Advantages of DCS over ACS. (2074 Bhadra).

CHAPTER - I

Introduction to Discrete-Time Control Systems

1.1 Introduction:

The rapid advances in use of digital controllers and computers in control system are the motivation for digital control or discrete time systems. Digital controls are used for maximum productivity, efficiency, precision control etc. The current trend towards digital rather than analog control of dynamic systems is mainly due to the availability of low-cost digital computers.

Types of Signals:

1. *Continuous-time (analog) signals:*

   It is defined over a continuous range of time. The Fig. 1.1 shows the continuous-time signal

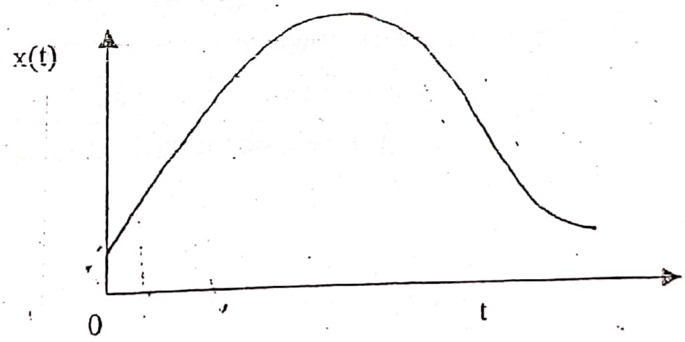

Fig 1.1

(Continuous-time analog signal)

In this type of signal the amplitude may assume a continuous range of values.

2. *Continuous-time quantized signals:*

   The continuous time signals being represented by a distinct set of values-quantized values (quantized in amplitude only not in time). The Fig. 1.2 shows the continuous – time quantized signal.

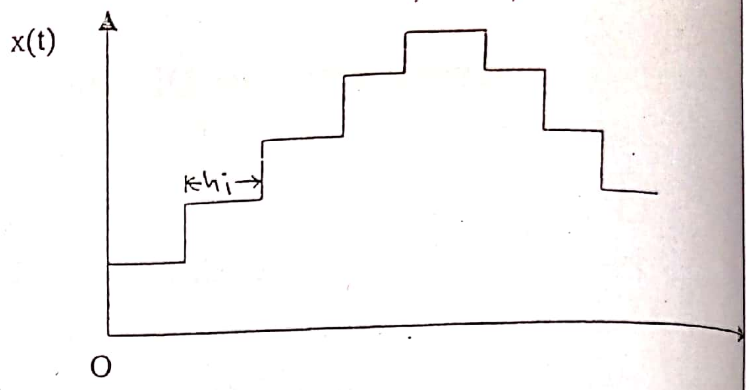

Fig 1.2

(Continuous-time quantized signal)

The range of magnitude is divided into a finite number of disjoint intervals hi wh
necessarily equal.

3. *Sampled-Data Signals:*

The signal which is defined at discrete intervals of time is called the dis signal. (That is, one in which independent variable t is quantized). If the amp take a continuous range of values, then the signal is called sampled-data sampled-data signal can be generated by sampling an analog signal at discrete of time. The Fig. 1.3 shows this type of signal.

Continuous time
quantized signal x(t)
→ quantized over amplitude
Digital
→ quantized in both time and amplitude

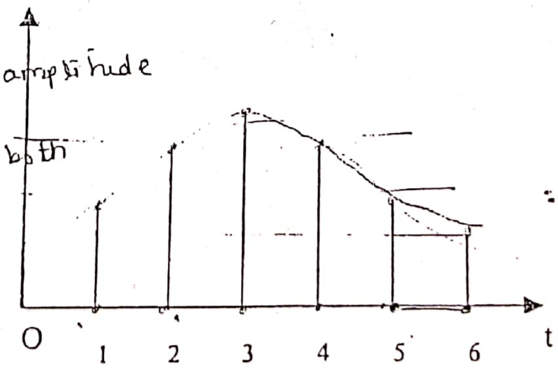

Fig 1.3

(Sampled-data signal)

4. *Digital Signals:*

It is a discrete-time signal with quantized amplitude. Clearly, it is a signal quantized both in amplitude and in time. The use of the digital controller requires quantization of signals both in amplitude and in time as shown in Fig. 1.4.

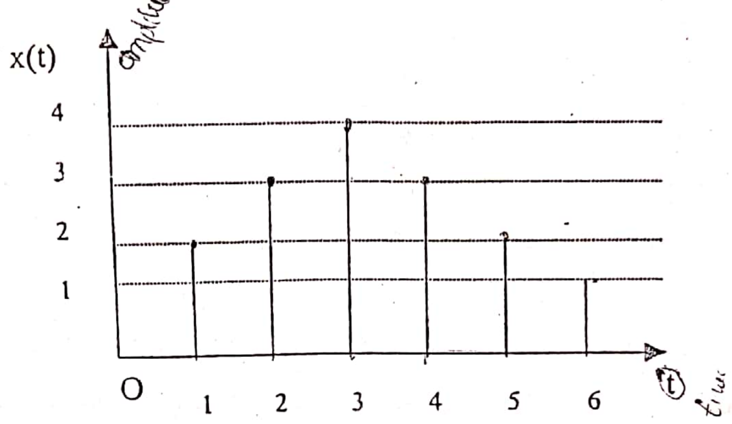

Fig 1.4

(Digital signal)

Most of the signals in the world are analog or continuous type so if the digital controllers are to be used then signal conversion from analog to digital becomes necessary.

Terminology such as discrete-time control systems sampled data control systems and digital control systems refer to the same type or very similar type of control systems.

The control systems we examine in this course are mostly linear and time-invariant. Linear system is one in which a principle of superposition applies. If y_1 be the response of the system to input x_1 and y_2 the response to input x_2, then the system is linear if and only if, for every scalar α and β, the response to input $\alpha x_1+\beta x_2$ be $\alpha y_1+\beta y_2$. A linear system may be described by linear differential equations. A linear time-invariant system is one in which the coefficients in the differential equation or difference equation do not vary with time i.e. one in which the properties of the system do not change with time.

Discrete-Time Control Systems and Continuous-Time Control Systems:

(i) Discrete-time control systems are those in which one or more variables can change only at discrete instant of time. These instants, which are denoted by kT or T_k (k=0, 1, 2, 3 ....).

(ii) Discrete-time control systems differ from continuous-time control systems i signals for discrete-time control systems are in sampled data form or in digita If a digital computer is involved in a control system as a digital control sampled data must be converted into digital data.

(iii) Continuos-time systems, whose signals are continuous in time, may be descr differential equations. But in the discrete-time systems, which involve signals or digital signals may be described by difference equations after app discretization of continuous time signals.

Sampling Process: Whenever a digital controller is involved, the signal used needs digital form. Since most real world signals are in analog form, thus signal nee converted into digital form. The process of converting analog signals to digital one i as sampling process. As for example, a sampling process is needed wherever a la controller or computer is time-shared by several plants in order to save cost. Then signal is sent out to each plant only periodically and thus signals become a samp signal.

The term "discretization" rather than "sampling", are frequently used in the an multiple input-multiple-output system, although both the terms mean the same thing.

It is important to note that occasionally the sampling operation or discretization i fictitious and has been introduced only to simplify the analysis of control syst actually contain only continuous-time signals. In fact, we often use a suitable disc model for a continuous-time system. An example is a digital computer simula continuous-time system. Such a digital computer simulated system can be analyzed parameters that will optimize a given performance index.

1.2 Digital Control Systems

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The configuration of a basic digital control system is shown in Fig. 1.5.

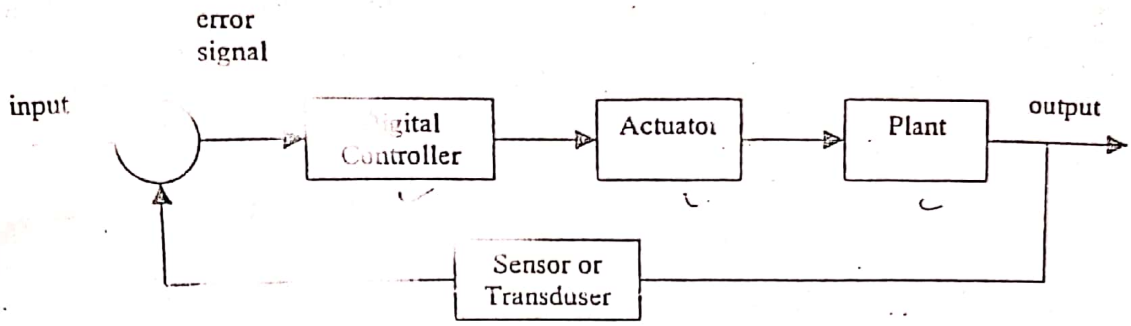

Fig 1.5

(Block diagram of a digital control system)

The input to ...

Digital Controller: The configuration of a digital controller is shown in Fig. 1.6.

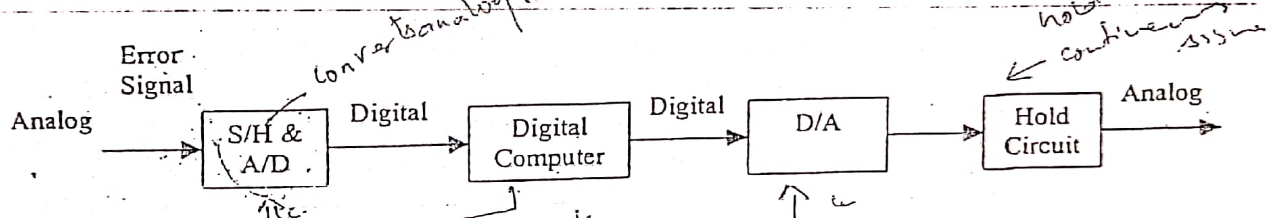

Fig 1.6

clock

(Block diagram of a digital controller)

The error signal is converted to the digital form by sample and hold circuit and analog to digital converter. The conversion is done at the sampling time. The digital computer processes the sequence of number by means of an algorithm and produces a new sequence of numbers.

The D/A converter and the hold circuit convert the sequence of numbers in numerical code into a piecewise continuous time signal. The output of the hold circuit, a continuous-time signal is fed to the plant, either directly or through the actuator, to control the dynamics.

Sampling or Discretization: The operation that transforms continuous-time sign[al into] discrete-time data is called sampling or discretization. The reverse operation, the op[eration] that transforms discrete-time data into a continuous-time signal is called data-hold. The S/H and A/D converter convert the analog signal into a sequence of numericall[y coded] binary words. Such conversion is called coding or encoding. The D/A conversion pr[ocess is] called decoding.

Some Definitions:

a) *Sample and Hold:* 2075

It describes a circuit that takes an analog input and holds it at a constant val[ue for a] specific period of time. Usually, the signal is electrical but the mechanical an[d] signals are also possible after converted into electrical signals.

b) *Analog-to-Digital Converter (A/D):*

It is also called an encoder that converts analog signals into digital signals, [i.e.] numerically coded signal. → (binary signal) A/D is needed as an interface between analog si[gnal and] digital computer. S/H is usually an integral part of A/D. A/D conv[ersion is an] approximation since analog signals can take infinite number of values [whereas] digital signals can not. The approximation is called the quantization.

c) *Digital-to-Analog Converter (D/A):*

It is also called a decoder which converts digital signals into an analog signa[l. It acts] as an interface between digital computer and analog component (either an a[ctuator or] a plant).

d) *Plant or Process:*

A plant is any physical system to be controlled. Examples: chemica[l plants,] spacecraft, machines, robots etc.

The operation to be controlled is called a process. Examples: chemical, e[tc.] and biological process.

e) *Transducer:*

A transducer is a device that converts an input signal into an output signal of another form. Such as a device that converts a pressure signal into a voltage output. The output signal, in general, depends on the past history of the input.

Classification of Transducers:

i) *Analog Transducer:* It is a transducer in which the input and output signals are continuous function of time.

ii) *Sampled-data Transducer:* It is one in which the input and output signals occur only at discrete instants of time, but the magnitudes of the signals are unquantized.

iii) *Digital Transducer:* A digital transducer is one in which the input and output signals occur only at discrete instants of time and the signal magnitudes are quantized.

Conversion of continuous time signal into discrete time signals obtained by taking samples of continuous time signal at discrete time instants.

Types of Sampling Operations:

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1. *Periodic Sampling:* In the case of periodic sampling, the sampling instants are equally spaced, or $t_k = kT$ where $k = 0,1,2.....$ It is the most convenient type of the sampling operation.

2. *Multiple-order Sampling:* The patron of the t_k's is repeated periodically, that is, $t_{k+r} - t_k$ is constant for all k.

3. *Multiple-rate Sampling:* In a control system having multiple loops, the largest time constant involved in one loop may be quite different from that of other loops. Hence it might be better to sample slowly in a loop involving large time constants. While in a loop involving only small time constants, the sampling rate must be fast. In this case, two concurrent sampling operation occur at $t_k = pT_1$ and qT_2, where T_1, T_2 are constants and p, q are integers.

4. *Random Sampling:* In this case, the sampling operation is random, or t_k is a random variable.

1.3 Quantizing and Quantization Error

The main functions involved in A/D conversion are sampling, quantizing and codi[ng]. The value of any sample falls between two adjacent 'permitted' output states, it mu[st] the value of any sample falls between two adjacent 'permitted' output states, it mu[st be] at the permitted state nearest the actual value of output signal. The process of repr[esenting a] continuous or analog signal by finite number of discrete state is called quantization.

The output state of each quantized state is then described by a numerical code, such [a process] is called encoding or coding. Encoding is a process of assigning a digital word [to] each discrete state.

Quantizing: The standard number system used for processing digital signal is t[he binary] number system. In this system, the code group consists of n pulses each indicat[ing] 'ON' (1) or 'OFF' (0). The n "ON-OFF" pulses can represent 2^n amplitude levels [or] states.

The quantization level Q is the range between two adjacent decision points and is gi[ven by]

$$Q = \frac{FSR}{2^n}$$

$$Q = \frac{FSR}{2^n}$$

Where, FSR is full scale range.

Left most bit is called most significant bit (MSB) and has the highest value whil[e the right] most bit called least significant bit (LSB) has the least weight. LSB is the quantiz[ation level] Q.

0 and ±½ Q

How it can be improved

Quantization Error: Due to the fact that the bits in a digital word is finite, A/D c[onversion] results in a finite resolution. The analog signals must be rounded off to a quantizat[ion level, causing] quantization error. The error varies between 0 and ± ½Q, No matter how many bit[s are used,] there is always some kind of quantization error when going A/D conversion.

