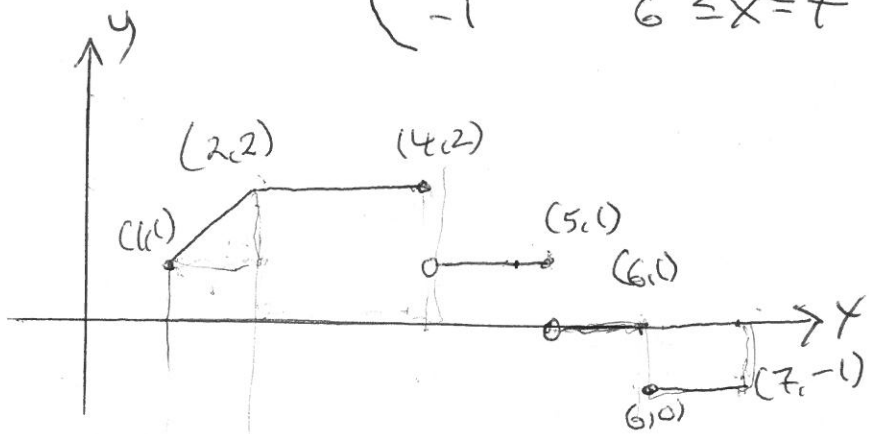
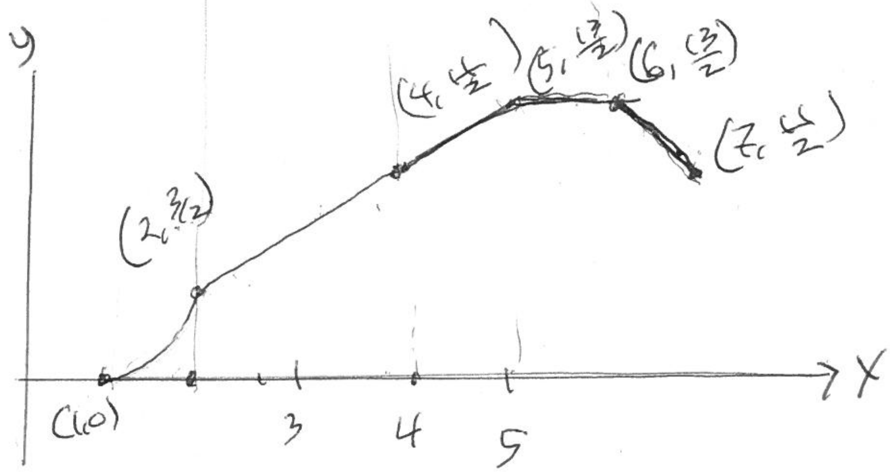
O21 Sec 5.3

The Fundamental Thm of Calculus

Let $y = f(x) = \begin{cases} x & 1 \le x \le 2 \\ 2 & 2 \le x \le 4 \\ 1 & 4 \le x < 5 \\ 0 & 5 \le x < 6 \\ -1 & 6 \le x \le 7 \end{cases}$



f(x)



graph the area under f(x) starting at x=1

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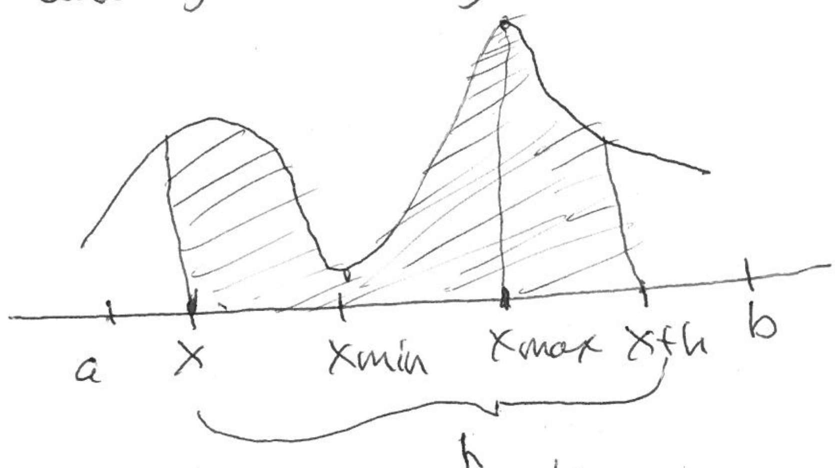
FTC Part I Let f be continuous on $[a,b]$

