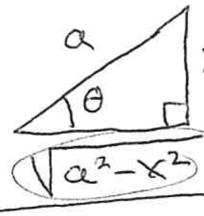
022 Sec 7.3 Friday, Jan 17, 2020 ①

If you email me, please include your first and last name and course number and section

Section 7.3 Trigonometric Substitution.

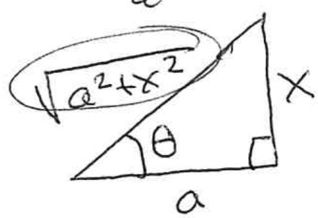
Expression in the integral	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$1-\sin^2\theta = \cos^2\theta$

$\frac{x}{a} = \sin\theta = \frac{opp}{hyp}$



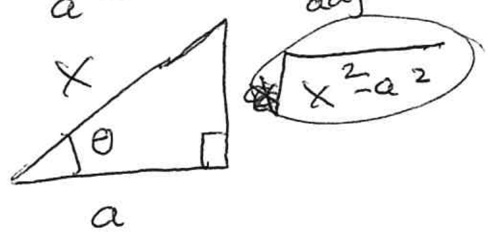
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$1+\tan^2\theta = \sec^2\theta$

$\frac{x}{a} = \tan\theta = \frac{opp}{adj}$



$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

$\frac{x}{a} = \sec\theta = \frac{hyp}{adj}$



Ex $I = \int \frac{dx}{x^2\sqrt{9-x^2}}$ here use $\sqrt{a^2-x^2}$ where $a = \sqrt{9} = 3$

Let $x = 3\sin\theta$

$\frac{x}{3} = \sin\theta$

$dx = 3\cos\theta \, d\theta$

Now, work with $\sqrt{9-x^2}$

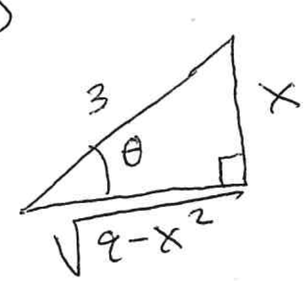
$$= \sqrt{9-(3\sin\theta)^2}$$
$$= \sqrt{9-9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)}$$
$$= 3\sqrt{1-\sin^2\theta} = 3\sqrt{\cos^2\theta}$$
$$= 3\cos\theta$$

Back to I

$$\int \frac{dx}{x^2\sqrt{9-x^2}} = \int \frac{3\cos\theta \, d\theta}{(3\sin\theta)^2 \cancel{3\cos\theta}} = \int \frac{d\theta}{9\sin^2\theta}$$

$$= \frac{1}{9}\int \frac{d\theta}{\sin^2\theta} = \frac{1}{9}\int \csc^2\theta \, d\theta = -\frac{1}{9}\cot\theta + C$$

Now, find θ in terms of x



So $\cot\theta = \frac{adj}{opp} = \frac{\sqrt{9-x^2}}{x}$

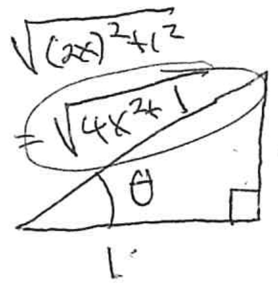
So $\int \frac{dx}{x^2\sqrt{9-x^2}} = -\frac{1}{9}\left(\frac{\sqrt{9-x^2}}{x}\right) + C$

$$= \frac{-\sqrt{9-x^2}}{9x} + C$$

Ex $I = \int \frac{dx}{\sqrt{4x^2+1}}$

solution $I = \int \frac{dx}{\sqrt{(2x)^2+1^2}}$

Let $u = 2x$, $a = 1$



So, $\frac{2x}{1} = 2x = \frac{opp}{adj} = \tan\theta$

$2x = \tan\theta$

$2dx = \sec^2\theta\, d\theta$

$dx = \frac{1}{2}\sec^2\theta\, d\theta$

So $\int \frac{1}{\sqrt{4x^2+1}}\, dx = \cancel{\frac{1}{2}\int}$?

To find ? work with $\sqrt{4x^2+1}$

$\sqrt{4x^2+1} = \sqrt{4\left(\frac{\tan\theta}{2}\right)^2+1} = \sqrt{\tan^2\theta+1} = \sec\theta$

So $\int \frac{1}{\sqrt{4x^2+1}}\, dx = \frac{1}{2}\int \frac{\sec^2\theta\, d\theta}{\sec\theta} = \frac{1}{2}\int \sec\theta\, d\theta$

$= \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$

Now find θ in terms of x

$\cancel{\sec\theta = \frac{hyp}{adj} = \frac{2x}{1} = 2x \text{ and } \tan\theta = \frac{opp}{adj} =}$

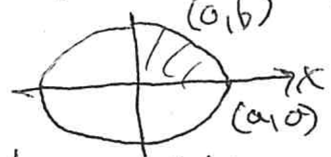
$\sec\theta = \frac{hyp}{adj} = \frac{\sqrt{4x^2+1}}{1} = \sqrt{4x^2+1}$

$\tan\theta = \frac{opp}{adj} = \frac{2x}{1} = 2x$

So $I = \frac{1}{2}\ln\left|\sqrt{4x^2+1} + 2x\right| + C$

<u>Application</u> Find the area enclosed by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



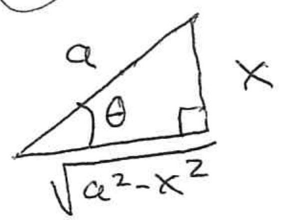
First find y in terms of x

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2}, \text{ or } y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

By symmetry we only need to find the area enclosed in the first quadrant, and then multiply by 4,

<u>Seek</u> $\int_0^a f(x)dx=\int_{x=0}^{x=a}\frac{b}{a}\sqrt{a^2-x^2}\,dx=\frac{b}{a}\int_0^a\sqrt{a^2-x^2}\,dx$

Let $x=a\sin\theta$



$\frac{x}{a}=\sin\theta=\frac{\text{opp}}{\text{hyp}}$

$dx=a\cos\theta\,d\theta$

So $I=\int\sqrt{a^2-x^2}\,dx=\int\sqrt{a^2-a^2\sin^2\theta}\;a\cos\theta\,d\theta$

Note $\sqrt{a^2-a^2\sin^2\theta}=\sqrt{a^2(1-\sin^2\theta)}=a\sqrt{\cos^2\theta}$
$=a\cos\theta$

$$I=\int a\cos\theta\,a\cos\theta\,d\theta=a^2\int\cos^2\theta\,d\theta$$

For our bounds

When $x=0$, $a\sin\theta=0$, so, $\sin\theta=0$, so $\theta=0$

" $x=a$, $a\sin\theta=a$, $\sin\theta=1$, so $\theta=\frac{\pi}{2}$

So, we want $\frac{b}{\cancel{a}}a^{\cancel{2}}\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$

$$= \cancel{ab\int_0^{\frac{\pi}{2}}}$$

$$= ab\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

$$= \frac{ab}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\, d\theta$$

$$= \frac{ab}{2}\left[\theta \Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos 2\theta \right] d\theta$$

$u = 2\theta$

$du = 2d\theta$

$$= \frac{\pi ab}{4}$$

So the total area is $4\left(\frac{\pi ab}{4}\right) = \pi ab$

In the special case of a circle : radius $r = a = b$

So the area enclosed by a circle is

$$A = \pi r^2$$