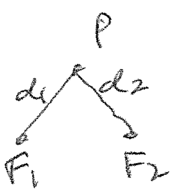
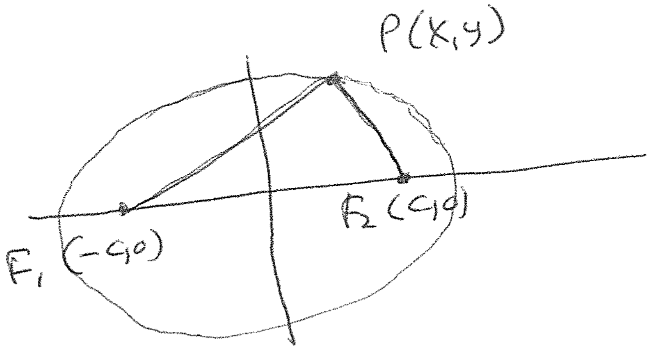
022 Sec 11.6 Wednesday, June 26, 2019
Start Class on Thursday, June 27 at 10 AM.
Conic Sections

Ellipse An ellipse is the set of points in a plane the sum of whose distances from those two fixed pts. is a constant



The two fixed points are called the foci

So $|F_1P| + |F_2P| = \text{constant}$



$$\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a.$$

So, $$\sqrt{(x-c)^2+y^2} = 2a - \sqrt{(x+c)^2+y^2}$$

Square both sides

$$x^2-2cx+c^2+y^2 = 4a^2 - 4a\sqrt{(x+c)^2+y^2} + x^2+2cx+c^2+y^2$$

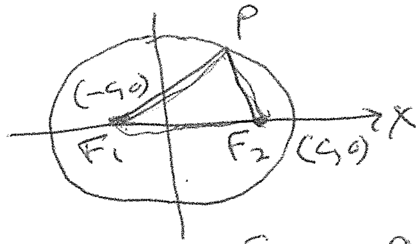
simplifies to

$$a\sqrt{(x+c)^2+y^2} = a^2+cx$$

Square again:

$$a^2(x^2+2cx+c^2+y^2) = a^4+2a^2cx+c^2x^2$$

i.e. $$(a^2-c^2)x^2+a^2y^2 = a^2(a^2-c^2)$$



From triangle F_1F_2P

$2c < 2a$, so $c < a$

so $a^2 - c^2 > 0$. Let $b^2 = a^2 - c^2$

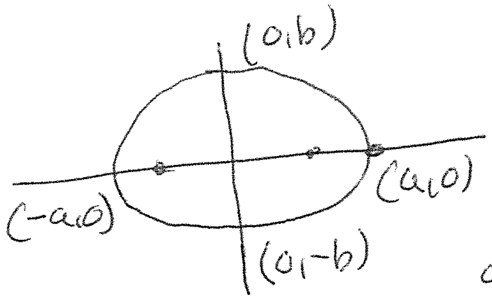
We now have the equation of an ellipse being

$b^2x^2 + a^2y^2 = a^2b^2$; divide by a^2b^2

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The x-intercepts are when $y = 0$, so $x = \pm a$

The y- " " " $x = 0$, so $y = \pm b$



The points $(-a,0)$, $(a,0)$ are the vertices of the ellipse and the lines segment from $(-a,0)$ to $(a,0)$ is the major axis.

The minor axis is the line segment from $(0,b)$ to $(0,-b)$

Note if $F_1 = F_2$ then $a = b$, and we have a circle center at the origin with radius $r = a = b$

To sum this up.

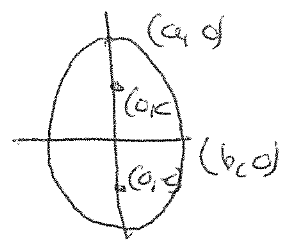
The ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\quad a \geq b > 0$

has foci $(\pm c, 0)$ where $c^2=a^2-b^2$ and vertices $(\pm a, 0)$

If the foci of an ellipse are at $(0, \pm c)$, i.e. on the y-axis, we just change x and y

