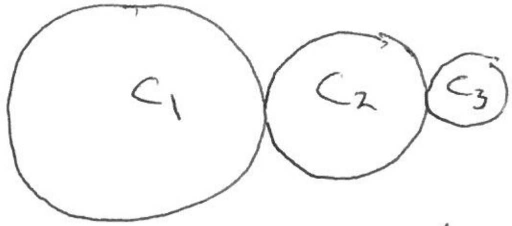
021 Sec 3.4

Section 3.4 The chain rule

Motivation



Let C_1 have 32 teeth
" C_2 " 8 "
" C_3 " 4 teeth

Assume that C_1 rotates at 5 revolutions per minute
so C_2 " " $5\left(\frac{32}{8}\right) = 20$ " " "
and C_3 rotates at $5\left(\frac{32}{8}\right)\left(\frac{8}{4}\right) = 40$ " "

<u>Note</u> $\frac{dC_2}{dC_1} = \frac{20}{5} = 4$, and $\frac{dC_3}{dC_2} = \frac{40}{20} = 2$

and $\frac{dC_3}{dC_1} = 8 = 4 \cdot 2 = \frac{dC_2}{dC_1}\frac{dC_3}{dC_2}$

Consider $\frac{d}{dx}(x^3+2x^2-7x+1)^{2019}$

In theory, we can compute this derivative with what we already know

---

Consider $\frac{d}{dx}\sqrt{3x+1}$

As of now, we can not find this

The chain rule will allow us to find the derivatives of both functions rather easily

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The Chain Rule

Let g be a differentiable function at X

Let f " " " " " $g(X)$

Let $F = f \circ g = f(g(x))$

Then $F'(X) = \left[f'(g(x))\right] g'(X)$

Other notation: If $y = f(u)$ and $u = g(X)$ are differentiable

Then $\frac{dy}{dX} = \frac{dy}{du}\frac{du}{dX}$

Ex $F(X) = (X^3 + 2X^2 - 7X + 1)^{2019}$ find $F'(X)$

Here: $F(X) = (f \circ g)X = f(g(X))$

where $g(X) = X^3 + 2X^2 - 7X + 1$

$f(X) = X^{2019}$

Now $F'(X) = f'(g(X))g'(X)$

$= 2019(g(X))^{2018} g'(X)$

$= 2019(X^3 + 2X^2 - 7X + 1)^{2018}(3X^2 + 4X - 7)$

Ex $G(t) = \sqrt{3t + 1}$

Here $G(t) = (f \circ g)(t) = f(g(t))$ where $g(t) = 3t + 1$

$f(t) = t^{\frac{1}{2}} = \sqrt{t}$

Now, $G'(t) = f'(g(t))g'(t)$

$= \frac{1}{2}(3t+1)^{-\frac{1}{2}} \frac{d}{dt}(3t+1)$

$= \frac{1}{2}(3t+1)^{-\frac{1}{2}}(3) = \frac{3}{2\sqrt{3t+1}}$

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Ex $\frac{d}{dx}(x^3+7x^5+\sin x)^{50}$

$= 50(x^3+7x^5+\sin x)^{49}\frac{d}{dx}(x^3+7x^5+\sin x)$

$= 50(x^3+7x^5+\sin x)^{49}(3x^2+35x^4+\cos x)$

Ex $\frac{d}{dt}\sqrt[3]{(4t^2+1)^7} = \frac{d}{dt}(4t^2+1)^{\frac{7}{3}}$

$= \frac{7}{3}(4t^2+1)^{\frac{4}{3}}\frac{d}{dt}(4t^2-1)$

$= \frac{7}{3}(4t^2+1)^{4/3}8t = \frac{56t}{3}(4t^2+1)^{\frac{4}{3}}$

Note The power rule is a special case of the chain rule

$\frac{d}{dx}x^n = nx^{n-1}$. Let $F(x)=(x)^n$

so, $F(x)=(f\circ g)x=f(g(x))$

with $g(x)=x$

$f(x)=x^n$

So $\frac{d}{dx}(x)^n = n(x)^{n-1}\frac{d}{dx}(x) = n(x)^{n-1}(1)$

$\frac{d}{dx}(x)^n = nx^{n-1}$

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$\frac{d}{dt} \sin(t^4)$

Here $F = f(g(t))$

where $f(t) = \sin t$

$g(t) = t^4$

So $\frac{d}{dt} \sin(t^4)$

$= \cos(t^4) \frac{d}{dt}(t^4)$

$= \cos(t^4) 4t^3$

$= 4t^3 \cos(t^4)$

$\frac{d}{dt} \sin^4(t) = \frac{d}{dt}[\sin t]^4$

Here $F = f(g(t))$

with $f(t) = t^4, g(t) = \sin t$

So $f'(g(t))g'(t)$

$= 4(\sin t)^3 g'(t)$

$= 4(\sin t)^3 \cos t$

$= 4 \sin^3 t \cos t$

Ex $\frac{d}{dx} \sin^5(x^8) = \frac{d}{dx}[\sin(x^8)]^5$

$= 5 [\sin(x^8)]^4 \frac{d}{dx} \sin(x^8)$

$= 5 \sin^4(x^8) \cos(x^8) \frac{d}{dx} x^8$

$= 5 \sin^4(x^8) \cos(x^8) 8x^7$

$= 40x^7 \sin^4(x^8) \cos(x^8)$

Ex $\frac{d}{dx} (x^5 + (3x^7 - 4x)^{10})^{12}$

$= 12[x^5 + (3x^7 - 4x)^{10}]^{11} \frac{d}{dx}[x^5 + (3x^7 - 4x)^{10}]$

$= 12[x^5 + (3x^7 - 4x)^{10}]^{11} [5x^4 + 10(3x^7 - 4x)^9 \frac{d}{dx}(3x^7 - 4x)]$

