A scaler equation of the plane through the point $P_0(x_0, y_0, z_0)$ with normal vector $n = \langle a, b, c\rangle$ is

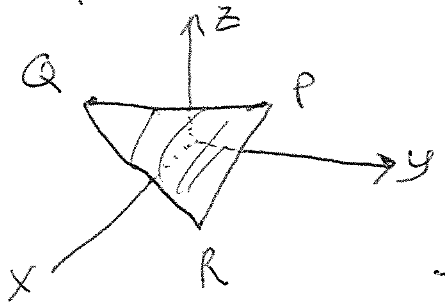
$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

We can rewrite

$$ax+by+cz+d=0$$

where $d=-(ax_0+by_0+cz_0)$

Ex Find an equation of the plane that passes through the pts $P(1,3,2)$, $Q=(3,-1,6)$ and $R(5,2,0)$



Let the vector a be the vector $\overrightarrow{PQ}$
" " " b " " " $\overrightarrow{PR}$

$$a=\langle 2,-4,4\rangle,\quad b=\langle 4,-1,-2\rangle$$

These both lie in the plane.

For a vector n, normal to the plane

$$n = a\times b = \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12i+20j+14k$$

For our pt, let $P = P(1,3,2)$, so an equation of the plane is

$$12(x-1)+20(y-3)+14(z-2)=0$$

$$6x+10y+7z=50$$

Ex Find the pt where the parametric equations

$x = 2+3t \quad y = -4t, \quad z = 5+t$

intersects the plane $4x+5y-2z = 18$

so substituting in

$4(2+3t)+5(-4t) - 2(5+t) = 18$

so $-10t = 20$, so $t = -2$

So, the point of intersection is

$$\left.\begin{aligned} x &= 2+3(-2) = -4 \\ y &= -4(-2) = 8 \\ z &= 5-2 = 3 \end{aligned}\right\} \text{pt of intersection is } (-4, 8, 3)$$

Two planes are parallel if their normal vectors are parallel.

Ex $\left.\begin{aligned} P_1&: x+2y-3z = 4 \\ P_2&: 2x+4y-6z = 3 \end{aligned}\right\} \begin{aligned} n_1 &= \langle 1, 2, -3\rangle \\ n_2 &= \langle 2, 4, -6\rangle = 2n_1 \end{aligned}$

so n_1 is parallel to n_2

so P_1 " " " P_2

If two planes are not parallel, they intersect in a line

The angle between two intersecting planes is defined to be the acute angle between their normal vectors.

Ex Find the angle between the planes

$P_1: x+y+z = 1$ and $P_2: x-2y+3z = 1$

The normal vectors are $n_1 = \langle 1,1,1\rangle$, $n_2 = \langle 1,-2,3\rangle$

Let θ be the angle between them.

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{1(1)+(1)(-2)+1(3)}{\sqrt{1+1+1}\,\sqrt{1+4+9}} = \frac{2}{\sqrt{42}}$$

So $\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ$

We now find parametric equations for the line L of intersection.

We do this, by first finding a point on L,

Ex where the line intersects the xy-plane by letting $z=0$ in the equations of both planes

so $\left.\begin{array}{l} x+y=1 \\ x-2y=1 \end{array}\right\} \Rightarrow x=1, y=0$

So, the point $(1,0,0)$ is on L,

Now L is in $P_1 \cap P_2$ so $L \perp n_1$, $L \perp n_2$

so a vector v parallel to L is given by

$$v = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5i - 2j - 3k$$

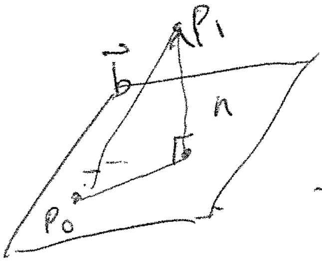
$$= i \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} - j \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + k \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= (3+2)i - (3-1)j + (-2-1)k$$

So, symmetric equations of L are

$$\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}$$

Distances Find a formula from the distance D from a point $P_1\ (x_1, y_1, z_1)$ to the plane $ax+by+cz+d=0$



Let $P_0 = (x_0, y_0, z_0)$ be any point in the given plane and let b be the vector corresponding to $\overrightarrow{P_0P_1}$

Then $b = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

The distance D from P_1 to the plane is equal to

$$D = |\mathrm{comp}_n b| = \frac{|n \cdot b|}{|n|}$$

$$= \frac{a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)}{\sqrt{a^2+b^2+c^2}}$$

$$D = \frac{|(ax_1+by_1+cz_1)-(ax_0+by_0+cz_0)|}{\sqrt{a^2+b^2+c^2}}$$

We can rewrite as follows

$$D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

Ex Find the distance between the parallel planes $P_1: 10x+2y-2z=5$ and $P_2: 5x+y-z=1$

Solution: We choose any pt in one plane and compute its distance to the other plane.

