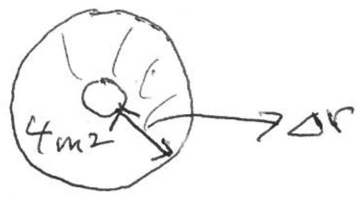
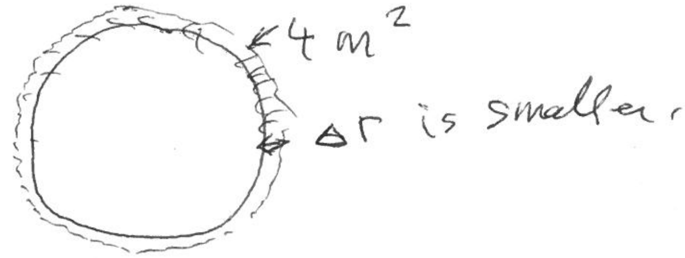
021 Sec

Sec 3.9 Related Rates

1) An oil spill in the shape of a circle is expanding at a constant rate of $4m^2/min$. How fast is the radius of the circle changing when the area of the circle is $200m^2$





<u>Know</u> $\frac{dA}{dt} = 4m^2/min$

<u>Seek</u> $\frac{dr}{dt}$ when $A = 200 m^2$

<u>Link</u>: $A = \pi r^2$

Differentiate the link wrt t.

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

so $$\frac{dr}{dt} = \frac{\frac{dA}{dt}}{2\pi r} = \frac{4}{2\pi r} = \frac{2}{\pi r}$$

To find r when $A = 200$, use $A = \pi r^2$

$200 = \pi r^2$, $\frac{200}{\pi} = r^2$; $r = \sqrt{\frac{200}{\pi}}$

So, $$\left.\frac{dr}{dt}\right|_{A=200} = \frac{2}{\pi\sqrt{\frac{200}{\pi}}} = \frac{2}{\sqrt{200\pi}} = \frac{1}{\sqrt{50\pi}}$$

i.e. $\frac{dr}{dt} \approx 0.079\ m/min$ when area $= 200m^2$

021 Sec

Ex A spherical balloon is expanding s.t. the radius increases at a constant rate of 2 inch/min.

At what rate is the volume changing, when the radius is 5 inches.

Seek: $\frac{dV}{dt}$, when $r = 5$ inches

Know: $\frac{dr}{dt} = 2$ inches/min

Link: $V = \frac{4}{3}\pi r^3$, Diff the link wrt t.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

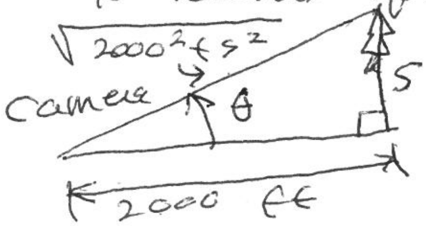
$$\left.\frac{dV}{dt}\right|_{r=5} = 4\pi(5^2)2 = 200\pi \text{ in}^3/\text{min}$$

So, when the radius is 5 inches, the volume is increasing at a rate of 200π in^3/min.

≈ 628 in^3/min

021 Sec 3.9

EX A changing angle of elevation.
A rocket is launched vertically. The position equation of the rocket is $s = 50t^2$ where s is in ft, t is in seconds.
A camera 2000 ft away from the launch site tracks the rocket.
Find the rate of change of the angle of elevation of the camera 10 seconds after launch.

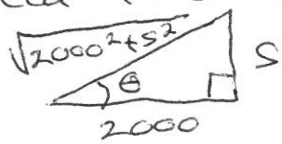


Seek $\frac{d\theta}{dt}$ when $t = 10$ seconds

Know the position (height) of the rocket at t seconds is $s(t) = 50t^2$

We will also need that $v(t) = s'(t) = 100t$

LINK: $\tan\theta = \frac{opp}{adj}$



$\tan\theta = \frac{s}{2000}$, now diff wrt t.