$= 12[x^5 + (3x^7 - 4x)^{10}]^{11} [5x^4 + 10(3x^7 - 4x)^9 (21x^6 - 4)]$

021 Sec 3.4

Recall $\frac{d}{dx} e^x = e^x$

We use the chain rule to find $\frac{d}{dx} e^{h(x)}$

Let $F(x) = f(g(x)) = e^{h(x)}$

where $g(x) = h(x)$

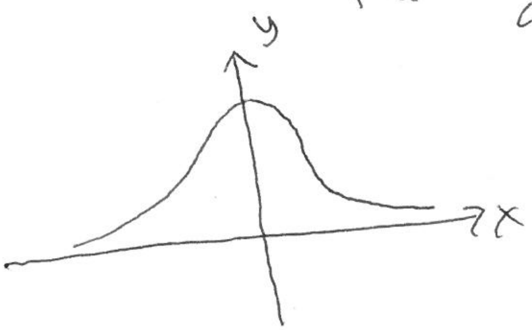
$f(x) = e^x$

so $\frac{d}{dx} e^{h(x)} = \left[e^{h(x)}\right] \frac{d}{dx} h(x)$

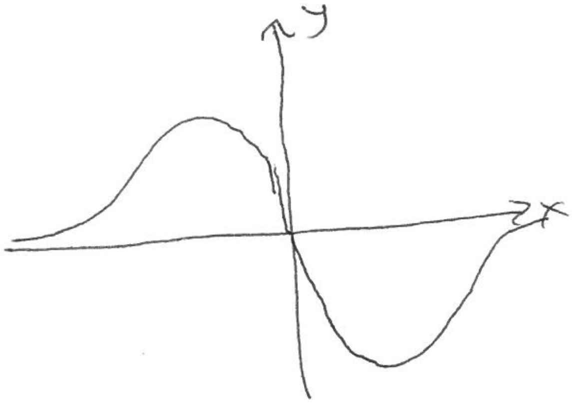
so $\frac{d}{dx} e^{h(x)} = h'(x) e^{h(x)}$ (a little more complicated for bases $\neq e$)

EX Find $f'(x)$ if $f(x) = e^{-x^2}$

$f'(x) = \frac{d}{dx}(-x^2) e^{-x^2} = -2x e^{-x^2}$



$f(x) = e^{-x^2}$



$f'(x) = -2x e^{-x^2}$

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We now use the chain rule to find $\frac{d}{dx} b^x$; $b>0$, $b \neq 1$

start with $b = e^{\ln b}$

$$\frac{d}{dx} b^x = \frac{d}{dx} e^{(\ln b)x} = \frac{d}{dx}[(\ln b)x] e^{(\ln b)x}$$

$$= (\ln b) e^{(\ln b)x} = (\ln b)\left[\left(e^{\ln b}\right)^x\right]$$

$$\frac{d}{dx} b^x = (\ln b)\, b^x$$

EX $\frac{d}{dx} 3^x = (\ln 3) 3^x$

Using the chain rule, we can show that

$$\frac{d}{dx} b^{f(x)} = (\ln b) f'(x) b^{f(x)}$$

EX $\frac{d}{dx} 5^{\tan x} = (\ln 5)(\sec^2 x) 5^{\tan x}$

To sum up:

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} b^x = (\ln b) b^x$$

$$\frac{d}{dx} e^{f(x)} = f'(x) e^{f(x)}$$

$$\frac{d}{dx} b^{f(x)} = (\ln b) f'(x) b^{f(x)}$$

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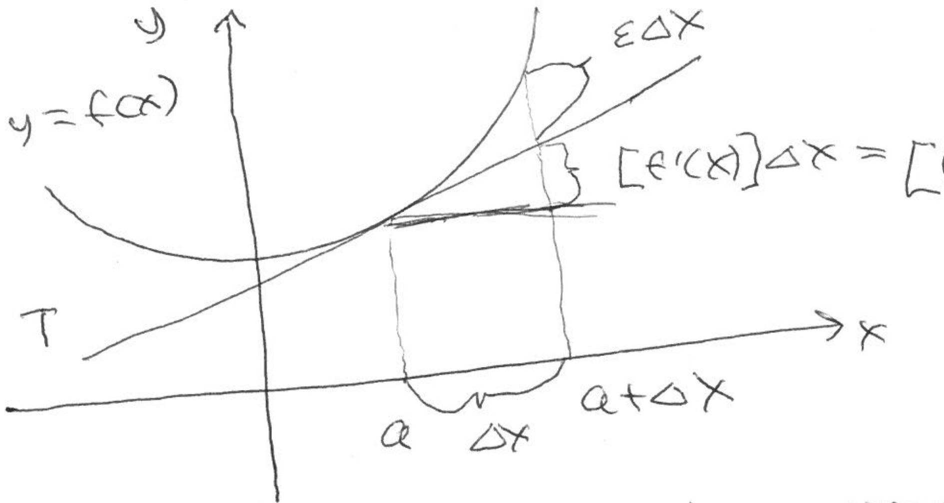
Background to a proof of the chain rule.



$[f'(x)]\Delta x = [f'(a)]\Delta x$

T is the tangent line to $y = f(x)$ at $x = a$

By defining $\varepsilon = 0$, when $\Delta x = 0$, we have that ε is a continuous function of Δx

Hence if f is differentiable at $x = a$,

$$\underbrace{\Delta y}_{\text{exact change in } y} = \underbrace{f'(a)\Delta x}_{\text{approximate change in } y} + \underbrace{\varepsilon \Delta x}_{\text{"error" in the approximation.}}$$

with $\varepsilon \to 0$ as $\Delta x \to 0$

021 Sec 3.4

<u>The Chain Rule</u>: Let g be diff at x, f is diff at $g(x)$

Let $F = f \circ g = f(g(x))$, Then F is diff at X and
$F'(x) = f'[g(x)]g'(x)$

<u>equivalently</u>: If $y = f(u)$ and $u = g(x)$ are differentiable functions then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

<u>Pf</u> Let $u = g(x)$ be diff at a and
$y = f(u)$ " " " $b = g(a)$

Let Δx is the change in x
Δu " " " " u
Δy " " " " y

So, $\Delta u = g'(a)\Delta x + \varepsilon_1 \Delta x$, where $\varepsilon_1 \to 0$ as $\Delta x \to 0$

$\Delta u = [g'(a) + \varepsilon_1]\Delta x$

Likewise. $\Delta y = f'(b)\Delta u + \varepsilon_2 \Delta u$ as $\varepsilon_2 \to 0$ as $\Delta u \to 0$

$\Delta y = [f'(b) + \varepsilon_2]\Delta u$

So, $\frac{\Delta y}{\Delta x} = \frac{[f'(b) + \varepsilon_2]\Delta u}{\Delta x} = \frac{[f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]\cancel{\Delta x}}{\cancel{\Delta x}}$

So $\frac{\Delta y}{\Delta x} = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]$

as $\Delta x \to 0$ and $\Delta u \to 0$ So $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$

So $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} [f'(b) + \varepsilon_1][g'(a) + \varepsilon_1]$

$\frac{dy}{dx} = f'(b)g'(a)$

$\frac{dy}{dx} = f'(g(a))g'(a)$

HW problem by request

$$\frac{d}{dx}\cos\left(\sqrt{\sin(\tan(7x))}\right)$$

$$= -\sin\left[(\sin(\tan(7x))\right]^{\frac{1}{2}} \frac{d}{dx}\left[\sin(\tan(7x))\right]^{\frac{1}{2}}$$

$$= -\sin\left[(\sin(\tan(7x))\right]^{\frac{1}{2}} \frac{1}{2}\left[\sin(\tan(7x))\right]^{-\frac{1}{2}} \frac{d}{dx}\sin(\tan(7x))$$

$$= -\sin\left[(\sin(\tan(7x))\right]^{\frac{1}{2}} \frac{1}{2}\left[\sin(\tan(7x))\right]^{-\frac{1}{2}} \cos(\tan(7x)) \frac{d}{dx}\tan(7x)$$

$$= -\sin\left[\sin(\tan(7x))\right]^{\frac{1}{2}} \frac{1}{2}\left[\sin(\tan(7x))\right]^{-\frac{1}{2}} \cos(\tan(7x)) \sec^2(7x)\, 7$$

$$= \frac{-7\cos(\tan(7x))\sec^2(7x)\sin\left[\sin(\tan(7x))\right]^{\frac{1}{2}}}{2\left[\sin(\tan(7x))\right]^{\frac{1}{2}}}$$

$$= \frac{-7\cos(\tan(7x))\sec^2(7x)\sin\left[\sin(\tan(7x))\right]^{\frac{1}{2}}}{2\sqrt{\sin(\tan(7x))}}$$

021 Section 3.5

Section 3.5 Implicit Differentiation

Explicit Function

where the function can be expressed in the form :

$y = f(x)$ ⟶ no y's, only variable is x

$f(x) = x^2$, $\quad f(x) = e^{\sin(\tan x)}$

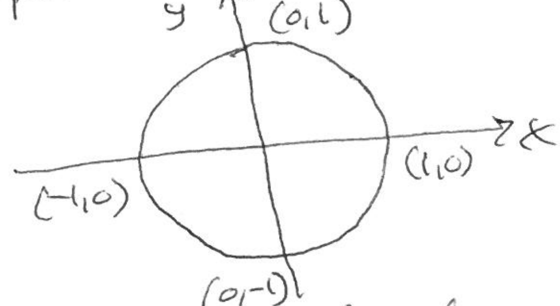
So far, we have only taken derivatives of functions given explicitly.

Implicit Functions The variables are mixed up

$x^2y^3 - \sin(\tan(xy)) = e^{\sin(\cos(x,y))}$

very difficult if not impossible to solve for y in terms of x.

Another Ex $\quad x^2 + y^2 = 1$



This is not a function

But from it, we can generate two functions

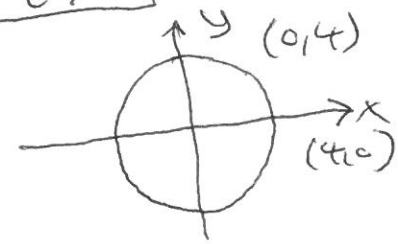
$y = \pm\sqrt{1-x^2}$, $\quad y = +\sqrt{1-x^2}$

$y = -\sqrt{1-x^2}$

021 Section 3.5

Finding Derivatives for functions given implicitly

Need $\boxed{\frac{d}{dx}x = \frac{dx}{dx} = 1}$ $\boxed{\frac{d}{dx}y = \frac{dy}{dx}}$



Ex Find $\frac{dy}{dx}$ if $x^2+y^2=16$

Solution Take $\frac{d}{dx}$ of both sides

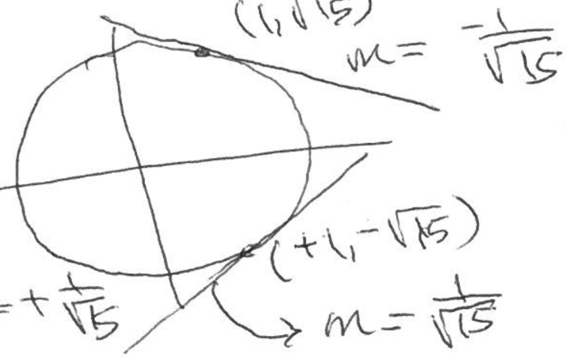
$$\frac{d}{dx}(x)^2 + \frac{d}{dx}(y)^2 = \frac{d}{dx}16$$

$$= 2(x)^1 \frac{d}{dx}x + 2(y)^1 \frac{d}{dx}y = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$x + y\frac{dy}{dx} = 0$, solve for $\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-x}{y}$, note the radius, $r=4$, does not enter into it.



~~Step~~ when $x=1$, $y=\sqrt{15}$, $y=-\sqrt{15}$

So $\frac{dy}{dx} = \frac{-1}{15}$, for $y = +\sqrt{15}$

If $x=1$, $y=-\sqrt{15}$, $\frac{dy}{dx} = \frac{-1}{-\sqrt{15}} = +\frac{1}{\sqrt{15}}$

Find $\frac{dy}{dx}$, for $x=6$, $y=7$

$$\frac{dy}{dx} = -\frac{x}{y} = \frac{-6}{7}$$

For this example, this answer has no meaning, because the point (6,7) is not on the graph.

021 Sec 3.5

<u>Find</u> $\dfrac{d^2y}{dx^2}$ if: $x^2+y^2=16$

given that $\dfrac{dy}{dx} = -\dfrac{x}{y}$

We want $\dfrac{d}{dx}\left(\dfrac{-x}{y}\right)$

$$\frac{d}{dx}\left(\frac{-x}{y}\right) = \frac{y\left(\frac{d}{dx}(-x)\right) - (-(x))\frac{d}{dx}y}{y^2}$$

$$= \frac{y(-1) + x\frac{dy}{dx}}{y^2}$$, now use $\dfrac{dy}{dx} = \dfrac{-x}{y}$

$$= \frac{y(-1)+x\left(\frac{-x}{y}\right)}{y^2} = \frac{-y - \frac{x^2}{y}}{y^2}$$

$$= \frac{-y^2-x^2}{y^3} = \frac{-(x^2+y^2)}{y^3}$$; now use $x^2+y^2=16$

$$\frac{d^2y}{dx^2} = \frac{-16}{y^3}$$

Note this answer is independ on the y-value. Also, the r^2 term $(r^2=16)$, which was not in the first derivative, reappears in the second derivative.

