021 Sec 6.4

~~You will need a scientific calculator for the test, Phones are not allowed~~

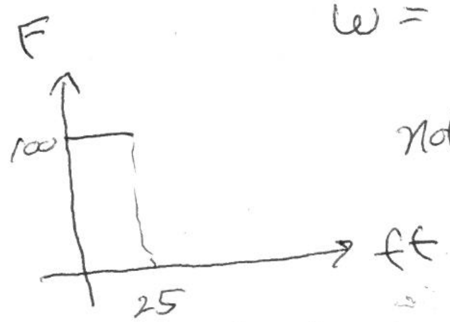
Section 6.4 Work

Def If a constant force of magnitude F is applied in the direction of motion of an object and the object moves a distance d, the work done by the force is $W = F \cdot d$

System	Force × Distance	Work
SI	Newton (N) × meters (M)	Joule J
CGS	dyne × centimeters	erg
BE	pound (lb) × feet	foot · lbs (ft · lbs)

Ex An object moves 25 ft along a line while subject to a constant force of 100 lbs in the direction of motion.

$W = (25\text{ ft})(100\text{ lbs}) = 2500$ ft · lbs

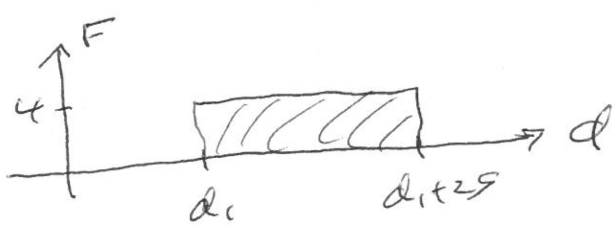


note - work is the area under the force function.

i.e. work is an integral

Ex An object moves 25 m along a straight line while subject to a force of 4 N in the direction of motion.

$W = 4(25) = 100$ N·M $= 100$ J



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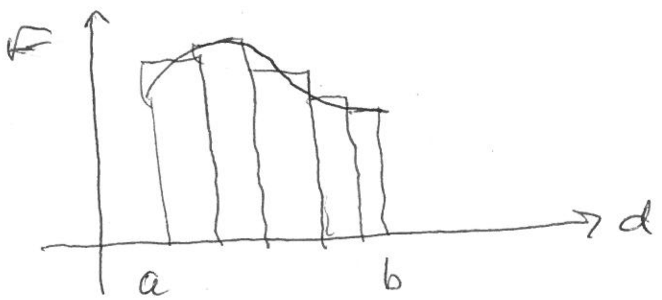
Generalize Suppose an object moves in a positive direction along a coordinate line while subject to a force $F(x)$ that varies in the direction of motion



We partition $[a,b]$ into n subintervals of equal lengths. Let x_i^* be in the i'th subinterval

We assume that in the i'th subinterval, the force has constant value $f(x_i^*)$

The work done on the i'th subinterval is about

$$f(x_i^*)\Delta x_i$$

The total work done is about

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

The exact value of the work done is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \int_a^b f(x)\,dx$$

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Hooke's Law A spring that is stretch x-units beyond its natural length pulls back with a force that is applied in the direction of motion

$F(x) = kx$, where k is a constant, called the spring constant

k has units of force per unit length.

Ex A spring exerts a force of 5N when stretched 1 meter beyond its natural length.

How much work is done when the spring is stretched 1.8 m beyond its natural length?

Solution First find k

We know $F(x) = 5N$ when $x=1$

so $5 = k \cdot 1$, so $k=5$

so $F(x) = 5x$

so $w = \int_0^{1.8} 5x\,dx = \frac{5x^2}{2}\Big|_0^{1.8} = 8.1 J.$

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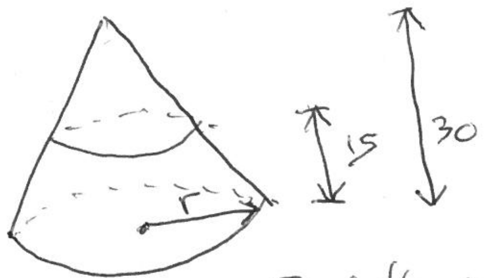
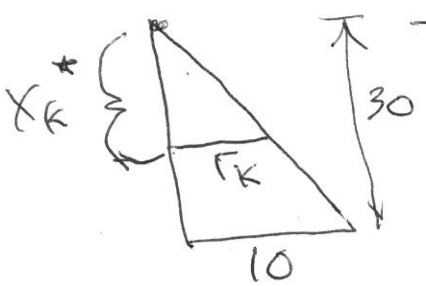


A water tank, in the shape of a cone has radius 10 ft, height 30 ft. It is filled to a depth of 15 ft. Find the work done in pumping all the water out of the tank, through a hole in the top of the tank.