Define a function g by $g(x) = \int_a^x f(t)\,dt$ for $t \in [a,b]$

Then g is continuous on $[a,b]$, g is diff. on (a,b)

and $g'(x) = f(x)$



By def of the function g, $g(x+h) - g(x)$ is the area under $y = f(x)$ on the interval $[x, x+h]$

By the extreme value thm,

There is a X_{min} s.t. f has a global min for $f(x)$ on $[x, x+h]$ at X_{min}

X_{max} " " " " " max " " " " " at X_{max}

By the comparison thms

$$h f(X_{min}) \leq g(x+h) - g(x) \leq h f(X_{max})$$

Now $h > 0$, so dividing by h does not change the direction of the inequalities

$$f(X_{min}) \leq \frac{g(x+h) - g(x)}{h} \leq f(X_{max})$$

As $h \to 0$, $x+h \to x$; this forces $X_{max} \to x$, so $f(X_{max}) \to f(x)$

Likewise $X_{min} \to x$, and $f(X_{min}) \to f(x)$

So $$\lim_{h\to 0} f(X_{min}) \leq \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \leq \lim_{h\to 0} f(X_{max})$$

$$f(x) \leq g'(x) \leq f(x)$$

So $f(x) = g'(x)$

021 Sec 5.3

Ex $\frac{d}{dx}\int_1^x \sec t\,dt = \cancel{\sec t}\ \sec x$

Ex $\frac{d}{dx}\int_1^{x^4} \sec t\,dt$, use a substitution. Let $u = x^4$

$= \frac{d}{dx}\int_1^u \sec t\,dt$. Now use the chain rule

$= \frac{d}{du}\left[\int_1^u \sec t\,dt\right]\frac{du}{dx}$

$= (\sec u)\frac{du}{dx} = (\sec u)4x^3 = \left[\sec(x^4)\right]4x^3$

$= 4x^3\sec(x^4)$

FTC Part II: If f is continuous on $[a,b]$

Then $\int_a^b f(x) = F(b) - F(a)$, where $F'(x) = f(x)$

"Abbreviated Proof"

Let $g(x) = \int_a^x f(t)dt$. By FTC part I, $g'(x) = f(x)$

So, a corollary of MVT, $F(x) = g(x) + C$ on $[a,b]$

That's basically it. For a full proof we need to consider one sided behavior at the endpts.

Ex $\int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$

Ex $\int_2^3 x^4 dx = \frac{x^5}{5}\Big|_2^3 = \frac{3^5}{5} - \frac{2^5}{5} = \frac{1}{5}\left[3^5 - 2^5\right]$

$= \frac{1}{5}\left[243 - 32\right] = \frac{211}{5}$

021 Sec 5.3

Comments

i) We only care about the values of the antiderivative $F(x)$ at the endpts. i.e. $F(b) - F(a)$

We do not care about the values of $F(x)$ in the interior of the interval.

