021 Sec B.1 Monday, Dec 2, 2019

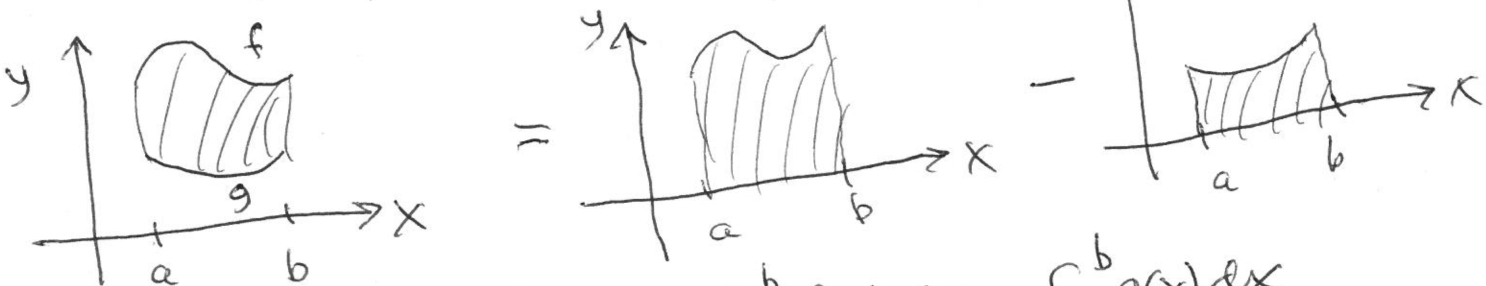
Section 6.1

We have already seen a special case of areas between curves, when we found the area under $f(x)$ on $[a,b]$, we were finding the area between $y=f(x)$ and $y=0$ on $[a,b]$

We now generalize this to find the area between

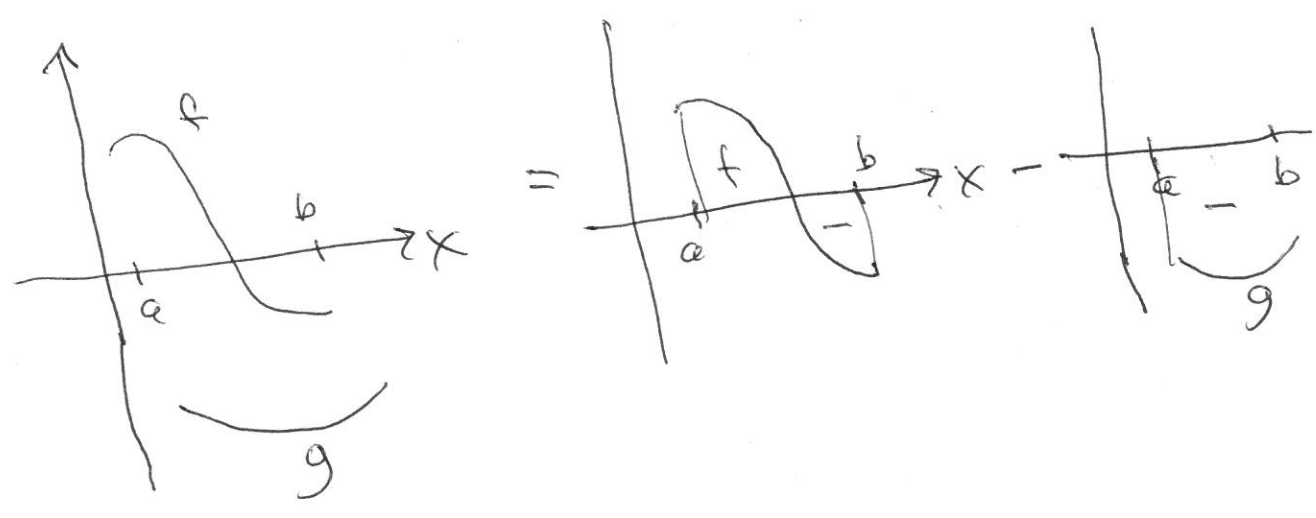
$y=f(x)$ and $y=g(x)$ on the closed interval $[a,b]$