G21 Sec 3.5

Find $\frac{dy}{dx}$ if $x^2y^4 = \sin(xy)$

Take $\frac{d}{dx}$ of both sides.

$$\left(2x\frac{d}{dx}x\right)y^4 + x^2\left(4y^3\frac{d}{dx}y\right) = \left[\cos(xy)\right]\left[\left(\frac{d}{dx}x\right)y + x\frac{d}{dx}y\right]$$

$$2xy^4 + 4x^2y^3\frac{dy}{dx} = \left[\cos(xy)\right]\left[y + x\frac{dy}{dx}\right]$$

Now, isolate $\frac{dy}{dx}$

$$2xy^4 + 4x^2y^3\frac{dy}{dx} = y\cos(xy) + x\cos(xy)\frac{dy}{dx}$$

$$4x^2y^3\frac{dy}{dx} - x\cos(xy)\frac{dy}{dx} = y\cos(xy) - 2xy^4$$

$$\left[\cancel{4x^2y^3\frac{dy}{dx}}\right.$$

$$\left[4x^2y^3 - x\cos(xy)\right]\frac{dy}{dx} = y\cos(xy) - 2xy^4$$

$$\frac{dy}{dx} = \frac{y\cos(xy) - 2xy^4}{4x^2y^3 - x\cos(xy)}$$

021 Sec 3.5

Find $\frac{d}{dx}$ for

$$x^2y^4 + \sin(xy^7) = x + 3y^5$$

Take $\frac{d}{dx}$ of both sides

$$\frac{d}{dx}(x^2y^4) + \frac{d}{dx}\sin(xy^7) = \frac{d}{dx}x + 3\frac{d}{dx}y^5$$

$$2xy^4 + 4x^2y^3\frac{dy}{dx} + \cos(xy^7)\frac{d}{dx}(xy^7) = 1 + 15y^4\frac{dy}{dx}$$

$$2xy^4 + 4x^2y^3\frac{dy}{dx} + [\cos(xy^7)]\left[y^7 + 7xy^6\frac{dy}{dx}\right] = 1 + 15y^4\frac{dy}{dx}$$

$$2xy^4 + 4x^2y^3\frac{dy}{dx} + y^7\cos(xy^7) + 7xy^6\cos(xy^7)\frac{dy}{dx} = 1 + 15y^4\frac{dy}{dx}$$

$$4x^2y^3\frac{dy}{dx} + 7xy^6\cos(xy^7)\frac{dy}{dx} - 15y^4\frac{dy}{dx} = 1 - 2xy^4 - y^7\cos(xy^7)$$

$$\frac{dy}{dx}\left[4x^2y^3 + 7xy^6\cos(xy^7) - 15y^4\right] = 1 - 2xy^4 - y^7\cos(xy^7)$$

$$\frac{dy}{dx} = \frac{1 - 2xy^4 - y^7\cos(xy^7)}{4x^2y^3 + 7xy^6\cos(xy^7) - 15y^4}$$

021 Sec 3.5

Two functions f and f^{-1} are inverses of each other iff $(f \circ f^{-1})x = (f^{-1} \circ f)x = x$ on their common domain

(Note. f has an inverse function iff f is 1 to 1.)

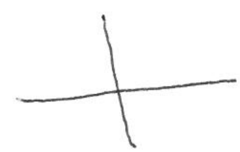
Ex $f(x) = x^3$ $\qquad f^{-1}(x) = x^{1/3}$

$$f(f^{-1}(x)) = f(x^{\frac{1}{3}}) = (x^{\frac{1}{3}})^3 = x$$

and $f^{-1}(f(x)) = f^{-1}(x^3) = (x^3)^{\frac{1}{3}} = x$

This is true for all x on $(-\infty, \infty)$

Ex $f(x) = x^2,$ $\qquad f^{-1}(x) = x^{\frac{1}{2}}$



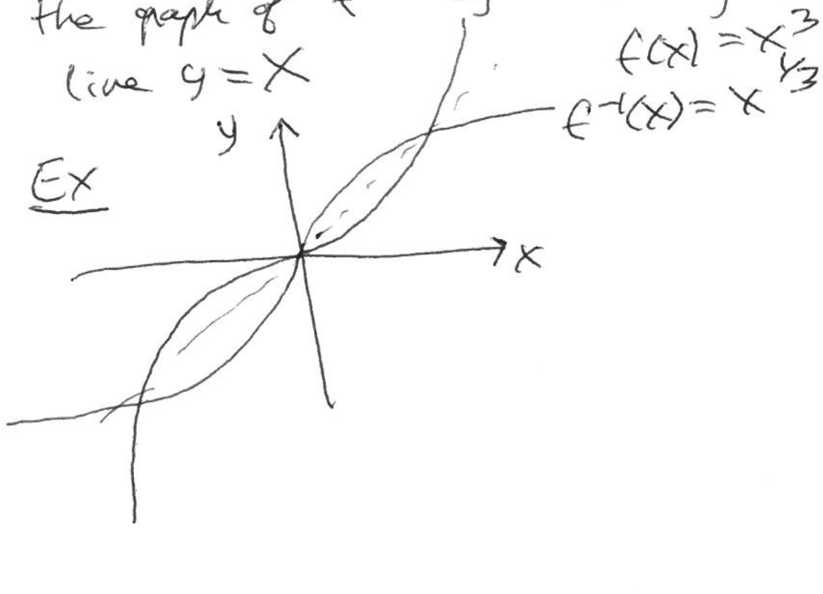
$$f(f^{-1}(x)) = f(x^{\frac{1}{2}}) = (x^{\frac{1}{2}})^2 = x$$

$$f^{-1}(f(x)) = f^{-1}(x^2) = (x^2)^{\frac{1}{2}} = x$$

for all $x \in [0, \infty)$

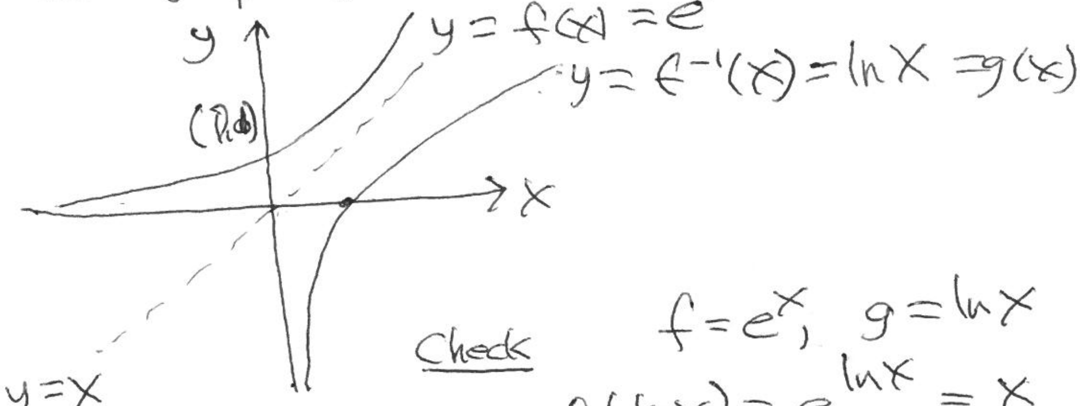
Property Given the graph of a 1 to 1 function f, we can obtain the graph of f^{-1} by reflecting the graph of f about the line $y = x$