(2) We are finding the "signed" area of f on the closed interval $[a,b]$ by evaluating an associated ~~function, (the antide~~ object (the antiderivative) on the boundary of the intervals (here the endpts)

Ex Find the area "under" $f(x) = 1 - x$ on $[0,2]$

Solution: $\displaystyle\int_0^2 (1-x)\,dx = \left(x - \frac{x^2}{2}\right)\Big|_0^2 = 0$

$= \left(2 - \frac{2^2}{2}\right) - \left(0 - \frac{0^2}{2}\right)$

Ex Find the area "between" $f(x) = 1 - x$ and the x-axis on $[0,2]$

Solution $\displaystyle\int_0^2 |f(x) - 0|\,dx = \int_0^2 |1-x|\,dx$

We need to find where $f(x) = 0$, $1 - x = 0$, $x = 1$

$$\int_0^1 [(1-x) - 0]\,dx + \int_1^2 [0 - (1-x)]\,dx$$

$$= \left(x - \frac{x^2}{2}\right)\Big|_0^1 + \left(\frac{x^2}{2} - x\right)\Big|_1^2$$

$$= \left[\left(1 - \frac{1}{2}\right) - (0 - 0)\right] + \left[(2-2) - \left(\frac{1}{2} - 1\right)\right]$$

$$= \frac{1}{2} + \left[0 - \left(-\frac{1}{2}\right)\right] = \frac{1}{2} + \frac{1}{2} = 1$$

G21 Sec 5.3/5.4

<u>Caution</u> $\int_{-1}^{1} \frac{1}{x^2}\,dx$ is $\underline{\underline{\text{not}}}$ $\int_{-1}^{1} x^{-2}\,dx = -x^{-1}\Big|_{x=-1}^{x=1} = \frac{-1}{1} + \frac{1}{1} = 0$

$f(x) = \frac{1}{x^2}$ is not continuous on the interval $[-1,1]$. ~~In fact~~ So the FTC does not apply

In fact, $y = \frac{1}{x^2}$ has a vertical asymptote at $x=0$

Keep in mind

1) $\int_a^b f(x)\,dx$ - the definite integral. It is a <u>number</u> that gives the signed (net) area under f on the interval (a,b)

2) $\int f(x)\,dx$ is a function $F(x)$. such that $F'(x) = f(x)$

The two are related by FTC ~~thm~~

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $\int f(x)\,dx = F(x)$

i.e. $F'(x) = f(x)$

021 Sec 5.4

General Principle

Every rule for finding a derivative gives a rule for finding an antiderivative and vice versa

Ex $\frac{d}{dx}x^n = nx^{n-1}$ so $\int nx^{n-1}\,dx = x^n + C$, for $n \neq 0$

This is usually written: $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$

We often have to do some work to put the integrand into usable form.

Ex $\int \frac{\sin\theta}{\cos^2\theta}\,d\theta = \int\left(\frac{1}{\cos\theta}\right)\left(\frac{\sin\theta}{\cos\theta}\right)d\theta = \int \sec\theta\tan\theta\,d\theta = \sec\theta + C$

note $\frac{d}{dx}\sec\theta = \tan\theta\sec\theta$

Ex $\int_1^2 \left[\frac{t^5 - 7t}{t}\right]dt = \int_1^2 \left[t^4 - 7\right]dt = \left(\frac{t^5}{5} - 7t\right)\Big|_{t=1}^{t=2}$

$= \left(\frac{2^5}{5} - 7(2)\right) - \left(\frac{1^5}{5} - 7(1)\right) = -\frac{4}{5}$

Non Ex $\int_{-2}^{2} \frac{t^5 - 7t}{t}\,dt$ is not doable, as of now, since $\frac{t^5 - 7t}{t}$ is undefined at $t = 0$

caution one might write

$\frac{t^5 - 7t}{t} = t^4 - 7$

but $\int_{-2}^{2} \frac{t^5 - 7t}{t}\,dt$ is undoable as of now

but $\int_{-2}^{2} (t^4 - 7)\,dt$ is doable

but $\int_{-2}^{2} \frac{t^5 - 7t}{7}\,dt$ will be solvable using techniques from calc 2.