The Fig. 1.7 shows a block diagram of a quantizer together with its in[put-output] characteristics. For an analog input x(t), the output y(t) takes on only a finite [number of] levels, which are integral multiples of the quantization level Q.

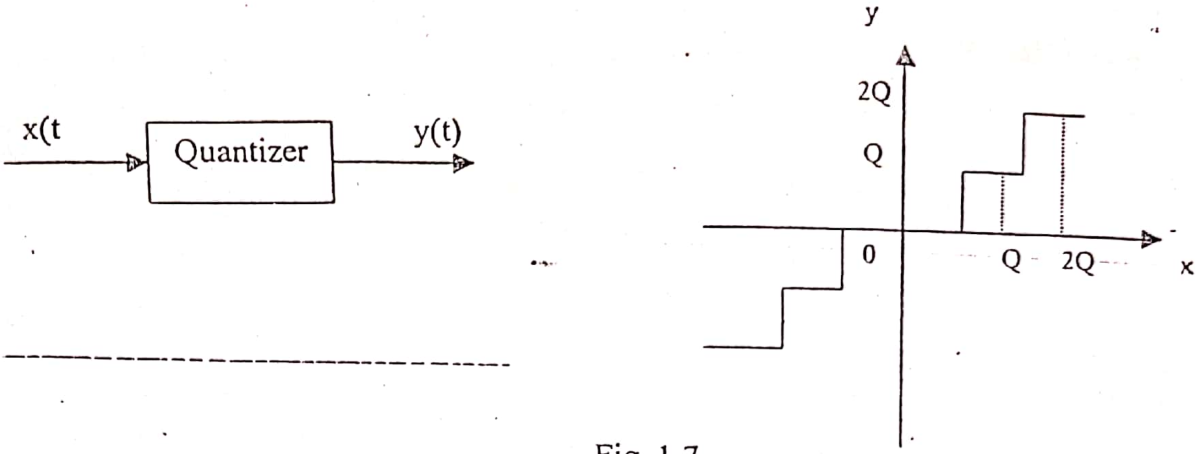

Fig. 1.7
(Block diagram of a quantizer and its input-output characteristics)

Since the quantizing process is an approximating process in that the analog quantity is approximated by a finite digital number, the quantization error is a round-off error. Clearly, the finer the quantization level is the smaller the round-off error.

The Fig. 1.8 shows an analog input x(t) and the discrete output y(t), which is in the form of staircase function. The quantization error e(t) is the difference between the input signal and the quantized output, or

$$e(t) = x(t) - y(t)$$

The magnitude of the quantized error is

$$0 \leq |e(t)| \leq \frac{1}{2}Q$$

For a small quantization level Q, the quantization error is like a random noise.

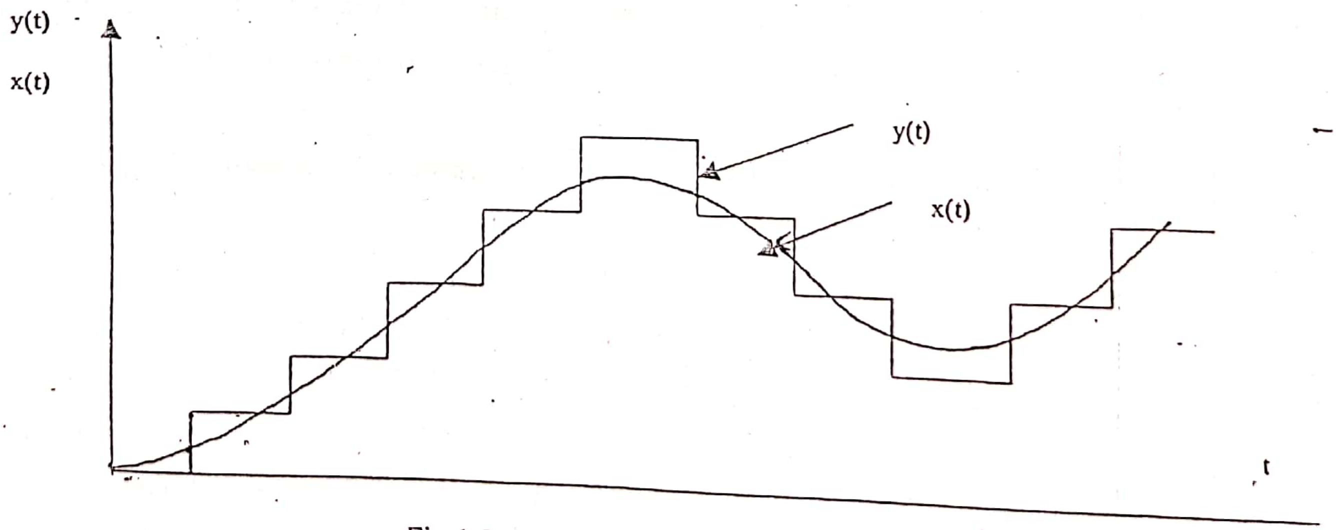

Fig 1.8
(Analog input x(t) and discrete output y(t))

1.4 2074 Bhadra

Data Acquisition and Data Distribution Systems:

The Fig. 1.9 shows the block diagram of a data acquisition system.

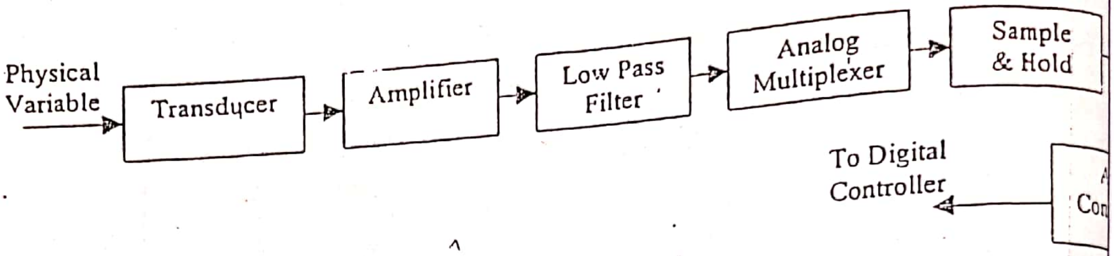

Fig 1.9

(Block diagram of data acquisition system)

In this system, the input to the system is a physical variable such as position, acceleration, temperature or pressure. Such a physical variable is first converted electrical signal (a voltage or current signal) by a appropriate transducer. The amp follows the transducer performs one or more of the following functions.

(i) It amplifies the voltage output of the transducer.

(ii) It converts the current signal into a voltage signal.

(iii) It buffers the signal.

The low pass fitter that follows the amplifier attenuates the high frequen components, such as noise signals (The electronic noises are random in nature ar reduced by low-pass filter). The output of the low-pass fitter is an analog signal. T is fed to the analog multiplexer. The output of the multiplexer is fed to the sample circuit, whose output is fed to analog to digital converter. The output of the conv digital form and it is fed to the digital computer (controller).

The Fig. 1.10 shows the block diagram of data distribution system.

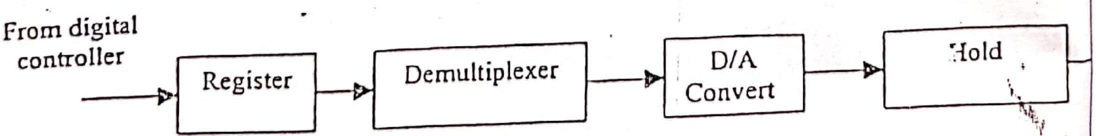

Fig 1.10

(Block diagram of data distribution system)

The reverse of the data-acquisition process is the data-distribution process. It consists of register, a demultiplexer, digital-to-analog converter and hold circuits. It converts the signal in digital form (binary numbers) into analog form. The output of D/A converter is fed to the hold circuit. The output of the hold circuit is fed to the analog actuator, which in turn, directly controls the plant under consideration.

1.4.1 *Analog Multiplexer:*

The analog multiplexer is a device that performs the function of time-sharing an A/D converter among many analog channels. If many signals are to be processed by a single digital controller, then these input signals must be fed to the controller through a multipexer.

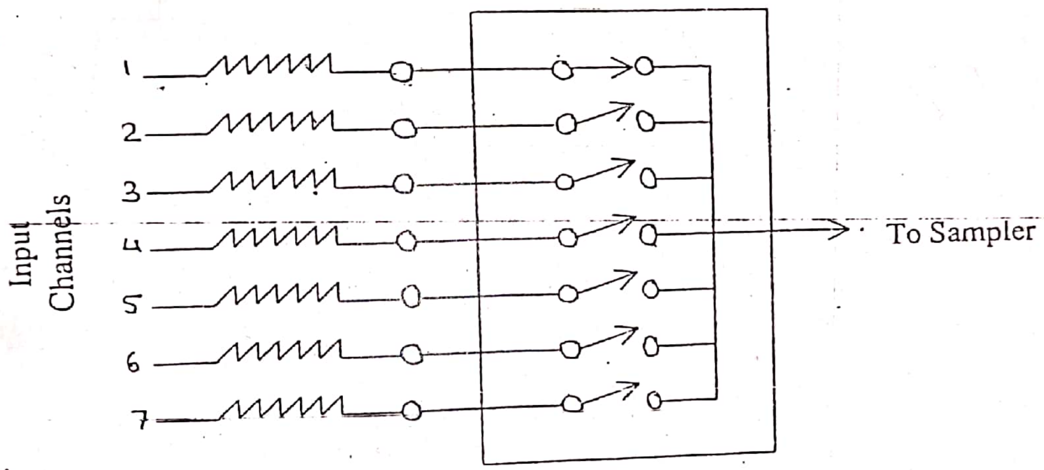

Fig 1.11

(Schematic diagram of an analog multiplexer)

The schematic diagram of an analog multiplexer is shown in Fig.1.11. It is a multiple switch (an electronic switch) that sequentially switches among many analog input channels in some prescribed fashion. When the switch is on in a given input channel, the input signal is connected to the output of the multiplexer for a specified period of time. During the connection time the sample and hold circuit samples the signal voltage (analog signal) and holds it s value, while the A/D converter converts the analog value into digital data.

1.4.2 *Demultiplexer:*

The opposite of multiplexer is demultiplexer which separates the composite output digital data from the digital controller into the original channels.

1.4.3 *Sample-and-Hold Circuit:*

A sampler in a digital system converts an analog signal into a train of amplitude-[illegible] pulses. The hold circuit holds the value of the sampled pulse signal over a specified time. The sample-and-hold is necessary in the A/D converter to produce a n[illegible] accurately represents the input signal at the sampling instant.

The Fig. 1.12 shows the simplified diagram of the sample-and-hold circuit. The S/[illegible] an analog circuit in which an input voltage is acquired and then stored on a hi[illegible] capacitor with low leakage and low dielectric absorption characteristics.

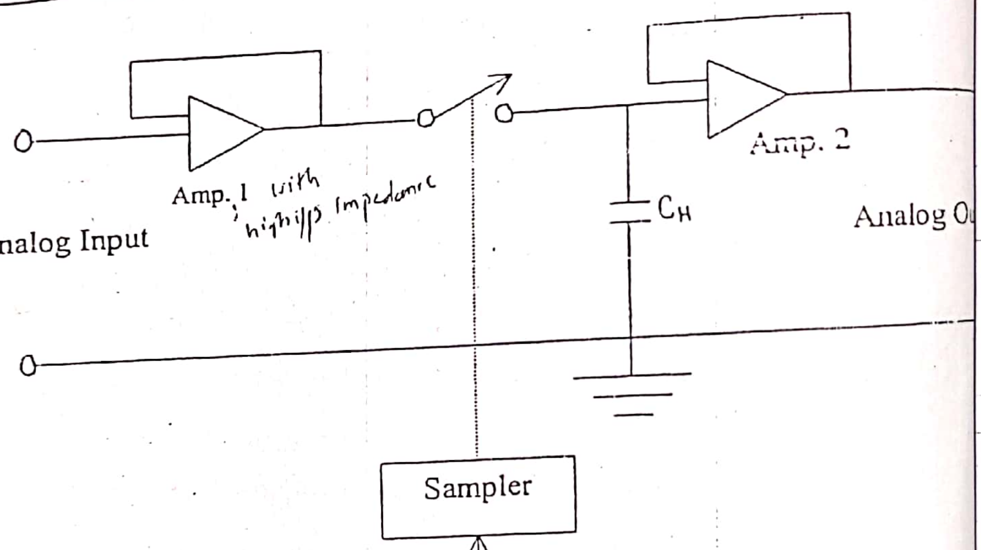

Fig 1.12

(Sample and hold circuit)

As shown in the Fig 1.12 an electronic switch is connected to the hold capacitor. [illegible] amplifier 1 is an input buffer amplifier with a high input impedance. Operational [illegible] is the output amplifier which buffers the voltage on the hold capacitor. When th[illegible] closed the capacitor charges up to the average level of the input signal that means [illegible] on the capacitor in the circuit tracks the input voltage. This mode of operation i[illegible] tracking mode. Now, when the switch is open the capacitor starts to discharge [illegible] through the stray leakage hence the capacitor voltage holds constant for a sp[illegible] period and this operation mode is known as hold mode. The Fig. 1.13 shows the tracking mode and hold mode.

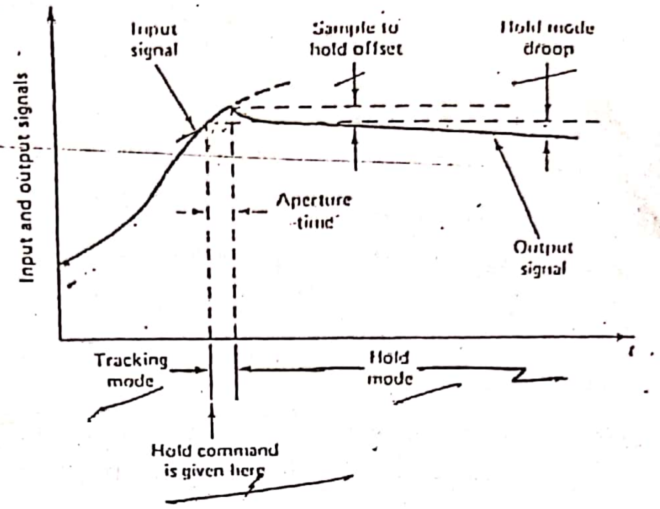

Fig 1.13

(Tracking mode and hold mode) 2075 (How this is employed) / 2074 held

In fact, the switching from the tracking mode to the hold mode is not instantaneous. When the hold command is given while the circuit is in the tracking mode, then the circuit will stay in the tracking mode for a short while before reacting to the hold command. The time interval during which the switching takes place is called the aperture time.

