Quiz – Thursday, Jan 23 on
1) Substitution
2) Integration by parts
3) Trigonometric Integrals.

More Examples of Trig Substitution

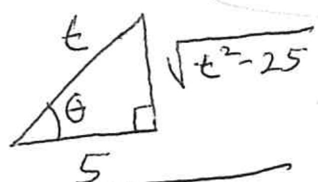
Ex $I = \int \frac{dt}{t^3\sqrt{t^2-25}}$

Solution: Let $t = 5\sec\theta$

$\frac{t}{5} = \sec\theta$

$dt = 5\sec\theta\tan\theta\, d\theta$

$\sec\theta = \frac{\text{hyp}}{\text{adj}}$




We also need $\sqrt{t^2-25}$

$= \sqrt{(5\sec\theta)^2-25} = \sqrt{25\sec^2\theta-25}$

$= 5\sqrt{\sec^2\theta-1} = 5\sqrt{\tan^2\theta}$

$= 5\tan\theta$

So $I = \int \frac{5\sec\theta\tan\theta\, d\theta}{(5\sec\theta)^3\, 5\tan\theta} = \frac{1}{125}\int \frac{d\theta}{\sec^2\theta} = \frac{1}{125}\int \cos^2\theta\, d\theta$

use $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$I = \frac{1}{250}\int (1+\cos 2\theta)\, d\theta = \frac{\theta}{250} + \frac{1}{500}\int (\cos 2\theta)\, 2d\theta$

$= \frac{\theta}{250} + \frac{1}{250}\int \cos 2\theta\, d\theta$

use $u = 2\theta$

$du = 2d\theta$

$= \frac{\theta}{250} + \frac{1}{500}\int \cos u\, du = \frac{\theta}{250} + \frac{1}{500}\sin u + C$

$= \frac{\theta}{250} + \frac{1}{500}\sin(2\theta) + C$

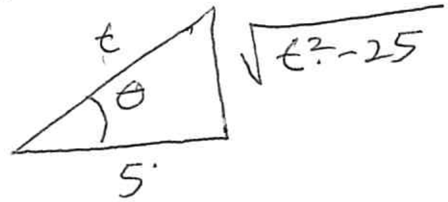
use $\sin(2\theta) = 2\sin\theta\cos\theta$

$$I = \frac{\theta}{250} + \frac{1}{500}\,2\sin\theta\cos\theta + C$$

$$I = \frac{\theta}{250} + \frac{1}{250}\sin\theta\cos\theta + C$$

$$I = \frac{1}{250}\left(\theta + \sin\theta\cos\theta\right) + C$$

now use




so, $\theta = \sec^{-1}\left(\frac{t}{5}\right)$

$$\sin\theta = \frac{\sqrt{t^2-25}}{t}$$

$$\cos\theta = \frac{5}{t}$$

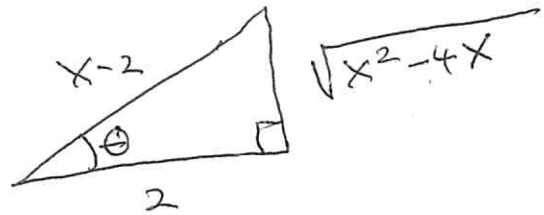
$$\text{so } I = \frac{1}{250}\left(\underbrace{\sec^{-1}\left(\frac{t}{5}\right)}_{\theta} + \underbrace{\left(\frac{\sqrt{t^2-25}}{t}\right)}_{\sin\theta}\underbrace{\left(\frac{5}{t}\right)}_{\cos\theta}\right)$$

Find $I = \int \frac{dx}{(x^2-4x)^{3/2}}$

Solution First rewrite the denominator by completing the square

$x^2 - 4x = (x-2)^2 - 2^2$

so $I = \int \frac{dx}{\left[(x-2)^2 - 2^2\right]^{3/2}}$




Let $x - 2 = 2\sec\theta$

so $\frac{x-2}{2} = \sec\theta$

$dx = 2\sec\theta\tan\theta\, d\theta$

Moreover, $\sqrt{(x-2)^2 - 2^2} = \sqrt{4\sec^2\theta - 4} = 2\sqrt{\sec^2 - 1}$

$= 2\sqrt{\tan^2\theta} = 2\tan\theta$

So $I = \int \frac{2\sec\theta\tan\theta\, d\theta}{(2\tan\theta)^3}$

$I = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$ : Now $\frac{\sec\theta}{\tan^2\theta} = \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}} = \left(\frac{1}{\cos\theta}\right)\frac{\cos^2\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin^2\theta}$

So $I = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$

Let $u = \sin\theta$

$du = \cos\theta$

$I = \frac{1}{4}\int u^{-2}\, du = \frac{1}{4}u^{-1}\left(\frac{1}{-1}\right) = \frac{-1}{4u} = \frac{-1}{4\sin\theta}$