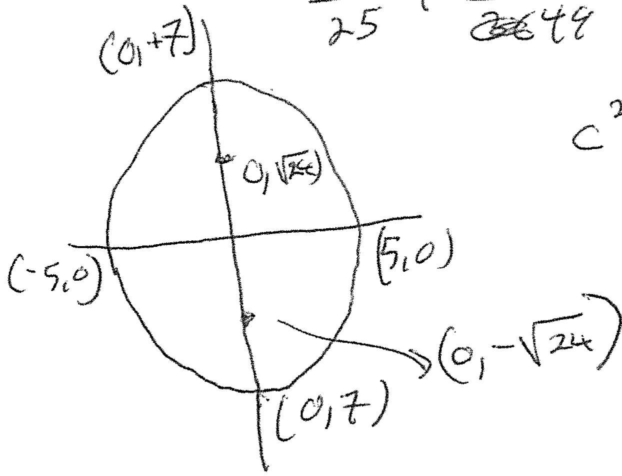
$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1, \quad a \geq b > 0$$

has foci $(0, \pm c)$, where $c^2=a^2-b^2$, and vertices $(0, \pm a)$



Sketch the graph of

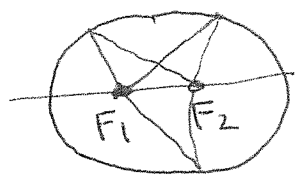
$$\frac{x^2}{25}+\frac{y^2}{49}=1, \text{ and find the foci}$$



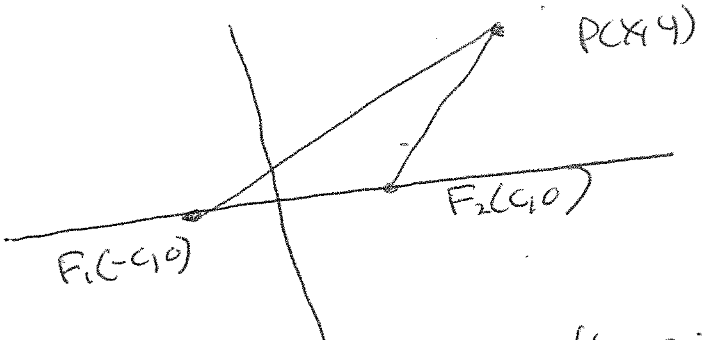
$c^2=a^2-b^2=49-25=24$

$c=\pm\sqrt{24}=2\sqrt{6}$

Reflection Property



Hyperbola



A hyperbola, is the set of points in the plane, such that that difference of the distances from two fixed points, the foci, and the pt P is constant

$$|PF_1| - |PF_2| = \pm 2a$$

If the two foci are at $(-c,0)$ and $(c,0)$

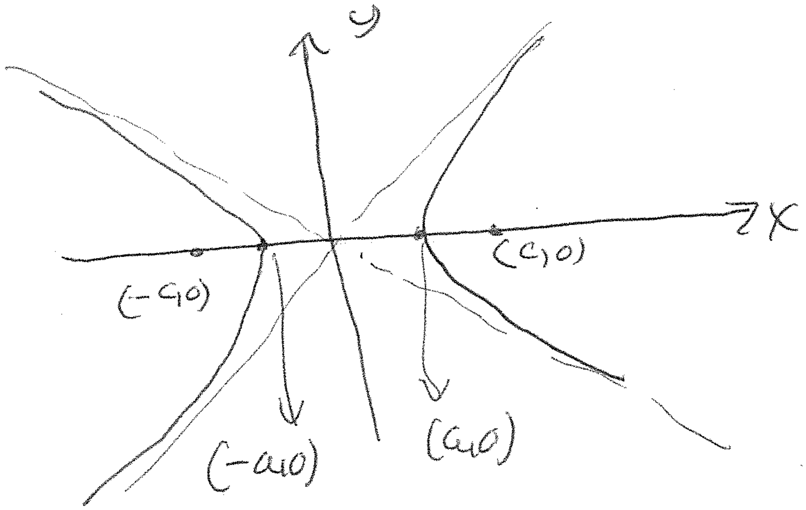
and the diffence of the distances

$$|PF_1| - |PF_2| = \pm 2a$$

Then the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

with $c^2 = a^2 + b^2$



A hyperbola has two branches

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geq 1$$

so $x^2 \geq a^2$

so $|x| = \sqrt{x^2} \geq a$

so $x \geq a$ or $x \leq -a$

The hyperbola has asymptotes

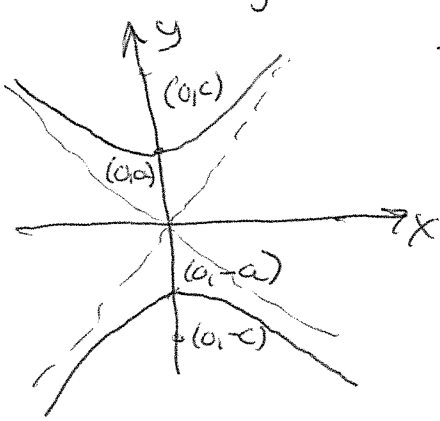
$$y = \left(\frac{b}{a}\right)x, \quad y = \left(-\frac{b}{a}\right)x$$