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{2000}\frac{ds}{dt}$$

So $\frac{d\theta}{dt} = (\cos^2\theta)\frac{1}{2000}\frac{ds}{dt}$

Now, $\cos\theta = \frac{adj}{hyp} = \frac{2000}{\sqrt{2000^2+s^2}}$

So $\frac{d\theta}{dt} = \frac{2000^{\cancel{2}}}{(2000^2+s^2)}\frac{1}{\cancel{2000}}\frac{ds}{dt}$,

$\frac{d\theta}{dt} = \frac{2000}{2000^2+s^2}\frac{ds}{dt}$,

Recall $s(t) = 50t^2$

$\frac{ds}{dt} = 100t$

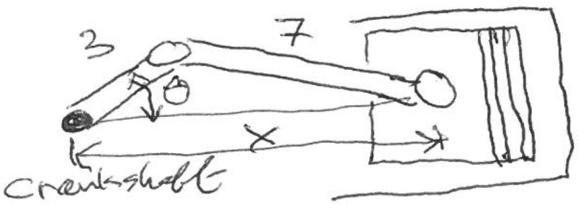
~~when~~ $s(10) = 50(10^2) = 5000$

$$\frac{d\theta}{dt} = \left(\frac{2000}{2000^2+5000^2}\right)100t$$

$$\left.\frac{d\theta}{dt}\right|_{t=10} = \frac{(2000)(100)(10)}{2000^2+5000^2} = \frac{2}{29} \text{ radians/sec}$$

021 Sec 3.9

<u>EX</u> Velocity of a piston

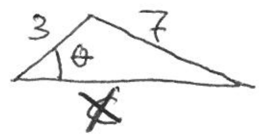


A 7 inch connecting rod is fastened to a 3 inch crank.

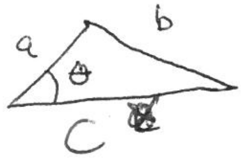
The crankshaft rotates counterclockwise at a constant rate of 200 Revolutions per minute. What is the velocity of the piston when $\theta = \frac{\pi}{3}$

<u>Know</u> $\frac{d\theta}{dt} = (200)(2\pi) = 400\pi$ radians / minute

<u>Seek</u> $\frac{dx}{dt}$ when $\theta = \frac{\pi}{3}$



<u>Link</u> : Law of Cosines



$$b^2 = a^2 + c^2 - (2ac)\cos\theta$$

For us. $7^2 = 3^2 + x^2 - 2(3)x\cos\theta$

Diff wrt t

$$0 = 2x\frac{dx}{dt} - 6\left[-x\sin\theta\frac{d\theta}{dt} + \cos\theta\frac{dx}{dt}\right]$$

$$\frac{dx}{dt} = \frac{6x\sin\theta}{6\cos\theta - 2x}\left(\frac{d\theta}{dt}\right)$$

We find x, by using the law of cosines again.

$$7^2 = 3^2 + x^2 - 2(3)x\underbrace{\cos\left(\tfrac{\pi}{3}\right)}_{=\frac{1}{2}}$$

$49 = 9 + x^2 - 3x$, $x^2 - 3x - 40 = 0$

$(x-8)(x+5) = 0$, $x = 8, -5$

use $x = 8$

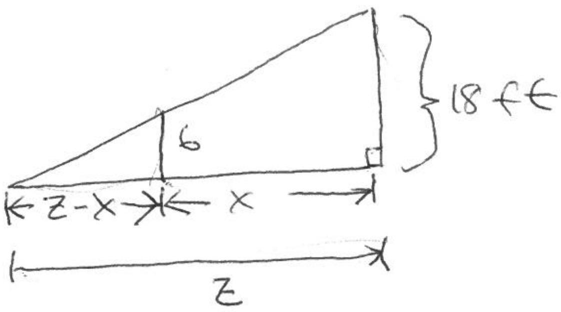
so $$\frac{dx}{dt} = \frac{6\cdot 8\sin\left(\frac{\pi}{3}\right)400\pi}{6\cos\left(\frac{\pi}{3}\right) - 2(8)} = \frac{6\cdot 8\frac{\sqrt{3}}{2}}{6\left(\frac{1}{2}\right) - 16}(400\pi)$$

$\frac{dx}{dt} = \frac{9600\pi\sqrt{3}}{-13}$ inch/min, $\frac{dx}{dt} \approx -4018$ inch/min

021 Section 3.9

More related rates

Ex A 6 ft tall person walks at 8 ft/sec away from a streetlight that is 18 ft tall



How fast is the tip of the person's shadow moving along the ground when the person is 100 ft. from the pole?

Know $\frac{dx}{dt} = 8$ ft/sec

Seek $\frac{dz}{dt}$ when $x = 100$ ft

Link Use similar triangles: $\frac{z}{18} = \frac{z-x}{6}$

so $6z = 18(z-x)$

$18x = 12z$

$3x = 2z$

Now, differentiate $3x = 2z$ wrt t.

$3\frac{dx}{dt} = 2\frac{dz}{dt}$

So $(3)(8) = 2\frac{dz}{dt}$

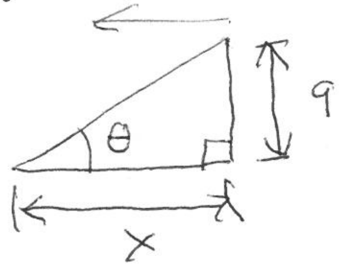
$\frac{dz}{dt} = 12$

So, the distance of tip of the shadow from the pole is increasing at 12 ft/sec, when the person is 100 ft away,

Note we never use the value of 100 ft.

021 Sec 3.9

Ex An airplane flies at a height of 9 km in the direction of an observer on the ground. The plane flies at a speed of 800 km/hr. Find the rate of change of the angle of elevation of the plane from the observer when the elevation angle is $\frac{\pi}{3}$



Seek $\frac{d\theta}{dt}$ when $\theta = \frac{\pi}{3}$

Know $\frac{dx}{dt} = -800$ km/hr.

Link $\cot\theta = \frac{x}{9}$, Diff the link wrt t

$$-\csc^2\theta \frac{d\theta}{dt} = \frac{1}{9}\frac{dx}{dt}$$

When $\theta = \frac{\pi}{3}$, $\csc\theta = \frac{2}{\sqrt{3}}$

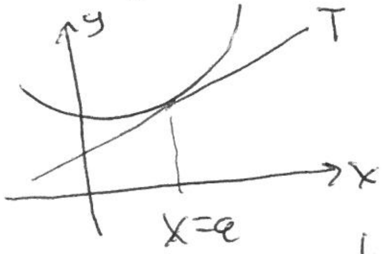
so $-\csc^2\theta = \frac{-4}{3}$

So, $\frac{d\theta}{dt} = \left(\frac{-800}{9}\right)\left(\frac{-3}{4}\right) = \frac{200}{3}$ $\frac{\text{radians}}{\text{hr}}$.

021 Section 3.10

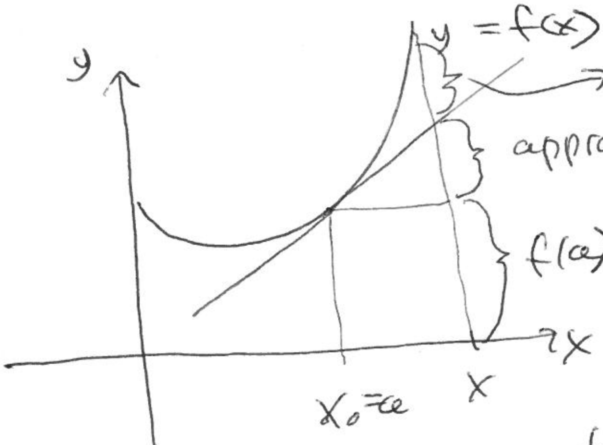
Section 3.10 Linear Approximations and Differentials

