022 sec 10.4 Friday, Feb 24 2020

Calculus with Polar Coordinates

Slopes and tangents in polar coordinates

To find the slope of a polar curve given as $r = f(\theta)$

Use the parametric form

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

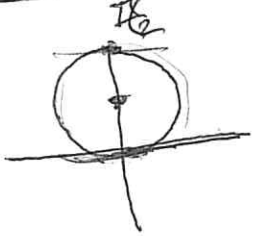
Now $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$, provided $\dfrac{dx}{d\theta} \neq 0$

Special Cases

We have horizontal tangents when $\dfrac{dy}{d\theta} = 0$ and $\dfrac{dx}{d\theta} \neq 0$

" " vertical tangents " $\dfrac{dx}{d\theta} = 0$ and $\dfrac{dy}{d\theta} \neq 0$

Ex Find the horizontal and vertical tangents of
$r = \sin\theta,\quad 0 \le \theta \le \pi$



Solution

$x = r\cos\theta = \cancel{r}\sin\theta\cos\theta$

$y = r\sin\theta = \cancel{r\sin\theta}\ \sin\theta\sin\theta = \sin^2\theta$

So, $\dfrac{dx}{d\theta} = \cos^2\theta - \sin^2\theta = \cos(2\theta)$

and $\dfrac{dx}{d\theta} = 0$ when $2\theta = \frac{\pi}{2},\ 2\theta = \frac{3\pi}{2}$
$\theta = \frac{\pi}{4},\ \theta = \frac{3\pi}{4}$.

so vertical tangents at $\theta = \frac{\pi}{4},\ \theta = \frac{3\pi}{4}$

now, $\dfrac{dy}{d\theta} = 2\sin\theta\cos\theta = \sin(2\theta)$. want $\sin(2\theta) = 0$

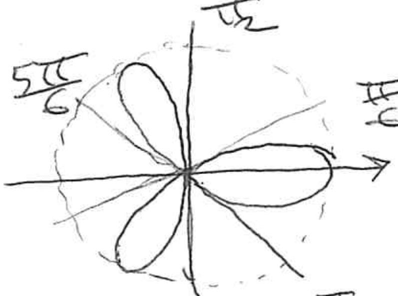
This is when $2\theta = 0,\ 2\theta = \pi$
$\theta = 0,\ \theta = \frac{\pi}{2}$

So horizontal tangents when $\theta = 0,\ \theta = \frac{\pi}{2}$

Also when $\theta = \pi$

Ex of Tangents at a pole

Consider $f(\theta) = 2\cos(3\theta)$



If $f(\alpha)=0$ and $f'(\alpha)\neq 0$, then the line $\theta=\alpha$ is tangent to the pole

In our example,

$f(\theta) = 2\cos(3\theta)$

$f(\theta)$ is 0 when $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

Moreover, $f'(\theta) = -6\sin(3\theta)$

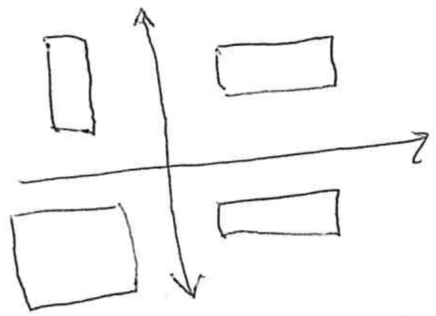
$f'(\theta) \neq 0$ at $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

So the lines $\theta = \frac{\pi}{6}$, $\theta = \frac{\pi}{2}$, $\theta = \frac{5\pi}{6}$ are all tangent to the graph at the pole

022 Section 10.4 Friday, Feb 21, 2021

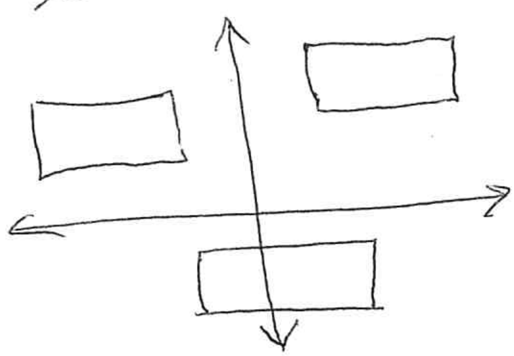
Areas and integrals in polar coordinates

Background: what is the area of rectangle whose sides are ~~both~~ parallel to the x axis and the y-axis