Ex



021 Sec 3.5

Ex graph $y = f(x) = e^x$



Check $f = e^x$, $g = \ln x$

$f(g(x)) = f(\ln x) = e^{\ln x} = x$

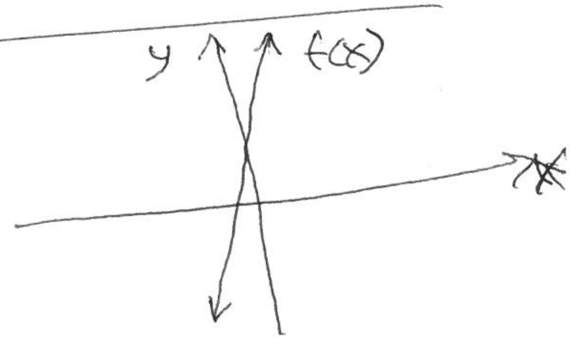
$g(f(x)) = g(e^x) = \ln e^x = x \ln e = x$

Note The roles of domain and range are switched with f and f^{-1}

i.e. The domain of f is the range of f^{-1}

The range of f " " domain of f^{-1}

How do we find the inverse of f



Ex Let $f(x) = 3x + 5$

f is 1 to 1, so f has an inverse function.

$f(x) = 3x + 5$

$y = 3x + 5$

$x = 3y + 5$ swap x and y's

Now solve for y in terms of x

$-3y = -x + 5$

$3y = x - 5$

$y = \frac{x-5}{3} = f^{-1}(x)$

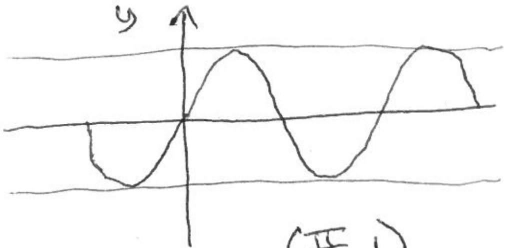
check: $f(f^{-1}(x)) = f\left(\frac{x-5}{3}\right) = \cancel{3}\left(\frac{x-5}{\cancel{3}}\right) + 5 = x - 5 + 5 = x$

$f^{-1}(f(x)) = f^{-1}(3x+5) = \frac{(3x+5)-5}{3} = \frac{3x}{3} = x$

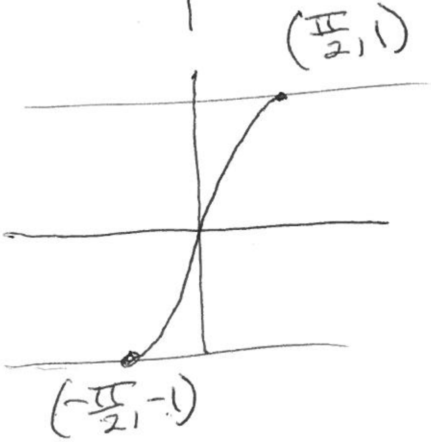
021 Sec 3.5

Inverse Trig Functions

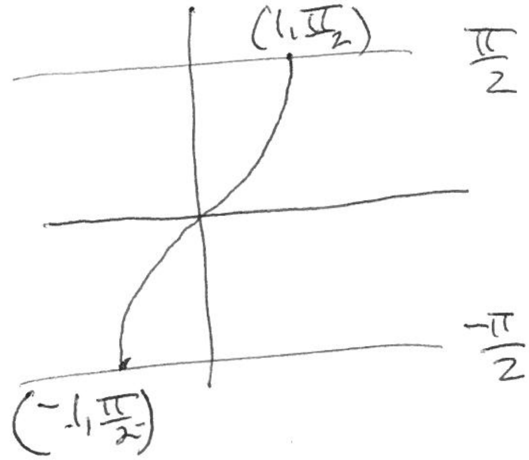
Graph of $y = f(x) = \sin x$



This is not 1 to 1.
In order to form $f^{-1}(x)$
we restrict x to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



now, reflect about the $\sin y = x$



$y = \sin^{-1} x$, $y = \arcsin x$

so $y = \sin^{-1} x$ iff $x = \sin y$

angle $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ | number $-1 \le x \le +1$ | number $-1 \le x \le +1$ | any angle

Ex $\sin\left(\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

We always have $\sin(\sin^{-1} x) = x$

We have to be careful with $\sin^{-1}(\sin x)$

Ex $\sin^{-1}(\sin 0) = \sin^{-1}(0) = 0$

$\sin^{-1}(\sin 4\pi) = \sin^{-1}(0) = 0$, not 4π

Ex $\underset{\text{number}}{\cos\left(\sin^{-1} 0\right)} = \underset{\text{angle}}{\cos(0)} = \underset{\text{number}}{1}$

021 Sec 3.5

We now show that $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

Start set $y=\sin^{-1}x$

~~We~~ have $y=\sin^{-1}x$ means $\sin y = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

We use implicit differ and the chain rule

$\sin y = x$, diff wrt x

$(\cos y)\frac{d}{dx}y = \frac{d}{dx}x$

$\cos y \frac{dy}{dx} = 1$ so $\frac{dy}{dx} = \frac{1}{\cos y}$

We now express $\cos y$ in terms of x

We assumed $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

We know that $\cos^2 y + \sin^2 y = 1$

$\cancel{\cos y = \sqrt{1-\sin^2 y}}$ $\quad \cos^2 y = 1-\sin^2 y$

$\cancel{\cos^2 y = \sqrt{1-x^2}}$

so $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

We now use the chain rule to find that $\frac{d}{dx}\sin^{-1}f(x) = \frac{f'(x)}{\sqrt{1-(f(x))^2}}$

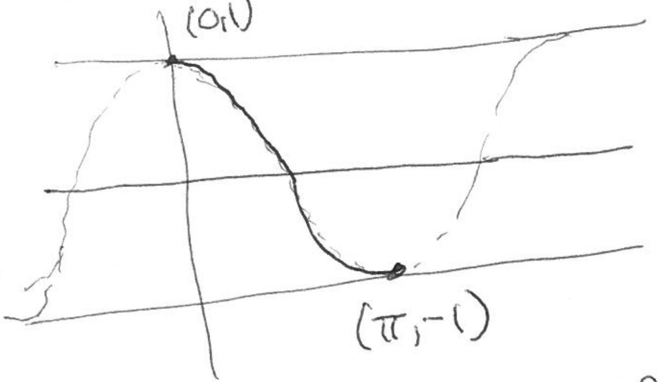
Pf $\frac{d}{dx}\sin^{-1}f(x) = \frac{1}{\sqrt{1-(f(x))^2}}f'(x)$

$= \frac{f'(x)}{\sqrt{1-(f(x))^2}}$

Ex $\frac{d}{dx}\sin^{-1}(x^3) = \frac{1}{\sqrt{1-(x^3)^2}}\frac{d}{dx}x^3 = \frac{3x^2}{\sqrt{1-x^6}}$

021

<u>Inverse Cosine</u> $y = f(x) = \cos x$



To define $y = \cos^{-1} x$
we restrict $y = \cos x$ to
$0 \le x \le \pi$

We define $y = \cos^{-1} x$ iff $\cos y = x$ and $0 \le y \le \pi$

any angle

scalar $-1 \le x \le 1$

We have :

$\cos^{-1}(\cos x) = x$ for $0 \le x \le \pi$

$\cos(\cos^{-1} x) = x$ for $-1 \le x \le +1$

Derivatives

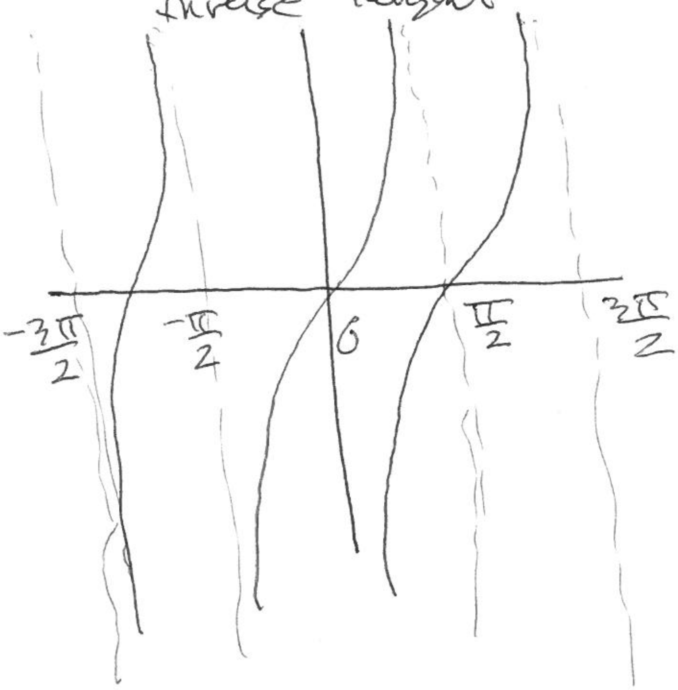
$$\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1 - x^2}}$$

Using the chain rule

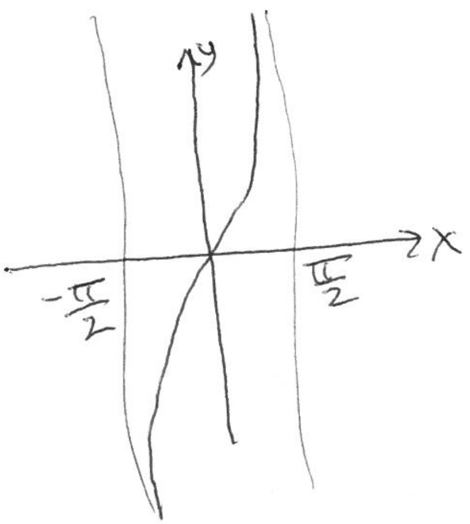
$$\frac{d}{dx} \cos^{-1}(h(x)) = \frac{-h'(x)}{\sqrt{1 - (h'(x))^2}}$$

021 Sec 3.5

Inverse Tangent

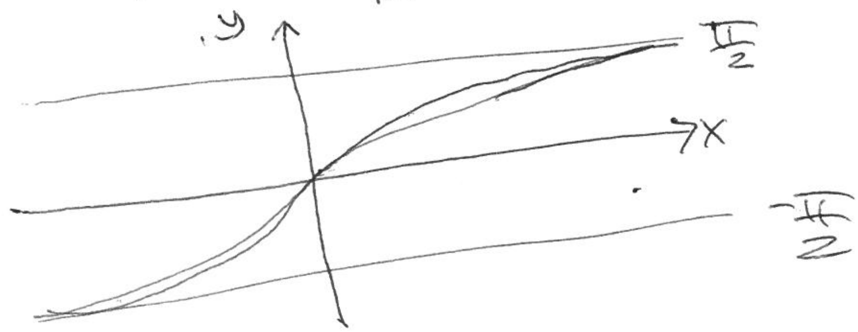


To make a 1 to 1 function, restrict $f(x) = \tan x$ to $(-\frac{\pi}{2}, \frac{\pi}{2})$



Graph $y = \tan^{-1} x$

label asymptotes



$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}$, $\frac{d}{dx} \tan^{-1} f(x) = \frac{f'(x)}{1+(f(x))^2}$

Ex Find $\frac{d}{dx} \tan^{-1}(5x) = \frac{\frac{d}{dx}(5x)}{1+(5x)^2} = \frac{5}{1+25x^2}$

We also have

$$\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\csc^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$$

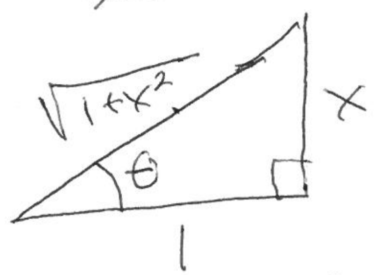
$$\frac{d}{dx}\sec^{-1}x = \frac{+1}{1+x^2}$$

Note, if the function begins with a c, there is a minus sign

Problem Find $\cos(\tan^{-1}x)$

i.e. Find the cosine of the angle θ s.t. $\tan\theta = x$

Solution Set up a right triangle, with angle θ s.t. $\tan\theta = x$



$$\tan\theta = \frac{opp}{adj} = x = \frac{x}{1}$$

We now have $\cos\theta = \frac{adj}{hyp} = \frac{1}{\sqrt{1+x^2}}$

We have $\cos(\tan^{-1}x) = \frac{1}{\sqrt{1+x^2}}$

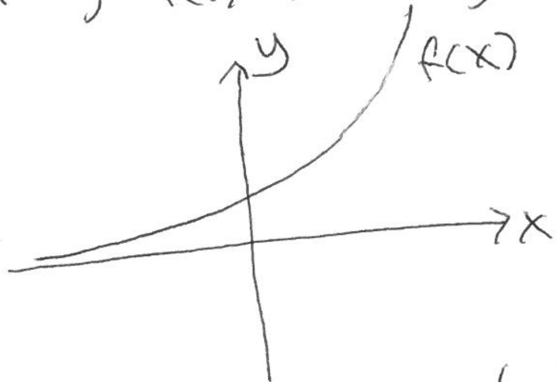
021 Section 3.6

Section 3.6 Derivatives of Log Functions.