Now $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{x^2-4x}}{x-2}$

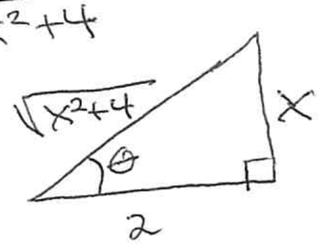
$I = -\frac{1}{4}\left(\frac{x-2}{\sqrt{x^2-4x}}\right) + C$

EX Find $I = \int \frac{dx}{x^2\sqrt{x^2+4}}$

Let $x = 2\tan\theta$

$\frac{x}{2} = \tan\theta$

$dx = 2\sec^2\theta\, d\theta$




now $\sqrt{x^2+4} = \sqrt{(2\tan\theta)^2+4} = \sqrt{4\tan^2\theta+4}$

$= \sqrt{4(\tan^2\theta+1)} = 2\sqrt{\tan^2\theta+1} = 2\sqrt{\sec^2\theta} = 2\sec\theta$

so $\sqrt{x^2+4} = 2\sec\theta$

So $I = \int \frac{\cancel{2}\sec^2\theta\, d\theta}{(2\tan\theta)^2\cancel{2}\sec\theta} = \frac{1}{4}\int \frac{\sec\theta\, d\theta}{\tan^2\theta}$

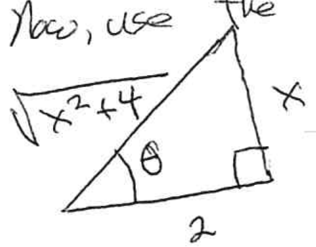
now $\frac{\sec\theta}{\tan^2\theta} = \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}} = \left(\frac{1}{\cos\theta}\right)\left(\frac{\cos^2\theta}{\sin^2\theta}\right) = \frac{\cos\theta}{\sin^2\theta}$

$I = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$. Use substitution with $u = \sin\theta$, $du = \cos\theta\, d\theta$

$I = \frac{1}{4}\int u^{-2}\, du = \frac{1}{4}\left(-\frac{1}{u}\right) + C$

$I = -\frac{1}{4}\left(\frac{1}{\sin\theta}\right) + C$

Now, use the triangle



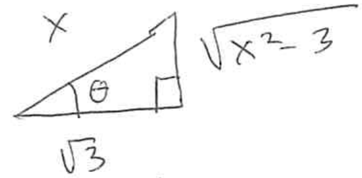

$\sin\theta = \frac{opp}{hyp} = \frac{x}{\sqrt{x^2+4}}$

$I = -\frac{1}{4}\left(\frac{\sqrt{x^2+4}}{x}\right) + C$

<u>Ex</u> $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx$

Let $x = \sqrt{3}\sec\theta$

$\frac{x}{\sqrt{3}} = \sec\theta$




$dx = \sqrt{3}\sec\theta\tan\theta\,d\theta$

$\sqrt{x^2-3} = \sqrt{3\sec^2\theta - 3} = \sqrt{3(\sec^2\theta-1)} = \sqrt{3}\tan\theta$

lower limit
when $x=\sqrt{3}$, $\sec\theta = 1$
and $\theta = 0$

upper limit
when $x>2$, $\sec\theta = \frac{2}{\sqrt{3}}$
$\theta = \frac{\pi}{6}$

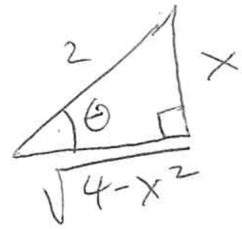
So, $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx = \int_0^{\frac{\pi}{6}} \frac{(\sqrt{3}\tan\theta)(\sqrt{3}\sec\theta\tan\theta)\,d\theta}{\sqrt{3}\sec\theta}$

$\displaystyle = \int_0^{\frac{\pi}{6}} \sqrt{3}\tan^2\theta)\,d\theta = \sqrt{3}\int_0^{\frac{\pi}{6}} (\sec^2\theta - 1)\,d\theta$

$\displaystyle = \sqrt{3}\left[\tan\theta - \theta\right]_0^{\frac{\pi}{6}} = \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right) = 1 - \frac{\sqrt{3}\pi}{6}$

$\approx 0.0931$

Ex $\displaystyle\int \frac{x^2\,dx}{(4-x^2)^{3/2}}$




$x = 2\sin\theta$

$\frac{x}{2} = \sin\theta$

$dx = 2\cos\theta\, d\theta$

$$I = \int \frac{(2\sin\theta)^2(2\cos\theta)\,d\theta}{(4\cos^2\theta)^{3/2}} = \int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta$$

$$= \tan^2\theta\, d\theta = \int (\sec^2\theta - 1)\,d\theta$$

$$= \tan\theta - \theta + C$$

Now $\tan\theta = \dfrac{x}{\sqrt{4-x^2}}$

$\theta = \sin^{-1}\left(\frac{x}{2}\right)$

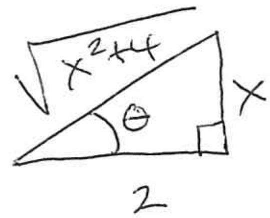
$$\int \frac{x^2\,dx}{(4-x^2)^{3/2}} = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C$$

