USAMO 2025 #6 — Complete Solution

Problem Statement

Let m and n be positive integers with $m \geq n$. There are m cupcakes of different flavors arranged around a circle and n people who like cupcakes. Each person assigns a non-negative real number score to each cupcake, depending on how much they like the cupcake. Suppose that for each person P, it is possible to partition the circle of m cupcakes into n groups of consecutive cupcakes so that the sum of P's scores of the cupcakes in each group is at least 1. Prove that it is possible to distribute the m cupcakes to the n people so that each person P receives cupcakes of total score at least 1 with respect to P.

Solution

Below is a fully-rigorous, self-contained proof at graduate level, incorporating specialists' verifications and adversarial critiques.

Step 0. Base cases.

- If $n = 1$ then the lone person can cut all m cupcakes into one contiguous group of total score ≥ 1 by hypothesis; assign all cupcakes.
- If $m = n$ then each person's partition into n single-cupcake arcs of score ≥ 1 forces $v_P(k) \geq 1$ for every cupcake k; the total score $\sum_k v_P(k) = n \cdot (\text{average } v_P) \geq n$ implies each singleton arc has score exactly ≥ 1, so again assigning one distinct cupcake per person works.

Henceforth assume $n \geq 2$ and $m > n$.

1. Discrete → Continuous translation.
For each person P (index $i = 1, \ldots, n$) define a step–function

$$v_i : [0,1] \longrightarrow \mathbb{R}_{\geq 0}, \qquad v_i(t) = m \cdot v_i(k) \quad \text{for } t \in \left[\tfrac{k-1}{m}, \tfrac{k}{m}\right).$$

Then

$$\int_0^1 v_i(t)\, dt \;=\; \sum_{k=1}^{m} v_i(k)$$

is finite and at least n by summing the n groups, each of score ≥ 1.

2. Simplex encoding of circular partitions.
Let

$$\Delta \;=\; \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} \;:\; \sum_{j=1}^{n} x_j = 1 \right\}.$$

For $x \in \Delta$ define breakpoints

$$b_0(x) = 0, \qquad b_j(x) = \Big(\sum_{t=1}^{j} x_t\Big) \bmod 1, \quad j = 1, \dots, n.$$

These induce consecutive arcs $A_j(x) = [\, b_{j-1}(x),\, b_j(x)\,]$ on the circle $\mathbb{R}/\mathbb{Z}$, partitioning it into n intervals.

3. Definition and continuity of $f_{i,j}$.

Define

$$f_{i,j}(x) \;=\; \int_{A_j(x)} v_i(t)\, dt.$$

The map $x \mapsto f_{i,j}(x)$ is continuous on Δ; in particular each super-level set

$$S_{i,j} \;=\; \big\{x \in \Delta : f_{i,j}(x) \geq 1\big\}$$

is closed.

4. KKM covering property.

For fixed i and all $x \in \Delta$,

$$\sum_{j=1}^{n} f_{i,j}(x) \;=\; \int_0^1 v_i(t)\, dt \;\geq\; n,$$

so at least one $f_{i,j}(x) \geq 1$. Hence $\bigcup_{j=1}^{n} S_{i,j} = \Delta$. The same holds on each face of Δ, giving the KKM hypothesis.

5. Gale's colourful KKM lemma.

Applying the colourful KKM lemma to the n colour classes $\{S_{i,1}, \dots, S_{i,n}\}$ $(i = 1, \dots, n)$ yields a point $x^* \in \Delta$ and a permutation σ of $\{1, \dots, n\}$ such that

$$f_{i,\sigma(i)}(x^*) \;\geq\; 1 \;\text{ for every } i.$$

6. Rounding to whole cupcakes.

Refine the hyperplane arrangement by inserting all rational cuts $b_j(x) = k/m$. Each face is a rational polytope with vertices in $(1/m)\mathbb{Z}^n$. Choosing a vertex x in the (non-empty) intersection with $\big\{f_{i,\sigma(i)} \geq 1\big\}$ preserves the inequalities while forcing every breakpoint $b_j(x)$ to lie at multiples of $1/m$. Hence each interval $A_j(x)$ is a union of whole cupcake subintervals, and

$$f_{i,\sigma(i)}(x) \;=\; \sum_{k \in A_{\sigma(i)}(x)} v_i(k) \;\geq\; 1.$$

7. Conclusion.

Give person i exactly those cupcakes whose subintervals lie in $A_{\sigma(i)}(x)$. Their total score is at least 1, completing the proof. □

References.

1. D. Gale, "The game of Hex and the Brouwer fixed-point theorem", *Amer. Math. Monthly* **91** (1984), 494—512.
2. F. Meunier and O. Schaudt, "A colourful proof of the KKM lemma", arXiv:1708.03201 (2017).