Let $y = f(x)$ be differentiable at $x=a$



near the pt. $(a, f(a))$ the curve can be closely approximated by the tangent line T

Generally, The closer we are to $x=a$, the better the approximation.



error

approximate change $= f'(a)[x-a] = f'(a)dx = dy$

$f(a)$

The total change or exact change is

approximate + error

$f(x) \approx f(a) +$ approx change

$f(x) \approx f(a) + f'(x)dx$, where $dx = x-a$

Some words on notation

Let $y=f(x)$. The differential of y is

$dy = f'(x)\Delta x = f'(x)dx$

Ex if $f(x) = y = x^4$, $dy = 4x^3dx$

The symbol Δ - always means the exact change

So Δx is the exact change in x

Δy " " " " y

etc

The letter "d" is an exact change if the variable that follows is an independent variable

The letter "d" is an approximate change if the letter follows is a dependent variable.

So in $y = f(x)$

x is the independent variable so $dx = \Delta x$

y " " dependent " " $dy \approx \Delta y$

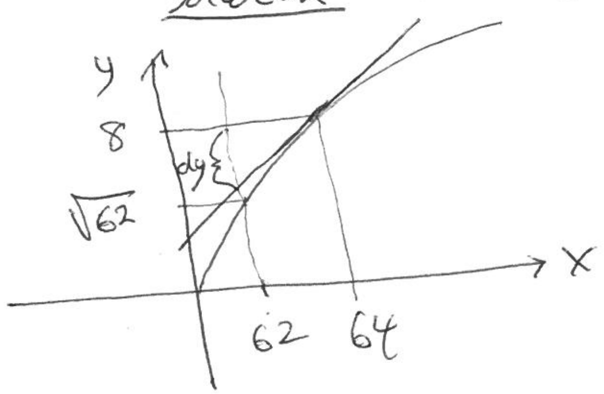
If we had $x = f(y)$, now

y is the independent variable, so $dy = \Delta y$

x is the dependent variable, so $dx \approx \Delta x$

Ex Approximate $\sqrt{62}$ using differentials

Solution Let $y = f(x) = \sqrt{x} = x^{\frac{1}{2}}$



$$dy = f'(x)\,dx$$
$$= \frac{1}{2\sqrt{x}}\,dx$$
$$= \frac{1}{2\sqrt{64}}(62-64)$$
$$= \frac{1}{2\sqrt{64}}(-2) = -\frac{1}{8}$$

So, $\sqrt{62} \approx \sqrt{64} + f'(64)\,dx$

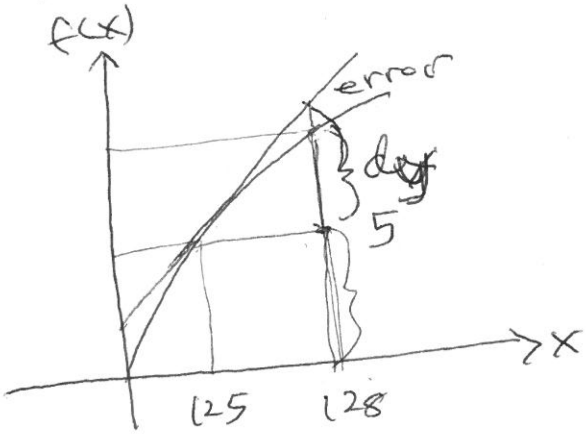
$\approx 8 - \frac{1}{8} = 7.875 \rightarrow$ by our calculations

To three decimal places $\rightarrow$ a better approx from a calculator.

$\sqrt{62} \approx 7.874$

Approximate $\sqrt[3]{128}$

Solution Let $f(x) = x^{\frac{1}{3}}$,



Know $\sqrt[3]{125} = 5$, i.e. $f(125) = 5$

$$f'(x) = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3x^{2/3}}$$

$$f'(125) = \frac{1}{3(125)^{2/3}} = \frac{1}{3 \cdot 5^2} = \frac{1}{75}$$

Now $dx = \Delta x = 128 - 125 = 3$, so $dy = \frac{1}{75} dx = \frac{1}{75}[128 - 125] = \frac{3}{75} = \frac{1}{25}$

So, $128^{\frac{1}{3}} \approx 125^{\frac{1}{3}} + \left(\frac{1}{3 \cdot 5^2}\right)(3)$

$\approx 128^{\frac{1}{3}} = 5 + \frac{1}{25} = 5.04$ our approximation

To four decimal places

$128^{\frac{1}{3}} \simeq 5.0267$, a better approximation from a calculator.

021 Sec 3.10 ④

Estimation of error.

The radius of a ball bearing is measured at 0.7 inches which is correct to within 0.01 Inch

i.e. $-0.01 \le \Delta r \le 0.01$

$.699 \le r \le 7.01$

Find bounds for the volume of the ball bearing.

<u>Solution</u> $V = \frac{4}{3}\pi r^3$

so $\Delta V \approx dV = 4\pi r^2 dr$

$= 4\pi(0.7)^2(\pm 0.01)$

$\simeq \pm 0.06158$ inch^3

so V is s.t.

$\frac{4}{3}\pi(0.7)^3 - 0.06685 \le V \le \frac{4}{3}\pi(0.7)^3 + 0.06158$

To decide if this is a big error or small error, we use relative error. In our problem we compute

$$\frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = 3\frac{dr}{r}$$

$$= \frac{3(\pm 0.01)}{0.7} \approx \pm 0.0429\%$$

The corresponding percentage error is

$$\frac{dV}{V} 100\% = 4.29\%$$

021

Section 3.11 The Hyperbolic Functions

Def $\sinh x = \dfrac{e^x - e^{-x}}{2}$

$\cosh x = \dfrac{e^x + e^{-x}}{2}$

$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x \cancel{+} e^{-x}}{e^x + e^{-x}}$

$\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

$\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

$\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

Identities

i) $\sinh(-x) = -\sinh x$, so $\sinh x$ is an odd function

Pf $\sinh(-x) = \dfrac{e^{-x} - e^{-(-x)}}{2} = \dfrac{e^{-x} - e^x}{2}$

$-\sinh(x) = -\dfrac{(e^x - e^{-x})}{2} = \dfrac{e^{-x} - e^x}{2}$

Also 2) ~~$\cosh^2 x$~~ $\cosh(-x) = \cosh(x)$, $\cosh x$ is an even function

3) i) $\cosh^2 x - \sinh^2 x = 1$

Pf $\left(\dfrac{e^x + e^{-x}}{2}\right)^2 - \left(\dfrac{e^x - e^{-x}}{2}\right)^2 \stackrel{?}{=} 1$

$\dfrac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} - \dfrac{e^{2x} - 2e^x e^{-x} + e^{2x}}{4} \stackrel{?}{=} 1$

$\dfrac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}} - e^{2x} + 2 - e^{2x}}{4} \stackrel{?}{=} 1$

$\dfrac{2+2}{4} = 1$

021 Sec 3.11

Derivatives of Hyperbolics

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^x}{2} - \frac{e^{-x}}{2}\right) = \frac{e^x}{2} + \frac{e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx}\cosh x = \frac{d}{dx}\left(\frac{e^x}{2} + \frac{e^{-x}}{2}\right) = \frac{e^x}{2} - \frac{e^{-x}}{2} = \sinh x$$

$$\frac{d}{dx}\tanh x = \frac{d}{dx}\frac{\sinh x}{\cosh x} = \frac{(\cosh x)\frac{d}{dx}\sinh x - \sinh x\frac{d}{dx}\cosh x}{\cosh^2 x}$$

$$= \frac{(\cosh x)(\cosh x) - (\sinh x)(\sinh x)}{\cosh^2 x}$$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$\frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}\coth x = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tan x$$