For now, assume $f(x) \geq g(x)$ for all x in $[a,b]$



$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

This is valid even if f and or g have negative values (we still want for now, $f \geq g$ $\forall x$)



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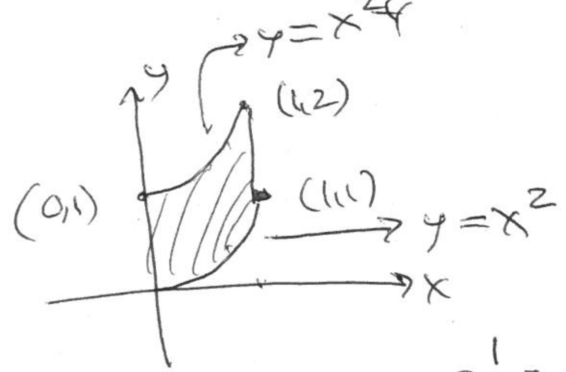
Def Let $f(x) \geq g(x)$ for all x in the closed interval $[a, b]$
The area of the region bounded by the curves given by
$y = f(x)$, $y = g(x)$ and the vertical lines $x = a$, $x = b$
is $A = \int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
provided that both right hand integrals exist

Ex Graph and find the area of the region R
bounded by the graphs of $f(x) = x^2 + 1$ on the interval $[0,1]$
$g(x) = x^2$



$$\text{Area} = \int_0^1 [(x^2+1) - x^2]\,dx$$
$$= \int_0^1 1\,dx = x\Big]_0^1 = 1 - 0 = 1$$

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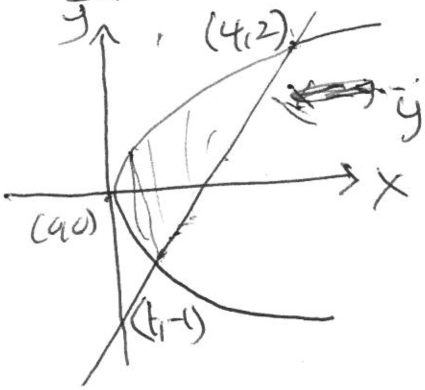
Ex Find the area enclosed by : $x = y^2$ and $y = x - 2$

$y = \pm\sqrt{x}$



$y = x - 2$

$x = y + 2$

Solution : Find pts of intersection

Set $y^2 = y + 2$

$y^2 - y - 2 = 0$

$(y-2)(y-1) = 0$

$y = -1, 2$

Integrating vertically

$$\text{Area} = \int_{x=0}^{x=1} (\sqrt{x} - (-\sqrt{x}))\,dx + \int_{x=1}^{x=4} (\sqrt{x} - (x-2)]\,dx$$

$$= \int_0^1 2x^{\frac{1}{2}}\,dx + \int_1^4 (x^{\frac{1}{2}} - x + 2)\,dx = \frac{9}{2}$$

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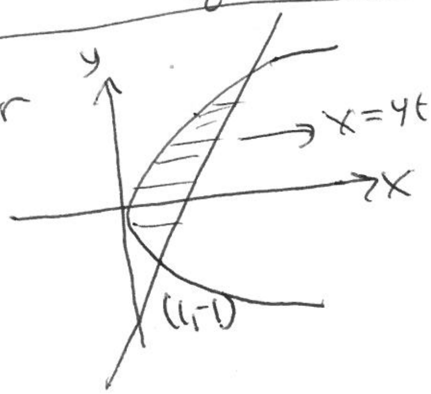
or



$\rightarrow x = y + 2$

$$\text{Area} = \int_{y=-1}^{y=+2} \left[(y+2) - y^2\right] \cancel{dx}\,dy$$

$$= \int_{-1}^{2} \left[-y^2 + y + 2\right] dy$$

$$= \left(-\frac{y^3}{3} + \frac{y^2}{2} + 2y\right)\Bigg|_{y=-1}^{y=2}$$

$$= \frac{9}{2}$$

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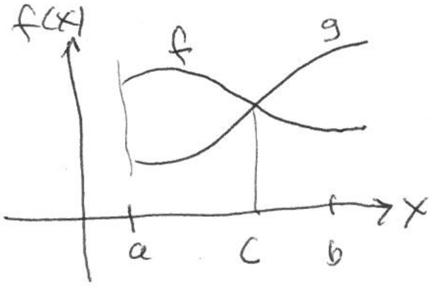
Final Exam