The output voltage during the hold mode may decrease slightly. The hold mode droop may be reduced by using a high-input-impedance output buffer amplifier. Such an output buffer amplifier must have very low-bias current.

The sample-and-hold operation is controlled by a periodic clock.

CHAPTER-II

The z-Transform

2.1 Introduction

A mathematical tool commonly used for the analysis and synthesis of discrete-time systems is the z transform. The role of the z transform in discrete-time systems is s that of the Laplace transforms in continuous-time systems.

In a linear discrete-time control system, a linear difference equation character dynamics of the system. To determine the systems response to a given input, difference equation must be solved with the z transform method, the solutions difference equation equations become algebraic in nature.

Discrete-Time Signals: The discrete-time signals arise if the system involves a operation of continuous-time signals. The sampled signal is $x(0)$, $x(T)$, $x(2T)$, ... T is the sampling period. Such a sequence of values arising from the sampling ope usually written as $x(kT)$. If the system involves an iterative process carried out by computer, the signal involved is a number sequence $x(0)$, $x(1)$, $x(2)$, ... The seq number is usually written as $x(k)$. Although x(k) is a number sequence, it can be co as a sampled signal of $x(t)$ when the sampling period T is 1 sec.

2.2 The z Transform

The z transform of a time function x (t), where t is nonnegative, or of a sequence x (kT), where k takes zero or positive integers and T is the sampling period, is define following equation:

$$X(z) = \mathcal{Z}[x(t)] = \mathcal{Z}[x(kT)] = \sum_{k=0}^{\infty} x(kT) z^{-k} \qquad \text{.............(2.1)}$$

For a sequence of numbers x(k), the z transform is defined by

$$X(z) = \mathcal{Z}[x(k)] = \sum_{k=0}^{\infty} x(k) z^{-k} \qquad \text{............(2.2)}$$

the power series has finite sum for any value of z when it converges and its region is where it converges. This region is called Region of Convergence.

The z transform defined by Eqs. (2.1) and (2.2) is referred to as the one-sided z transform. The symbol $\mathcal{Z}$ denotes "The z transform of". In the one-sided z transform, we assume $x(t) = 0$ for $t < 0$ or $x(k) = 0$ for $k < 0$.

The z transform of x (t), where $-\infty < t < \infty$, or of x (k), where k takes integer values (0, ± 1, ± 2, ...), is defined by

$$X(z) = \mathcal{Z}[x(t)] = \mathcal{Z}[x(kT)] = \sum_{k=-\infty}^{\infty} x(kT)z^{-k} \qquad \text{..............(2.3)}$$

$$X(z) = \mathcal{Z}[x(k)] = \mathcal{Z}[x(k)] = \sum_{k=-\infty}^{\infty} x(k)z^{-k} \qquad \text{..............(2.4)}$$

The z transform defined by Eqs. (2.3) and (2.4) is referred to as the two-sided z transform. In the two-sided z transform, the time function x (t) is assumed to be nonzero for $t < 0$ and the sequence x (k) is considered to have nonzero values for $k < 0$. Both the one-sided and two-sided z transforms are series in powers of z^{-1}.

z^{-1}

2.2.1. Region of Convergent for z-Transform :

As we know that

$$X(z) = \mathcal{Z}[x(t)] = \sum_{k=0}^{\infty} x(kT)z^{-k}$$

$z[x(t)] =$

is a power series which may not converge for all values of z. That is, the infinite sum may not be finite. For any given sequence, the set of values of z for which the z transform converges is called the region of convergence (ROC).

Example 1. Find the region of convergence for the function $x(k) = 1, \quad k = 0,1,2, \ldots\ldots$

Solution: As we know that,

$$X(z) = \mathcal{Z}[x(k)] = \frac{1}{1-z^{-1}}$$

$\frac{1}{1-z^{-1}}$

converges if $|z^{-1}| < 1$ i.e. $|z| > 1$ (ROC)

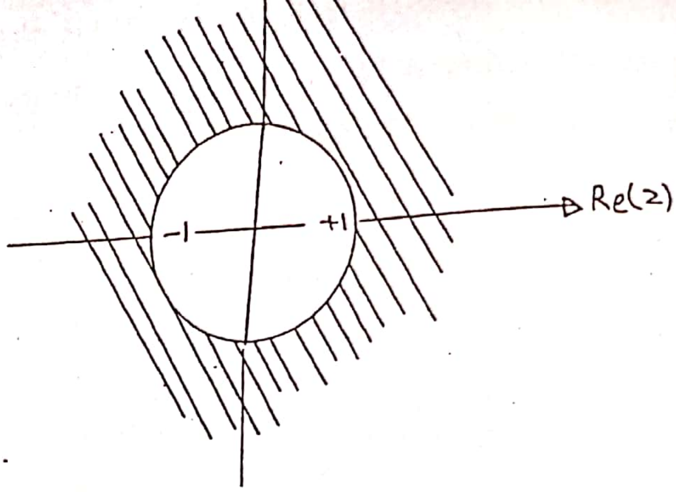

Fig. 2.1(Region of convergence)

Example 2. x (k) = a^k u (k), where u (k) is a unit step function.

Solution. Now, $Z\left[x(k)\right] = \dfrac{1}{1 - az^{-1}}$ $\quad = \dfrac{1}{1-az^{-1}}$

ROC is $\left|az^{-1}\right| \angle 1$

i.e. $|z| \rangle a$

Example 3. x (k) = $(1/2)^k$ u (k) + $(-1/3)^k$ u (k), where u (k) is the unit step function

Now, $Z\left[x(k)\right] = \dfrac{1}{1 - \frac{1}{2}z^{-1}} + \dfrac{1}{1 + \frac{1}{3}z^{-1}}$

ROC's are

$$\left|\frac{1}{2}z^{-1}\right| \angle 1 \;\&\; \left|-\frac{1}{3}z^{-1}\right| \angle 1$$

i.e. $|z| \rangle \dfrac{1}{2},\ |z| \rangle \dfrac{1}{-3}$

Hence, the region of convergence for this function is $|z| > \dfrac{1}{3}$ $\qquad \frac{1}{2}$??

2.3 z Transforms of Elementary Function

(i) *Unit-step function*: Let us find the z transform of the unit-step function

$(1+x)^{-1} =$

$$X(t) = \begin{cases} 1(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

The z transform is give by,

$$X(z) = \mathcal{Z}[1(t)] = \sum_{k=0}^{\infty} 1 z^{-k} = \sum_{k=0}^{\infty} z^{-k}$$

$S_n = \frac{a}{1-r}$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + \ldots\ldots\ldots\ldots$$

$$= \frac{1}{1-z^{-1}}$$

$$= \frac{z}{z-1}$$

(ii) ***Unit-Ramp Function:*** Let us find the z transform of the unit ramp function

$$x(t) = \begin{cases} t\,; & t \geq 0 \\ 0\,; & t < 0 \end{cases}$$

The z transform of unit ramp function is give by,

$$X(z) = \mathcal{Z}[t] = \sum x(kT)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} kTz^{-k} = T\sum_{k=0}^{\infty} kz^{-k}$$

$(1-z)^{-2}$

$= 1+2z+3z^2+4z^3+ \cdots$

$$= T\,(z^{-1} + 2z^{-2} + 3z^{-3} + \ldots\ldots\ldots)$$

$$= T\frac{z^{-1}}{(1-z^{-1})^2}$$

$$= T\frac{z}{(z-1)^2}$$

The figure below shows the sampled unit-ramp signal.

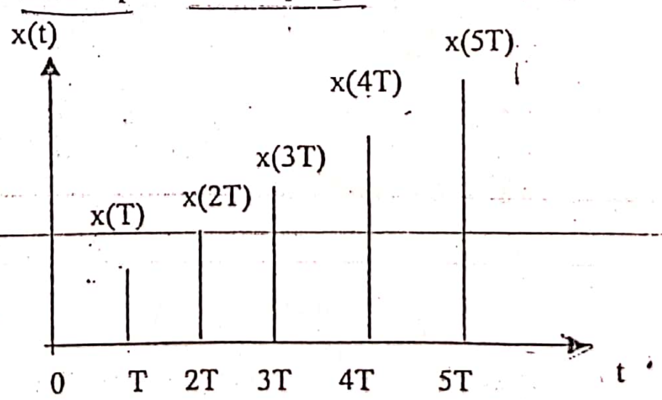

Fig 2.2 (Sample unit-ramp signal)

Polynomial Function a^k: Let us obtain the z transform of x (k) as defined by

$$x(k)=\begin{cases} a^k, & k=0,1,2,\ldots\ldots \\ 0, & k<0 \end{cases}$$

Where a is a constant.

The z transform is then given by,

$$X(z)=\mathcal{Z}[a^k]=\sum_0^\infty x(k)z^{-k}=\sum_0^\infty a^k z^{-k}$$

$$=1+az^{-1}+a^2z^{-2}+a^3z^{-3}+\ldots\ldots\ldots$$

$$=\frac{1}{1-az^{-1}}$$

$$=\frac{z}{z-a}$$

(iii) *Exponential Function*: Let us find the z-transform of

$$x(t)=\begin{cases} e^{-at}; & t\geq 0 \\ 0 ; & t<0 \end{cases}$$

Since $x(kT)=e^{-akT}$, $k=0,1,2,3,\ldots\ldots$

we have

$$X(z)=\mathcal{Z}[e^{-at}]=\sum_{k=0}^{\infty}x(kT)z^{-k}$$

$$=\sum_{k=0}^{\infty}e^{-akT}z^{-k}$$

$$=1+e^{-aT}z^{-1}+e^{-2aT}z^{-2}+e^{-3aT}z^{-3}+\ldots.$$

$$=\frac{1}{1-e^{-aT}z^{-1}}$$

$$=\frac{z}{z-e^{-aT}}$$

(iv) *Sinusoidal Function*: Consider the sinusoidal function

$$x(t)=\begin{cases}\sin\omega t & ,t\geq 0 \\ 0 & ,t<0\end{cases}$$

As we know that

$$e^{j\omega t} = \cos \omega t + \cancel{j \sin \omega t}$$

$$e^{-j\omega t} = \cos \omega t - j \sin \omega t.$$

Thus we have

$$\sin \omega t = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

Since the z transform of the exponential function is

$$\mathcal{Z}[e^{-at}] = \frac{1}{1 - e^{-aT} z^{-1}}$$

We can write

$$X(z) = \mathcal{Z}[\sin \omega T] = \mathcal{Z}\Big[\frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})\Big]$$

$$= \frac{1}{2j}\Big(\frac{1}{1 - e^{j\omega T} z^{-1}} - \frac{1}{1 - e^{-j\omega t} z^{-1}}\Big)$$

$$= \frac{1}{2j} \frac{(e^{j\omega T} - e^{-j\omega T}) z^{-1}}{1 - (e^{j\omega T} + e^{-j\omega T}) z^{-1} + z^{-2}}$$

$$= \frac{z^{-1} \sin \omega T}{1 - 2z^{-1} \cos \omega T + z^{-2}}$$

$$= \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}$$

Example 4. Obtain the z transform of the cosine function

$$x(t) = \begin{cases} \cos \omega t & , t \geq 0 \\ 0 & , t < 0 \end{cases}$$

Solution. The z transform is given by

$$X(z) = \mathcal{Z}[\cos\omega t] = \frac{1}{2}\mathcal{Z}[e^{j\omega t} + e^{-j\omega t}]$$

$$= \frac{1}{2}\left(\frac{1}{1-e^{j\omega T}z^{-1}} + \frac{1}{1-e^{-j\omega T}z^{-1}}\right)$$

$$= \frac{1}{2}\left(\frac{2-(e^{-j\omega T}+e^{j\omega T})z^{-1}}{1-(e^{j\omega T}+e^{-j\omega T})z^{-1}+z^{-2}}\right)$$

$$= \frac{1-z^{-1}\cos\omega T}{1-2z^{-1}\cos\omega T+z^{-2}}$$

$$= \frac{z^2 - z\cos\omega T}{z^2 - 2z\cos\omega T + 1}$$

Example 5. Obtain the z transform of

$$X(s) = \frac{1}{s(s+1)}$$

Solution. Whenever a given function is in s-domain then there are mainly two app determine the z transform. The first approach is to convert x (s) into x (t) and then transform of x (t). And the second approach is to expand x (s) into partial fractions z transform table to find the z transform of the expanded terms. There are approaches, will be explained later.

$\frac{1}{s} - \frac{1}{s+1}$

The inverse Laplace transform of x (s) is $x(t) = 1 - e^{-t}$

$$x(t) = 1 - e^{-t}, \quad t \geq 0.$$

Hence, $Z[x(t)] = \sum_{k=0}^{\infty} (1 - e^{-kt}) z^{-k}$

$$X(z) = \mathcal{Z}[1-e^{-t}] = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-T}z^{-1}}$$

$= \sum_{k=0}^{\infty} z^{-k} - \sum_{k=0}^{\infty} e^{-kt}$

$$= \frac{(1-e^{-T})z^{-1}}{(1-z^{-1})(1-e^{-T}z^{-1})}$$

$= 1 + z^{-1} + z^{-2} + \cdots - ($

$$= \frac{(1-e^{-T})z}{(z-1)(z-e^{-T})}$$

$= \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-T}z^{-1}}$

$= \frac{\cancel{1} - e^{-T}z^{-1} - \cancel{1} + z^{-1}}{(1-z^{-1})(1-e^{-T}z^{-1})}$

$= \frac{z^{+1}(1-e^{-T})}{z-1(z-e^{-T})}$

Example 6. Find the z transform of function x (t) = t^2.