<u>Ex</u> $\frac{d}{dx}\tanh^3\sqrt{x} = \frac{d}{dx}\left[\tanh(x^{\frac{1}{2}})\right]^3$

$$= 3\left[\tanh(x^{\frac{1}{2}})\right]^2 \frac{d}{dx}\left[\tanh(x^{\frac{1}{2}})\right]$$

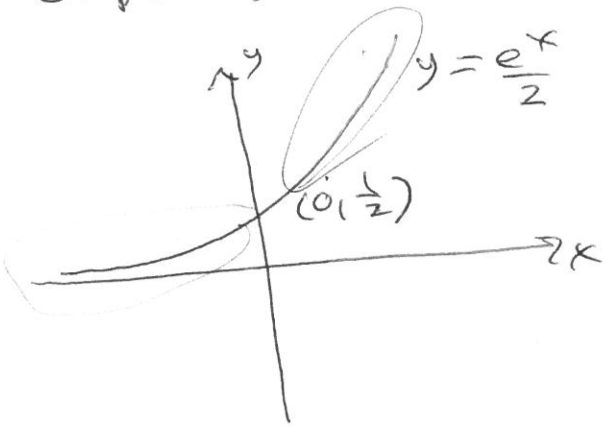
$$= \left(3\tanh^2\sqrt{x}\right)\operatorname{sech}^2(x^{\frac{1}{2}})\frac{d}{dx}x^{\frac{1}{2}}$$

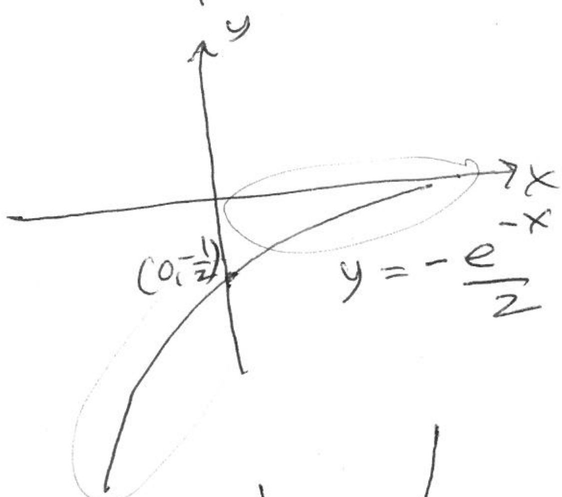
$$= 3\tanh^2\sqrt{x}\operatorname{sech}^2\sqrt{x}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

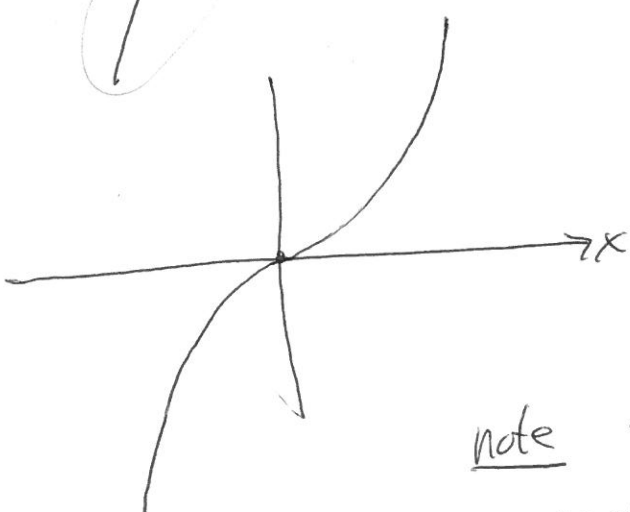
$$= \frac{3}{2\sqrt{x}}\tanh^2\sqrt{x}\operatorname{sech}^2\sqrt{x}$$

