022 Sec 12.3 Tuesday, June 18, 2019 ①

Next quiz, quiz 6 on Friday, June 21
(not Wed, June 19th)

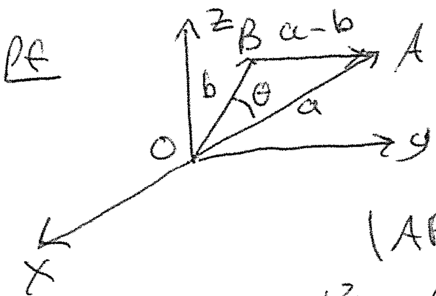
Recall, that if $a = \langle a_1, a_2, a_3, \dots, a_n \rangle$

$$b = \langle b_1, b_2, b_3, \dots, b_n \rangle$$

$$a \cdot b = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

Thm If θ is the acute angle between nonzero vectors a and b then $a \cdot b = |a||b|\cos\theta$ so $\cos\theta = \frac{a \cdot b}{|a||b|}$, $\theta = \cos^{-1}\frac{a \cdot b}{|a||b|}$

Pf



Pf Apply the law of cosines to the triangle OAB to obtain

$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB|\cos\theta$$

from $|a|^2 = a \cdot a$

Now, $|a-b|^2 = (a-b)\cdot(a-b)$,

$$= a \cdot a - a \cdot b - b \cdot a + b \cdot b$$

$$= |a|^2 - 2a \cdot b + |b|^2$$

so: $\cancel{|a|^2} - 2a \cdot b \cancel{+|b|^2} = \cancel{|a|^2 + |b|^2} - 2|a||b|\cos\theta$

$$-2ab = -2|a||b|\cos\theta$$

$$a \cdot b = |a||b|\cos\theta$$

Two nonzero vectors are perpendicular, $\perp$ or othogonal if $a \cdot b = 0$

The $\vec{0}$ vector is said to be orthogonal to all vectors.

~~so~~ We can check that $i \perp j$, $i \perp k$, $j \perp k$

$$i \cdot j = \langle 1,0,0\rangle \cdot \langle 0,1,0\rangle = 1\cdot 0 + 0 \cdot 1 + 0 \cdot 0 = 0$$

$$i \cdot k = \langle 1,0,0\rangle \cdot \langle 0,0,1\rangle = 1\cdot 0 + 0\cdot 0 + 0 \cdot 1 = 0$$

$$j \cdot k = \langle 0,1,0\rangle \cdot \langle 0,0,1\rangle = 0\cdot 0 + 1\cdot 0 + 0 \cdot 1 = 0$$

Since $\cos\theta > 0$ if $0 \le \theta \le \frac{\pi}{2}$

and $\cos\theta < 0$ if $\frac{\pi}{2} < \theta \le \pi$

We have $a \cdot b > 0$ if $\theta < \frac{\pi}{2}$

$a \cdot b < 0$ if $\theta > \frac{\pi}{2}$

So, roughly speaking, $a \cdot b > 0$ if a and b point in the same general direction

$a \cdot b = 0$ if $a \perp b$

$a \cdot b < 0$ if a and b point in roughly opposite directions.

<u>Note</u> If $a \cdot b = |a||b|$, then $\cos\theta = 1$, and $\theta = 0$

Direction Angles and Direction Cosines

The direction angles of a vector $\vec{a}$ are α, β, γ in $[0, \pi]$ s.t.

α is the angle a makes with the positive x-axis

β " " " " " " " " y-axis

γ " " " " " " " " z-axis

The directional cosines of the vector $\vec{a}$ are the cosines of these angles.

$a \cdot i = \langle a_1, a_2, a_3 \rangle \cdot \langle 1, 0, 0 \rangle$
$= a_1 \cdot 1 + a_2 \cdot 0 + a_3 \cdot 0 = a_1$

$|i| = \sqrt{1^2 + 0^2 + 0^2} = 1$

so $\cos\alpha = \frac{a \cdot i}{|a||i|} = \frac{a_1}{|a|}$

$\cos\beta = \frac{a_2}{|a|}$, $\cos\gamma = \frac{a_3}{|a|}$

We have $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\frac{a_1^2}{|a|^2} + \frac{a_2^2}{|a|^2} + \frac{a_3^2}{|a|^2} = 1$

We can write:

$a = |a| \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$

So the directional cosines of a, are the component of the unit vector pointing in the direction of a

Ex Find the directional angles of $a = \langle 1, 2, 3 \rangle$

<u>Solution</u> $|a| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

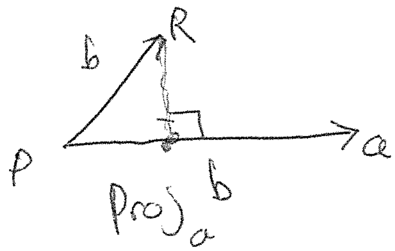
$\cos\alpha = \frac{1}{\sqrt{14}}$, $\cos\beta = \frac{2}{\sqrt{14}}$, $\cos\gamma = \frac{3}{\sqrt{14}}$