~~021 A - Mon Dec 9th 7:30 AM Voley 205~~

~~021 B - Fri Dec 13th 10:30 AM Voley 205~~

~~021 C - Tues Dec 10th 1:30 PM Voley 205~~

We now consider the case of 2 continuous functions f and g on $I = [a,b]$ where $f > g$ on part of I and $g > f$ on another part of I.

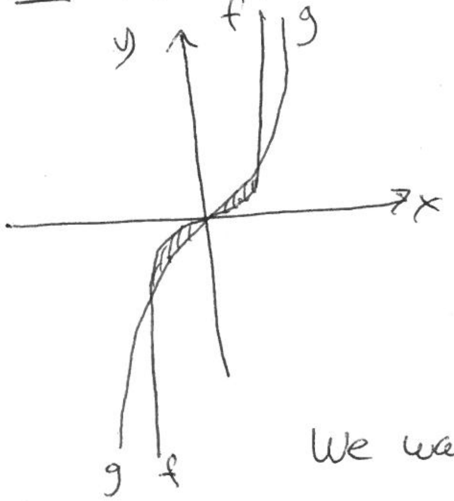


The area is not $\int_a^b [f(x) - g(x)]dx$

The area is: $\int_a^c [f(x) - g(x)]dx + \int_c^b [g(x) - f(x)]dx$

note $A = \int_a^b |f(x) - g(x)|dx = \int_a^b |g(x) - f(x)|dx$

Ex Find the area between $f(x) = x^5$ and $g(x) = x^3$



First find where f and g intersect.

$$x^5 = x^3$$
$$x^5 - x^3 = 0$$
$$x^3(x^2-1) = 0$$
$$x^3(x-1)(x+1) = 0$$
$$x = -1, 0, 1$$

We want: $\int_{-1}^0 (x^5 - x^3)dx + \int_0^1 (x^3 - x^5)dx$

$$= \left(\frac{x^6}{6} - \frac{x^4}{4}\right)\Big|_{-1}^0 + \left(\frac{x^4}{4} - \frac{x^6}{6}\right)\Big|_0^1 = \frac{1}{6}$$

slightly easier way; Set $H(x) = x^5 - x^3$ so H is an odd function

so $\int_{-1}^1 |H(x)|dx = 2\int_0^1 |H(x)|dx = 2\left(\frac{1}{12}\right) = \frac{1}{6}$

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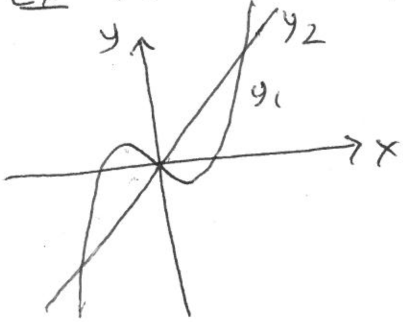
EX Find the area between: $y_1 = x^3 - x$, $y_2 = 3x$



Find the values of x at the points of intersection

$$x^3 - x = 3x$$
$$x^3 - x - 3x = 0$$
$$x^3 - 4x = 0$$
$$x(x^2 - 4) = 0$$
$$x(x-2)(x+2) = 0$$
$$x = 0, -2, 2$$