<u>Solution</u> Divide the water into layers (slabs)

Assume that all the water in a layer is located on the axis of the cone.

We have to multiply the weight of a layer (volume · density) by the distance to the top of the cone. We integrate this product



The work required to raise the K'th layer to the top is $W_K \approx F_K \cdot X_K^*$

The force needed to lift the K'th layer is the force needed to overcome gravity

By similar triangles $\dfrac{r_K}{X_K^*} = \dfrac{10}{30}$, so $r_K = \dfrac{X_K^*}{3}$

The volume of the K'th layer

$$V_K \approx \underbrace{\pi (r_K^*)^2}_{\text{area}} \underbrace{\Delta X_K}_{\text{thickness}} = \pi\left(\frac{X_K^*}{3}\right)^2 \Delta X_K = \frac{\pi}{9}(X_K^*)^2 \Delta X_K$$

The density of water is about $62.4\ \text{lbs}/\text{ft}^3$

So the weight, i.e. the Force of the K'th layer is

$$F_K \approx \underset{\text{density}}{(62.4)}\underbrace{\left(\tfrac{\pi}{9}x_K^{*}\right)^2 \Delta x_K}_{\text{volume}}$$

Now $W_K \approx F_K \cdot x_K$

$$W_K = \underbrace{\left[62.4\,\tfrac{\pi}{9}(x_K^{*})^2 \Delta x_K\right]}_{\text{mass = weight = force}} \cdot \underbrace{x_K^{*}}_{\text{distance}}$$

So, the total work in pumping out all the water is

$$\approx \sum_{K=1}^{n} \frac{62.4\pi}{9}(x_K^{*})^3 \Delta x_K$$

$$\text{work} = \int_{15}^{30} \frac{62.4\pi}{9} x^3\,dx$$

$$= 1{,}316{,}250\pi \text{ ft/lbs}$$

$$\approx 4{,}135{,}000 \text{ ft/lbs}$$

<u>Ex</u> A space probe of mass 5.00×10^8 kg travels through space subject only to the force of its engines.

The starting speed is $V = 1.1 \times 10^4$ m/s

The engine fires continuously over a distance of 2.50×10^6 m with a constant force of 4.00×10^5 N in the direction of motion. What is the final speed of the probe

<u>Solution</u> $W = F \times D = (4.00 \times 10^5 N)(2.50 \times 10^6 m)$

$W = 1.00 \times 10^{12} J$

We know $K_f = \frac{1}{2} m V_f^2$ can be expressed in terms of the work W and the initial kinetic energy,

$K_i = \frac{1}{2} m v_i^2, \quad K_f = W + K_i$

So $K_f = (1.00 \times 10^{12} J) + \frac{1}{2} \underbrace{(5.00 \times 10^8 kg)}_{mass} \underbrace{(1.10 \times 10^4 m/s)^2}_{accel}$

$K_f = 4.025 \times 10^{12} J$

Now use $K_f = \frac{1}{2} m V_f^2$

Now solve for $V_f = \sqrt{\frac{2K_f}{m}}$

$$V_f = \sqrt{\frac{2(4.025 \times 10^{12})}{5.00 \times 10^4}}$$

$V_f = 1.27 \times 10^4$ m/s

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Work-Energy relationship

We assume that an object moves in the positive direction on a straight line in the direction of motion.

Let $x = x(t)$ be the position function

so $v = v(t) = x'(t)$ is velocity

and $a = v'(t) = x''(t)$ is acceleration.