To sum up

(7) The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has foci $(\pm c, 0)$ with $c^2 = a^2 + b^2$; vertices $(\pm a, 0)$ and asymptotes

$$y = \pm\left(\frac{b}{a}\right)x$$

To get a hyperbola that opens up and down just switch the roles of x and y



The hyperbola: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

has foci $(0, \pm c)$, $c^2 = a^2 + b^2$

vertices $(0, \pm a)$

asymptotes $y = \pm\left(\frac{a}{b}\right)x$

Ex Find the foci and asymptotes of the hyperbola

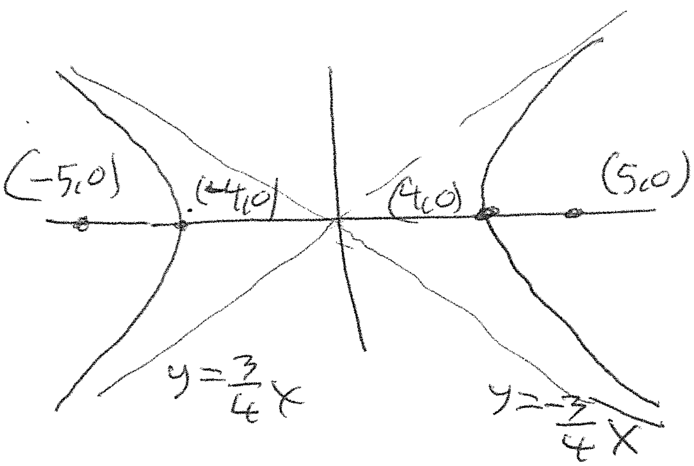
$9x^2 - 16y^2 = 144$ and sketch the graph

$$\frac{x^2}{16} - \frac{y^2}{9} = 1, \quad \frac{x^2}{4^2} - \frac{y^2}{3^2} = 1$$

so $a = 4, b = 3$; since $c^2 = 16 + 9 = 25$

The foci are $(\pm 5, 0)$, The asymptotes are the lines

$y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$



Shifted Conics

So far all of our conics and our equations have been "centered" at (0,0).

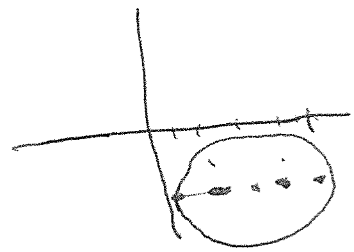
We can center the conics by replacing

x with $x-h$

y " $y-k$

Ex Find an equation of the ellipse with foci $(2,-2)$, $(4,-2)$ and vertices $(1,-2)$ and $(5,-2)$

Solution



The major axis is the line segment joining the vertices $(1,-2)$ and $(5,-2)$

it has length 4, so $a=2$

The distance between the foci is 2, so $c=1$,

so $b^2 = a^2 - c^2 = 2^2 - 1^2 = 3$,

The center of the ellipse is the midpoint of the major axis

in this case, $(3,-2)$.

Replace x by $x-3$

y by $y+2$

$$\frac{(x-3)^2}{4} + \frac{(y+2)^2}{3} = 1$$

Sketch the conic

$$9x^2-4y^2-72x+8y+176=0$$

<u>Solution</u> complete the squares

$$4(y^2-2y)-9(x^2-8x)=176$$

$$4(y^2-2y+1)-9(x^2-8x+16)=\cancel{176+\ldots}$$
$$176+4(1)+(-9)(16)$$

$$4(y-1)^2-9(x-4)^2=36$$

$$\frac{(y-1)^2}{9}-\frac{(x-4)^2}{4}=1$$

This is a hyperbola, opens up/down (y^2 is positive)

center $x=4, y=1$

$a^2=9$, $b^2=4$, $c^2=9+4=13$, foci are $(4, 1+\sqrt{13})$, $(4, 1-\sqrt{13})$

The vertices are $(4,4)$ and $(4,-2)$

The asymptotes are $y-1=\pm\frac{3}{2}(x-4)$