We want

$$A = \int_{-2}^{0} [(x^3 - x) - 3x]\,dx + \int_0^2 [3x - (x^3 - x)]\,dx$$

$$A = \int_{-2}^{0} [x^3 - 4x]\,dx + \int_0^2 [4x - x^3]\,dx$$

$$= \left(\frac{x^4}{4} - 2x^2\right)\Big|_{-2}^{0} + \left(2x^2 - \frac{x^4}{4}\right)\Big|_0^2$$

$$= 4 + 4 = 8$$

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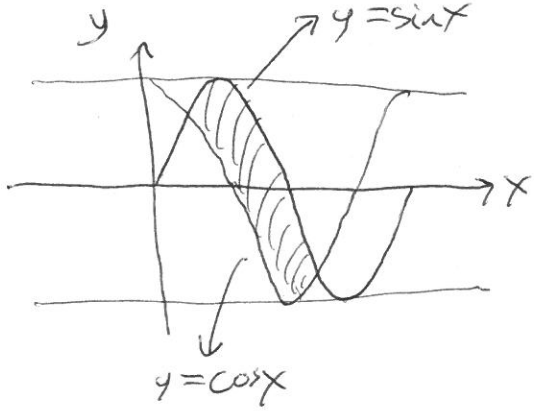
Ex Find the area of the region enclosed by one cycle of

$y = \sin x, \quad y = \cos x$



Find pts of intersection on $[0, 2\pi]$

$\sin x = \cos x$

$\frac{\sin x}{\cos x} = 1$

$\tan x = 1$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

$$\text{Area} = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} [\sin x - \cos x]\,dx$$

$$= (-\cos x - \sin x) \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= \left[-\left(\frac{-1}{\sqrt{2}}\right) - \left(\frac{-1}{\sqrt{2}}\right)\right] - \left[\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right]$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

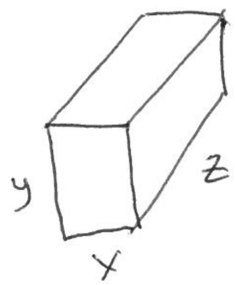
$$= \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

021 Sec 6.2

Section 6.2 Volumes

A (right) circular cylinder

Volume is: $V = (\pi r^2) h$ = (area of the base)(height)

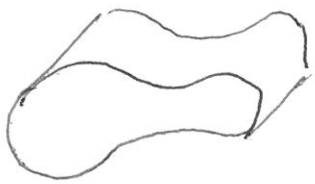


rectangular box

$V = (xy)(z) =$ (area of base) (height)

<u>Def</u> A cylinder is a solid with a top and bottom which are congruent shapes, The top and bottom are in parallel planes,

Volume = (area of the base) × height



To approximate the volume of the sections of a solid where x is in the ith subinterval, we compute the volume of the approximating cylinder S_i

$$V(S_i) \approx \underbrace{A(x_i^*)}_{\text{area of a cross section}} \underbrace{\Delta x}_{\text{height}}$$

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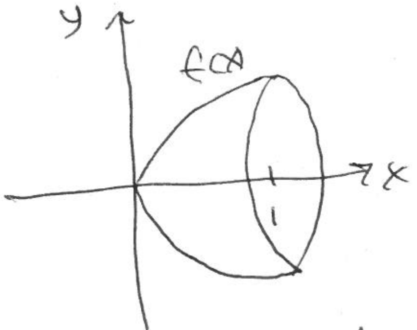
<u>Def</u> Let S be a solid that lies between $x=a$ and $x=b$

If the cross-sectional area of S in the plane P_x through x is perpendicular to the x-axis is $A(x)$, with A a continuous function, then the volume V is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*) \Delta x = \int_a^b A(x)\,dx$$

<u>Ex</u> Find the volume of the solid obtained by revolving the graph of $f(x) = \sqrt{x}$ about the x-axis between $x=0$ and $x=1$



The cross-sections $A(x)$ are circles with radius $f(x) = \sqrt{x}$

So $A(x) = \pi(\sqrt{x})^2 = \pi x$

So the volume is

$$V = \int_0^1 \pi x\,dx = \pi \int_0^1 x\,dx = \pi \frac{x^2}{2}\Big|_0^1 = \frac{\pi}{2}$$

<u>Aside</u>

In finding area (2 dim idea) we integrate line segments (1 dim)

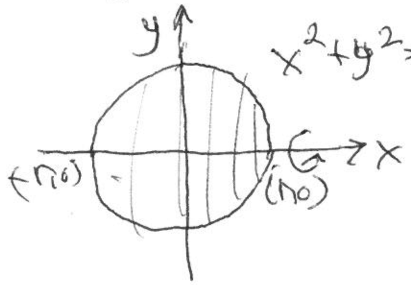
" " volume (3 dim idea) " " areas (2 dim)