021 Sec 5.4

<u>Ex</u> $\int_1^2 \left(\frac{t^5-7}{t}\right)dt = \int_1^2 \left(t^4 - \frac{7}{t}\right)dt = \left(\frac{t^5}{5} - 7\ln t\right)\Big|_{t=1}^{t=2}$

$= \left(\frac{2^5}{5} - 7\ln 2\right) - \left(\frac{1^5}{5} - 7\ln 1\right)$

$= \frac{32}{5} - 7\ln 2 - \frac{1}{5} = \frac{31}{5} - 7\ln 2$

<u>The Net Change Thm</u> For motivation, consider the following.

Let $A(t)$ be the amount of money in an account at time t months

You start with \$500 and deposit \$30 per month (no interest)

so $A(t) = \underbrace{A_0}_{\text{initial amount}} + \underbrace{[A'(t)]}_{\text{rate of change per month}} \underbrace{t}_{\text{number of months.}}$

$A(t) = 500 + 30t$

So after 6 months, you have

$A(6) = 500 + 30(6) = \$680$

The net change in the account is: final value − initial value

\$680 − \$500 = \$180

note $\int_0^6 A'(t)\,dt = \int_0^6 30\,dt = 30t\Big]_{t=0}^{t=6} = \$180$

Drawback, flaw in this example is that you do not deposit money in a continuous fashion

<u>Ex</u> A pool has an initial volume of 500L

It is being filled at a constant rate of 30L per minute

$A(t) = A_0 + A'(t)t = 500 + 30t$

Net change thm: The integral of a rate of change is the net change.

net change = final value − initial value

i.e. $\int_a^b F'(x)dx = \underbrace{F(b) - F(a)}_{\text{net change}}$

$F'(x)$ ↓ rate of change

Ex If $P(t)$ is the population at time t, then

$\int_a^b P'(t)dt = P(b) - P(a)$

= Population at time b − Population at time a

= net change in the population.

Ex If m is the mass of a rod, one common way to denote linear density is $\rho(x) = m'(x)$

so $\int_a^b \rho(x)dx = m(b) - m(a)$

which is the mass of the rod between pts a and b.

021 Sec 5.4

The Net Change Thm

A pool has 500 L of water.
It is being filled at a rate of 20 L minute
After 10 minutes, there are

$500 + 20(10) = 700$ L

The net change is $700 - 500 = 200$

Note if we integrate the rate of change

$$\int_0^{10} 20\,dt = 20t\Big|_0^{10} = 200 = \text{net change}$$

The Net Change Thm

The integral of the rate of change is the net change

$$\int_a^b F'(t)\,dt = F(b) - F(a)$$

021 Sec 5.4

<u>EX</u> If an object has position function $s(t)$.

Then $s'(t) = \text{velocity} = v(t)$

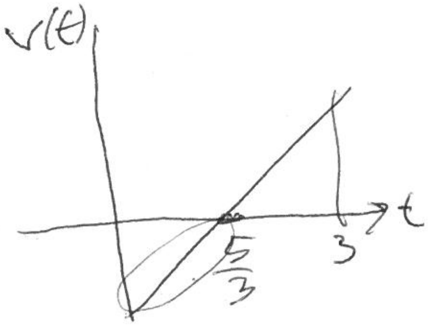
so $\int_a^b s'(t)\,dt = \int_a^b v(t)\,dt = s(b) - s(a)$

This is the net change in the position between $x=a$ and $x=b$

Diffence between displacement and distance traveled.

<u>EX</u> Suppose that the velocity of a particle at time t is:

$v(t) = 3t - 5$



The displacement is the net change

So on $[0,3]$, the displacement is.

$$\int_0^3 (3t-5)\,dt = \left(\frac{3t^2}{2} - 5t\right)\Big|_{t=0}^{t=3} = \left(\frac{27}{2} - 15\right) - 0 = \frac{-3}{2}$$

The Distance traveled is $\int_a^b |v(t)|\,dt$

In our example, the distance traveled is:

$$\int_0^3 |v(t)|\,dt = \int_0^{5/3} (5-3t)\,dt + \int_{5/3}^3 (3t-5)\,dt = \frac{41}{6}$$

In general: $\int_a^b f(x)\,dx \le \left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx$

In our example $-\frac{3}{2} \le +\frac{3}{2} \le \frac{41}{6}$

021 Sec 5.5

Section 5.5 The Substitution Rule

Recall that every rule for finding a derivative gives a rule for finding an antiderivative.