By Newton's Second Law

$F(x(t)) = m v'(t)$, m is the mass of the object

Assume that $v(t_0) = v_i$ is the initial velocity, $x(t_0) = a$

and $v(t_1) = v_f$ is the final velocity, $x(t_1) = b$

So
$$W = \int_a^b F(x)\,dx = \int_a^b F(x)\,dx = \int_{x(t_0)}^{x(t_1)} F(x)\,dx$$

$$\underset{=}{\text{substitution}} \int_{t_0}^{t_1} F(x(t))\,x'(t)\,dt$$

$$= \int_{v(t_0)}^{v(t_1)} m v\,dv = \int_{v_i}^{v_f} m v\,dv = \frac{1}{2} m v^2 \Big|_{v=v_i}^{v=v_f}$$

To sum up $W = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2$

<u>Hence</u> work is the change in the kinetic energy of the object.

The units of work are the same units use in the kinetic energy

$$W = \frac{1}{2} m \left(v_f^2 - v_i^2\right)$$

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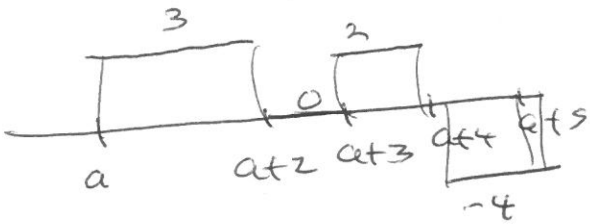
Section 6.5 Average Value

What is the arithmetic mean of $3, 3, 0, 2, -4$

$$\frac{3+3+0+2+(-4)}{5} = \frac{4}{5}$$

graph $f(x) = \begin{cases} 3 & \text{if } a \le x < a+2 \\ 0 & \text{if } a+2 \le x < a+3 \\ 2 & \text{if } a+3 \le x < a+4 \\ -4 & \text{if } a+4 \le x \le a+5 \end{cases}$



What is $\int_a^{a+5} f(x)\,dx$ = area under the curve = 4

Now find $\dfrac{1}{(a+5)-a}(4) = \dfrac{4}{5}$

Def Let $f(x)$ be defined on $[a,b]$
The average value of $f(x)$ on $[a,b]$ is

$$A.V. = \frac{1}{b-a}\int_a^b f(x)\,dx$$

EX Find the average value of $f(x) = x^2$ on $[1,4]$

$$A.V = \frac{1}{4-1}\int_1^4 x^2\,dx = \frac{1}{3}\left[\frac{x^3}{3}\right]_1^4 = \frac{1}{9}\left[4^3 - 1^3\right] = \frac{63}{9} = 7$$

As our motivating example showed, a function need not equal its average value

Mean Value Theorem for Continuous functions

If $f(x)$ is continuous on $[a,b]$, then there is at least one value of c s.t. $a \le c \le b$ and

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

i.e. there is a c in (a,b) s.t. $f(c)$ equals the average value of f on $[a,b]$

Continuing our last example $f(x) = x^2$ on $[1,4]$, av value is 7

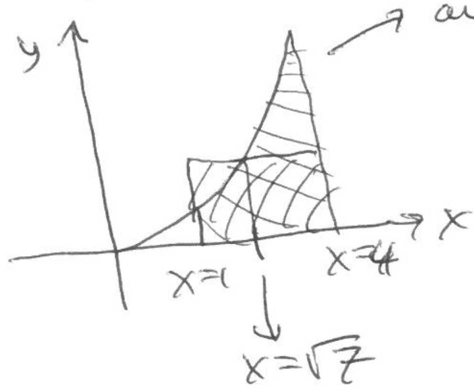
~~set~~ Since $f(x) = x^2$, set $x^2 = 7$, $x = \pm\sqrt{7}$, use $x = +\sqrt{7}$

Note $f(\sqrt{7}) = (\sqrt{7})^2 = 7 =$ average value

and $1 < \sqrt{7} < 4$, $\sqrt{7} \approx 2.65$

Let's look at the above geometrically

area is $\int_1^4 x^2\,dx = \frac{x^3}{3}\Big|_1^4 = \frac{4^3}{3} - \frac{1^3}{3} = \frac{63}{3} = 21$



area of the rectangle is $b \cdot h = (4-1)\, f(\sqrt{7})$

$= 3 \cdot 7 = 21$