A review of Logs

First graph $y=f(x)=2^x$ using a table,

x	2^x
0	1
1	2
2	4
3	8
-1	$\frac{1}{2}$
-2	$\frac{1}{4}$
-3	$\frac{1}{8}$

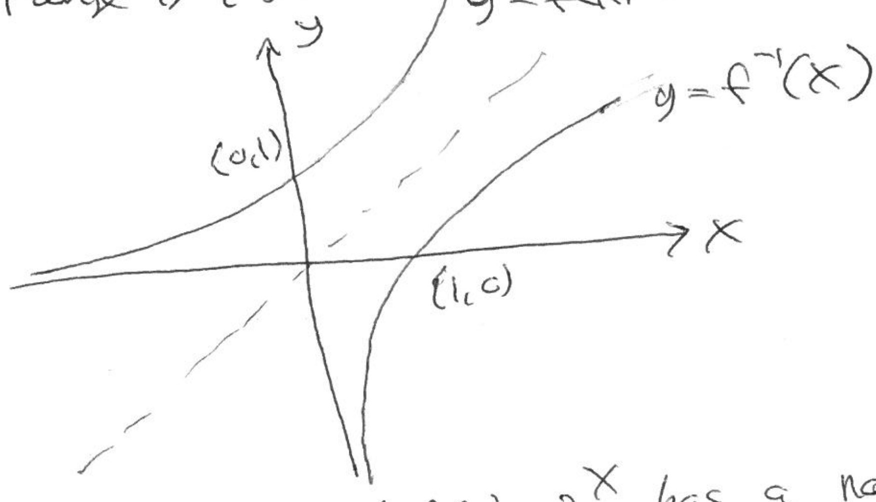


Note f is 1 to 1, so it has an inverse function.

$f\begin{cases} \text{domain is } \{x \mid -\infty < x < +\infty\} \\ \text{range is } \{y \mid 0 < y < \infty\} \end{cases}$

for $f^{-1}\begin{cases} \text{domain is } \{x \mid 0 < x < \infty \\ \text{range is } \{y \mid -\infty < y < +\infty \end{cases}$

x	$f^{-1}(x)$
1	0
2	1
4	2
8	3
$\frac{1}{2}$	-1
$\frac{1}{4}$	-2
$\frac{1}{8}$	-3



The inverse function of $f(x)=2^x$ has a name

$y=f^{-1}(x)=\log_2 x$

so $y=\log_2 x$ iff $2^y = x$

021 Sec 3.6

In general, $a>0$, $a \neq 1$

$y = \log_a x$ and $y = a^x$ are inverse functions

Pf Let $f^{-1}(x) = \log_a x$, $f(x) = a^x$

compute $(f \circ f^{-1})(x) = f(\log_a x) = a^{\log_a x} = x$

$(f^{-1} \circ f)(x) = f^{-1}(a^x) = \log_a a^x = x \log_a a = x \cdot 1 = x$

Ex of use of logs

Solve! $e^{6+5x} = 10$ (note ln means $\log_e$)

Solution Take ln of both sides,

$\ln(e^{6+5x}) = \ln 10$

$(6+5x) \ln e = \ln 10$, note $\ln e = 1$

$(6+5x) = \ln 10$

$x = \frac{(\ln 10) - 6}{5}$

Another solution $e^{6+5x} = 10$

$e^6 e^{5x} = 10$

$e^{5x} = \frac{10}{e^6}$, now take ln of both sides

$\ln(e^{5x}) = \ln\left(\frac{10}{e^6}\right)$

$5x = (\ln 10) - \ln(e^6) = (\ln 10) - 6$

$x = \frac{(\ln 10) - 6}{5}$

021 Sec 3.6

Let $f(x) = a^x$, $a > 0$, $a \neq 1$. Then $f^{-1}(x) = \ln_a x$

The domain of $f(x)$ is $(-\infty, \infty)$

" range of $f(x)$ is $(0, \infty)$

The domain of $f^{-1}(x)$ is $(0, \infty)$

The range of $f^{-1}(x)$ is $(-\infty, \infty)$

Properties of logs: Let $x, y > 0$, $a > 0$, $a \neq 1$

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a x^r = r \log_a x$$

Notation

a) $\ln a$ always means $\log_e a$, the natural logs

b) $\log_a x$, $(a > 0, a \neq 1)$

i) $a = e$ base e

ii) base 10 $\log_{10} x$, useful because we have a base 10 number system.

iii) base 2 $\log_2 x$

In most elementary texts $\log x$ means $\log_{10} x$

" " advanced " $\log x$ " $\ln x$

621 Sec 3.6

Thm $\frac{d}{dx}\log_a x = \frac{1}{x\ln a}$

special case of the thm is $\frac{d}{dx}\log_e x = \frac{1}{x\ln e} = \frac{1}{x\cdot 1} = \frac{1}{x}$

Pf of $\frac{d}{dx}\log_a x = \frac{1}{x\ln a}$

Start with $y = \log_a x$ so $a^y = x$

Now use implicit differentiation wrt x on $a^y = x$

$$a^y = x$$

$$(\ln a)\left(\frac{dy}{dx}\right)a^y = \frac{d}{dx}x = 1$$

So $\frac{dy}{dx} = \frac{1}{a^y \ln a}$, but $a^y = x$

$$\frac{dy}{dx} = \frac{1}{x\ln a}$$

If $a=e$ $\frac{d}{dx}\ln x = \frac{1}{x}$

Using the chain rule

$$\frac{d}{dx}\log_a |f(x)| = \frac{1}{(\ln a)f(x)}\frac{d}{dx}f(x)$$

$$\frac{d}{dx}\log_a |f(x)| = \frac{f'(x)}{(\ln a)f(x)}$$

Special case $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$

021 Sec 3.6

Ex $\frac{d}{dx} \ln(x^2+7x) = \frac{\frac{d}{dx}(x^2+7x)}{x^2+7x} = \frac{2x+7}{x^2+7x}$