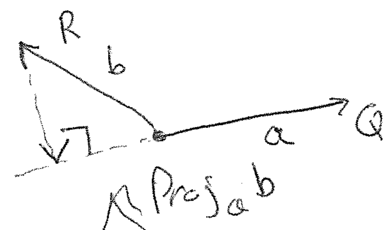
$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right)$, $\alpha \approx 74°$

$\beta \approx 58°$, $\gamma \approx \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) \approx 37°$

Projections

consider





The scalar projection of b onto a, (aka the component of b along a) is defined to be the signed magnitude of the vector projection and equals $|b|\cos\theta$, where θ is the angle between a and b,

This is denoted $\text{comp}_a b$.

<u>note</u> if $\frac{\pi}{2} < \theta < \pi$, $\text{comp}_a b < 0$

We have $a \cdot b = |a||b|\cos\theta = |a| \left(|b|\cos\theta\right)$

$= |a| \, \text{comp}_a b$

since $|b|\cos\theta = \frac{a \cdot b}{a} = \frac{a}{|a|} \cdot b$, we have

scalar projection of b onto a $\quad \text{comp}_a b = \frac{a \cdot b}{|a|}$

vector projection of b onto a, $\quad \text{proj}_a b = \left(\frac{a \cdot b}{|a|}\right)\frac{a}{|a|} = \left(\frac{a \cdot b}{|a|^2}\right)a$

EX Let $a = \langle 2,3,-4 \rangle$, $b = \langle 1,1,2 \rangle$

$$|b| = \sqrt{1^2+1^2+2^2} = \sqrt{6}$$

so $\frac{1}{|b|} b = \frac{1}{\sqrt{6}} \langle 1,1,2 \rangle$

so $\text{comp}_b\, a = \langle 2,3,-4 \rangle \cdot \frac{1}{\sqrt{6}} \langle 1,1,2 \rangle = \frac{-3}{\sqrt{6}}$

and $\text{comp}_a\, b = b \cdot \left(\frac{1}{|a|} a \right)$, with $|a| = \sqrt{29}$

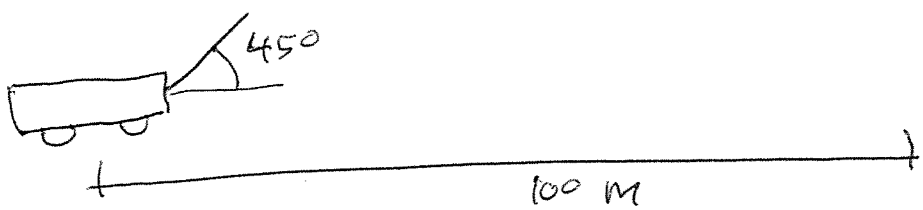
so $\frac{1}{|a|} \langle a \rangle = \frac{1}{\sqrt{29}} \langle 2,3,-4 \rangle$, so $\text{comp}_a\, b = \langle 1,1,2 \rangle \cdot \frac{1}{\sqrt{29}} \langle 2,3,- \rangle$

$$= \frac{-3}{\sqrt{29}}$$

and the $\text{Proj}_a\, b = \frac{a \cdot b}{|a|^2} a$

$$= \frac{\langle 2,3,-4 \rangle \cdot \langle 1,1,2 \rangle}{29} \langle 2,3,-4 \rangle = \frac{-3}{\sqrt{29}} \langle 2,3,-4 \rangle$$

EX A wagon is pulled 100 m along a horizontal path with a constant force of 80 N. The handle is at an angle of 45° above the horizontal. Find the work done by the force



$$W = F \cdot D = |F|\,|D| \cos 45^\circ$$

$$\approx (80)(100)(0.707) \approx 5656.8\ J.$$

Section 12.4 The Cross Product

~~Intr~~ "Review" of determinants

The determinant of a square matrix is a number.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Ex $$\begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 3(0)-(-1)(4) = 0+4 = 4$$

