We have seen that if $a_n \to 0$, then $\sum a_n$ might converge or $\sum a_n$ might diverge

Divergence Test If $a_n \not\to 0$, as $n \to \infty$ then $\sum_{n=1}^{\infty} a_n$ diverges

Ex $\sum_{n=1}^{\infty} \frac{3n^3}{4n^3+7n^2}$, $\lim_{n\to\infty} \frac{3n^3}{4n^3+7n^2} = \frac{3}{4}$, and $\frac{3}{4} \neq 0$

So $\sum_{n=1}^{\infty} \frac{3n^3}{4n^3+7n^2}$ diverges.

Thm Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both be ~~divergent~~ convergent series.
Let c and d be constants. Then

i) $\sum_{i=1}^{\infty} c a_n = c \sum_{i=1}^{\infty} a_n$

ii) $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

Thm If $\sum_{n=1}^{\infty} a_n$ is a divergent series, then $\sum_{n=1}^{\infty} c a_n$ diverges.

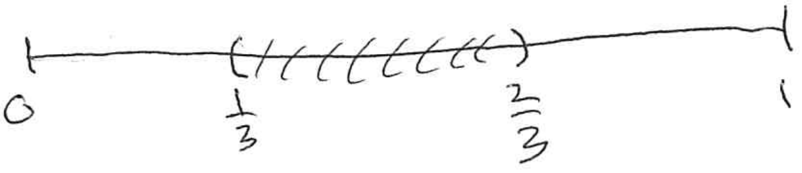
Caution: The sum of two ~~convergent~~ divergent series might be convergent or it might be divergent.

Ex $\sum_{n=1}^{\infty} a_n$, where $a_n = 1$ for all n
$\sum_{n=1}^{\infty} b_n$ where $b_n = 1$ " " "
$\sum_{n=1}^{\infty} c_n$ where $c_n = -1$ for all n
all divergent series

Then $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} (1+1) = \sum_{n=1}^{\infty} 2 = +\infty$ diverges

but $\sum_{n=1}^{\infty} (a_n + c_n) = \sum_{n=1}^{\infty} (1-1) = \sum_{n=1}^{\infty} 0 = 0$, converges

The Cantor Set - Start with the closed interval $[0,1]$



Take away the open middle third

The 2nd iteration; we remove the middle third of the 2 pieces left from above

The first iteration, we remove a length of $\frac{1}{3} = \frac{2^0}{3^1}$

" second " , " " " " $2\left(\frac{1}{3^2}\right) = \frac{2^1}{3^2}$

" third " , " " " " " $4\left(\frac{1}{3^3}\right) = \frac{2^2}{3^3}$

In total we ~~...~~ remove $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n$

This is a geometric series, with $a = \frac{1}{3}$, $r = \frac{2}{3}$

So, the total of the same is $\dfrac{a}{1-r} = \dfrac{\frac{1}{3}}{1-\frac{2}{3}} = \dfrac{\frac{1}{3}}{\frac{1}{3}} = 1$

Section 11.3 The integral test.

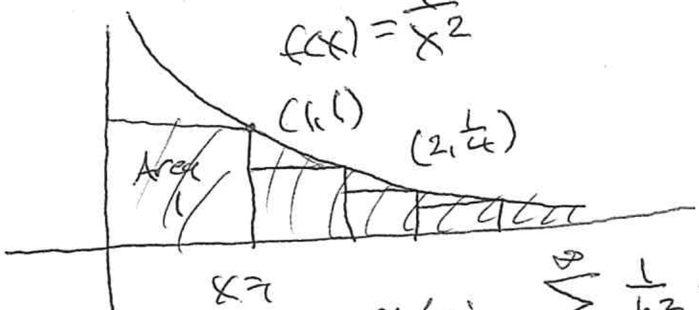
We start by asking, what is the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$

n	S_n
5	1.4636
10	1.5498
100	1.6350

We state, but do not prove

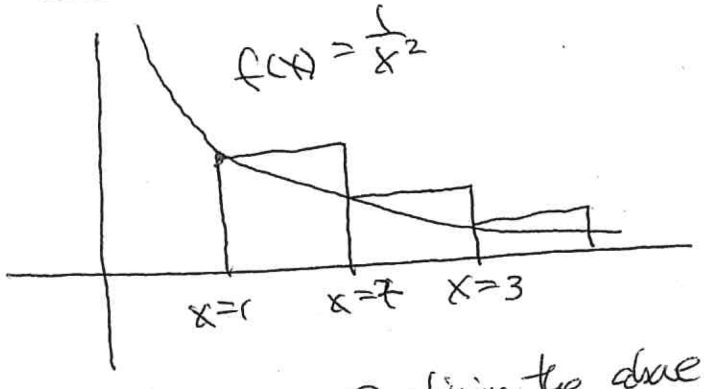
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

We can bound the sum, $\sum \frac{1}{n^2}$ be looking at indefinite integrals.



Note: $\sum_{n=1}^{\infty} \frac{1}{n^2} < 1^2 + \int_1^{\infty} \frac{1}{x^2}\,dx = 1+1$

Likewise



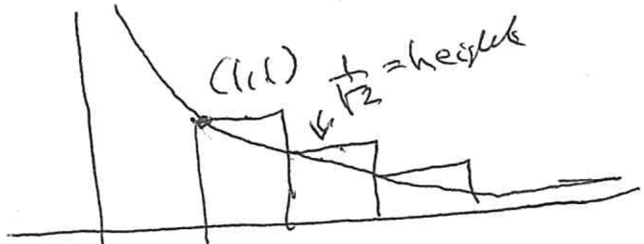
$$\sum_{n=1}^{\infty} \frac{1}{n^2} > \int_1^{\infty} \frac{1}{x^2}\,dx = 1$$

Combining the above: $1 < \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$

Note $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges

Now consider $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Compare with $\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$, which we know diverges to ∞



Note $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is the sum of the areas of the rectangles.

So $0 < \int_1^{\infty} \frac{1}{\sqrt{x}}\,dx \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Since $\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$ diverges to ∞, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ also diverges to ∞

The (improper) integral test.

Let $f(x)$ be positive, continuous, decreasing on $[1,\infty)$

Define $\{a_n\}$ by $a_n = f(n)$

~~Then Σ~~

Then $\sum_{n=1}^{\infty} a_n$ is convergent iff $\int_1^{\infty} f(x)\,dx$ is convergent

__Caution__ We have that even if $\sum_1^{\infty} a_n$ converges,

then $\int_1^{\infty} f(x)\,dx$ converges, but usually

$$\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x)\,dx$$

Ex Determine if $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converges or diverges

solution Set $f(x) = \frac{x}{x^2+1}$

so $f(x) > 0$ on $(1, \infty)$

$f(x)$ is continuous on $(1, \infty)$

now $f'(x) = \frac{-x^2+1}{(x^2+1)^2}$. So $f'(x) < 0$ if $x > 1$

so $f(x)$ is decreasing on $(1, \infty)$

so, the integral test applies.

$I = \int_1^{\infty} \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_1^{\infty} \frac{2x\,dx}{x^2+1}$, use $u = x^2+1$, $du = 2x\,dx$

$I = \frac{1}{2}\lim_{b\to\infty}\left[\ln(x^2+1)\right]_1^b$

$= \frac{1}{2}\lim_{b\to\infty}\left[\ln(b^2+1) - \ln 2\right] = \infty$

Hence $\int_1^{\infty} \frac{x}{x^2+1}\,dx$ diverges

so, $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges