<u>Areas</u> If a curve is traced out by the parametric equations

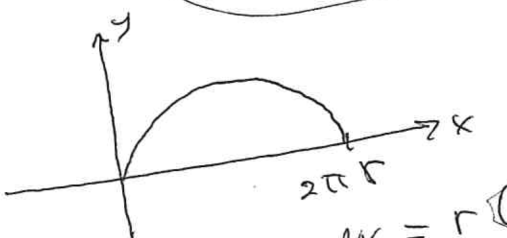
$$x = f(t) \quad \text{for} \quad \alpha \le t \le \beta$$
$$y = g(t)$$

<u>Then</u> the area under the curve is

$$A = \int_\alpha^\beta g(t)\, f'(t)\, dt$$ - this is a consequence of the substitution rule

<u>Ex</u> Find the area under one arch of the cycloid

$$x = r(\theta - \sin\theta)$$
$$y = r(1 - \cos\theta)$$



solution $dx = r(1-\cos\theta)\,d\theta$

$$\text{So} \quad A = \int_0^{2\pi} y\,dx = \int_0^{2\pi} \underbrace{r(1-\cos\theta)}_{y}\, \underbrace{r(1-\cos\theta)\,d\theta}_{dx}$$

$$A = r^2 \int_0^{2\pi} (1-\cos^2\theta)^2\, d\theta = r^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= r^2 \int_0^{2\pi} \left[1 - 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right] d\theta$$

$u = 2\theta$
$du = 2\,d\theta$

$$= r^2 \left[\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= r^2 \left(\tfrac{3}{2}\cdot 2\pi\right) = 3\pi r^2$$

Arclength for curves given parametrically

Recall that the arclength of a curve C given by $y = h(x)$ over the interval $[x_0, x_1]$ is

$$s = \int_{x_0}^{x_1} \sqrt{1 + (h'(x))^2}\, dx$$

If C is represented by the parametric equations

$x = f(t)$; $y = g(t)$ for $a \le t \le b$.

and if $\frac{dx}{dt} = f'(t) > 0$

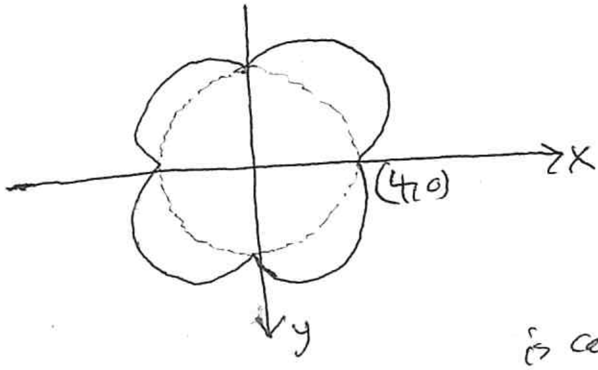
Then
$$s = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, dx$$

$$= \int_a^b \sqrt{\frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}}\, \frac{dx}{dt}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

Ex $\int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$



Large circle of radius 4
Roll a circle of radius 1,
along the circumference of
the larger circle. The curve traced out ~~is called~~ on the edge of the smaller circle
is called an epicycloid

Given that $x = 5\cos t - \cos(5t)$
$y = 5\sin t - \sin(5t)$

Arclength is: $L = 4\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= 4\int_0^{\frac{\pi}{2}} \sqrt{(-5\sin t + 5\sin(5t))^2 + (5\cos t - 5\cos(5t))^2}\, dt$

$= 20\int_0^{\frac{\pi}{2}} \sqrt{2 - 2\sin t \sin 5t - 2\cos t\cos(5t)}\, dt$

trig fn $\cos(\alpha + \beta)$

$= 20\int_0^{\frac{\pi}{2}} \sqrt{2 - 2\cos 4t}\, dt = 20\int_0^{\frac{\pi}{2}} \sqrt{4\sin^2(2t)}\, dt$

$= 40\int_0^{\frac{\pi}{2}} \sin(2t)\, dt = 20\int_0^{\pi} \sin u\, du$

$u = 2t$
$du = 2dt$

$= 20(-\cos u)\Big|_0^{\pi}$

$= -20[\cos\pi - \cos 0] = -20(-1-(-1))$

$= 40$

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Surface area of a solid of revolution

Let a curve C, for (x,y), $y \geq 0$

be given by $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$

be rotated about the x-axis

f' and g' continuous

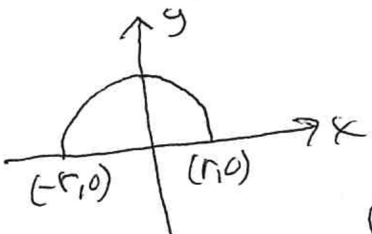
The surface area of the resulting solid is

$$SA = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Note. y acts like a radius, so $2\pi y$ is like a circumference

Ex A sphere of radius r

Let $x = r\cos t$, $y = r\sin t$, $0 \leq t \leq \pi$



so $\frac{dx}{dt} = -r\sin t$, $\frac{dy}{dt} = r\cos t$

$$SA = 2\pi r \int_0^\pi \underbrace{\sin t}_{y} \sqrt{\underbrace{r^2\sin^2(t)}_{\left(\frac{dx}{dt}\right)^2} + \underbrace{r^2\cos^2(t)}_{\left(\frac{dy}{dt}\right)^2}}\, dt$$

$$= 2\pi r \int_0^\pi \sin t \sqrt{r^2}\, dt$$

$$= 2\pi r^2 \int_0^\pi \sin t\, dt = 2\pi r^2 (-\cos t)\Big|_{t=0}^{t=\pi}$$

$$= 2\pi r^2 (1+1)$$

$$= 4\pi r^2$$