So if we have $y=z=0$ in the first plane, we get $10x=5$, so $(\frac{1}{2}, 0, 0)$ is in P_1

so the distance between $(\frac{1}{2}, 0, 0)$ and the plane $5x+y-z-1=0$ is

$$D = \frac{|5(\frac{1}{2})+1(0)-1(0)-1|}{\sqrt{5^2+1^2+(-1)^2}} = \frac{\frac{3}{2}}{3\sqrt{3}} = \frac{\sqrt{3}}{6}$$

Now two skew lines will lie in parallel planes,
We define the distance between two skew lines to be the distance between those two parallel planes.

Ex We showed that the lines:

L_1: $x = 1+t$ $\quad y = -2+3t \quad z = 4-t$

L_2: $x = 2s \quad y = 3+s \quad z = -3+4s$

are skew, find the distance between them,

A common normal vector to both planes must be orthogonal to both $v_1 = \langle 1, 3, -1\rangle$ the direction of L_1 and to $v_2 = \langle 2, 1, 4\rangle$, the direction of L_2.

so, let $n = v_1 \times v_2 = \begin{vmatrix} i & j & k \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 13i - 6j - 5k$

Letting $s = 0$, we get the pt $(0, 3, -3)$ is on L_2

an equation for P_2 is: $13(x-0) - 6(y-3) - 5(z+3) = 0$

$13x - 6y - 5z + 3 = 0$

Likewise set $t = 0$ in L_1 to get $(1, -2, 4)$ on P_1

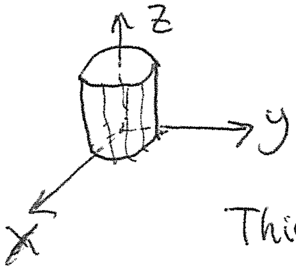
So the distance between L_1 and L_2 equals the distance from $(1, -2, 4)$ to $13x - 6y - 5z + 3 = 0$

By formula 9, we have

$$D = \frac{|13(1) - 6(-2) - 5(4) + 3|}{\sqrt{13^2 + (-6)^2 + (-5)^2}} = \frac{8}{\sqrt{230}} \approx 0.53$$

Section 12.6 Cylinders and Quadric Surfaces

When you see the word cylinder, you probably think of.



This has equation: $x^2+y^2=1$

Note no z, so z can be anything.

Think of this cylinder as the set of lines which are parallel to the z-axis and are perpendicular to the circle $x^2+y^2=1$

In general: a cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line. in our example, the z-axis, and pass through a given plane curve

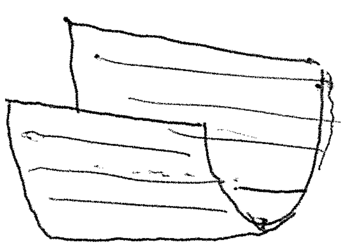
In our example, $x^2+y^2=1$ (a circle)

our example is a circular cylinder

Ex The surface $z=x^2$

The given line is the y-axis

and the plane curve is the parabola $z=x^2$ (in the xz-plane)



This is a parabolic cylinder

Quadratic Surfaces

These are the graphs of the general 2nd degree equation,

$$Ax^2+By^2+Cz^2+Dxy+Eyz+Fxz+Gx+Hy+Iz+J=0$$

With some work, you can put the above into one of the two simpler forms

$$Ax^2+By^2+Cz^2+J=0 \quad \text{or} \quad Ax^2+By^2+Iz=0$$

For lack of a better term, these are the "conic surfaces"

The traces of a surface is the curve of intersection of the surface with a plane that is parallel to one ~~#~~ of the coordinate planes.

<u>Ex</u> Use traces to sketch the quadric surface with equation: $x^2+\frac{y^2}{9}+\frac{z^2}{4}=1$, <u>note</u> the surface is closed and bounded

Letting $z=0$, we have that the trace in the xy-plane is

$$x^2+\frac{y^2}{9}=1, \text{ an ellipse}$$

In general if $z=k$, the trace in the plane $z=k$ is

$$x^2+\frac{y^2}{9}=1-\frac{k^2}{4}, \quad \text{with } z=k,$$

these are ellipses and are defined for $-2<x<2$

<u>Likewise</u> Vertical traces parallel to the yz and xz planes are ~~al~~ also ellipses

for $x=k$, $\frac{y^2}{9}+\frac{z^2}{4}=1-k^2$, $-1\le k\le 1$

for $y=k$, $x^2+\frac{z^2}{4}=1-\frac{k^2}{9}$ $\quad -3<k<3$

We get an ellipsoid

Ex Sketch $z = 4x^2 + y^2$ unbounded, not closed

$x = 0$, we have $z = y^2$. So the intersection in the yz-plane is a parabola

If $\cancel{x} = k$, we have $z = y^2 + 4k^2$

So our vertical slice (trace) is a parabola (k is a constant)

If $z = k$, we have $z = 4x^2 + y^2$: ellipses

We have an elliptic paraboloid.

Ex $z = y^2 - x^2$

In the vertical plane $x = k$, we have paraboles

$z = y^2 - k^2$ open up

The traces in the vertical plane $y = k$ are parabolas

$z = -x^2 + k^2$ which open down.

The horizontal traces. : $y^2 - x^2 = k$ – family of parabolas.

Put them all together to get

a hyperbolic paraboloid