Solution. The z transform of x (t) is given by

$$X(z) = \mathcal{Z}[x(t)] = \sum_{k=0}^{\infty} T^2k^2z^{-k}$$

$$= T^2\sum_{k=0}^{\infty} k^2z^{-k}$$

$$= T^2\,[z^{-1} + 4z^{-2} + 9z^{-3} + 16z^{-4} + ...]$$

$$= T^2z^{-1}\,[1 + 4z^{-1} + 9z^{-2} + 16z^{-3} + ...]$$

$$= \frac{T^2z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$$

$$= \frac{T^2z(z+1)}{(z-1)^3}$$

$\frac{n!}{s^{n+1}}$ = 6n

$1^2 + 2^2z^{-1} + 3^2z^{-2}$

Table of z transforms:

	X (s)	x (t)	X (kT) or x (k)	X (z)
1.	-	-	Kronecker delta δ_0 (k) 1, k = 0 0, k ≠ 0	1
2.	-	-	δ_0 (n – k) 1, n = k 0, n ≠ k	z^{-k}
3.	$\frac{1}{s}$	1(t)	1(k)	$\frac{1}{1-z^{-1}}$ $\frac{z}{z-1}$
4.	$\frac{1}{s+a}$	e^{-at}	e^{-akT}	$\frac{1}{1-e^{-aT}z^{-1}}$ $\frac{z}{z-e^{-aT}}$
5.	$\frac{1}{s^2}$	t	KT	$\frac{Tz^{-1}}{(1-z^{-1})^2}$ $\frac{Tz}{(z-1)^2}$

$X(s)$ $x(t)$

6.	$\frac{2}{s^3}$	t^2	$(kT)^2$	$\frac{T^2 z^{-1}(1+z}{(1-z^{-1})}$
7.	$\frac{6}{s^4}$	t^3	$(kT)^3$	$\frac{T^3 z^{-1}(1+4z^{-1}}{(1-z^{-1})}$
8.	$\frac{a}{s(s+a)}$	$1-e^{-at}$	$1-e^{-akt}$	$\frac{(1-e^{-aT})z}{(1-z^{-1})(1-e}$
9.	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$e^{-akt}-e^{-bkt}$	$\frac{(e^{-aT}-e^{-bT})z}{(1-e^{-aT}z^{-1})(1-e}$
10.	$\frac{1}{(s+a)^2}$	te^{-at}	KTe^{-akT}	$\frac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1}}$
11.	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	$(1-akT)\,e^{-akt}$	$\frac{1-(1+aT)e}{(1-e^{-aT}z^{-1}}$
12	$\frac{2}{(s+a)^3}$	$t^2 e^{-at}$	$(kT)^2 e^{-akT}$	$\frac{T^2 e^{-aT}(1+e^{-aT}z^{-1})z}{(1-e^{-aT}z^{-1})^3}$
13.	$\frac{a^2}{s^2(s+a)}$	$at-1+e^{-at}$	$AkT-1+e^{-akT}$	$\frac{[(aT-1+e^{-aT})+(1-e^{-aT}-aT}{(1-z^{-1})^2(1-e^{-aT}z}$
14.	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\frac{z^{-1}\sin\omega T}{1-2z^{-1}\cos\omega T+z}$
15.	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\frac{1-z^{-1}\cos\omega T}{1-2z^{-1}\cos\omega T+z}$
16.	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin \omega t$	$e^{-akt}\sin \omega kT$	$\frac{e^{-aT}z^{-1}\sin\omega T}{1-2e^{-aT}z^{-1}\cos\omega T+e}$
17.	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$e^{-akt}\cos \omega kT$	$\frac{1-e^{-aT}z^{-1}\cos\omega T}{1-2e^{-aT}z^{-1}\cos\omega T+e}$
18.			a^k	$\left(\frac{z}{z-a}\right)$ $\frac{1}{1-az^{-1}}$
19.			a^{k-1} $k=1,2,3,\ldots$	$\left(\frac{1}{z-a}\right)$ $\frac{z^{-1}}{1-az^{-1}}$

$a^k u(k) \rightarrow \frac{1}{1-az^{-1}}$

$a^{k-1} u(k-1) \rightarrow z^{-1}$

20.			ka^{k-1}	$\frac{z^{-1}}{(1-az^{-1})^2}$ $\frac{z}{(z-a)^2}$
21.			$k^2 a^{k-1}$	$\frac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$
22.			$k^3 a^{k-1}$	$\frac{z^{-1}(1+4az^{-1}+a^2z^{-2})}{(1-az^{-1})^4}$
23.			$k^4 a^{k-1}$	$\frac{z^{-1}(1+11az^{-1}+11a^2z^{-2}+a^3z^{-3})}{(1-az^{-1})^5}$
24.			$a^k \cos k\pi$	$\frac{1}{1+az^{-1}}$
25.			$\frac{k(k-1)}{2!}$	$\frac{z^{-2}}{(1-z^{-1})^3}$
26.			$\frac{k(k-1)\ldots(k-m+2)}{(m-1)!}$	$\frac{z^{-m+1}}{(1-z^{-1})^m}$
27.			$\frac{k(k-1)}{2!}a^{k-2}$	$\frac{z^{-2}}{(1-az^{-1})^3}$
28.			$\frac{k(k-1)\ldots(k-m+2)}{(m-1)!}a^{k-m+1}$	$\frac{z^{-m+1}}{(1-az^{-1})^m}$

$x(t) = 0$, for $t<0$.

$x(kT) = x(k) = 0$, for $k<0$.

Unless otherwise noted, $k = 0,1,2,3,\ldots\ldots$

2.4 Important Properties and Theorems of the z Transform

(i) ***Multiplication by a constant***: If X (z) is the z transform of x (t), then

$$\mathcal{Z}[ax(t)] = a\,\mathcal{Z}[x(t)] = aX(z).$$

Where a is a constant.

Proof: By definition of z transform, we have

$$\mathcal{Z}[ax(t)] = \sum_{k=0}^{\infty} ax(kT)z^{-k} = a\sum_{k=0}^{\infty} x(kT)z^{-k}$$
$$= aX(z)$$

$$Z\left[\frac{x(k)}{k}\right] = -\int_0^z z^{-1}X(z)\,dz$$

$$Z[t^n] = -Tz\frac{d}{dz}\left[Z(t^{n-1})\right]$$

$$Z[k\,x(k)] = -z\frac{d}{dz}X(z),$$

(ii) ***Linearity of the z transform***: The z transform possesses an important linearity. If f (k) and g (k) are z transformable and α and β are scalars, then x (k) for linear combination $x(k) = \alpha f(k) + \beta g(k)$ has the z transform

$$X(z) = \alpha F(z) + \beta G(z).$$

Where F (z) and G (z) are the z transforms of f (k) and g (k), respectively.

Proof: The z transform of function x (k) is given by

$$\begin{aligned} X(z) &= \mathcal{Z}[x(k)] = \mathcal{Z}[\alpha f(k) + \beta g(k)] \\ &= \sum_{k=0}^{\infty}[\alpha f(k) + \beta g(k)]z^{-k} \\ &= \alpha \mathcal{Z}[f(k)] + \beta \mathcal{Z}[g(k)] \\ &= \alpha F(z) + \beta G(z) \end{aligned}$$

(iii) ***Multiplication by a^k***: If X (z) is the z transform of x (k), then the z tra… $a^k x(k)$ can be given by $X(a^{-1} z)$:

$$\mathcal{Z}[a^k x(k)] = X(a^{-1} z).$$

Proof: As we know that

$$\begin{aligned} \mathcal{Z}[a^k x(k)] &= \sum_{k=0}^{\infty} a^k x(k) z^{-k} \\ &= \sum_{k=0}^{\infty} x(k)(a^{-1}z)^{-k} \\ &= X(a^{-1}z) \end{aligned}$$

(iv) ***Shifting Theorem***: This theorem is also referred as the real translation t… $x(t) = 0$ for $t < 0$ and x (t) has the z transform X (z), then

$$\mathcal{Z}[x(t - nT)] = z^{-n} X(z) \qquad \text{..........(2.5)}$$

and $$\mathcal{Z}[x(t + nT)] = z^{n}[X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k}] \qquad \text{..........(2.6)}$$

where n is zero or positive integer.

Proof: As we know that

$$\mathcal{Z}[x(t-nT)] = \sum_{k=0}^{\infty} x(kT-nT)z^{-k}$$

$$= z^{-n}\sum_{k=0}^{\infty} x(kT-nT)z^{-(k-n)}$$

Let $\quad m = k - n$

$\quad \therefore k = m + n$

when $\quad k = 0,\ m = -n$

$\quad k = \infty,\ m = \infty$

So, $\quad \mathcal{Z}[x(t-nT)] = z^{-n}\sum_{m=-n}^{\infty} x(mT)z^{-m}$

Since, x (mT) = 0 for m < 0, so we can change the lower limit of m = -n to m = 0

Hence,

$$\mathcal{Z}[x(t-nT)] = z^{-n}\sum_{m=0}^{\infty} x(mT)z^{-m}$$

$$= z^{n}X(z)$$

Now, to prove Eq. (2.6).

As we know that,

$$\mathcal{Z}[x(t+nT)] = \sum_{k=0}^{\infty} x(kT+nT)z^{-k}$$

$$= z^{n}\left[\sum_{k=0}^{\infty} x(kT+nT)z^{-(k+n)}\right]$$

$$= z^{n}\left[\sum_{m=n}^{\infty} x(mT)z^{-m}\right]$$

Let
m = k+n
⇒ k = m−n

where k + n = m (suppose)

so that when k = 0, m = n and

when k = ∞, m = ∞.

$= z^{n}\left(\sum_{m=0}^{\infty} x(mT)z^{-m} - \sum_{k=0}^{n-1} x(kT)z^{-k}\right)$

$= z^{n}\left(X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k}\right)$

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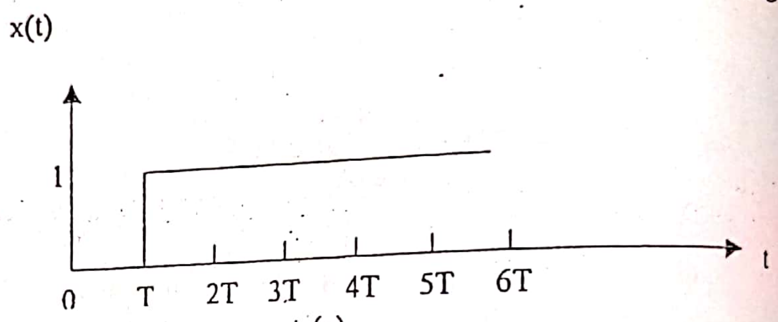
$$\mathcal{Z}[x(t+nT)] = z^{n}\left[\sum_{m=0}^{\infty} x(mT)z^{-m} - \sum_{m=0}^{n-1} x(mT)z^{-m}\right]$$

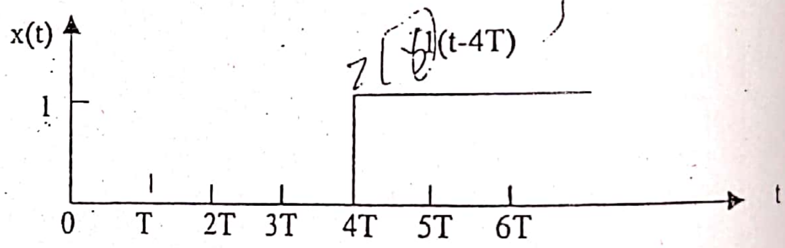
$$= z^{n}\left[X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k}\right] \qquad \text{(proved)}$$

Example 7. Find the z transforms of unit-step functions that are delayed by 1 period and 4 sampling periods, respectively, as shown in Fig.2.3 (a) a



(a)

(Unit-step function delayed by 1 sampling period)



(b)

(Unit-step function delayed by 4 sampling period)

Fig 2.3

Solution. Using the shifting theorem we have

$$\mathcal{Z}[1(t-T)] = z^{-1}\mathcal{Z}[1(t)] = z^{-1}\cdot\frac{1}{1-z^{-1}} = \frac{z^{-1}}{1-z^{-1}}$$

Also,

$$\mathcal{Z}[1(t-4T)] = z^{-4}\mathcal{Z}[1(t)] = z^{-4}\frac{1}{1-z^{-1}} = \frac{z^{-4}}{1-z^{-1}}$$

Example 8. Obtain the z transform of

$$f(k) = a^{k-1}, \quad k = 1,2,3,........$$
$$= 0, \quad k \leq 0$$

Solution. The z transform of a^k is

$$\mathcal{Z}\left[a^k\right] = \frac{1}{1-az^{-1}}$$

$$\mathcal{Z}\left[a^{k-1}\right] = \sum_{k=1}^{\infty} a^{k-1} z^{-k}$$
$$= z^{-1} + a z^{-2} + a^2 z^{-3} + a^3 z^{-4} + \cdots$$
$$= z^{-1}\left(1 + a z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots\right)$$
$$= z^{-1} \cdot \frac{1}{1-az^{-1}} = \frac{z^{-1}}{1-az^{-1}}$$

and so

$$\mathcal{Z}\left[f(k)\right] = \mathcal{Z}\left[a^{k-1}\right] = z^{-1}\frac{1}{1-az^{-1}}$$ [By Shifting theorem]

$$= \frac{z^{-1}}{1-az^{-1}}$$ Where k = 1, 2, 3, ...

Example 9. Consider the function y (k), which is a sum of functions x (h), where h = 0,1,2,.....k, such that

$$y(k) = \sum_{h=0}^{k} x(h), \qquad k = 0, 1, 2, ...$$

where y (k) = 0 for k < 0. Obtain the z transform of y (k).

Solution. Given that,

$$y(k) = \sum_{h=0}^{k} x(h)$$
$$= x(0) + x(1) + ... + x(k-1) + x(k)$$

and also, $y(k-1) = x(0) + x(1) + ... + x(k-1)$

Hence, y (k) - y (k - 1) = x (k), k = 0,1,2,.....

$$\mathcal{Z}\left[y(k) - y(k-1)\right] = \mathcal{Z}\left[x(k)\right]$$

or

$$Y(z) - z^{-1}Y(z) = X(z)$$

$$\therefore Y(z) = \frac{1}{(1-z^{-1})}X(z)$$

(v) *Complex Translation Theorem*: If x (t) has the z transform X (z), then the z tr[...] e^{-at} x (t) can be given by X ($z e^{aT}$). This is known as the complex translation theorem.

Proof: As we know that

$$\mathcal{Z}\left[e^{-at}x(t)\right] = \sum_{k=0}^{\infty} x(kT)e^{-akT}z^{-k}$$

$$= \sum_{k=0}^{\infty} x(kT)(ze^{aT})^{-k}$$

$$= X(ze^{aT})$$

Thus, we can conclude that to find the z transform of e^{-at} x (t), we can just replace z by $z e^{aT}$.

Example 10. Obtain the z transforms of e^{-at} sin ωt and e^{-at} cosωt, respectively, b[y] complex translation theorem.

