x-8 \leq 1$ and $|x-8| \leq 1$

So Taylor's Inequality gives

$$|R_2(x)| \leq \frac{0.0021}{3!} \cdot 1^3 = \frac{0.0021}{6} < 0.0004$$

So for $7 \leq x \leq 9$, the error in using the approximation is at most 0.0004

__Ex__ Find the max ~~error~~ possible error in using
$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$, where $-0.3 \le x \le 0.3$

Find $\sin 12^\circ$ to 6 correct decimal places.

Now $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

__is an__ alternating series ~~test~~, and the terms $\to 0$
so use the alternating series test.
The error is bounded by the 4'th term

$$\left|\frac{x^7}{7!}\right| = \frac{|x|^7}{5040}. \quad \text{With } -0.3 \le x \le 0.3,\ |x| \le 0.3$$

so |the error| is bounded by

$$\frac{(0.3)^7}{5040} \approx 4.3 \times 10^{-8}$$

The find $\sin 12^\circ$, we convert to radian measure

$$\sin(12^\circ) = \sin\left(\frac{12\pi}{180}\right) = \sin\left(\frac{\pi}{15}\right)$$

$$\approx \frac{\pi}{15} - \left(\frac{\pi}{15}\right)^3 \frac{1}{3!} + \left(\frac{\pi}{15}\right)^5 \frac{1}{5!} \approx 0.20791169$$

so to six decimal places
$\sin 12^\circ \approx 0.207912$

For what values of x is our approximation accurate to within 0.00005

Solution: We need $\frac{|x|^7}{5040} \le 0.00005$

or $|x| < (0.252)^{\frac{1}{7}} \approx 0.821$

So accurate to within 0.00005 if $|x| < 0.82$

Application

In special relativity, mass

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

m_0 is the mass at rest, c is the speed of light.

Kinetic energy

$$K = mc^2 - m_0c^2$$

a) Show that when v is small compared to c, we

$$K = \frac{1}{2}m_0v^2$$

Solution

$$K = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0c^2\left[\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} - 1\right]$$

when $X = \frac{-v^2}{c^2}$, the maclaurin series for $(1+X)^{-\frac{1}{2}}$ is a binominal series with $K = -\frac{1}{2}$,

$$\text{so } (1+X)^{-\frac{1}{2}} = 1 - \frac{1}{2}X + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}X^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}X^3 + \cdots$$

$$= 1 - \frac{1}{2}X + \frac{3}{8}X^2 - \frac{5}{16}X^3 + \cdots$$

and $K = m_0c^2\left[\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right) - 1\right]$

$$K = m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right)$$

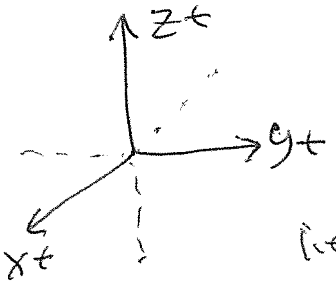
if $v \ll c$, then all the terms after the first are close to 0,

so $K \simeq m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2}\right) = \frac{1}{2}m_0v^2$

Section 12.1 Three-Dimensional Coordinate Systems

To locate a point on the number line, you need 1 number.
To locate a point " " plane you need two numbers,
an x-value and a y-value.

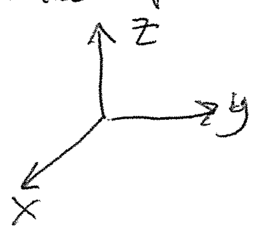
To locate a point in 3 space, we need three numbers
an x-value, a y-value, and a z-value.



We use the right-hand rule to obtain the orientation of z.

i.e. curl the fingers of your right hand
around the z-axis going from the positive x-axis to
the positive y-axis (i.e. counterclockwise) your right
thumb will point in the direction of the positive z-axis.

Coordinate planes



(a, b, c)

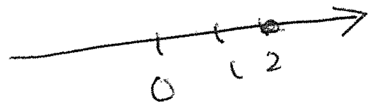
a → x-coordinate
b → y-coordinate
c → z-coordinate

All points of the form (a,b,0) is the xy-plane
" " " " " (0,b,c) is the yz-plane
" " " " " (a,0,c) " " xz-plane
" " " " "

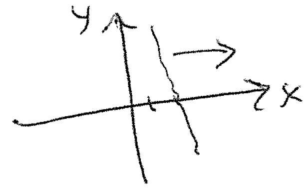
What type of object is $X=2$?

Answer, it depends on the context

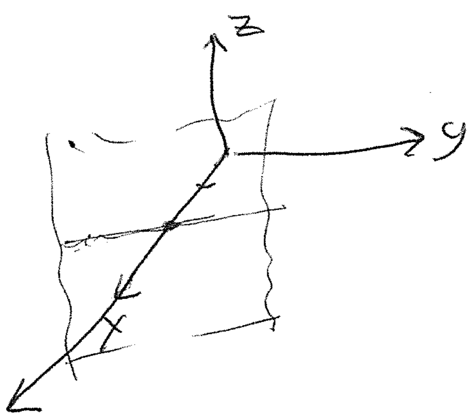
On the number line $X=2$ is a point



In the xy-plane $X=2$ is a vertical line
a point in the $(2,y)$, with
y - being any number.



In $\mathbb{R}^3$, $X=2$ is a plane $(2,y,z)$; where y and z ~~are~~ both range over the real numbers



Distance Formula, in three dimensions; The distance $|P_1P_2|$ between $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ is

$$|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

This really is the Pythagorean Theorem.

Ex $P=(2,-4,0)$, $Q(-2,4,0)$

$$|P_1P_2| = \sqrt{(2-(-2))^2+(-4-4)^2+(0+0)^2}$$
$$= \sqrt{4^2+(-8)^2+0^2} = \sqrt{16+64} = \sqrt{80}$$

<u>Spheres</u>

A sphere in $\mathbb{R}^3$ is the set of all points equidistant from a center point.

The equation of a sphere with center $C(h,k,l)$ and radius r is: $(x-h)^2+(y-k)^2+(z-l)^2=r^2$

<u>Ex</u> Find the center of the sphere with equation

$$x^2+y^2+z^2-6x+10y=150$$

$$(x-3)^2+(y+5)^2+z^2=150+9+25$$

$$(x-3)^2+(y+5)^2+z^2=184$$

So, the center is $(3,-5,0)$: radius is $\sqrt{184}$

Octants. in $\mathbb{R}^3$ we have 8 octants

x	y	z	
+	+	+	1st octant
+	+	−	
+	−	+	
+	−	−	
−	+	+	
−	+	−	
−	−	+	
−	−	−	

Mathematica,
math 21, Lab 5, Newtons method

Hand in part b, <u>with</u> the output

I want to see the graphs

Due June 21