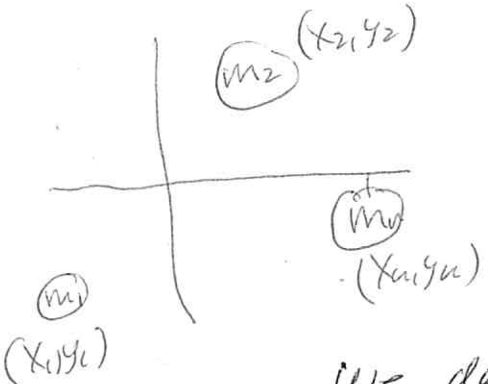
Center of Mass in a 2-dimen System



We extend the idea of a moment to two dimensions by looking at a system of masses in the xy plane at (x_i, y_i)

We define two moments, one wrt x-axis
" " y-axis

Def Let point masses $m_1, m_2, \dots, m_n$ be located at $(x_1, y_1), (x_2, y_2), \dots (x_n, y_n)$

i) The moment about the ~~y~~-axis is
$$M_y = \Sigma m_i x_i$$

2) The moment ~~wrt~~ about the x-axis is
$$M_x = \Sigma m_i y_i$$

Let $m = \Sigma m_i$ be the total mass of the system

The center of mass $(\bar{x}, \bar{y})$ (a center of gravity) is
$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}$$

The moment of a system of masses in the plane can be taken about any horizontal or vertical line by taking the sum of the product of the masses and the directed distances from the points to the line.

$$\text{Moment} = m_1(y_1 - b) + m_2(y_2 - b) + \dots + m_n(y_n - b)$$
~~hori~~ horizontal line $y = b$

$$\text{Moment} = m_1(x_1 - a) + m_2(x_2 - a) + \dots + m_n(x_n - a)$$
vertical line $x = a$

Ex Find the center of mass when

$$m_1 = 6 \quad \text{at } (3,-2)$$
$$m_2 = 3 \quad \text{at } (0,0)$$
$$m_3 = 2 \quad \text{at } (-5,3)$$
$$\cancel{m = 6 + 3 m_4 + 9 = \text{at } (4,2)}$$
$$m_4 = 9 \quad \text{at } (4,2)$$

$m = 6 + 3 + 2 + 9 = 20$ mass s

$M_y = 6(3) + 3(0) + 2(-5) + 9(4) = 44$ moment about y-axis

$M_x = 6(-2) + 3(0) + 2(3) + 9(2) = 12$ " " x-axis

so $\bar{x} = \dfrac{M_y}{m} = \dfrac{44}{20} = \dfrac{11}{5}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{12}{20} = \dfrac{3}{5}$

center of mass is $\left(\dfrac{11}{5}, \dfrac{3}{5}\right)$

Recall that the center of mass is $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}$$

We now consider a thin ~~sheet~~ flat plate called a lamina of uniform density.

Recall that density is a measure of mass per unit volume

Ex $\frac{gr}{cm^3}$, $\frac{lbs}{(ft)^3}$

Density is often denoted by the greek letter ρ

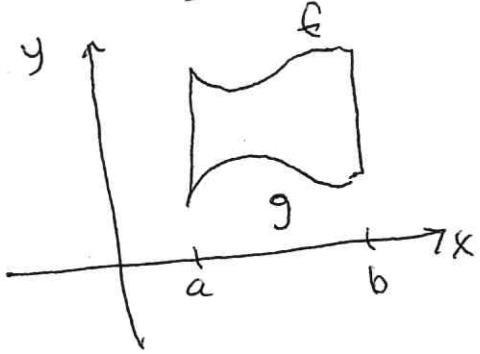
Note mass = density · volume.

Suppose we have a rectangle of uniform density then the center of mass, is located at the center of the rectangle

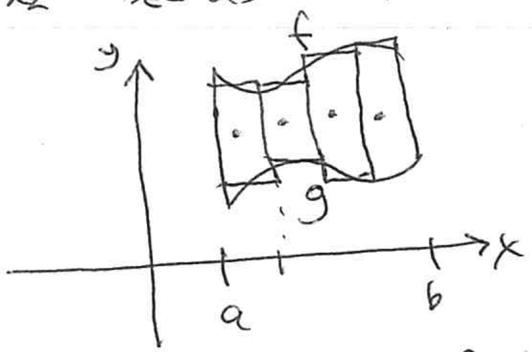
Consider an irregularly shaped planar region of uniform constant density ρ, bounded by the graphs

$y = f(x)$ $a \le x \le b$, Let $g(x) < f(x)$

$y = g(x)$



We seek the center of mass of the lamina.