Solution. As we know that

$$\mathcal{Z}[\sin\omega t] = \frac{z^{-1}\sin\omega T}{1 - 2z^{-1}\cos\omega T + z^{-2}}$$

Let us now substitute $z e^{aT}$ for z to obtain the z transform of e^{-at} sin ωt, as follows

$$\mathcal{Z}\left[e^{-at}\sin\omega t\right] = \frac{e^{-aT}z^{-1}\sin\omega T}{1 - 2e^{-aT}z^{-1}\cos\omega T + e^{-2aT}z^{-2}}$$

Similarly,

$$\mathcal{Z}[\cos\omega t] = \frac{1 - z^{-1}\cos\omega T}{1 - 2z^{-1}\cos\omega T + z^{-2}}$$

Let us now substitute $z e^{aT}$ for z to obtain the z transform of e^{-at} cos ωt, as follows

$$\mathcal{Z}\left[e^{-at}\cos\omega t\right] = \frac{1 - e^{-aT}z^{-1}\cos\omega T}{1 - 2e^{-aT}z^{-1}\cos\omega T + e^{-2aT}z^{-2}}$$

Example 11. Obtain the z transforms of te^{-at}.

Solution. As we know that

$$\mathcal{Z}[t] = \frac{Tz^{-1}}{(1-z^{-1})^2} = X(z)$$

Thus, $$\mathcal{Z}[te^{-at}] = X(ze^{aT}) = \frac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$$

(vi) ***Initial Value Theorem:*** If x (t) has the z transform X (z) and if $\lim_{z\to\infty} X(z)$ exists, then the initial value x (0) of x (t) or x (k) is given by

$$x(0) = \lim_{z\to\infty} X(z)$$

Proof: As we know that

$$X(z) = \sum_{k=0}^{\infty} x(k)z^{-k}$$
$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots\ldots$$

Letting $z\to\infty$, we get

$$\lim_{z\to\infty} X(z) = x(0)$$

Hence, the behavior of the signal in the neighbourhood of t = 0 or k = 0 can thus be determined by the behavior of X(z) at z = ∞. This theorem is useful for checking z transform calculations for possible errors. Since x (0) is always known, a check for the initial value by $\lim_{z\to\infty} X(z)$ can easily spot errors in X (z), if any exist

Example 12. Determine the initial value x (0) if the z transform of x (t) is given b

$$X(z) = \frac{(1-e^{-T})z^{-1}}{(1-z^{-1})(1-e^{-T}z^{-1})}$$

Solution. By using the initial value theorem, we find

$$x(0) = \lim_{z\to\infty} \frac{(1-e^{-T})z^{-1}}{(1-z^{-1})(1-e^{-T}z^{-1})} = 0$$

Note: The given X (z) was the z transform of x (t) = 1 - e^{-t}. So, when t = 0, x agrees with the result obtained by initial value theorem.

ख(v) ***Final Value Theorem***: Suppose that x (k), where x (k) = 0 for k < transform X (z) and that all the poles of X (z) lie inside the unit circle, with exception of a simple pole at z = 1 (condition for stability). Then, the final value is, the value of x (k) as k approaches infinity, can be given by

$$\boxed{\lim_{k\to\infty} x\,(k) = x\,(\infty) = \lim_{z\to 1} [(1 - z^{-1})\, X\,(z)]}$$

$x(\infty) = \lim_{Z\to 1} (Z-1)$

Proof: As we know that

$$\mathcal{Z}\,[x(k)] = X(z) = \sum_{k=0}^{\infty} x(k)z^{-k} \qquad \text{.........(2.7)}$$

and

$$\mathcal{Z}\,[x(k-1)] = z^{-1}X(z) = \sum_{k=0}^{\infty} x(k-1)z^{-k} \qquad \text{.........(2.8)}$$

Hence, subtracting Eq. (2.8) from (2.7), we get

$$\sum_{k=0}^{\infty} x(k)z^{-k} - \sum_{k=0}^{\infty} x(k-1)z^{-k} = X(z) - z^{-1}X(z) \qquad \text{.........(2.9)}$$

Now, taking the limit as z → 1 to the both sides, we get

$$\lim_{z \to 1}\left[\sum_{k=0}^{\infty} x(k)z^{-k} - \sum_{k=0}^{\infty} x(k-1)z^{-k}\right] = \lim_{z \to 1}\left[(1-z^{-1})X(z)\right] \quad \text{.......(2.10)}$$

The left-hand side of Eq. (2.10) becomes,

$$[x(0) - x(-1)] + [x(1) - x(0)] + [x(2) - x(1)] + \ldots\ldots = x(\infty) = \lim_{k\to\infty} x(k)$$

Hence,

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1}\,[(1 - z^{-1})\,X(z)] \quad \text{..... (2.11)}$$

This Eq (2.11) proves the final value theorem. The final value theorem is useful in determining the behavior of x (k) as $k\to\infty$ from its z transform X (z).

Example 13. Obtain the final value $x(\infty)$ of

$$X(z) = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}}, \quad a > 0$$

by using the final value theorem.

Solution. By using the final value theorem.

$$x(\infty) = \lim_{z\to 1}\left[(1-z^{-1})X(z)\right]$$

$$= \lim_{z\to 1}\left[(1-z^{-1})\left(\frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}}\right)\right] = \lim_{z\to 1} \frac{(1-z^{-1})}{(1-z^{-1})} - \frac{(1-z^{-1})}{1-e^{-aT}z^{-1}}$$

$$= 1 \qquad = 1-0$$

$$= 1$$

Note: The given X (z) was the z transform of $x(t) = 1 - e^{-at}$. So, $x(\infty) = 1$. Which agrees with the result obtained by final value theorem.

2.5 The Inverse z Transform

The notation for the inverse z transform is $\mathcal{Z}^{-1}$. The inverse z transform of X (z) y
corresponding time sequence x (k).

When X (z), the z transform of x (kT) or x (k) is given, the operation that determ
corresponding x (kT) or x (k) is known as inverse z transformation and is denoted by

$$x(k) = \mathcal{Z}^{-1}[X(z)]$$

It should be noted that the inverse z-transform yields the sampled function x (k)
continuous-time function x (t).

Here, four methods, by which we can determine the inverse z transform, are explaine

(i) Direct Division Method: The defining equation of the z transform i.e. $X(z) = \sum_{k=0}^{\infty} x$
suggests that the inverse z transform can be determined simply by expanding the
into an infinite series in powers of z^{-1}. This method is useful when it is desired to f
the first several terms of x (kT) or x (k).

As we know that

$$X(z) = \sum_{k=0}^{\infty} x(kT)z^{-k}$$

$$= x(0) + x(T)z^{-1} + x(2T)z^{-2} + \ldots\ldots + x(kT)z^{-k} + \ldots\ldots$$

or,

$$X(z) = \sum_{k=0}^{\infty} x(k)z^{-k}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots\ldots\ldots + x(k)z^{-k} + \ldots\ldots$$

Then x (kT) or x (k) is the coefficients of z^{-k} terms. Hence, the values of x (kT) or x
$k = 0, 1, 2, \ldots\ldots$ can be determined by inspection.

Example 14. Find x (k) for k = 0,1,2, ... when X (z) is given by

$$X(z) = \frac{z}{\left(z+\frac{1}{2}\right)(z-1)}$$

Solution. The function X(z) can be written as a ratio of polynomials in z^{-1} as follows,

$$X(z) = \frac{z}{z^2 - 0.5z - 0.5}$$

$$= \frac{z^{-1}}{1 - 0.5z^{-1} - 0.5z^{-2}}$$

And divided as follows:

$$\begin{array}{r|l}
 & z^{-1} + 0.5z^{-2} + 0.75z^{-3} \\
\hline
1 - 0.5z^{-1} - 0.5z^{-2} & z^{-1} \\
 & \underline{z^{-1} - 0.5z^{-2} - 0.5z^{-3}} \\
 & 0.5z^{-2} + 0.5z^{-3} \\
 & \underline{0.5z^{-2} - 0.25z^{-3} - 0.25z^{-4}} \\
 & 0.75z^{-3} + 0.25z^{-4} \\
 & \underline{0.75z^{-3} - \cdots\cdots}
\end{array}$$

Thus, $X(z) = z^{-1} + 0.5\,z^{-2} + 0.75\,z^{-3} + \ldots\ldots$

By comparing this infinite power series expansion of X (z) with

$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$, we

$x(0) = 0$

$x(1) = 1$

$x(2) = 0.5$

$x(3) = 0.75$

$\vdots$

Example 15. Find x(k), when X(z) is given by

$$X(z) = \frac{1}{z+1}$$

Solution. X(z) can be rewritten in the power series in z^{-1} as follows:

$$X(z) = \frac{z^{-1}}{1+z^{-1}} = z^{-1} - z^{-2} + z^{-3} - z^{-4} + \ldots\ldots$$

By comparing the infinite series expansion of X(z) with

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}, \text{ we obtain}$$

$$x(0) = 0$$
$$x(1) = +1$$
$$x(2) = -1$$
$$x(3) = +1$$
$$x(4) = -1$$
$$\vdots$$

This is an alternating signal of 1 and -1, which starts from k = 1. The Fig.2.4 belo[w] plot of this signal.

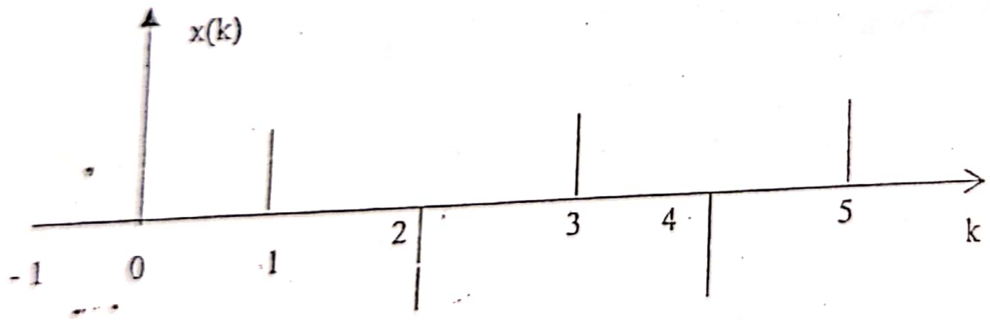


Fig 2.4

(Alternating signal of 1 and –1 starting from k=1.)

Example 16. Obtain the inverse z transform of $X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$.

Solution. Since X (z) has a finite number of terms, it corresponds to a signal of fini[te]

By inspection, we get

$x(0) = 1$

$x(1) = 2$

$x(2) = 3$

$x(3) = 4$ and all other x (k) values are zero.

(iii) *Partial-Fraction- Expansion Method:*

(a) For simple poles:

Let us consider X (z) as given by

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \dots\dots + b_{m-1} z + b_m}{z^n + a_1 z^{n-1} + \dots\dots + a_{n-1} z + a_n}, \qquad m \le n$$

Let us now-factorize the denominator of X(z) and find the poles of X(z):

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \dots\dots + b_{m-1} z + b_m}{(z-p_1)(z-p_2)\dots\dots(z-p_n)}$$

We then expand $\frac{X(z)}{z}$ into partial fractions so that each term is easily recognizable in a table of z transform.

$$\frac{X(z)}{z} = \frac{a_1}{z-p_1} + \frac{a_2}{z-p_2} + \dots\dots\dots + \frac{a_n}{z-p_n}$$

where

$$a_i = \left[(z-p_i)\frac{X(z)}{z}\right]_{z=p_i}$$

(b) *For multiple poles:* If $\frac{X(z)}{z}$ involves a multiple pole, for example, a double pole at $z = p_1$ and no other poles, then $\frac{X(z)}{z}$ will have the form

$$\frac{X(z)}{z} = \frac{c_1}{(z-p_1)^2} + \frac{c_2}{z-p_1}$$

Now, the coefficients c_1 and c_2 are determined from

$$c_1 = \left[(z-p_1)^2 \frac{X(z)}{z}\right]_{z=p_1}$$

$$c_2 = \left\{\frac{d}{dz}\left[(z-p_1)^2 \frac{X(z)}{z}\right]\right\}_{z=p_1}$$

Example 17. Obtain the inverse z transform of

$$X(z) = \frac{2z^3 + z}{(z-2)^2(z-1)}$$

by use of the partial-fraction expansion method.

Solution. Let us expand X(z) into partial fractions as follows:

$$\frac{X(z)}{z} = \frac{2z^2+1}{(z-2)^2(z-1)} = \frac{k_1}{(z-2)^2} + \frac{k_2}{(z-2)} + \frac{k_3}{(z-1)}$$

$$\text{Where} \quad k_1 = \left[\frac{2z^2+1}{(z-2)^2(z-1)}(z-2)^2\right]_{z=2}$$

$$= \left.\frac{2z^2+1}{z-1}\right|_{z=2}$$

$$= 8+1$$

$$= 9$$

$$k_2 = \left\{\frac{d}{dz}\left[\frac{2z^2+1}{(z-1)}\right]\right\}_{z=2}$$

$$= \left.\frac{(z-1)(4z)-(2z^2+1)}{(z-1)^2}\right|_{z=2}$$

$$= \frac{8-9}{1}$$

$$= -1$$

$$k_3 = \frac{2z^2+1}{(z-2)^2}\Big|_{z=1}$$

$$= 3$$

$$\frac{X(z)}{z} = \frac{9}{(z-2)^2} - \frac{1}{z-2} + \frac{3}{z-1}$$

$$\text{or, } X(z) = \frac{9z}{(z-2)^2} - \frac{z}{z-2} + \frac{3z}{z-1}$$

$x(k) = \frac{9}{2} k 2^k - 2^k + 3$

From the table of z transform, we can write

$$x(kT) = 9k(2^{k-1}) - 2^k + 3 \Rightarrow$$

$9k\, 2^{k-1} u(k-1) - 2^k u(k) + 3$

$9k\, 2^{k-1} u(k) - 2^k u(k) + 3$

$9k\, 2^{k-1} - 2^k + 3$, $k = 0,1,2,3,\ldots$

where $k = 0,1,2,3,\ldots\ldots$

imp **Example18.** Obtain the inverse z transform of $X(z) = \dfrac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$, where a is a constant and T is the sampling period, by use of partial fraction expansion method.