The chain rule is a method for finding derivatives of functions that are compositions of other functions,

The substitution rule is the corresponding rule for finding antiderivatives.

Consider two Integrals

① $\int [x^5-7x^4+3x+1]^{2019} [5x^4-28x^3+3]\,dx$

In theory, we can solve this now, but impractical

The substitution rule will make it easy

② $\int \sqrt{2x-1}\,dx$. As of now, we can't do this.

The substitution rule will make it easy

021 Sec 5.5

The substitution rule

Suppose that we have an integral of the form

$\int f(g(x))g'(x)dx$, If $F' = f$

Then $\int F'(g(x))g'(x)dx = F(g(x)) + C$

Pf By the chain rule: $\frac{d}{dx}\left[F(g(x))+C\right] = F'(g(x))g'(x)$ ✓

Ex $I = \int (x^5-7x^4+3x+1)^{2019}(5x^4-28x^3+3)dx$

In this problem, $F'(x) = x^{2019}$

$g(x) = x^5-7x^4+3x+1$

$g'(x) = 5x^4-28x^3+3$

So, I is of the form $I = \frac{(x^5-7x^4+3x+1)^{2020}}{2020}$

Check $\frac{d}{dx}\left[\frac{(x^5-7x^4+3x+1)^{2020}}{2020}+C\right]$

$= \cancel{2020}\frac{(x^5-7x^4+3x+1)^{2019}}{\cancel{2020}}\frac{d}{dx}(x^5-7x^4+3x+1)$

$= (x^5-7x^4+3x+1)^{2019}(5x^4-28x^3+3)$

021 Sec 5.5

Ex $I = \int \sqrt{2x+1}\, dx$

$= \int (2x+1)^{\frac{1}{2}} dx$

$u = 2x+1$

$du = 2dx$

$I = \frac{1}{2} \int \underbrace{(2x+1)^{\frac{1}{2}}}_{u^{1/2}} \underbrace{2dx}_{du}$

$I = \frac{1}{2} \int u^{1/2} du = \frac{1}{2}\left[\frac{2u^{3/2}}{3}\right] = \frac{1}{3} u^{3/2} + C$