For a 3 by 3 matrix - cofactor expansion

$$\begin{vmatrix} 3 & -2 & 0 \\ 4 & -3 & 2 \\ 1 & 6 & -5 \end{vmatrix}$$, expand along the first row

$$= (-1)^{1+1} 3 \begin{vmatrix} -3 & 2 \\ 6 & -5 \end{vmatrix} + (-1)^{1+2}(-2)\begin{vmatrix} 4 & 2 \\ 1 & -5 \end{vmatrix} + (-1)^{1+3} \cdot 0 \begin{vmatrix} 4 & -3 \\ 1 & 6 \end{vmatrix}$$

$$= +3\begin{vmatrix} -3 & 2 \\ 6 & -5 \end{vmatrix} + 2\begin{vmatrix} 4 & 2 \\ 1 & -5 \end{vmatrix} - 0\begin{vmatrix} 4 & -3 \\ 1 & 6 \end{vmatrix}$$

$$= +3\left[(-3)(-5) - 6(2)\right] + 2\left[4(-5) - (1)(2)\right]$$

$$= +3\left[15 - 12\right] + 2\left[-20-2\right] = +3\left[3\right] + 2\left[-22\right]$$

$$= +9 - 44 = -35$$

$$\begin{vmatrix} 3 & -2 & 0 \\ 4 & -3 & \cancel{2} \\ 1 & 6 & -5 \end{vmatrix} \qquad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

The value of a determinant is independent of which row, or column one expands along

The Cross-Product

Let $a = \langle a_1, a_2, a_3 \rangle$, $b = \langle b_1, b_2, b_3 \rangle$

Then $a \times b = \langle \underset{i}{a_2 b_3 - a_3 b_2}, \underset{j}{a_3 b_1 - a_1 b_3}, \underset{k}{a_1 b_2 - a_2 b_1} \rangle$

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

ex $a = \langle 4, -2, 5 \rangle$, $b = \langle 3, 1, -1 \rangle$

$$a \times b = \begin{vmatrix} i & j & k \\ 4 & -2 & 5 \\ 3 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -2 & 5 \\ 1 & -1 \end{vmatrix} i - \begin{vmatrix} 4 & 5 \\ 3 & -1 \end{vmatrix} j + \begin{vmatrix} 4 & -2 \\ 3 & 1 \end{vmatrix} k$$

$$= -3i + 19j + 10k$$

If $a = \langle a_1, a_2, a_3 \rangle$, then $a \times a = \vec{0}$

$i \times i = \vec{0}$, $j \times j = \vec{0}$, $k \times k = \vec{0}$

In particular

$i \times j = k$ $\quad j \times k = i$ $\quad k \times i = j$

and

$j \times i = -k$ $\quad k \times j = -i$ $\quad i \times k = -j$

Thm $\vec{a}\times\vec{b}$ is orthogonal to $\vec{a}$ and also to $\vec{b}$

Pf Show that $(a\times b)\cdot a = 0$

$$(a\times b)\cdot a = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - a_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3$$

Expand out and cancel to see that $(a\times b)\cdot a = 0$. Likewise to show that $(a\times b)\cdot b = 0$

We have $a\times b$ is a vector which is perpendicular to $\vec{a}$ and $\vec{b}$ and hence $a\times b$ is perpendicular to the plane formed by $\vec{a}$ and $\vec{b}$

The direction of $a\times b$ is given by the right hand rule so $a\times b$ and $b\times a$ point in opposite directions

Thm If θ is the angle between a and b, $(0\le\theta\le\pi)$ then $|a\times b| = |a||b|\sin\theta$

Pf $|a\times b|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$

$= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2 + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2$

$= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$

$= |a|^2|b|^2 - (a\cdot b)^2 = |a|^2|b|^2 - |a|^2|b|^2\cos^2\theta$ (thm 12.3.3)

$= |a|^2|b|^2(1-\cos^2\theta) = |a|^2|b|^2\sin^2\theta$

so $|a\times b|^2 = |a|^2|b|^2\sin^2\theta$

So $|a\times b| = |a||b|\sin\theta$, note $0\le\theta\le\pi$

To sum up

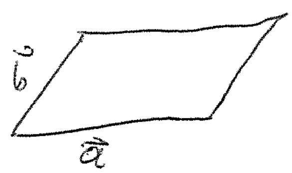
$a \times b$ is a vector with it
being perpendicular to $\vec{a}$ and to $\vec{b}$,
orientation determined by the right-hand rule
magnitude $= |a||b| \sin\theta$

<u>Corollary</u> Two non-zero vectors a and b are parallel iff $a \times b = 0$

<u>Pf</u> Two non-zero vectors a and b are parallel iff the angle
between them (θ) $\theta = 0$ or $\theta = \pi$
if $\theta = \pi$, a and b are going in opposite directions

In either case $\sin\theta = 0$, so $|a \times b| = 0$, hence $a \times b = 0$

Note



$|a \times b|$ equals the area of the parallelogram
determined by a and b.

Ex Show that $a = \langle 2, 1, -1 \rangle$ and $b = \langle -6, -3, +3 \rangle$
are parallel

$$a \times b = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ -6 & -3 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ -3 & 3 \end{vmatrix} i - \begin{vmatrix} 2 & -1 \\ -6 & 3 \end{vmatrix} j + \begin{vmatrix} 2 & 1 \\ -6 & -3 \end{vmatrix} k$$

$$= 0i + 0j + 0k = 0$$