$A = (\Delta x)(\Delta y)$

Let's have these rectangles all have the same $l = \Delta x$ and the same $\Delta y = h$



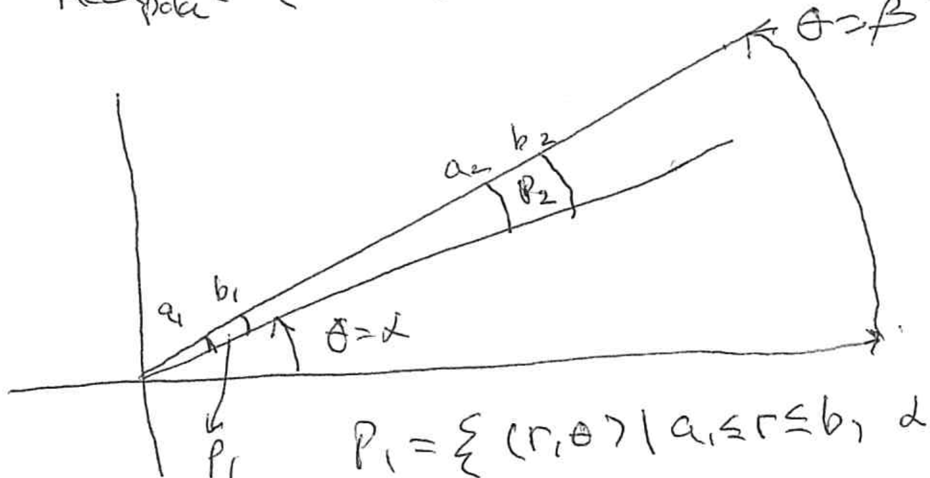
Note that all these rectangles have the same area.
No matter where they are located.

$R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$

$b-a$ stays the same for all rectangles
$d-c$ " " " " "

A polar rectangle is where r is between a pair of constants
θ " " " " " "

$Rect_{polar} = \{(r,\theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$



$P_1 = \{(r,\theta) \mid a_1 \le r \le b_1,\ \alpha \le \theta \le \beta\}$

$P_2 = \{(r,\theta) \mid a_2 \le r \le b_2,\ \alpha \le \theta \le \beta\}$

note $b_1 - a_1 = b_2 - a_2$

But area $P_2 >$ area P_1

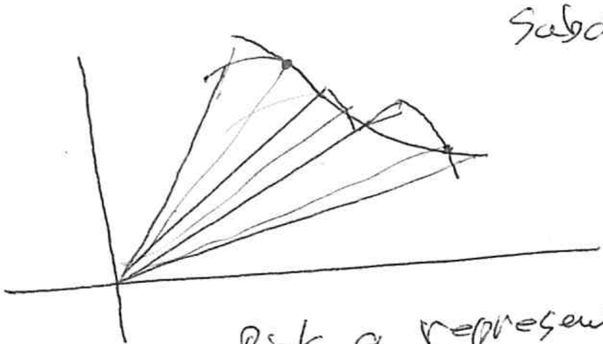
The area of a polar rectangle depends on Δr, $\Delta\theta$, and how far the center of the rectangle is from the origin

Integrating in Polar Coordinates

Area of a polar region

First consider the area of a sector of a circle

$$A = \frac{1}{2}\theta r^2$$



Subdivide the main arc into subarcs

We approx a subarea by a sector of a circle.

Pick a representative pt in each subarc

Form the Riemann Sum.

$$A \approx \sum_{i=1}^{n} \left(\frac{1}{2}\right)(\Delta\theta)\left[f(\theta_i)\right]^2$$

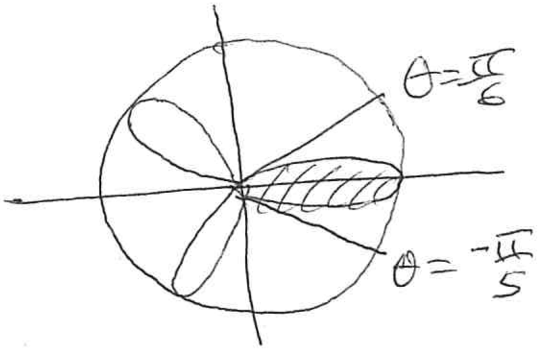
Take a limit as $n \to \infty$ and obtain

$$\text{Area} = \lim_{n\to\infty} \frac{1}{2} \sum_{i=1}^{n} \left[f(\theta_i)\right]^2 \Delta\theta$$