Ex $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

$x = 2\tan\theta$

$\frac{x}{2} = \tan\theta$

$dx = 2\sec^2\theta\,d\theta$




$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\,d\theta}{4\tan^2\theta\cdot 2\sec\theta} = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

Now $\frac{\sec\theta}{\tan^2\theta} = \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin^2\theta}$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta \quad = \frac{1}{4}\int \frac{du}{u^2} = \frac{1}{4}\int u^{-2}\,du$$

Let $u = \sin\theta$

$du = \cos\theta\,d\theta$

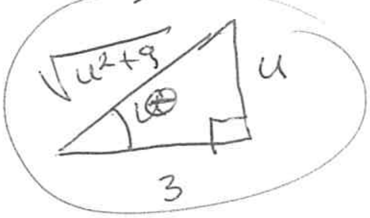
$$= \frac{1}{4}\left(-\frac{1}{u}\right) + C = \frac{-1}{4\sin\theta} + C = \frac{-\csc\theta}{4} + C$$

Now $\csc\theta = \frac{\sqrt{x^2+4}}{x}$

So $-\frac{\csc\theta}{4} + C = \frac{-\sqrt{x^2+4}}{4x} + C$

Ex $$I = \int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$$

First Find $$\int \frac{x^3}{\left(\sqrt{(2x)^2+3^2}\right)^3}\,dx$$



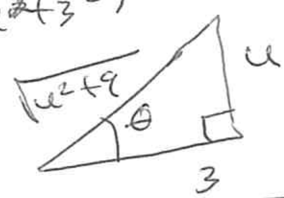

$u = 2x$

$x = \frac{u}{2}$

$du = 2dx$

$dx = \frac{du}{2}$

$$I = \frac{1}{8}\int \frac{u^3\,dx}{\left(\sqrt{u^2+3^2}\right)^3} = \frac{1}{16}\int \frac{u^3\,du}{\left(\sqrt{u^2+3^2}\right)^3}$$




Now use $u = 3\tan\theta$

$du = 3\sec^2\theta\,d\theta$

$\frac{u}{3} = \tan\theta$

$$\sqrt{(3\tan\theta)^2+3^2} = \sqrt{9\tan^2\theta+9} = 3\sqrt{\tan^2\theta+1} = 3\sec\theta$$

$$I = \frac{1}{16}\int_0^{\frac{\pi}{3}} \frac{(3\tan\theta)^3\, 3\sec^2\theta\,d\theta}{3^3\sec^3\theta}$$

$$I = \frac{81}{16}\int_0^{\frac{\pi}{3}} \frac{\tan^3\theta\sec^2\theta\,d\theta}{(9\sec^2\theta)^{3/2}}$$

$$= \frac{81}{16}\int \frac{\tan^3\theta\sec^2\theta\,d\theta}{3^3\sec^3\theta} = \frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{\tan^3\theta}{\sec\theta}\,d\theta$$

$$= \frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{\sin^3\theta}{\cos^2\theta}\,d\theta$$

use $\sin^3\theta = \sin^2\theta\sin\theta$

$= (1-\cos^2\theta)\sin\theta$

$$I = \frac{3}{16}\int \frac{1-\cos^2\theta}{\cos^2\theta}\sin\theta \, d\theta$$

use a substitution

$$v = \cos\theta$$
$$dv = -\sin\theta \, d\theta$$

$$I = -\frac{3}{16}\int_1^{\frac{1}{2}} \frac{1-v^2}{v^2}\, dv = \frac{3}{16}\int_1^{\frac{1}{2}} \frac{v^2-1}{v^2}\, dv$$

$$= \frac{3}{16}\int_1^{\frac{1}{2}} (1-v^{-2})\, dv = \frac{3}{16}\left(v+\frac{1}{v}\right)\Big|_{v=1}^{v=\frac{1}{2}}$$

$$= \frac{3}{16}\left[\left(\frac{1}{2}+\frac{1}{\frac{1}{2}}\right)-\left(1+\frac{1}{1}\right)\right] = \frac{3}{32}$$

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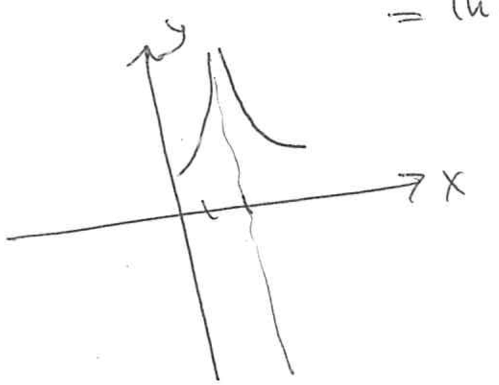
$$\int_1^4 \frac{dx}{x-2} \neq \ln|x-2|\Big|_1^4$$

$$= \ln 2 - \ln 1 = \ln 2$$




We have a vertical asymptote at $x=2$