Quiz on Friday, June 21

① Vectors - basic operations sec 12.2

② Dot product sec 12.3

③ Cross product sec 12.4

Recall - Let $a = \langle a_1, a_2, a_3 \rangle$, $b = \langle b_1, b_2, b_3 \rangle$

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Ex Find a vector perpendicular to the plane passing through the points $P(1,4,6)$, $Q(-2,5,-1)$, $R(1,-1,1)$

$$\overrightarrow{PQ} = (-2-1)i + (5-4)j + (-1-6)k = -3i + j - 7k$$

$$\overrightarrow{PR} = (1-1)i + (-1-4)j + (1-6)k = -5j - 5k$$

We want

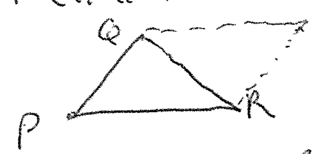
$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = (-5, -35)i + (15-0)j + (15-0)k$$

$$= -40i - 15j + 15k$$

Any nonzero multiple is also perpendicular to the plane.

Ex Find the area of the triangle with vertices $P(1,4,6)$, $Q(-2,5,-1)$, $R(1,-1,1)$

Solution



We find the area of the parallelogram formed by PQR and divide by 2

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(-40)^2 + (-15)^2 + 15^2} = 5\sqrt{82}$$

So, the area of the rectangle is $\frac{5}{2}\sqrt{82}$

022 Sec 12.4 Wednesday, June 29, 2019

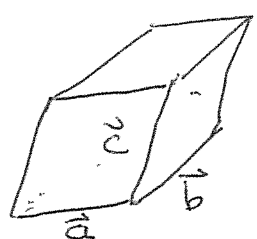
Properties of the cross product

Let a, b, c be vectors and c a scalar, then

1) $a \times b = b \times a$
2) $(ca) \times b = c(a \times b) = a \times (cb)$
3) $a \times (b + c) = a \times b + a \times c$
4) $(a + b) \times c = a \times c + b \times c$
5) $a \cdot (b \times c) = (a \times b) \cdot c$
6) $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$

Triple Products (triple scalar product)

$$a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



Let S be the solid with defined by the edges $\vec{a}, \vec{b}, \vec{c}$, Then the volume of S is $|a \cdot (b \times c)|$

Ex Show the vectors $a = \langle 1, 4, -7 \rangle$, $b = \langle 2, -1, 4 \rangle$, $c = \langle 0, -9, 18 \rangle$ are coplanar - i.e. are all in the same plane.

Solution Show that the volume of the solid formed by a, b, c is 0. i.e. height is 0.

$$a \cdot (b \times c) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7 \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$$

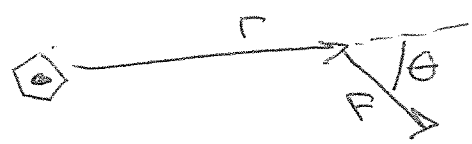
$$= 1(18) - 4(36) - 7(-18) = 0$$

Torque - consider a force F acting on a rigid body at a point given by a position vector r.

The torque τ is defined as

$$\tau = r \times F$$

Torque measures the tendency of the body to rotate about the origin.



We have that $|\tau| = |r \times F| = |r||F| \sin\theta$

with θ the angle between the position and force vectors.

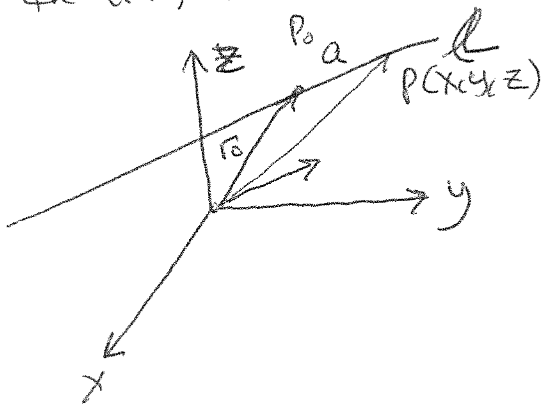
Ex A bolt is tightened by applying a 40-N force to a 0.25 m wrench. Find the magnitude of the torque about the center of the bolt.

Solution $|\tau| = |r \times F| = |r||F| \sin 75^\circ$
$= (0.25)(40) \sin 75^\circ = 10 \sin 75^\circ \approx 9.66 \text{ N}\cdot\text{m}$

Section 12.5 Equations of Lines and Planes.

In $\mathbb{R}^2$ a line can be determined by a point and a direction (slope)

In $\mathbb{R}^3$, the case is the same but more complicated.



fixed

Let $P_0(x_0,y_0,z_0)$ be on L

Let r_0 be the position vector for P_0

Let $P(x,y,z)$ be another, non fixed, point on L.

Let a be the vector from P_0 to P

Let v be a vector starting at the origin

with v parallel to ℓ.

So tv as t ranges over the real numbers is a line parallel to L.

So we add $\vec{r_0}$ to each pt in tv we get L

So $\boxed{r = r_0 + tv}$ is a vector equation of L.

Each value of the parameter t gives a point on L.