$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} \left[f(\theta)\right]^2 d\theta$$

with $0 < \beta - \alpha \leq 2\pi$

Ex Find the area of one petal of the rose curve given by $r = 3\cos(3\theta)$



For our limits of integration we use our results on tangents

We have a tangent line to the pole at $\theta = \frac{\pi}{6}$, $\theta = -\frac{\pi}{6}$

$$\text{area} = A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} [3\cos(3\theta)]^2\, d\theta$$

$$= \frac{9}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2(3\theta)\, d\theta$$

$$= \frac{9}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1+\cos(6\theta)}{2}\, d\theta$$

$$= \frac{9}{2}\left[\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}\, d\theta + \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos 6\theta}{2}\, d\theta\right]$$

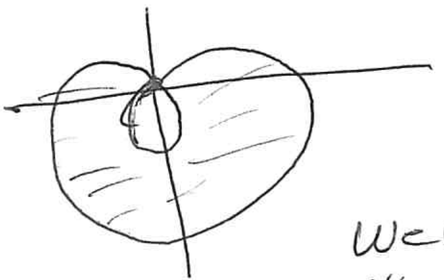
$u = 6\theta$
$du = 6\, d\theta$

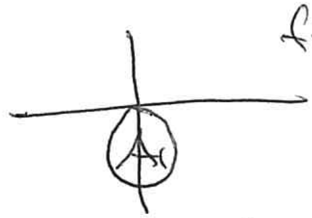
$$= \frac{9}{4}\left[\theta + \frac{\sin 6\theta}{6}\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= \frac{9}{4}\left(\frac{\pi}{6} + \frac{\pi}{6}\right) = \frac{3\pi}{4}$$

Ex Find the area of the region lying between the inner and outer loops of the limacon $r = 1 - 2\sin\theta$

first find this area





We need to find the limits of integration. With a little work, using the result on tangent line at the pole. We have

$$A_1 = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1-2\sin\theta)^2\, d\theta$$

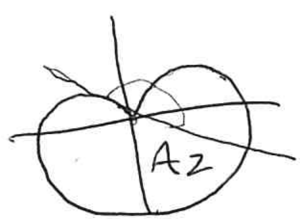
$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1 - 4\sin\theta + 4\sin^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left[1 - 4\sin\theta + 4\left(\frac{1-\cos 2\theta}{2}\right)\right] d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left[3 - 4\sin\theta - 2\cos 2\theta\right] d\theta$$

$$= \frac{1}{2}\left[3\theta - 4\cos\theta - 2\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} = \frac{1}{2}(2\pi - 3\sqrt{3}) = \pi - \frac{3\sqrt{3}}{2}$$

The entire area



$$A_2 = \int_{\frac{5\pi}{6}}^{\frac{13\pi}{6}} (1-2\sin\theta)^2\, d\theta = 2\pi + \left(\frac{3\sqrt{3}}{2}\right)$$

The area between the inner and outer loops is

$$A = A_1 - A_2 = \left(2\pi + \frac{3\sqrt{3}}{2}\right) - \left(\pi - \frac{3\sqrt{3}}{2}\right)$$

$$= \pi + 3\sqrt{3} \approx 8.34$$

Points of Intersection of Polar Graphs

A pt can can have different polar representations.

We need to be careful

<u>Ex</u> Find the pts of intersection of

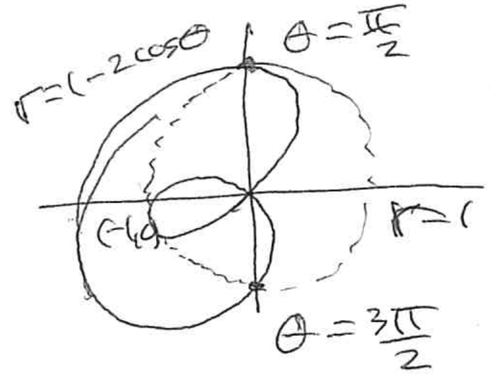
$r = 1 - 2\cos\theta$ and $r = 1$

Setting them equal

$1 = 1 - 2\cos\theta$

$-2\cos\theta = 0, \quad \cos\theta = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$



But we missed the pt of intersection at $(-1, 0)$,

Because this pt does not have the same polar coordinates for both graphs.