Solution. Let us expand $\dfrac{X(z)}{z}$ into partial fractions as follows.

$$X(z) = \frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})} = \frac{k_1}{z-1} + \frac{k_2}{z-e^{-aT}}$$

Where,

$$k_1 = \frac{1-e^{-aT}}{z-e^{-aT}}\Big|_{z=1}$$

$$= 1$$

and,

$$k_2 = \frac{1-e^{-aT}}{z-1}\Big|_{z=e^{-aT}}$$

$$= -1$$

$$\frac{X(z)}{z} = \frac{1}{z-1} - \frac{1}{z-e^{-aT}}$$

By use of z transform Table, we can write

$$x(kT) = 1 - e^{-akT} \quad \text{where } k = 0,1,2,\ldots\ldots$$

(iii) *Inversion Integral Method*: As we know that

$$X(z) = \sum_{k=0}^{\infty} x(kT)z^{-k} \quad \ldots\ldots($$

$$= x(0) + x(T)z^{-1} + x(2T)z^{-2} + \ldots.. + x(kT)z^{-k} \quad \ldots\ldots$$

By multiplying both sides by z^{k-1} we get

$$X(z)\,z^{k-1} = x(0)\,z^{k-1} + x(T)\,z^{k-2} + x(2T)\,z^{k-3} + \ldots.. + x(kT)\,z^{-1} + \ldots \quad \ldots..$$

Let us integrate both sides along with the circle in counterclockwise direction,

$$\oint_c X(z)z^{k-1}\,dz = \oint_c x(0)z^{k-1}\,dz + \ldots + \oint_c x(kT)z^{-1}\,dz + \ldots. \quad \ldots\ldots\ldots$$

where c is a circle with its center at the origin of the z-plane such that all poles are inside it.

Now, applying Cauchy's theorem, all the terms on the right-hand side of Eq. (2. zero except

$$\oint x(kT)z^{-k}dz$$

$$\oint x(z)z^{k-1}dz = \oint x(kT)z^{-k}dz \quad \ldots\ldots\ldots$$

From which we obtain

$$x(kT) = \frac{1}{2\pi j}\oint X(z)z^{k-1}dz \quad \ldots\ldots($$

The coefficient x (kT) associated with the term z^{-1} is the residue.

The Eq. (2.17) is the inversion integral for the z transform, which is equivalent to sta

$x(kT) = \Sigma$ [residues of $X(z)\, z^{k-1}$ at the poles of $X(z)$].

Let us now evaluate residues:

(a) *For a simple pole*: If the denominator of $X(z)z^{k-1}$ contains a simple pole $z = z_i$ then the corresponding residue k is

$$\boxed{k = \lim_{z \to z_i} [(z - z_i)\, X(z)\, z^{k-1}]}$$

(b) *For a multiple pole:* If $X(z)z^{k-1}$ contains a multiple pole z_j of order q , then the residue k is given by

$$\boxed{k = \frac{1}{(q-1)!} \lim_{z \to z_j} \frac{d^{q-1}}{dz^{q-1}} \left[(z - z_j)^q X(z) z^{k-1}\right]}$$

Example 19. Obtain the inverse z transform of $X(z) = \dfrac{z^2}{(z-1)^2 (z - e^{-aT})}$ by using inversion integral method.

Solution. $x(k) = \Sigma$ residues of $X(z)z^{k-1}$ at the poles of $X(z)$.

Now, $X(z)z^{k-1} = \dfrac{z^{k+1}}{(z-1)^2 (z - e^{-aT})}$

Here, $X(z)\, z^{k-1}$ has a simple pole at $z = e^{-aT}$ and a double pole at $z = 1$.

$\therefore x(kT) = k_1 + k_2$

Now, the residue k_1 can be found as,

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$$k_1 = \left[\text{residue of } \frac{z^{k+1}}{(z-1)^2(z-e^{-aT})} \text{ at pole } z = e^{-aT}\right]$$

$$= \lim_{z \to e^{-aT}} \left[\frac{z^{k+1}}{(z-1)^2}\right]$$

$$= \frac{e^{-a(k+1)T}}{(1-e^{-aT})^2}$$

$$k_2 = \text{residue of } \frac{z^{k+1}}{(z-1)^2(z-e^{-aT})} \text{ at double pole } z = 1$$

$$= \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d}{dz}\left[\frac{z^{k+1}}{(z-e^{-aT})}\right]$$

$$= \lim_{z \to 1}\left[\frac{(z-e^{-aT})(k+1)z^k - z^{k+1}}{(z-e^{-aT})^2}\right]$$

$$= \frac{k}{1-e^{-aT}} - \frac{e^{-aT}}{(1-e^{-aT})^2}$$

Hence,

$$x(kT) = k_1 + k_2 = \frac{e^{-aT}e^{-akT}}{(1-e^{-aT})^2} + \frac{k}{1-e^{-aT}} - \frac{e^{-aT}}{(1-e^{-aT})^2}$$

$$= \frac{kT}{T(1-e^{-aT})} - \frac{e^{-aT}(1-e^{-akT})}{(1-e^{-aT})^2}, \quad k = 0, 1, 2, \ldots$$

Example 20. Obtain the inverse z transform of $X(z) = \dfrac{10z}{(z-1)(z-2)}$ by using integral method.

Solution. Since $X(z)\, z^{k-1}$ has two poles at $z = 1$ and $z = 2$. So, there are two residu k_2.

Now, to evaluate the residue k_1,

$$k_1 = \lim_{z \to 1}\left[\frac{10z(z-1)z^{k-1}}{(z-1)(z-2)}\right]$$

$$= \lim_{z \to 1}\left[\frac{10z^k}{(z-2)}\right]$$

$$= -10$$

Similarly,

$$k_2 = \lim_{z \to 2}\left[\frac{10z(z-2)z^{k-1}}{(z-1)(z-2)}\right]$$

$$= \lim_{z \to 2}\left[\frac{10z^k}{(z-1)}\right]$$

$$= 102^k$$

Hence, $x(k) = -10 + 10(2^k)$

$$= 10(-1 + 2^k), \qquad k = 0,1,2, \ldots\ldots.$$

Example 21. Obtain x (kT) by using the inversion integral method when X(z) is given by

$$X(z) = \frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$$

Solution. $x(kT) = \Sigma$[Residues of $X(z)\, z^{k-1}$ at the poles of X(z)].

Here the poles are at $z = 1$, and $z = e^{-aT}$

$\therefore\ x(kT) = k_1 + k_2$

Now, let us evaluate k_1,

$$k_1 = \lim_{z \to 1}\left[\frac{z^k(1-e^{-aT})}{z-e^{-aT}}\right]$$

$$= 1$$

and,

$$k_2 = \lim_{z \to e^{-aT}}\left[\frac{z^k(1-e^{-aT})}{z-1}\right]$$

$$= -e^{-akT}$$

$$\therefore\ x(kT) = 1 - e^{-akT}, \qquad k = 0,1,2, \ldots\ldots$$

Table of Important Properties and Theorems of the z-Transform:

	$x(t)$ or $x(k)$	$Z[x(t)$ or $Z[x(k)]$
1.	$ax(t)$	$AX(z)$.
2.	$ax_1(t)+bx_2(t)$	$aX_1(z)+bX_2(z)$
3.	$x(t+T)$ or $x(k+1)$	$zX(z)-zx(0)$
4.	$x(t+2T)$	$z^2X(z)-z^2x(0)-zx(T)$
5.	$x(k+2)$	$z^2X(z)-z^2x(0)-zx(1)$
6.	$x(t+kT)$	$z^kX(z)-z^kx(0)-z^{k-1}x(T)-\dots-zx(kT-T)$
7.	$x(t-kT)$	$z^{-k}X(z)$
8.	$x(n+k)$	$z^kX(z)-z^kx(0)-z^{k-1}x(1)-\dots\dots-zx(k-1$
9.	$x(n-k)$	$z^{-k}X(z)$
10.	$tx(t)$	$-Tz\frac{d}{dz}X(z)$
11.	$kx(k)$	$-z\frac{d}{dz}X(z)$
12.	$e^{-at}x(t)$	$X(ze^{aT})$
13.	$e^{-ak}x(k)$	$X(ze^{a})$
14.	$a^kx(k)$	$X\left(\frac{z}{a}\right)$
15.	$ka^kx(k)$	$-z\frac{d}{dz}X\left(\frac{z}{a}\right)$
16.	$x(0)$	$\lim_{z\to\infty} X(z)$ if the limit exists
17.	$x(\infty)$	$\lim_{z\to 1} [(1-z^{-1})X(z)]$ if $(1-z^{-1})X(z)$ is analytic and outside the unit circle
18.	$\nabla x(k)=x(k)-x(k-1)$	$(1-z^{-1})X(z)$
19.	$\Delta x(k)=x(k+1)-x(k)$	$(z-1)X(z)-zx(0)$
20.	$\sum_{k=0}^{n} x(k)$	$\frac{1}{1-z^{-1}}X(z)$
21.	$\frac{\partial}{\partial a}x(t,a)$	$\frac{\partial}{\partial a}x(z,a)$

22.	$k^m x(k)$	$\left(-z\frac{d}{dz}\right)^m X(z)$
23.	$\sum_{k=0}^{n} x(kT)y(nT-kT)$	$X(z)Y(z)$
24.	$\sum_{k=0}^{\infty} x(k)$	$X(1)$

2.6 z-Transform Method for Solving Difference Equations

Review: we have already derived

$$\mathcal{Z}\, x(k-n) = z^{-n} X(z) \text{ and}$$

$$\mathcal{Z}\, x(k+n) = z^{n}\left[X(z) - \sum_{k=0}^{n-1} x(k) z^{-k}\right]$$

Then x (k + 1), x (k + 2), x (k + 3), ... and x (k - 1), x (k -2), x (k - 3), ... can be expressed in terms of X(z) and the initial conditions as follows;

(i) $\mathcal{Z}[x(k+4)] = z^4 X(z) - z^4 x(0) - z^3 x(1) - z^2 x(2) - z\, x(3)$

$= z^4 \left[X(z) - x(0) - z^{-1} x(1) - z^{-2} x(2) - z^{-3} x(3)\right]$

(ii) $\mathcal{Z}[x(k+3)] = z^3 X(z) - z^3 x(0) - z^2 x(1) - z\, x(2)$

(iii) $\mathcal{Z}[x(k+2)] = z^2 X(z) - z^2 x(0) - z\, x(1)$

(iv) $\mathcal{Z}[x(k+1)] = z\, X(z) - z\, x(0)$

(v) $\mathcal{Z}[x(k)] = X(z)$

$(-z)^m \quad \frac{d}{dz} \quad x(2)$

(vi) $\mathcal{Z}[x(k-1)] = z^{-1} X(z)$

(vii) $\mathcal{Z}[x(k-2)] = z^{-2} X(z)$

(viii) $\mathcal{Z}[x(k-3)] = z^{-3} X(z)$

(ix) $\mathcal{Z}[x(k-4)] = z^{-4} X(z)$

Example 22. Solve the following difference equation by use of z transform met

$$x(k+2) + 3x(k+1) + 2x(k) = 0. \text{ Given that } x(0) = 0, x(1) = 1$$

Solution. As we know that

$$\mathcal{Z}[x(k+2)] = z^2 X(z) - z^2 x(0) - z\,x(1)$$

$$\mathcal{Z}[x(k+1)] = z X(z) - z\,x(0)$$

$$\mathcal{Z}[x(k)] = X(z)$$

Now, taking z transform to the both sides of given difference equation, we get

$$z^2 X(z) - z^2 x(0) - z\,x(1) + 3z X(z) - 3z\,x(0) + 2 X(z) = 0$$

or,

$$(z^2 + 3z + 2) X(z) - z = 0$$

(putting $x(0) = 0$ and $x(1) = 1$)

or,

$$X(z) = \frac{z}{z^2 + 3z + 2} = \frac{z}{z+1} - \frac{z}{z+2}$$

Taking inverse z transform, we get

$$x(k) = (-1)^k - (-2)^k, \qquad k = 0,1,2, \ldots$$

Example 23. Solve the following differential equation

$x(k+2) - x(k+1) + 0.25\,x(k) = u(k+2)$ where $x(0) = 1$, $x(1) = 2$, $u(k)$ is inp and is given by $u(k) = 1$, $k = 0,1,2, \ldots$

Solution. The given difference equation is

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$$x(k+2) - x(k+1) + 0.25\,x(k) = u(k+2)$$

By taking z transform to the both sides; we get

$$z^2 X(z) - z^2 x(0) - z\,x(1) - z\,X(z) + z\,x(0) + 0.25\,X(z) = z^2 U(z) - z^2 u(0) - z\,u(1)$$

or, $z^2 X(z) - \cancel{z^2} - \cancel{2z} - zX(z) + \cancel{z} + 0.25\,X(z) = z^2\,U(z) - \cancel{z^2} - \cancel{z}$

or, $X(z)\,[z^2 - z + 0.25] = \cancel{2z} + z^2\,\dfrac{z}{z-1}$

$$= \frac{z^3 + 2z^2 - 2z}{z-1}$$

or, $\dfrac{X(z)}{z} = \dfrac{z^2 + 2z - 2}{(z-1)(z^2 - z + 0.25)}$

or, $\dfrac{X(z)}{z} = \dfrac{z^2 + 2z - 2}{(z-1)\left(z - \frac{1}{2}\right)^2}$

$z^2X(z) - z - zX(z) + 0.25X(z) - z^2U(z) - z^2/z$

$z^2X(z) - z - zX(z) + 0.25X(z) = z^2U(z) - z^2 - z$

$X(z)$

$z^2X(z) - z - zX(z) + 0.25X(z) = \dfrac{z^3}{(z-1)} - z^2$

$z^2X(z) - zX(z) + 0.25X(z) = z^3 - z^3, z^2$

$z^2X(z) - zX(z) + 0.25X(z) =$

Let us now expand X(z)/z into partial fraction,

$k-1$

$$\frac{X(z)}{z} = \frac{k_1}{z-1} + \frac{k_2}{z - \frac{1}{2}} + \frac{k_3}{\left(z - \frac{1}{2}\right)^2}$$

$k\left(\frac{1}{2}\right)$

To evaluate k_1, k_2 and k_3, we get

$$k_1 = 4, k_2 = -3, k_3 = \frac{3}{2}$$

$$\text{So, } X(z) = \frac{4z}{z-1} - \frac{3z}{z - \frac{1}{2}} + \frac{3}{2}\frac{z}{\left(z - \frac{1}{2}\right)^2}$$

On taking inverse z transform,

$$x(k) = 4 - 3\left(\frac{1}{2}\right)^k + \frac{3}{2}k\left(\frac{1}{2}\right)^{k-1}$$

Where $k = 0, 1, 2, \ldots$

2.7 Real Convolution Theorem

Let us consider the functions $x_1(t)$ and $x_2(t)$, where

$x_1(t) = 0, \quad$ for $t < 0$

$x_2(t) = 0, \quad$ for $t < 0$

Assume that $x_1(t)$ and $x_2(t)$ are z transformable and their z transforms are $X_1(z)$ respectively. Then

$$X_1(z)X_2(z) = \mathcal{Z}\left[\sum_{h=0}^{k} x_1(hT)x_2(kT-hT)\right] \quad \text{..........(2.18)}$$

$$= \mathcal{Z}[x_1(k) * x_2(k)]$$

This equation is called the real convolution theorem.