$I = \frac{1}{3}(2x+1)^{3/2} + C$

Check $\frac{d}{dx}\left[\frac{(2x+1)^{3/2}}{3} + C\right]$

$= \frac{3}{2} \frac{(2x+1)^{\frac{1}{2}}}{\cancel{3}} \frac{d}{dx}(2x+1)$

$= \frac{(2x+1)^{1/2} \cancel{2}}{\cancel{2}}$

$= \sqrt{2x+1}$

021 Sec 5.5

Ex $I = \int \tan x \, dx$

$= \int \frac{\sin x}{\cos x} dx$

Let $u = \cos x$

$du = -\sin x \, dx$

$I = -\int \frac{-\sin x}{\cos x} dx$

$= -\int \frac{du}{u} = -\ln u + C$

$I = -\ln|\cos x| + C$

$I = \ln|\sec x| + C$

Recall

$\ln x^a = a \ln x$

so $-\ln|\cos x| + C$

$= (-1)\ln|\cos x| + C$

$= \ln|\cos x|^{-1} + C$

$= \ln\frac{1}{|\cos x|} + C$

$= \ln|\sec x| + C$

021 Sec 5.5

Ex $\int \frac{x+1}{x}\,dx = \int \left(1+\frac{1}{x}\right)dx = x + \ln|x| + C$

Ex
$I = \int \frac{x}{x+1}\,dx = ?$

$u = x+1$

$du = dx$

$x = u-1$

$$I = \int \frac{u-1}{u}\,du = \int\left(1-\frac{1}{u}\right)du$$

$$= u - \ln u + C_0$$

$$= (x+1) - \ln|x+1| + C_0$$

$$I = x - \ln|x+1| + C \quad \text{where} \quad C = C_0 + 1$$

<u>check</u> $\frac{d}{dx}\left[x - \ln|x+1| + C\right]$

$$= 1 - \frac{1}{x+1} = \frac{(x+1)-1}{x+1} = \frac{x}{x+1}$$

021 sec 5.5

Finding Definite Integrals

Ex Find $I_d = \displaystyle\int_{x=1}^{x=5} \frac{x}{\sqrt{2x-1}}\,dx$

Solution First find $I = \displaystyle\int \frac{x}{\sqrt{2x-1}}\,dx$

Let $u = \sqrt{2x-1}$

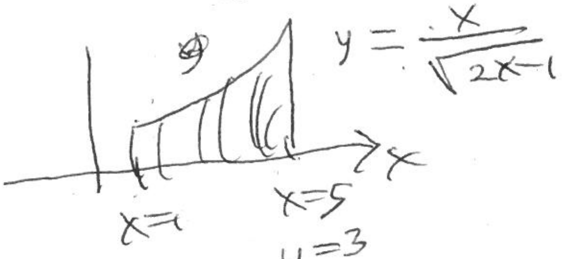
so $u^2 = 2x-1$

$x = \dfrac{u^2+1}{2}$

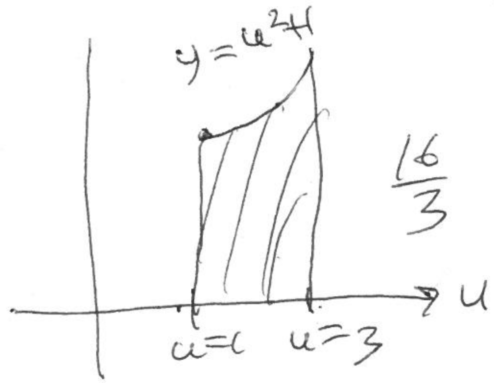
$dx = \dfrac{2u\,du}{2} = u\,du$

So $I = \displaystyle\int \frac{1}{u}\left(\frac{u^2+1}{2}\right) u\,du = \int \frac{u^2+1}{2}\,du = \frac{1}{2}\left(\frac{u^3}{3}+u\right)+C$

So $I_d = \dfrac{1}{2}\left[\dfrac{(2x-1)^{3/2}}{3} + \sqrt{2x-1}\right]_{x=1}^{x=5} = \dfrac{16}{3}$



or $I_d = \displaystyle\int_{u=1}^{u=3} \frac{u^2+1}{2}\,du = \frac{1}{2}\left(\frac{u^3}{3}+u\right)\Big|_{u=1}^{u=3} = \frac{16}{3}$



Ex Find $I_d = \displaystyle\int_{t=0}^{t=\frac{\pi}{4}} \frac{\cos(2t)}{1+\sin(2t)}\,dt$

$u = 1+\sin(2t)$
$du = 2\cos(2t)\,dt$

$I = \frac{1}{2}\displaystyle\int \frac{2\cos(2t)}{1+\sin(2t)}\,dt$

$= \frac{1}{2}\displaystyle\int \frac{du}{u}$

$= \frac{1}{2}\ln|u| + C$

back to x's

$\frac{1}{2}\ln|1+\sin(2t)|\Big|_{t=0}^{t=\frac{\pi}{4}}$

$= \frac{1}{2}\left[\ln\left(1+\sin(2)\frac{\pi}{4}\right) - \ln(1+\sin(2)(0))\right]$

