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More on the integral test

Ex Use the integral test to determine if

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \text{ converges}$$

Solution Let $f(x) = \frac{1}{x^2+1}$

Note $f(x)$ is positive, continuous and on $(0,\infty)$ decreasing

We can apply the integral test.

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \tan^{-1} x \Big]_{x=1}^{x=\infty} = \tan^{-1}\infty - \tan^{-1}(1)$$

$$\frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

So, $\int_1^{\infty} \frac{1}{x^2+1}\,dx$ converges to $\frac{\pi}{4}$

So $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges, but not to $\frac{\pi}{4}$

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p-series. A series in the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a p-series

A p-series will converge if $p > 1$

" " " " diverge if $0 < p \le 1$

Pf We use the integral test on, $\int_1^{\infty} \frac{1}{x^p}\,dx$

The improper integral converges if $p > 1$, diverges if $p \le 1$

Ex $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges since $3 > 1$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ diverges since $\frac{1}{3} < 1$

Estimating the sum of a series.

If $\sum_{n=1}^{\infty} a_n$ converges to $S$, finding the exact value of $S$ can be really hard. However, we can estimate $S$ by

$$S \approx \sum_{i=1}^{n} a_i = S_n$$

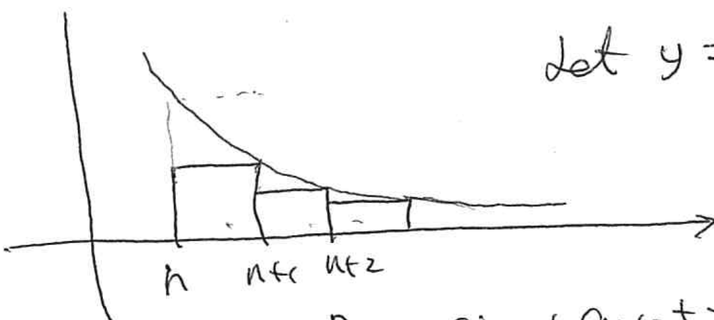
The error in $S$, called the remainder, is

$$R_n = S - S_n = a_{n+1} + a_{n+2} + \cdots + \cdots = \sum_{i=n+1}^{\infty} a_i$$

For a convergent series, as $n \to \infty$, $R_n \to 0$

Now, let $a_n > 0$, and $a_{n+1} < a_n$ (decreasing), $\lim_{n\to\infty} a_n = 0$

Let $y = f(x)$, where $f(n) = a_n$




$$R_n = a_{n+1} + a_{n+2} + \cdots \leq \int_n^{\infty} f(x)\,dx$$


$$R_n = a_{n+1} + a_{n+2} + \cdots \geq \int_{n+1}^{\infty} f(x)\,dx$$

Remainder Estimate for the integral test.

Suppose $f(n) = a_n$, where $f$ is a positive, continuous, decreasing function for $x \geq n$. Assume $\sum a_n$ is convergent, $\sum a_n = S$

Let $R_n = S - S_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$$, adding $S_n$ to all sides

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \leq S_n + R_n = S \leq \int_n^{\infty} f(x)\,dx + S_n$$

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Ex Estimate $\sum_{n=1}^{\infty} \frac{1}{n^3}$ using the first 10 terms

note - this is a p-series with $p=3>1$, so it converges

Now $\sum_{n=1}^{\infty} \frac{1}{n^3} \approx S_{10} = \frac{1}{1^3} + \frac{1}{2^3} + \cdots + \frac{1}{10^3} \approx 1.1975$

By the remainder theorem,

$$R_n \leq \int_{10}^{\infty} \frac{1}{x^3}\,dx = \lim_{b \to \infty} \int_{10}^{b} x^{-3}\,dx$$

$$= \lim_{b \to \infty} \left. \frac{-1}{2x^2} \right]_{10}^{b} = \lim_{b \to \infty} \frac{-1}{2b^2} + \frac{1}{2(10)^2} = \frac{1}{200} = 0.005$$

so $1.1975 - .005 \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \leq 1.1975 + .005$

$$1.20664 \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \leq 1.202534$$

note $1.1975 = S_{10}$

$.005 \approx \sum_{n=11}^{\infty} \frac{1}{n^3}$

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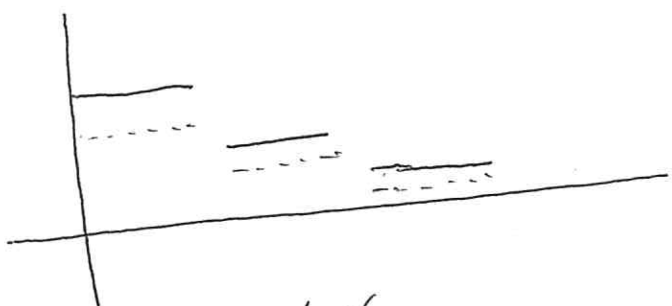
Sec 11.4 The comparsion test. Sort of a discrete form of the integral test

Compare: $A = \sum_{n=1}^{\infty} \frac{1}{3^n+5}$ with $B = \sum_{n=1}^{\infty} \frac{1}{3^n}$

We know that $B$ converges, because $B$ is geometric series with $a = \frac{1}{3}$, $r = \frac{1}{3}$. $B = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$

Both series have <u>only positive</u> terms

For every $i$, $a_i < b_i$



— $\sum \frac{1}{3^n}$ converge

..... $\sum \frac{1}{3^n+5}$ must converge

The Comparsion test

Let $\sum a_n$ $\sum b_n$ be series with non negative terms.

1) If $a_n \le b_n$ for all sufficiently large $n$, and $\sum b_n$ converges then $\sum a_n$ must converge

2) If $b_n \le a_n$ for all large $n$, and $\sum b_n$ diverges then $\sum a_n$ diverges

<u>Caution</u> If $a_n < b_n$ for all large $n$ and $\sum a_n$ converges Then $\sum b_n$ might converge or might diverge.

Likewise if $a_n < b_n$ and $\sum b_n$ diverges then $\sum a_n$ might diverge or might converge

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{2+n^{1/2}}$

compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ which we know diverges by the p-test

Now, $\dfrac{1}{2+\sqrt{n}} < \dfrac{1}{\sqrt{n}}$ for $n>1$

So no help.

We now compare $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$ with the harmonic series,

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, $a_n = \frac{1}{n}$. We know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Now $a_n = \dfrac{1}{n} \leq \dfrac{1}{2+\sqrt{n}}$ (note for $n>4$ we have $n > 2+\sqrt{n}$ so $\frac{1}{n} < \frac{1}{2+\sqrt{n}}$ for $n>4$)

Since, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

We have that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$ diverges