Proof: As we know that

$$\mathcal{Z}\left[\sum_{h=0}^{k} x_1(hT)x_2(kT-hT)\right]$$

$$= \sum_{k=0}^{\infty} \sum_{h=0}^{k} x_1(hT)x_2(kT-hT)z^{-k}$$

$$= \sum_{k=0}^{\infty} \sum_{h=0}^{k} x_1(hT)x_2(kT-hT)z^{-k} \quad \text{..........(2.19)}$$

Where the condition has been used is $x_2(kT - hT) = 0$ for $h > k$. Now, let us define such that when $k = 0$, $m = -h$ and when $k = \infty$, $m = \infty$. Then

$$\mathcal{Z}\left[\sum_{h=0}^{k} x_1(hT)x_2(kT-hT)\right] = \sum_{h=0}^{\infty} x_1(hT)z^{-h} \sum_{m=-h}^{\infty} x_2(mT)z^{-m} \quad \text{..........(2.20)}$$

Since $x_2(mT) = 0$ for $m < 0$. Hence we can write,

$$\mathcal{Z}\left[\sum_{h=0}^{k} x_1(hT)x_2(kT-hT)\right]$$

$$= \sum_{h=0}^{\infty} x_1(hT)z^{-h} \sum_{m=0}^{\infty} x(mT)z^{-m}$$

$$= X_1(z)X_2(z) \quad \text{..........(2.21)}$$

2.8. Complex Convolution Theorem

The complex convolution theorem is useful in obtaining the z transform of the product of two sequences $x_1(k)$ and $x_2(k)$. Suppose both $x_1(k)$ and $x_2(k)$ are zero for $k < 0$. Assume that

$$X_1(z) = \mathcal{Z}[x_1(k)], \quad |z| > R_1$$

$$X_2(z) = \mathcal{Z}[x_2(k)], \quad |z| > R_2$$

where R_1 and R_2 are the radii of absolute convergence for $x_1(k)$ and $x_2(k)$, respectively. Then the z transform of the product of $x_1(k)$ and $x_2(k)$ can be given by

$$\mathcal{Z}[x_1(k)x_2(k)] = \frac{1}{2\pi j}\oint \delta^{-1} X_2(\delta) X_1(\delta^{-1}z)\, d\delta \qquad \text{.........(2.22)}$$

Where $R_2 < |\delta| < \frac{|z|}{R_1}$

Proof: As we know that the z transform of $x_1(k)\, x_2(k)$ is,

$$\mathcal{Z}[x_1(k)x_2(k)] = \sum_{k=0}^{\infty} x_1(k)x_2(k)z^{-k} \qquad \text{.........(2.23)}$$

The series on the right hand side of Eq (2.23) converges for $|z| > R$, where R is the radius of absolute convergence for $x_1(k)\, x_2(k)$.

From inversion integral method,

$$x_2(k) = \frac{1}{2\pi j}\oint X_2(z) z^{k-1} dz$$

$$= \frac{1}{2\pi j}\oint X_2(\delta)\delta^{k-1} d\delta \qquad \text{.........(2.24)}$$

Substituting $x_2(k)$ from Eqs. (2.24) into(2.23), we get

$$Z[x_1(k)\,x_2(k)] = \frac{1}{2\pi j}\sum_{k=0}^{\infty} \oint X_2(\delta)\delta^{k-1}x_1(k)z^{-k}d\delta \qquad \ldots\ldots$$

Noting that Eq (2.23) converges uniformly for the region $|z| > R$, we may int
order of summation and integration, then Eq. (2.25) can be written as,

$$Z[x_1(k)x_2(k)] = \frac{1}{2\pi j}\oint \delta^{-1}X_2(\delta)\sum_{k=0}^{\infty} x_1(k)(\delta^{-1}z)^{-k}d\delta \qquad \ldots\ldots($$

Since,

$$\sum_{k=0}^{\infty} x_1(k)(\delta^{-1}z)^{-k} = X_1(\delta^{-1}z)$$

We have,

$$Z[x_1(k)x_2(k)] = \frac{1}{2\pi j}\oint \delta^{-1}X_2(\delta)X_1(\delta^{-1}z)\,d\delta. \qquad \ldots\ldots(2.$$

Where c is a contour (a circle with centre at origin) which lies in the region given
$|\delta| > R_2$ and $|\delta^{-1} z| > R_1$ or

$$R_2 < |\delta| < \frac{|z|}{R_1} \qquad \ldots\ldots (2.2$$

2.9. Parseval's Theorem

The inequality derived in the previous theorem is

$$R_2 < |\delta| \angle \frac{|z|}{R_1}$$

If this inequality satisfies for $|z|=1$ or $R_2 < |\delta| \angle \frac{1}{R_1}$

Then by substituting $|z| = 1$ in Eq (2.27) of the previous theorem, we get

$$\mathcal{Z}[x_1(k)x_2(k)] = \frac{1}{2\pi j}\oint \delta^{-1}X_2(\delta)X_1(\delta^{-1})d\delta$$

Let us set $x_1(k) = x_2(k) = x(k)$ then

$$\mathcal{Z}[x^2(k)] = \frac{1}{2\pi j}\oint \delta^{-1}X(\delta)X(\delta^{-1})d\delta$$

$$= \frac{1}{2\pi j}\oint z^{-1}X(z)X(z^{-1})dz \qquad \text{.........(2.29)}$$

or,

$$\sum_{k=0}^{\infty} x^2(k) = \frac{1}{2\pi j}\oint z^{-1}X(z)X(z^{-1})dz \qquad \text{..........(2.30)}$$

This Eq. (2.30) is known as Parseval's theorem and it is useful for obtaining the summation of $x^2(k)$.

2.10 Some Examples

Example 1. Obtain the z transform of k^2.

$z[k] = \frac{z^{-1}}{(1-z^{-1})^2}$

Solution. By definition of z transform,

$z[k^2] = \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$

$$\mathcal{Z}[k^2] = \sum_{k=0}^{\infty} k^2 z^{-k} = z^{-1} + 4z^{-2} + 9z^{-3} + \ldots..$$

$$= z^{-1}(1+z^{-1})(1+3z^{-1}+6z^{-2}+10z^{-3}+\ldots\ldots)$$

$$= \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$$

$z[k^2] = -z\frac{d}{dz}X(z)$

$X(z) = \frac{z^{-1}}{(1-z^{-1})^2}$

$z[k \cdot x(k)] = -z\frac{d}{dz}X(z)$

$z[k \cdot x(k)] = -z\frac{d}{dz}X(z)$

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do

Example 2. Obtain the z transform of the curve x(t) shown in Fig. 2.5. Assu[me] sampling period T is 1 sec.

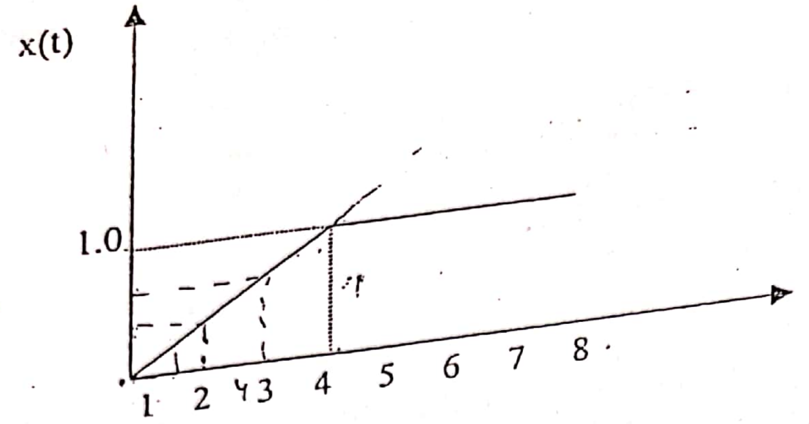

Fig 2.5

(Curve of x(t))

Solution. From the above figure we obtain

$$x(0) = 0$$
$$x(1) = 0.25$$
$$x(2) = 0.5$$
$$x(3) = 0.75$$
$$x(4) = 1, \qquad k = 4, 5, 6, \ldots\ldots$$

Then the z transform of x(k) can be given by

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k} \qquad\qquad z^{-4}(1+$$

$$= 0.25z^{-1} + 0.5z^{-2} + 0.75z^{-3} + z^{-4} + z^{-5} + z^{-6} + \ldots\ldots \qquad = z^{-4}$$

$$= 0.25(z^{-1} + 2z^{-2} + 3z^{-3}) + z^{-4}\frac{1}{1-z^{-1}}$$

$$= \frac{z^{-1} + z^{-2} + z^{-3} + z^{-4}}{4(1-z^{-1})} \qquad \frac{a(1-r^N)}{1-r}$$

$$= \frac{1}{4}\frac{z^{-1}(1-z^{-4})}{(1-z^{-1})^2} \qquad \frac{1}{4}$$

We can verify this by the following way. As we know that the curve x(t) can be writ[ten]

$$x(t) = \frac{1}{4}t - \frac{1}{4}(t-4)1(t-4)$$

where 1(t -4) is the unit-step function occurring at t = 4. Since the sampling period T = 1 sec, the z transform of x(t) can be given by,

$$X(z) = \mathcal{Z}[x(t)] = \mathcal{Z}\left[\frac{1}{4}t\right] - \mathcal{Z}\left[\frac{1}{4}(t-4)1(t-4)\right]$$

$$= \frac{1}{4}\frac{z^{-1}}{(1-z^{-1})^2} - \frac{1}{4}\frac{z^{-4}z^{-1}}{(1-z^{-1})^2}$$

$$= \frac{1}{4}\frac{z^{-1}(1-z^{-4})}{(1-z^{-1})^2}$$

(same result as obtained earlier).

Example 3. Obtain the inverse z transform of $X(z) = \frac{z(z+2)}{(z-1)^2}$ by use of all the methods described in this chapter.

Solution.

Method 1 (Direct division method): Let us write X(z) as a ratio of two polynomials in z^{-1}.

$$X(z) = \frac{1+2z^{-1}}{(1-z^{-1})^2} = \frac{1+2z^{-1}}{1-2z^{-1}+z^{-2}}$$

Dividing the numerator by the denominator, we get

$$X(z) = 1 + 4z^{-1} + 7z^{-2} + 10z^{-3} + \ldots\ldots$$

Hence,

$x(0) = 1$

$x(1) = 4$

$x(2) = 7$

$x(3) = 10$

$\vdots$

Method 2 (Partial-fraction-expansion method): Let us expand X(z) into partial fraction,

$$X(z) = \frac{z(z+2)}{(z-1)^2} = 1 + \frac{3z}{(z-1)^2} + \frac{1}{z-1}$$

$$= 1 + \frac{3z^{-1}}{(1-z^{-1})^2} + \frac{z^{-1}}{1-z^{-1}}$$

Now,

$$\mathcal{Z}^{-1}[1] = \begin{cases} 1, & k = 0 \\ 0, & k = 1,2,3,.... \end{cases}$$

$$\mathcal{Z}^{-1}\left[\frac{z^{-1}}{(1-z^{-1})^2}\right] = k, \quad k = 0,1,2,3,......$$

$$\mathcal{Z}^{-1}\left[\frac{z^{-1}}{1-z^{-1}}\right] = \begin{cases} 1, & k = 1,2,3,... \\ 0, & k \le 0 \end{cases}$$

Hence we obtain,

$$x(0) = 1$$

$$x(k) = 3k + 1, \qquad k = 1,2,3,.......$$

from which we can combine as

$$x(k) = 3k + 1, \qquad k = 0,1,2, ...$$

Method 3(Inversion integral method):

Here, we can write

$$X(z)z^{k-1} = \frac{(z+2)z^k}{(z-1)^2}$$

Since $X(z)\, z^{k-1}$ has a double pole at $z = 1$. So, we can find x(k) by;

$$x(k) = \left[\text{residue of } \frac{(z+2)z^k}{(z-1)^2} \text{ at double pole } z = 1\right]$$

So,

$$x(k) = \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d}{dz}\left[\frac{(z-1)^2(z+2)z^k}{(z-1)^2}\right]$$

$$= \lim_{z \to 1} \frac{d}{dz}\left[(z+2)z^k\right]$$

$$= 3k+1, \qquad k = 0, 1, 2, \ldots\ldots\ldots$$

Example 4. Consider the difference equation imp!

$$x(k+2) - 1.379\, x(k+1) + 0.3679\, x(k) = 0.3679\, u(k+1) + 0.2642\, u(k)$$

where $x(k)$ is the output and $x(k) = 0$ for $k \le 0$ and $u(k)$ is the input and is given by

$$u(k) = 0, \qquad k < 0$$

$$u(0) = 1$$

$$u(1) = 0.2142$$

$$u(2) = -0.2142$$

$$u(k) = 0, \qquad k = 3, 4, 5, \ldots\ldots$$

Determine the output $x(k)$.