Ex $\frac{d}{dx} \ln x = \frac{1}{x}$

$\frac{d}{dx} \ln x^2 = \frac{2x}{x^2} = \frac{2}{x}$

$\frac{d}{dx} \ln x^3 = \frac{3x^2}{x^3} = \frac{3}{x}$

$\vdots$

$\frac{d}{dx} \ln x^n = \frac{nx^{n-1}}{x^n} = \frac{n}{x}$

$\frac{d}{dx} \cancel{\ln} \ln x^2 = \frac{d}{dx} 2\ln x$

$= 2\frac{d}{dx} \ln x$

$= 2(\frac{1}{x}) = \frac{2}{x}$

$\frac{d}{dx} \ln x^n = \frac{d}{dx} n \ln x$

$= n \frac{d}{dx} \ln x = \frac{n}{x}$

Ex Find $\frac{d}{dx}\sqrt{\ln x} = \frac{d}{dx}(\ln x)^{\frac{1}{2}}$. Now use the chain rule.

$\frac{d}{dx}(\ln x)^{\frac{1}{2}} = \frac{1}{2}(\ln x)^{\frac{1}{2}-1} \frac{d}{dx} \ln x = \frac{1}{2}(\ln x)^{-\frac{1}{2}} \frac{1}{x}$

$\frac{d}{dx}\sqrt{\ln x} = \frac{1}{2x\sqrt{\ln x}}$

Change of Basis formula, Let $a>0$, $a \neq 1$

Then $\log_a x = \frac{\ln x}{\ln a}$

Pf Let $y = \log_a x$, so $a^y = x$

Take ln of both sides

$\ln a^y = \ln x$

$y \ln a = \ln x$

$y = \frac{\ln x}{\ln a}$, so $\log_a x = \frac{\ln x}{\ln a}$

Ex $\log_{3.5} 17 = \frac{\ln 17}{\ln 3.5} = \frac{2.833}{1.25} = 2.26$

check $3.5^{2.26} = 16.87$

021 Sec 3.6

~~Quiz Tues, Oct 8~~
~~1) Derivatives involving trig, exponentials, logs chain rule~~
~~2) Implicit Differentiation.~~

Logarithmic Differentiation

Find $\frac{dy}{dx}$ if $y = \frac{x^{2/3}\sqrt[5]{6x^3-7}}{(6x+4)^9}$

In theory, we can use the quotient rule, the product rule, the chain rule, to find $\frac{dy}{dx}$. The answer will be long and messy. Logarithmic Differentiation gives an easier route to the same long and messy.

Solution: First take ln of both sides

$$\ln y = \ln \frac{x^{2/3}\sqrt[5]{6x^3-7}}{(6x+4)^9}$$

Using the rules for logs, expand out the RHS

$$\ln y = \ln x^{2/3} + \ln(6x^3-7)^{\frac{1}{5}} - \ln(6x+4)^9$$

$$\ln y = \frac{2}{3}\ln x + \frac{1}{5}\ln(6x^3-7) - 9\ln(6x+4)$$

Now, take $\frac{d}{dx}$ of both sides, using the chain rule for LHS

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{3}\left(\frac{1}{x}\right) + \frac{1}{5}\left(\frac{18x^2}{6x^3-7}\right) - 9\left(\frac{6}{6x+4}\right)$$

$$\frac{dy}{dx} = y\left[\frac{2}{3x} + \frac{18x^2}{30x^3-35} - \frac{54}{6x+4}\right]$$

$$\frac{dy}{dx} = \left[\frac{x^{2/3}\sqrt[5]{6x^3-7}}{(6x+4)^9}\right]\left[\frac{2}{3x} + \frac{18x^2}{30x^3-35} - \frac{54}{6x+4}\right]$$

021 Sec 3.6

Ex Find $\frac{dy}{dx}$ if $y = \frac{(3x-4)^{10}\,(6x^3+x^2)^5}{\sqrt[4]{(5x^6-7)^3}}$

Solution : ~~lny = ln~~ ln

$$\ln y = 10\ln(3x-4) + 5\ln(6x^3+x^2) - \frac{3}{4}\ln(5x^6-7)$$

$$\frac{1}{y}\frac{dy}{dx} = 10\left(\frac{3}{3x-4}\right) + 5\left(\frac{18x^2+2x}{6x^3+x^2}\right) - \frac{3}{4}\left(\frac{30x^5}{5x^6-7}\right)$$

$$\frac{dy}{dx} = \left[\frac{(3x-4)^{10}(6x^3+x^2)^5}{\sqrt[4]{(5x^6-7)^3}}\right]\left[\frac{30}{3x-4} + \frac{90x^2+10x}{6x^3+x^2} - \frac{90x^5}{20x^6-28}\right]$$

Ex Find $\frac{dy}{dx}$ if $y = x^x$, use logarithmic differentiation

Take ln of both sides

$$\ln y = \ln x^x = x\ln x$$

Take $\frac{d}{dx}$ of both sides

$$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + 1\cdot\ln x = 1+\ln x$$

$$\frac{dy}{dx} = y\left[1+\ln x\right] = x^x\left[1+\ln x\right]$$

Note For $x>1$, $\ln x > 0$, so $1+\ln x > 1$

so $x^x(1+\ln x) > x^x$

so for $x>1$, $f'(x) > f(x)$

3.6

The number e as a limit

We know if $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$

so $f'(1) = \frac{1}{1} = 1$

We use this to show that

$$e = \lim_{x \to 0} (1+x)^{\frac{1}{x}}$$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x}$$

$$= \lim_{x \to 0} \frac{\ln(1+x) - \overbrace{\ln 1}^{=0}}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1+x)$$

$$= \lim_{x \to 0} \ln(1+x)^{\frac{1}{x}}$$

Since $f'(1) = 1$, we have

$$\lim_{x \to 0} \ln(1+x)^{\frac{1}{x}} = 1$$

Now $e = e^1 = e^{\lim_{x \to 0} \ln(1+x)^{\frac{1}{x}}} = \lim_{x \to 0} e^{\ln(1+x)^{\frac{1}{x}}}$

$$= \lim_{x \to 0} (1+x)^{\frac{1}{x}}$$

i.e. $\boxed{e = \lim_{x \to 0} (1+x)^{\frac{1}{x}}}$

Now, let $n = \frac{1}{x}$, so as $x \to 0^+$, $n \to \infty$

$$\boxed{e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n}$$