If the vector v that gives the direction of L is $v = \langle a,b,c\rangle$, then $tv = \langle ta, tb, tc\rangle$

with $r = \langle x,y,z\rangle$ and $r_0 = \langle x_0,y_0,z_0\rangle$

$\langle x,y,z\rangle = \langle x_0+ta,\ y_0+tb,\ z_0+tc\rangle$

so $x = x_0+ta$, $y = y_0+tb$, $z = z_0+tc$

This are the parametric equations of L through P_0 and parallel to $v = \langle a,b,c\rangle$

Ex Let $P_0 = (2,4,6)$, so $r_0 = \langle 2,4,6\rangle$

Let $v = \langle 3,5,7\rangle$. Find the equation of the line L through P and parallel to v

Solution $r = (2i + 4j + 6k) + t(3i + 5j + 7k)$

$r = (2+3t)i + (4+5t)j + (6+7t)k$

so $x = 2+3t \qquad y = 4+5t \qquad z = 6+7t$

are a set of parametric equations for the line L.

The set of parametric equations is not unique.

Another set of parametric equations for the same line is

set $t = 1$

$x = 5+3t, \qquad y = 9+5t \qquad z = 13+7t$

If a vector $v = \langle a,b,c\rangle$ is parallel to a line L, then the numbers a,b,c are called the direction numbers of L.

We can also describe a line by eliminating the parameter t to get

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

These are the symmetric equations of L.

Ex Find parametric and symmetric equations of the line passing through $A(1,0,1)$ and $B(2,4,5)$

Solution: We must have $v = \langle 2-1, 4-0, 5-1\rangle = \langle 1, 4, 4\rangle$

The direction numbers are $a=1, b=4, c=4$

so $x = 2+1\cdot t$, $y = 4+4t$; $z = 5+4t$

The symmetric equations are:

$$\frac{x-2}{1} = \frac{y-4}{4} = \frac{z-5}{4}$$

When will this line intersect the ~~xy-plane~~ yz-plane?

~~so~~ Solution the x-value of the pt of intersection is $x=0$

$$-\frac{2}{1} = \frac{y-4}{4} = \frac{z-5}{4}$$

so $-2 = \frac{y-4}{4}$ or $-8 = y-4$ so $y=-4$

$-2 = \frac{z-5}{4}$ or $-8 = z-5$, so $-3 = z$

so the pt of intersection is $(0,-4,-3)$

The line segment from r_0 to r_1 is given by

$r(t) = (1-t)r_0 + tr_1$ for $0 \le t \le 1$

Def Two lines L_1 and L_2 are skew if

1) L_1 is not parallel to L_2

and

2) L_1 does not intersect L_2

Ex Show that

L_1: $x = 1 + t$ $\quad y = -2 + 3t$ $\quad z = 4 - t$

L_2: $x = 2s$ $\quad y = 3 + s$ $\quad z = -3 + 4s$

are skew

We first show that L_1 and L_2 are not parallel

If $L_1 \parallel L_2$ then the coeff of L_1 would have to be a constant multiple of L_2.

L_1. $(1, 3, -1)$ for the coeff of x to be a multiple of each other

L_2 $(2, 1, 4)$

the multiplier must be 2,

But for the coeff of the y's we need a multiplier of 3

There is no constant c s.t.

$$c\langle 1, 3, -1\rangle = \langle 2, 1, 4\rangle$$

If L_1 and L_2 intersected, then there are values of s and t s.t.

1) $1 + t = 2s$

2) $-2 + 3t = 3 + s$

~~3) $4 - t = -3 + 4s$~~

3) $4 - t = -3 + 4s$

Solve 1) and 2) for s and t

$1 + t = 2s \Rightarrow t = 2s - 1$

so $-2 + 3(2s - 1) = 3 + s$

$-2 + 6s - 3 = 3 + s$

$5s = 8$, $s = \frac{8}{5}$

so $t = 2\left(\frac{8}{5}\right) - 1 = \frac{11}{5}$

Substitute into (3). $4 - \frac{11}{5} \stackrel{?}{=} -3 + 4\left(\frac{8}{5}\right)$, no

so no point of intersection

So L_1 and L_2 are skew.

<u>Planes</u> A plane in space is determined by

a) A pt in the plane, call it P_0

b) A vector n, called a normal vector that is orthogonal to the plane.

i.e. n is orthogonal to every vector in the plane.

Let $P(x, y, z)$ be a point in the plane.

Let r_0 and r be the position vectors of P_0 and P

We have that the vector $r - r_0$ is represented by $\overrightarrow{P_0P}$

<u>Note that</u> $n \perp (r - r_0)$ so

$$\boxed{\begin{array}{c} n \cdot (r - r_0) = 0 \\ \text{i.e.} \quad n \cdot r = n \cdot r_0 \end{array}}$$

Either of these two equation is a vector equation of the plane.

We now seek a scalar equation for the plane.

Writing $n = \langle a, b, c \rangle$, $r = \langle x, y, z \rangle$, $r_0 = \langle x_0, y_0, z_0 \rangle$

$n \cdot (r - r_0) = 0$ becomes

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

or

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0} \quad (2)$$

This is a scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $n = \langle a, b, c \rangle$

<u>Ex</u> Find an equation of the plane through the pt $(2, 4, -1)$ with normal $n = \langle 2, 3, 4 \rangle$

$$2(x-2) + 3(y-4) + 4(z-(-1)) = 0$$

$$2x + 3y + 4z = 12$$

Any non-zero multiple of this equation will be an equation of the plane.