<u>Limit Laws for Sequences</u>

If $\{a_n\}$, $\{b_n\}$ both converge (i.e. have a finite limit)

Let c a constant: Then

1) $\lim\limits_{n\to\infty} \{a_n \pm b_n\} = \lim\limits_{n\to\infty} \{a_n\} \pm \lim\limits_{n\to\infty} \{b_n\}$

2) $\lim\limits_{n\to\infty} c a_n = c \lim\limits_{n\to\infty} a_n$

3) $\lim\limits_{n\to\infty} (a_n b_n) = \lim\limits_{n\to\infty} a_n \lim\limits_{n\to\infty} b_n$

4) $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, provided $\lim\limits_{n\to\infty} b_n \neq 0$

5) $\lim\limits_{n\to\infty} a_n{}^p = \left[\lim\limits_{n\to\infty} a_n\right]^p$, for $p>0$, $a_n > 0$

A squeeze, (pinching) thm for sequences

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$

and if $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$

In general: if $\lim\limits_{n\to\infty} |a_n| = 5$ we do not have that $\lim\limits_{n\to\infty} a_n = 5$

Ex $a_n = -5$ for all n, $\lim\limits_{n\to\infty} |a_n| = 5$ but $\lim\limits_{n\to\infty} a_n = -5$

<u>However</u> if $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$, note $-0 = 0$

Let's Consider $\{a_n\}$, $a_n = \{3 + (-1)^n\}$

The first few terms are: 2, 4, 2, 4, 2, 4, ...

So $\lim\limits_{n\to\infty} a_n$ does not exist.

Consider the subsequence consisting of the odd number terms

$a_1, a_3, a_5, a_7 = 2, 2, 2, 2, \ldots$

So this subsequence does converge to 2

Consider the subsequence of all even number terms.

$a_0, a_2, a_3, \ldots = 4, 4, 4, \ldots$

This subsequence converges to 4.

This example shows that you can have subsequences converging but the original sequence does not converge

But If a sequence converges to L, then any subsequence (with infinitely many terms) converges to L,

~~EX~~ ~~If~~ .

EX Find $\lim_{n\to\infty} \frac{n}{1-2n}$

Solution Form the function defined on $x>0$

$$f(x) = \frac{x}{1-2x}$$

If $f(x) \to L$, then the seq converges to L,

$$\lim_{x\to\infty} \frac{x}{1-2x} \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{1}{\frac{1}{x}-2} = \frac{-1}{2}$$

So $f(x) \to \frac{-1}{2}$, this forces $a_n \to \frac{-1}{2}$

EX Does $\{a_n\}$, converge, where $a_n = \frac{n^2}{2^n-1}$

Solution Use L'Hospital's Rule. Let $f(x) = \frac{x^2}{2^x-1}$

$$\lim_{x\to\infty} \frac{x^2}{2^x-1} \overset{H}{=} \lim_{x\to\infty} \frac{2x}{(\ln 2)2^x} \overset{H}{=} \lim_{x\to\infty} \frac{2}{(\ln 2)^2 2^x} = 0$$

$0! = 1$

$1! = 1$

$2! = 1\cdot 2$

$3! = 1\cdot 2\cdot 3$

$\vdots$

$(n-1)! = 1\cdot 2\cdots (n-1)$

$n! = 1\cdot 2\cdots (n-1)n$

$n! = (n-1)!\,n$

n	2^n	$n!$
1	2	1
2	4	2
3	8	6
4	16	24
5	32	120

note if $n \geq 4$, $n! > 2^n$

so $n \geq 4$ $\frac{1}{n!} < \frac{1}{2^n}$

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Thm If $\lim_{n\to\infty} a_n = L$ and f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$

Ex $\lim_{n\to\infty}\left(\sin\frac{\pi}{n}\right) = \lim_{x\to\infty}\left(\sin\frac{\pi}{x}\right) = \sin\left(\lim_{n\to\infty}\frac{\pi}{n}\right) = \sin 0 = 0$

Ex Find $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n!}{n^n}$.

Note a_n is defined for only positive integers, so we cannot use L'Hospital's Rule.

Consider: $a_n = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdot \cdots \cdot n} = \frac{1}{n}\underbrace{\left(\frac{2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot \cdots \cdot n}\right)}_{\text{less than } 1}$

so $\lim_{n\to\infty}\frac{n!}{n^n} = 0$

Thm $\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1<r<1 \\ 1 & \text{if } r=1 \end{cases}$

$\lim_{n\to\infty} r^n$ diverges if $n \geq 1$, $n \leq -1$

Def A seq is increasing if $a_n < a_{n+1}$ $\forall n$.

" " " decreasing if $a_n > a_{n+1}$

A sequence is monotone if it is either increasing or decreasing.

Ex Show that $a_n = \frac{n}{n+1}$ is increasing. Note $a_{n+1} = \frac{n+1}{(n+1)+1}$

Solution 1 $a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)(n+1) - n(n+2)}{(n+2)(n+1)}$

$= \frac{n^2+2n+1-n^2-2n}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} > 0$

i.e. $a_n < a_{n+1}$

Solution 2 Let $f(x) = \frac{x}{x+1}$.

so $f'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2} > 0$

so $f'(x)$ is positive, so f is increasing

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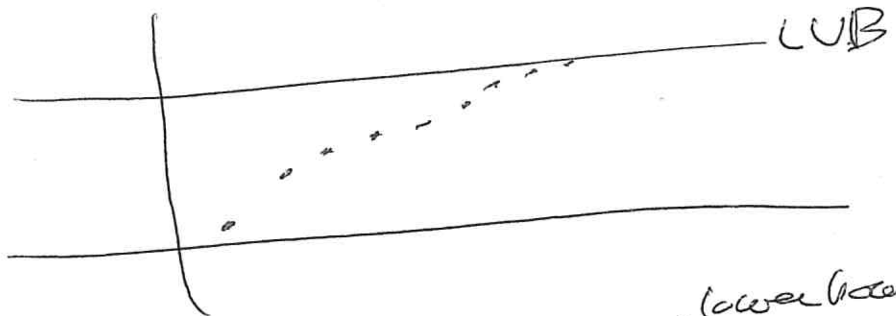
Def A seq $\{a_n\}$ is bounded above is $\exists$ a real number M s.t. $a_n \le M \quad \forall n \ge 1$

~~The~~ M is called an upper bound.

The smallest such M is the least upper bound LUB

Thm A increasing sequence that is bounded above converges to it LUB

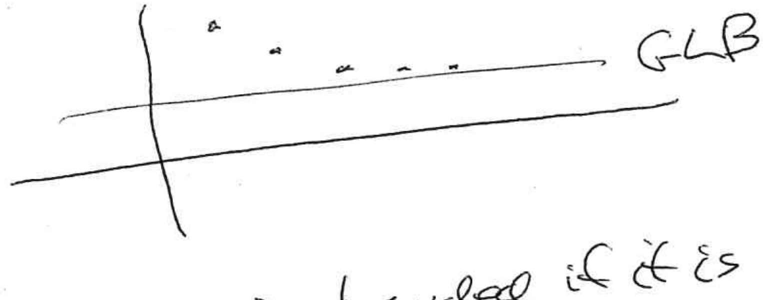


$\{a_n\}$ is bounded below if $\exists\, m$ s.t. $a_n \ge m \quad \forall n$

lower bound

The largest lower bound is the greatest lower bound GLB

Thm A decreasing ~~bounded seq~~ seq which is bounded below converges to its GLB



Def A sequence is bounded if it is bounded above and bounded below.