$= \frac{1}{2}\ln\left(1+\sin\left(\frac{\pi}{2}\right)\right) - \ln(1)$

$= \frac{1}{2}\ln 2$

stay in u's

$\frac{1}{2}\ln|u|\Big|_{u=1}^{u=2}$

$= \frac{1}{2}\left[\ln 2 - \ln 1\right]$

$= \frac{1}{2}\ln 2$

Ex $\quad I_d = \int_{x=0}^{x=1} 6x(3x^2+7)^{20}\,dx$

$$\int 6x(3x^2+7)^{20}\,dx$$

$u = 3x^2+7$

$du = 6x\,dx$

$$I = \int u^{20}\,du$$

$$= \frac{u^{21}}{21}$$

back to x's

$$\left.\frac{(3x^2+7)^{21}}{21}\right|_{x=0}^{x=1}$$

$$\frac{(3\cdot 1^2+7)^{21}}{21} - \frac{(3\cdot 0^2+7)^{21}}{21}$$

$$\frac{10^{21}}{21} - \frac{7^{21}}{21}$$

$$= \frac{10^{21}-7^{21}}{21}$$

stay in u's

$$\left.\frac{u^{21}}{21}\right|_{u=7}^{u=10}$$

$$= \frac{10^{21}}{21} - \frac{7^{21}}{21}$$

$$= \frac{10^{21}-7^{21}}{21}$$

021 Sec 5.5

Ex $I = \int x e^{-x^2} dx$

$u = -x^2$

$du = -2x\,dx$

$I = -\frac{1}{2}\int (-2x) e^{-x^2} dx$

$= -\frac{1}{2}\int e^u du = \frac{-1}{2} e^u + c$

$= -\frac{1}{2} e^{-x^2} + c$

or

$u = e^{-x^2}$

$du = -2x e^{-x^2} dx$

$I = -\frac{1}{2}\int (-2x) e^{-x^2} dx$

$= \frac{-1}{2}\int du$

$= -\frac{1}{2}u + c$

$= -\frac{1}{2} e^{-x^2} + c$

Ex $I = 2\int_{x=1}^{x=e} \frac{\ln x}{x} dx$

Let $u = \ln x$

$du = \frac{1}{x} dx$

$I = 2\int_{u=0}^{u=1} u\,du$

$= 2\left[\frac{u^2}{2}\right]_0^1$

$= 2\left[\frac{1^2}{2} - \frac{0^2}{2}\right]$

$= 2\left(\frac{1}{2}\right)$

$= 1$

or $2\left[\frac{(\ln x)^2}{2}\right]_{x=1}^{x=e}$

$= 2\left[\frac{(\ln e)^2}{2} - \frac{(\ln 1)^2}{2}\right]$

$= 2\left[\frac{1^2}{2} - \frac{0^2}{2}\right]$

$= 2\left(\frac{1}{2}\right)$

$= 1$

821 Sec 5.5

Symmetry: Let f be continuous on $[-a, a]$

If

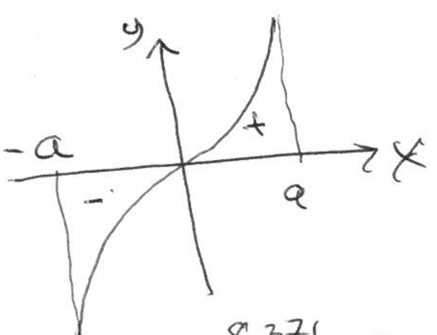
a) f is even i.e. $f(x) = f(-x)$

so the graph of f is symmetric wrt the y-axis

Then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

b) If f is odd, $f(x) = -f(-x)$

The graph of f is symmetric wrt the origin



Then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$

Ex $\displaystyle\int_{-8.371}^{8.371} \left[16.3x^{387} - 6x^{23} - \pi^{e} \cdot x^{11} - 1.83x^{3}\right] dx = 0$