The root test:

a) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and hence convergent

b) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = +\infty$

the series $\sum_{n=1}^{\infty} a_n$ is divergent

c) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the root test is inconclusive.

Ex $\sum_{n=0}^{\infty} \left(\frac{5n+7}{6n+3}\right)^n$

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{5n+7}{6n+3}\right)^n}$$

$$= \lim_{n\to\infty} \frac{5n+7}{6n+3} = \frac{5}{6} < 1$$

so $\sum_{n=0}^{\infty} \left(\frac{5n+7}{6n+3}\right)^n$ converges

Given any <u>finite</u> series

$$\sum_{i=1}^{n} a_i,$$ the series always converges, and

we get the same sum, no matter what order we sum the terms.

A rearrangement of a series, is simply rearranging the order of the terms of some original series.

~~A divergent series~~

If $\sum_{i=0}^{n} a_n$ is an alternating series with $|a_n| \to 0$, $a_n \to 0$ and $a_n > 0$ $\forall n$, then the order in which we add the terms will change the sum.

<u>Thm</u> Riemann

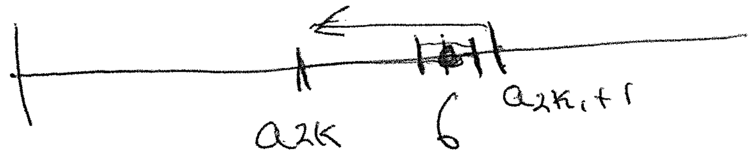
if $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series and r is any real number, then there is a rearrangement of $\sum a_n$ which converges to r.

Idea of the proof. Since $\sum a_n$ is conditionally convergent we have $\sum |a_n|$ diverges

This implies that

$a_1 + a_3 + a_5 + \cdots \to +\infty$, and $a_{2n+1} \to 0$

$-a_2 + a_4 - a_6 - \cdots \to -\infty$ and $a_{2n} \to 0$



Quiz on Wed
1) Sum a geometric series
2) Convergence of p-series
3) show that the harmonic series dies
4) Error Estimates for alternating Series

Section 11.8 Power Series

A power series centered at 0 is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable
The c_i's are constants

A power series is defined by the c_i's. For a given set of c_i's, a power series will converge for some values of x and diverge for other values of x

Ex if $c_0 = c_1 = c_2 = \cdots = 1$

$$\sum_{n=0}^{\infty} c_n x^n = 1 + x + x^2 + x^3 + \cdots$$

This is a geometric series and converges if $|x| < 1$,

A power series, centered at a, or in $x-a$ is

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Key Theorem For a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ one of the following is true.

i) The series converges only for $x = a$
ii) " " " for all x, $x \in \mathbb{R}$
iii) There is a positive number R s.t. the series converges if $|x-a| < R$ and diverges if $|x-a| > R$

R is called the radius of convergence.

The interval of convergence is one of the following

$(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$, $[a-R, a+R]$

<u>EX</u> Find the interval of convergence for

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$, use the ratio test

We want $\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| < 1$

We have $\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0$ for all x

The interval of convergence is, $(-\infty, \infty)$ and

The radius of convergence $R = +\infty$

<u>EX</u> $\sum_{n=0}^{\infty} 3(x-2)^n$. ~~For~~ let $a_n = 3(x-2)^n$

consider $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left| \frac{3(x-2)^{n+1}}{3(x-2)^n} \right| = \lim_{n\to\infty} |x-2|$

we need $|x-2| < 1$

$-1 < x-2 < 1$

$1 < x < 3$. Radius of convergence is 1.