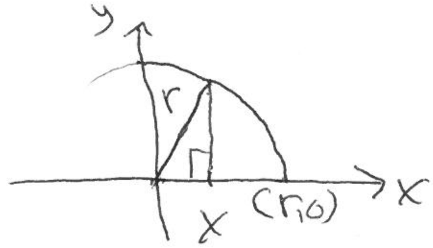
In a sense, integration "increases" the dimension by 1,

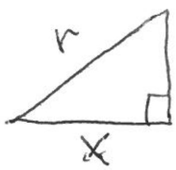
021 Sec 6.2

<u>Ex</u> Find the volume of a sphere of radius r

wlog (without loss of generality), assume that the sphere is centered at the origin







$\sqrt{r^2-x^2}$ = radius of the cross section

The area of a cross-section is $\pi r^2 = \pi \left[f(x)\right]^2$

$$= \pi\left(\sqrt{r^2-x^2}\right)^2$$

$$= \pi\left(r^2-x^2\right)$$

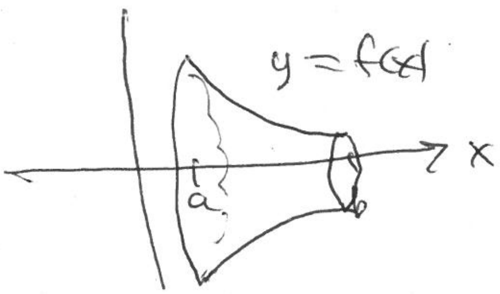
$$V = \pi\int_{-r}^{r}(r^2-x^2)\,dx$$

so $V = 2\pi\int_0^r (r^2-x^2)\,dx$, note $A(x) = (r^2-x^2)\pi$ is an even function

$$V = 2\pi\left[r^2x - \frac{x^3}{3}\right]_{x=0}^{x=r} = 2\pi\left[r^3-\frac{r^3}{3}\right] = \frac{4}{3}\pi r^3$$

The sphere is an example of a solid of revolution

Let $f(x) \geq 0$ on $[a,b]$, Revolve $f(x)$ about the x-axis.



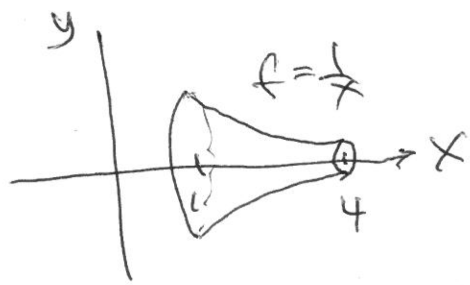
A cross-section at x will be a circle of radius $f(x)$

The area of a cross-section is

$$\pi r^2 = \pi [f(x)]^2$$

$$\text{so Volume} = \int_a^b A(x)\,dx = \int_a^b \pi [f(x)]^2 dx = \pi \int_a^b [f(x)]^2 dx$$

<u>EX</u> Find the volume of the solid generated when $f(x) = \frac{1}{x}$ on $[1,4]$ is revolved about the x-axis.



$$V = \pi \int_1^4 \left(\frac{1}{x}\right)^2 dx$$

$$= \pi \int_1^4 x^{-2} dx$$

$$V = \pi \frac{x^{-1}}{-1}\Big|_{x=1}^{x=4} = -\frac{\pi}{x}\Big|_1^4 = \pi\left[-\frac{1}{4}+1\right]$$

$$V = \frac{3\pi}{4}$$

If a solid is <u>not</u> a solid of revolution it can be tricky to find a formula for the area of a cross-section.