021 Sec 3.11

Graph $y = f(x) = \sinh x = \frac{e^x}{2} - \frac{e^{-x}}{2}$







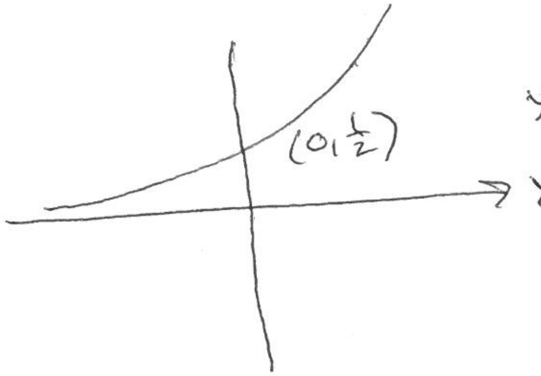
$y = \frac{e^x - e^{-x}}{2} = \sinh x$

note $\sinh x = y$ is 1 to 1

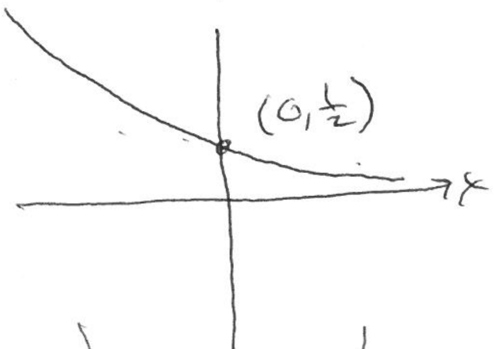
so $y = \sinh x$ has an inverse function

021 Sec 3.11

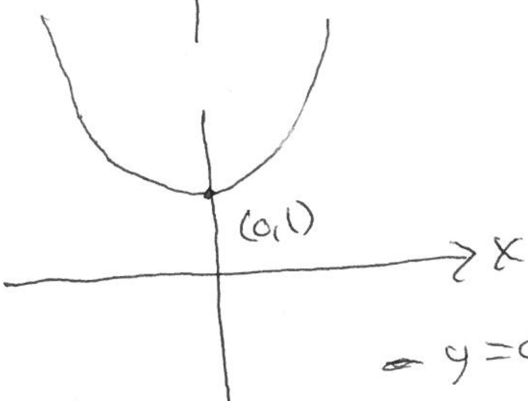
Graph $f(x) = \cosh x = \frac{e^x}{2} + \frac{e^{-x}}{2}$



$y = \frac{e^x}{2}$



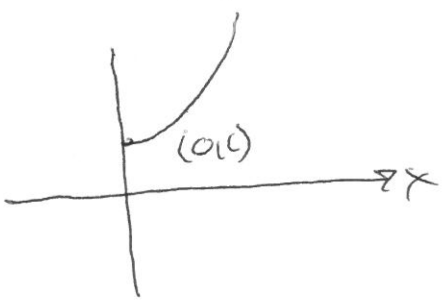
$y = \frac{e^{-x}}{2}$



$y = \frac{e^x}{2} + \frac{e^{-x}}{2} = \cosh x$

$y = \cosh x$ is not 1-1

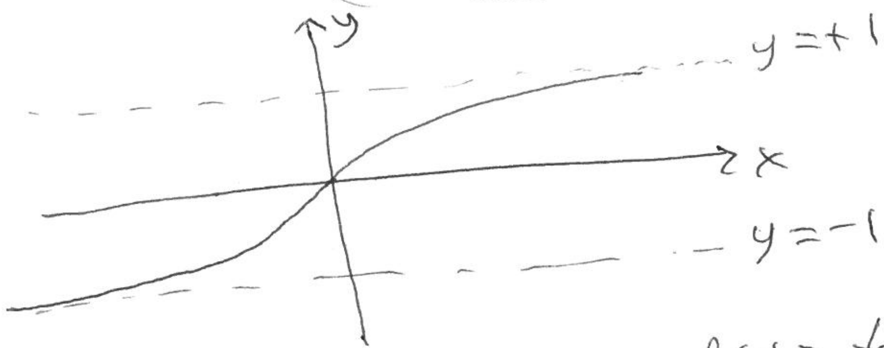
So, we restrict the function to $[0, \infty)$