We check the endpts separately

If $x=1$, $\sum 3(x-2)^n$ is $3\sum(-1)^n$, which diverges

If $x=3$, $\sum 3(x-2)^n$ is $3\sum(+1)^n$ which diverges

So, the interval of convergence is $(1,3)$

<u>Ex</u> Find the interval of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}\right| = \lim_{n\to\infty} \left|\frac{nx}{n+1}\right|$$

$$= |x| \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = |x|(1) = x$$

So the radius of convergence is 1

and our interval of convergence is one of the following

$[-1,1]$ or $[-1,1)$ or $(-1,1]$ or $(-1,1)$

In our original series $\sum_{n=0}^{\infty} \frac{x^n}{n}$,

If $x=1$, we have $\sum \frac{1}{n}$, diverges

If $x=-1$ " " $\sum \frac{(-1)^n}{n}$, converges

so the interval of convergence is $[-1,1)$

<u>Ex</u> $\sum_{n=0}^{\infty} n!x^n$ $\quad \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right|$

$$= \lim_{n\to\infty} \left|\frac{(n+1)x}{1}\right| = \lim_{n\to\infty} |(n+1)x| = +\infty \quad \forall x$$

except $x=0$,

So, the radius of convergence is 0

and the interval of convergence is $[0]$

022 Tues, June 11, 2024
sec 11.8

Bessel Function, of order 0

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$, Find the interval of convergence

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2}\cdot\frac{2^{2n}(n!)^2}{x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{2n+2}}{2^{2n+2}(n+1)^2(n!)^2}\cdot\frac{2^{2n}(n!)^2}{x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{4(n+1)^2}\right| \to 0 \text{ for all } x$$

so Radius of convergence is $+\infty$

and the interval of convergence is $(-\infty, +\infty)$

<u>Ex</u> $\sum_{k=1}^{\infty} \frac{1}{k^2} x^k$, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{k^2}{(k+1)^2}\frac{|x|^{k+1}}{|x|^k}\right|$

$$= \left(\frac{k}{k+1}\right)^2 |x| \to |x| \text{ as } k\to\infty$$

so we need $|x|<1$, so Radius of convergence is 1.

<u>Test Ends</u> For $x=-1$, we have $\sum \frac{1}{k^2}x^k$ is $\sum \frac{(-1)^k}{k^2}$. Converges by alternating series test

For $x=+1$, we have $\sum \frac{1}{k^2}x^k$ is $\sum_{k=1}^{\infty}\frac{1}{k^2}$, converge by the p-test $p=2>1$

so, the interval of convergence is $[-1, 1]$

Section 11.9 Representation of Functions ~~of~~ as Power Series

Start with $\sum_{n=0}^{\infty} x^n = 1+x+x^2+\cdots = \frac{1}{1-x}$, converges for $|x|<1$

$$\frac{1}{1-x} = 1+x+x^2+\cdots$$

EX Write $\frac{1}{1+x^2}$ as a power series and find the interval of convergence

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1-x^2+x^4-x^6+x^8-\cdots$$

This is a power series and hence converge when $|-x^2|<1$
i.e. $x^2<1$ so Radius of convergence is 1,
The interval of convergence is $(-1,1)$
when $x=1$ or $x=-1$, we have $1-1+1-1+1\ldots$ diverges

Ex Find a power series representation for $\frac{1}{x+2}$

Sol $$\frac{1}{2+x} = \frac{1}{2(1+\frac{x}{2})} = \frac{1}{2\left[1-\left(-\frac{x}{2}\right)\right]} = \frac{1}{2}\left[\frac{1}{1-\left(-\frac{x}{2}\right)}\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

This series converges when $\left|-\frac{x}{2}\right|<1$, i.e. $|x|<2$
so the interval of convergence is $(-2,2)$

Differentiation and Integration of Power Series.

Thm If the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has a radius of convergence $R>0$ then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is differentiable, and hence continuous, on the interval $(a-R, a+R)$

and $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

also $\int f(x)dx = C + c_0(x-a) + c_1 \dfrac{(x-a)^2}{2} + c_2 \dfrac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

in the interval $(a-R, a+R)$

The upside is that finding derivatives and antiderivatives of a power series is pretty easy.

The downside is that there are questions of convergence and if we only differentiate or integrate a finite number of terms, we have error.

We have

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n(x-a)^n\right]$$

$$\int \left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] dx = \sum_{n=0}^{\infty} \int c_n (x-a)^n \, dx$$

<u>Caution</u> The Radius of Convergence R is the same for a function f(x) ~~and~~ in its power series and for its derivative f'(x) and antiderivative $\int f(x)\,dx$ } as power series

But convergence or divergence at the endpts might change.