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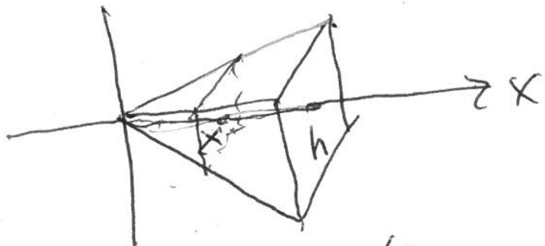
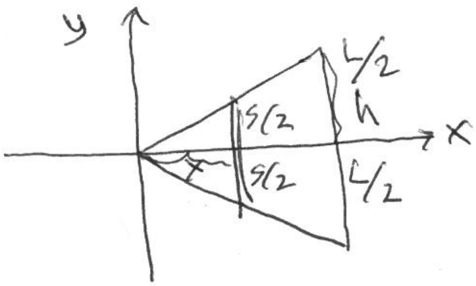
<u>Ex</u> Find the area of a square base pyramid by integrating cross-sections that are perpendicular to the x-axis.



The center of the base is on the x-axis

We need a formula for the area of a cross-section at x, $A(x)$

Let the pyramid have a base length of L, true height of h and a slant height of S



Using similar triangles, we have

$$\frac{x}{h} = \frac{\frac{s}{2}}{\frac{L}{2}} = \frac{s}{L}, \text{ so } s = \frac{Lx}{h}$$

The area of a cross-section is s^2

$$A(s) = s^2 = \left(\frac{Lx}{h}\right)^2 = \frac{L^2}{h^2}x^2$$

<u>Note</u> L, h are constants, the only variable is x

$$\text{so } V = \int_0^h A(x)\,dx = \int_0^h \frac{L^2}{h^2}x^2\,dx = \frac{L^2}{h^2}\int_0^h x^2\,dx$$

$$V = \frac{L^2}{h^2}\,\frac{x^3}{3}\Big|_{x=0}^{x=h} = \frac{1}{3}L^2h$$

$$V = \frac{1}{3}(\text{area of base})(\text{height})$$

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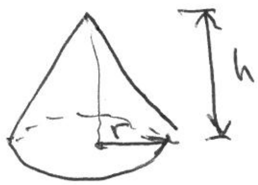
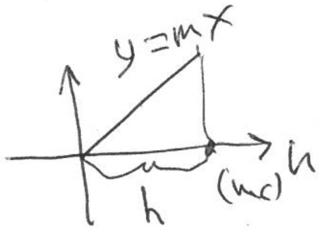
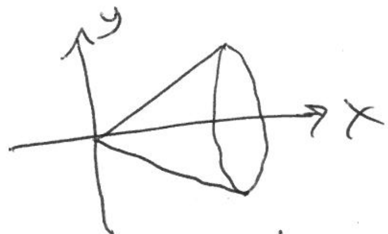
A cone is a solid whose base is in a plane. The solid tapers to a pt. So the pyramid is a cone



$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}(\text{area of base})(\text{height})$

We now show the above is true





$$V = \int_0^h \pi\,[f(x)]^2\,dx = \pi \int_0^h (mx)^2\,dx$$

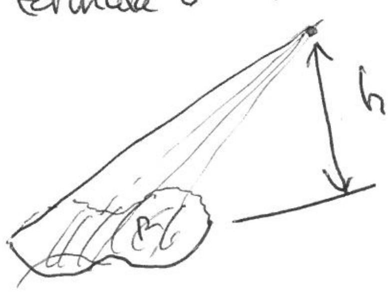
$$V = m^2\pi \int_0^h x^2\,dx = m^2\pi \frac{x^3}{3}\Big|_{x=0}^{x=h}$$

$$V = \frac{1}{3}\pi m^2 x^3 = \frac{1}{3}(\pi m x^2)x$$

$$= \frac{1}{3}(\text{area of base})(\text{height})$$

Note $y = mx$, so $x = \frac{y}{m}$

The formula is true for any cone



Volume $= \frac{1}{3}(\text{area of base})(\text{height})$

021. Sec 6.3 Volumes by cylindrical shells ①

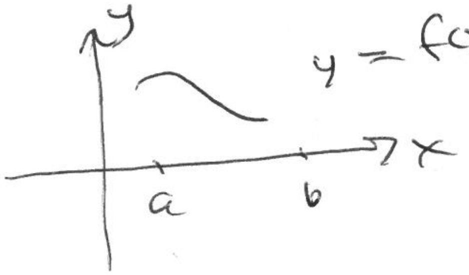
Def Let f be continuous and non-neg on $[a,b]$, $a \geq 0$