So the restriction of $\cosh x$ on $[0, \infty)$ is 1-1 and hence has an inverse

021 Sec 3.11

$$y = \tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$



This is 1 to 1, so $f(x) = \tanh x$ has an inverse function

021 Section 3.11

$\sinh x = \dfrac{e^x - e^{-x}}{2}$ $\qquad \cosh x = \dfrac{e^x + e^{-x}}{2}$

We now define the inverse hyperbolics

$y = \sinh^{-1} x$ iff $\sinh y = x$

$y = \cosh^{-1} x$ iff $\cosh y = x,\ y \geq 0$

$y = \tanh^{-1} x$ iff $\tanh y = x$

We have $y = \sinh^{-1} x = \ln(x + \sqrt{x^2+1}),\ x \in \mathbb{R}$

$\cosh^{-1} x = \ln(x + \sqrt{x^2-1})\quad x \geq 1$

$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right),\ -1 < x < 1$

We now show that $y = \sinh^{-1} x = \ln(x + \sqrt{x^2+1})$

$y = \sinh^{-1} x$ means $x = \sinh y$

$x = \dfrac{e^y - e^{-y}}{2}$, so $2x = e^y - e^{-y}$

multiply through by e^y

$e^y - 2x - e^{-y} = 0$,

$e^y e^y - e^y 2x - e^y e^{-y}$ $(e^y)^2 - (2x)e^y - 1 = 0$

$e^{2y} - e^y 2x - 1$, $(e^y)^2 - (2x)e^y - 1$

This is a quadratic in e^y. Use the quadratic formula

$ax^2 + bx + c = 0,\quad x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$e^y = \dfrac{2x \pm \sqrt{(-2x)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \dfrac{2x \pm \sqrt{4x^2+4}}{2} = \dfrac{2x \pm 2\sqrt{x^2+1}}{2}$

$e^y = x \pm \sqrt{x^2+1}$; is this a "+", or a "−"

$e^y > 0$, so $x \pm \sqrt{x^2+1}$ would have to be > 0.

Consider $x^2 + 1 > x^2$

$\sqrt{x^2+1} > \sqrt{x^2} = x$, so $\sqrt{x^2+1} > x$

So $x - \sqrt{x^2+1} < 0$, we don't use this solution

and $x + \sqrt{x^2+1} > 0$, so, we use this one

so $\quad e^y = x + \sqrt{x^2+1}$, take ln of both sides

$\ln e^y = \ln(x + \sqrt{x^2+1})$

$y = \ln(x + \sqrt{x^2+1})$

$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$

021 Sec 3.11

We now can show that

$$\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}, \quad x>1$$

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}, \quad -1<x<1$$

We will show that $\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}$

<u>Pf</u> $\frac{d}{dx}\cosh^{-1}(x) = \frac{d}{dx}\ln(x+\sqrt{x^2-1}), \quad x>1$

$$= \frac{\frac{d}{dx}(x+\sqrt{x^2-1})}{x+\sqrt{x^2-1}} = \frac{1+\frac{2x}{2\sqrt{x^2-1}}}{x+\sqrt{x^2-1}}$$

$$= \frac{\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}} = \left(\frac{\cancel{\sqrt{x^2-1}+x}}{\sqrt{x^2-1}}\right)\left(\frac{1}{\cancel{\sqrt{x^2-1}+x}}\right)$$

$$= \frac{1}{\sqrt{x^2-1}}$$

<u>Ex</u> $\frac{d}{dx}\cosh^{-1}(5x^3)$

$$= \frac{1}{\sqrt{(5x^3)^2-1}}\frac{d}{dx}5x^3 = \frac{15x^2}{\sqrt{25x^6-1}}$$

<u>Also</u> $\frac{d}{dx}\operatorname{csch}^{-1}x = \frac{1}{|x|\sqrt{x^2+1}}$

$$\frac{d}{dx}\operatorname{sech}^{-1}x = \frac{-1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}\coth^{-1}x = \frac{1}{1-x^2}, \quad x>1$$