We approximate the region with rectangles as indicated
For each rectangle, the center of mass is the center of that rectangle

So, we have a finite number of point masses,
So, we are back to our previous examples.
We then let the number of rectangles become infinite.

Let f and g be continuous functions s.t.

$$f(x) \geq g(x) \quad \text{on } [a,b]$$

Consider the planar lamina of uniform density ρ bounded by the graphs of $y = f(x)$, $y = g(x)$ for $a \leq x \leq b$

① The moment about the x-axis is,

$$M_x = \rho \int_a^b \left[\frac{f(x)+g(x)}{2}\right]\left[f(x)-g(x)\right] dx$$

② The moment about the y-axis is

$$M_y = \rho \int_a^b x \left[f(x) - g(x)\right] dx$$

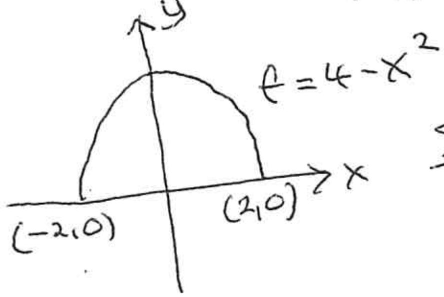
③ The center of mass $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}$$

where $m = \rho \int_a^b \left[f(x) - g(x)\right] dx$
density · area

m = total mass of the lamina

Ex Find the center of mass of the lamina of constant density ρ bounded by the graphs of $f(x) = 4 - x^2$ and the x-axis, so $g(x) = 0$



Solution First, find the mass

$$m = \rho \int_{-2}^{2} (4 - x^2)\,dx = \rho \left[4x - \frac{x^3}{3} \right]_{-2}^{2} = \frac{32\rho}{3}$$

Also,

$$M_x = \rho \int_{-2}^{2} \left(\frac{4 - x^2}{2}\right)(4 - x^2)\,dx = \rho \int_{-2}^{2} (16 - 8x^2 + x^4)\,dx$$

$$= \frac{\rho}{2} \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_{-2}^{2}$$

$$M_x = \frac{256\rho}{15}$$

so $$\bar{y} = \frac{M_x}{m} = \frac{\frac{256\rho}{15}}{\frac{32\rho}{3}} = \frac{8}{5}$$

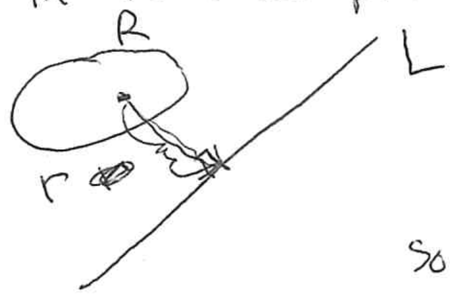
We can find $\bar{x}$ using our formula, or just note that our region is symmetric wrt the y-axis, and the density is constant so $\bar{x}$ must be on the y-axis, so $\bar{x} = 0$

Center of mass is $\left(0, \frac{8}{5}\right)$

Note The center of mass of the region in a plane is called the centroid of the region

The Thm of Pappus

Let R be a region in a plane. Let L be a line in the same plane that does not intersect the interior of R



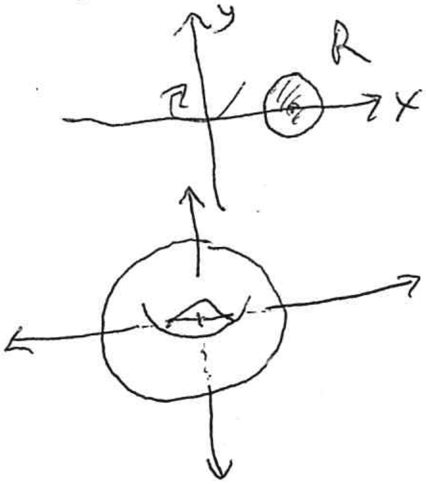
The volume of the solid of revolution formed by revolving R about L

so $V = 2\pi r A$

where r is the distance from the centroid of R to L and A is the area of R

so $V = (\text{distance traveled by the centroid})(\text{area of } R)$

<u>Ex</u> Find the volume of the torus formed by revolving the disk inside $\underbrace{(x-2)^2+y^2=1}_{\text{circle radius 1, center at } (2,0)}$ about the y-axis



The centroid of R is $(2,0)$

The area of the region is

$A = \pi r^2 = \pi(1^2) = \pi$

The volume of the torus is

$V = (2\pi r)A = 2\pi(2)\pi$

$V = 4\pi^2 \approx 39.5$