Let R be the region bounded above by $y=f(x)$

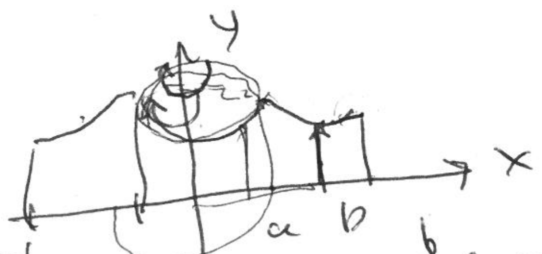
bounded below by ~~y=0~~ $y=0$ (x-axis)

and on the sides by the vertical line segments

$x=a$, $x=b$.



Then the volume of the solid of revolution that is generated by revolving R about the y-axis is



$$V = \int_a^b \underbrace{(2\pi x)}_{\text{circum}} \underbrace{f(x)}_{\text{height}} \underbrace{dx}_{\text{thickness}}$$

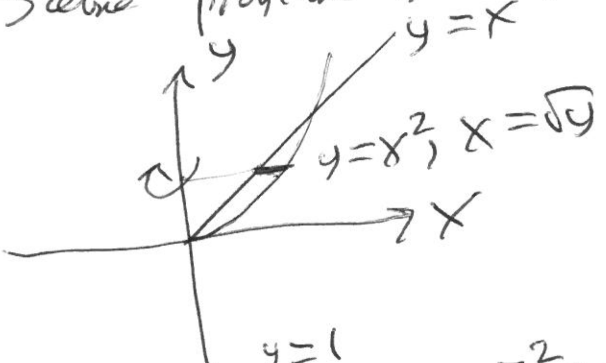
Ex Let D be the region between $y=x$ and $y=x^2$. Find the volume of the solid generated when D is revolved about the y-axis.

To find a and b, set $x=x^2$, $x^2-x=0$, $x(x-1)=0$, $x=0,1$

$$V = \int_0^1 \underbrace{(2\pi x)}_{\text{radius}} \underbrace{x}_{f(x)} dx - \int_0^1 2\pi x \cdot x^2 dx$$

$$V = 2\pi \int_0^1 (x^2 dx) - \int_0 (x^3 dx) = \frac{\pi}{6}$$

Same problem use cylinders (disks) from 6.2



$$V = \int_{y=0}^{y=1} \pi \left[\sqrt{y}\right]^2 dy - \pi \int_{y=0}^{1} (y)^2 dy$$

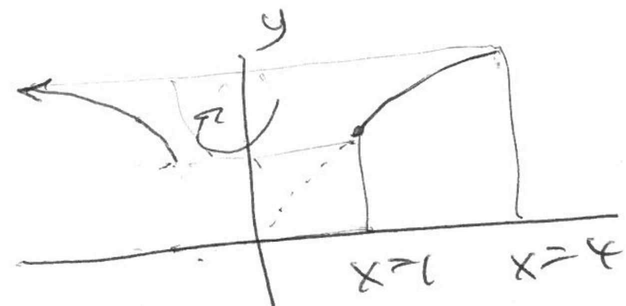
$$= \pi \left[\int_0^1 y\, dy - \int_0^1 y^2 dy \right] = \frac{\pi}{2}$$

using disks our line segments are perpendicular to the ~~axis~~ axis of rotation.

Using shells - our line segments are parallel to the axis of rotation.

Oxl 6.3 (3)

Ex Use shells to find the volume of the solid generated when the region enclosed by $y=\sqrt{x}$, $x=1$, $x=4$, $y=0$ is revolved about the y-axis:



Using disks we would need two integrals

$$V=\int_{x=1}^{x=4} 2\pi x\, f(x)\,dx = \int_1^4 2\pi x\left(x^{\frac{1}{2}}\right)dx$$

$$=\int_1^4 2\pi x^{3/2}\,dx = 2\pi x^{5/2}\left(\tfrac{2}{5}\right)\Big|_1^4$$

$$=\frac{124\pi}{5}$$