Solution. By taking the z transform of the given difference equation, we obtain

$$[z^2 X(z) - z^2 x(0) - z\, x(1)] - 1.3679[z\, X(z) - z\, x(0)] + 0.3679\, X(z)$$
$$= 0.3679\, [z\, U(z) - z\, u(0)] + 0.2642\, U(z) \qquad \ldots\ldots\ldots\ (2.31)$$

By substituting $k = -1$ into the given difference equation, we find

$$x(1) - 1.3679\, x(0) + 0.3679\, x(-1) = 0.3679\, u(0) + 0.2642\, u(-1)$$

Since $x(0) = x(-1) = 0$, $u(-1) = 0$ and $u(0) = 1$, we obtain

$$x(1) = 0.3679\, u(0) = 0.3679$$

By substituting the initial data

$x(0) = 0$, $x(1) = 0.3679$, $u(0) = 1$ into Eq (2.31), we get

$$z^2 X(z) - 0.3679z - 1.3679zX(z) + 0.3679X(z) = 0.3679U(z) - 0.3679z + 0.2642\, U(z)$$

Solving for X(z), we get

$$X(z) = \frac{0.3679z + 0.2642}{z^2 - 1.3679z + 0.3679}\ U(z)$$

The z transform of the input u(k) is

$$U(z) = \mathcal{Z}[\,u(k)\,] = 1 + 0.2142\, z^{-1} - 0.2142\, z^{-2}$$

$U(z) = z[u_k] = 1$

Hence,

$$X(z) = \frac{0.3679z + 0.2642}{z^2 - 1.3679z + 0.3679}\ (1 + 0.2142z^{-1} - 0.2142z^{-2})$$

$$= \frac{0.3679z^{-1} + 0.3430z^{-2} - 0.02221z^{-3} - 0.05659z^{-4}}{1 - 1.3679z^{-1} + 0.3679z^{-2}}$$

$$= 0.3679z^{-1} + 0.8463z^{-2} + z^{-3} + z^{-4} + z^{-5} + \ldots\ldots$$

Thus, the inverse z transform of X(z) gives

$x(0) = 0$

$x(1) = 0.3679$

$x(2) = 0.8463$

$x(k) = 1; \qquad k = 3, 4, 5, \ldots\ldots$

Example 5. Consider the difference equation $x(k+2) = x(k+1) + x(k)$ where $x(0) = 0$ and $x(1) = 1$. Note that $x(2) = 1$, $x(3) = 2$, $x(4) = 3$, ... The 2, 3, 5, 8, 13, ... is known as the Fibonacci series. Obtain the general soluti closed form.

Solution. By taking the z transform of this difference equation, we get

$$z^2 X(z) - z^2 x(0) - z\,x(1) = z\,X(z) - z\,x(0) + X(z)$$

Solving for X(z) gives

$$X(z) = \frac{z^2 x(0) + zx(1) - zx(0)}{z^2 - z - 1}$$

Now, by substituting the initial data $x(0) = 0$ and $x(1) = 1$ into this last equation, w

$$X(z) = \frac{z}{z^2 - z - 1}$$

$$= \frac{1}{\sqrt{5}}\left(\frac{z}{z - \frac{1+\sqrt{5}}{2}} + \frac{z}{z - \frac{1-\sqrt{5}}{2}}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1}{1 - \frac{1+\sqrt{5}}{2}z^{-1}} + \frac{1}{z - \frac{1-\sqrt{5}}{2}z^{-1}}\right)$$

The inverse z transform of X(z) is

$$x(k) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$$

$k = 0, 1, 2, \ldots\ldots\ldots\ldots$

2.11 Problems

1. Obtain the z transform for the following functions.

(a) $x(t) = t^2$

(b) $x(t) = t^4$

(c) $x(t) = te^{at}$

(d) $x(t) = t^3 e^{-at}$

(e) $x(t) = t^2 e^{-at}$

(f) $x(t) = 1 - e^{-at}$

(g) $x(t) = e^{-at} - e^{-bt}$

(h) $x(t) = (1 - at)\, e^{-at}$

(i) $x(t) = at - 1 + e^{-at}$

(j) $x(k) = a^{k-1},\ k = 1, 2, \ldots$

(k) $x(k) = k\, a^{k-1}$

(l) $x(k) = k^2 a^{k-1}$

(m) $x(k) = a^k \cos k\pi$

(n) $x(k) = k^3 a^{k-1}$

Ans.

(a) $\dfrac{T^2 z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}$

(b) $\dfrac{T^4 z^{-1}(1 + 11z^{-1} + 11z^{-2} + z^{-3})}{(1 - z^{-1})^5}$

(c) $\dfrac{T\ e^{aT}\ z^{-1}}{(1 - e^{aT} z^{-1})^2}$

(d) $\dfrac{T^3 z^{-1} e^{-aT} (1 + 4 z^{-1} e^{-aT} + \ldots}{(1 - z^{-1} e^{-aT})^4}$

(e) $\dfrac{T^2 e^{-aT} (1 + e^{-aT} z^{-1}) z^{-1}}{(1 - e^{-aT} z^{-1})^3}$

(f) $\dfrac{(1 - e^{-aT}) z^{-1}}{(1 - z^{-1})(1 - e^{-aT} z^{-1})}$

(g) $\dfrac{(e^{-aT} - e^{-bT})\ z^{-1}}{(1 - e^{-aT} z^{-1})(1 - e^{-bT} z^{-1})}$

(h) $\dfrac{1 - (1 + aT) e^{-aT} z^{-1}}{(1 - e^{-aT} z^{-1})^2}$

(i) $\dfrac{[(aT - 1 + e^{-aT}) + (1 - e^{-aT} - aT e^{-aT}) z^{-1}] z^{-1}}{(1 - z^{-1})^2 (1 - e^{-aT} z^{-1})}$

(j) $\dfrac{z^{-1}}{1 - a z^{-1}}$

(k) $\dfrac{z^{-1}}{(1 - az^{-1})^2}$

(l) $\dfrac{z^{-1}(1 + az^{-1})}{(1 - az^{-1})^3}$

(m) $\dfrac{1}{1 - az^{-1}}$

(n) $\dfrac{z^{-1}(1 + 4az^{-1} + a^2 z^{-2})}{(1 - az^{-1})^4}$

$\rightarrow \dfrac{1}{1 + az^{-1}}$?

2. Find the inverse z transform of the following:

(a) $\dfrac{z}{z + a}$

(b) $\dfrac{1}{z + a}$

(c) $\dfrac{1}{z - a}$

(d) $\dfrac{3z^2 + 2z + 1}{z^2 - 3z + 2}$

(e) $\dfrac{3z^2 + 2z + 1}{z^2 + 3z + 2}$

(f) $\dfrac{2z}{(2z - 1)^2}$

(g) $\dfrac{z - 0.4}{z^2 + z + 2}$

(h) $\dfrac{z^{-1}}{(1 - az^{-1})^2}$

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(i) $\dfrac{z-4}{(z-1)(z-2)^2}$

Ans.

(a) $(-a)^k$

(b) $(-a)^{k-1}u(k-1)$

(c) $a^{k-1}u(k-1)$

(d) $3\delta(k)-6u(k-1)+17(2)^{k-1}u(k-1)$

(e) $3\delta(k)+2(-1)^{k-1}u(k-1)-9(-2)^{k-1}u(k-1)$

(f) $k(1/2)^k$

$17(2)^{k-1}u(k-1)-6(1)^{k-1}u(k-1)$

(g) $-\dfrac{2}{\sqrt{7}}(-\sqrt{2})^k\sin\theta_k+\dfrac{0.8}{\sqrt{7}}(-\sqrt{2})^{k-1}\sin\theta_{k-1}u(k-1)$

$-7(-2)^{k-1}u(k-1)+2(-1)^{k-1}u(k-1)$

$\theta=\tan^{-1}\sqrt{7}$

(h) $\dfrac{1}{a}ka^k$

(i) $-3u(k-1)+3(2)^{k-1}u(k-1)-(k-1)(2)^{k-1}u(k-1)$

3. Obtain the solution of the following difference equation in terms of x(0) and x(1):
x(k+2) + (a+b) x(k+1) +ab x(k) =0 where a and b are constants and k=0, 1, 2, ….
Also assume case (i) a=b; case (ii) a≠b.

Ans. (i) $x(k)=x(0)(-a)^k+[ax(0)+x(1)]k(-a)^{k-1}$

(ii) $x(k)=\dfrac{bx(0)+x(1)}{b-a}(-a)^k+\dfrac{ax(0)+x(1)}{a-b}(-b)^k$

4. Solve the difference equation x(k+2)+3x(k+1)+2x(k)=u(k); x(0)=1, x(k)=0 for k< 0.

$u(k)=0$

Ans.

$u(k)=1$

$$x(k)=\frac{1}{6}-\frac{3}{2}(-1)^k+\frac{7}{3}(-2)^k$$

5. Solve the difference equation

$$x(k+2)-3x(k+1)+2x(k)=4^k;\ x(0)=0,x(1)=1$$

Ans. $\left[-1+(2)^k\right]+\left[\frac{1}{3}-\frac{1}{2}(2)^k+\frac{1}{6}(4)^k\right]$

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3.1 Impulse Sampling

Let us consider a fictitious sampler commonly called an impulse sampler. The sampler is considered to be a train of impulses that begins with $t = 0$, with period equal to T and the strength of each impulse equal to the sampled continuous-time signal at the corresponding sampling instant. A pictorial diagra sampler is shown in figure.

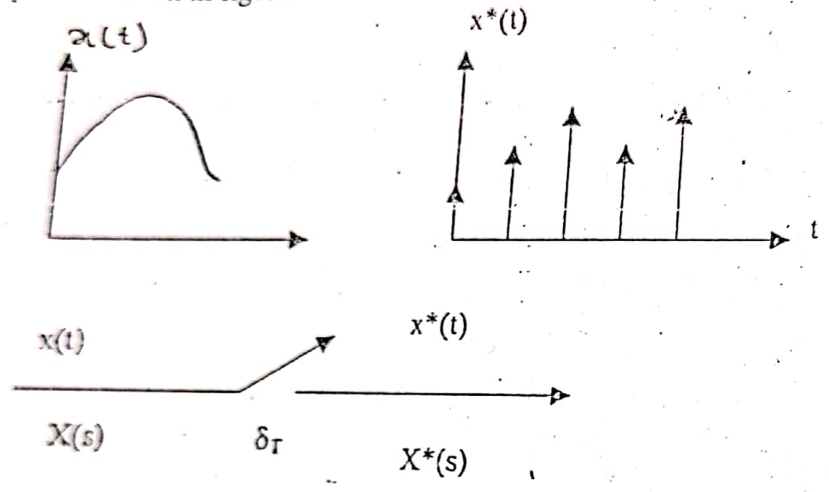


Fig 3.1 (Impulse Sampler)

(Mathematically, an impulse is defined as having an infinite magnitude with zero graphically represented by an arrow with an amplitude representing the strength of

The impulse-sampled output is a sequence of impulses with the strength of eac equal to the magnitude of x(t) at the corresponding instant of time. That is at time t impulse is $x(kT)\,\delta(t-kT)$. The notation to represent the impulse-sampled output is sampled signal $x^*(t)$, a train of impulses, can thus be represented by the infinite sum

$$x^*(t) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT)$$

$$\text{or, } x^*(t) = x(0)\delta(t) + x(T)\delta(t-T) + \ldots + x(kT)\delta(t-kT) + \ldots \qquad \ldots(3.1)$$

δ (t -kT

Let us define a train of unit impulses as $\delta_T(t)$.

or,

$$\delta_T(t) = \sum_{k=0}^{\infty} \delta(t - kT)$$

Hence, the sampler output is equal to the product of the continuous-time input x(t) and the train of unit impulses $\delta_T(t)$.

The sampler may be considered a modulator with the input x(t) as the modulating signal and the train of unit impulses $\delta_T(t)$, as the carrier as shown in figure below.

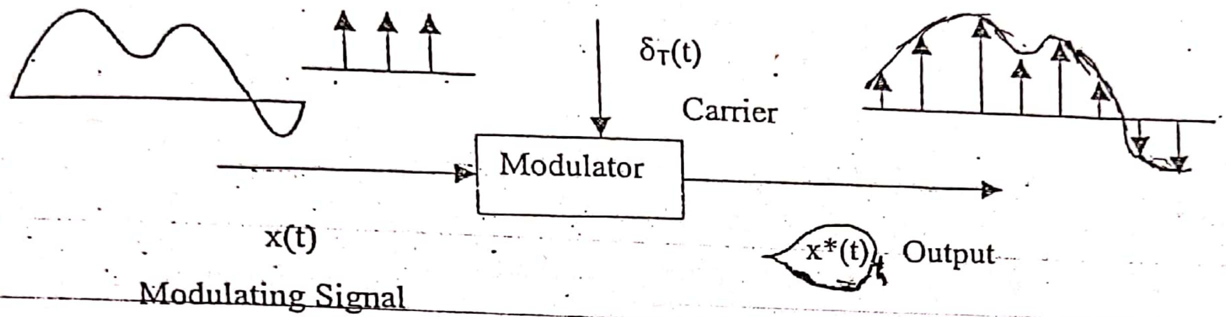


Fig 3.2 (Impulse sampler as a modulator)

Taking the Laplace Transform of Eq.(3.1), we get

$$X^*(S) = \mathcal{L}[x^*(t)] = x(0)\mathcal{L}[\delta(t)] + x(T)\mathcal{L}[\delta(t-T)] + x(2T)\mathcal{L}[\delta(t-2T)] + \ldots\ldots\ldots\ldots$$

$$= x(0) + x(T)e^{-Ts} + x(2T)e^{-2Ts} + \ldots\ldots\ldots$$

$$= \sum_{k=0}^{\infty} x(kT)e^{-kTs} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3.2)$$

Let us define $\qquad e^{Ts} = z$

$$s = \frac{1}{T}\ln z$$

This Eq.(3.2) becomes

$$X^*(s)\Big|_{s=\frac{1}{T}\ln z} = \sum_{k=0}^{\infty} x(kT)z^{-k} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.3)$$

Hence we may write,

$$X^*(s)\Big|_{s=\frac{1}{T}\ln z} = X(z)$$

Hence we can conclude,

$$X^*(s) = X^*\left(\frac{1}{T}\ln z\right) = X(z) = \sum_{k=0}^{\infty} x(kT)z^{-k}$$

3.2 Data-Hold Circuit

Data-hold is a process of generating a continuous-time signal h(t) from a d sequence x(kT). A hold circuit converts the sampled signal into a continuous- which approximately reproduces the signal applied to the sampler.

The signal h(t) during the time interval $kT \le t \le (k+1)T$ may be approxi polynomial in τ as follows:

$$h(kT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \ldots\ldots + a_1\tau + a_0 \qquad \ldots\ldots\ldots\ldots (3$$

Where $0 \le \tau < T$

The signal h(kT) must be equal to x(kT).

or,

$$h(kT) = x(kT)$$

Hence, Eq.(3.5) can be written as,

$$h(kT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \ldots\ldots + a_1\tau + x(kT) \qquad \ldots\ldots\ldots\ldots ($$

If the data-hold circuit is an nth-order polynomial, it is called an nth-order hold. T it is called a first order hold.

The simplest data-hold is obtained when n = 0 in Eq.(3.6) that is when,

$$h(kT+\tau) = x(kT) \qquad \text{.......................(3.7)}$$

3.2.1 *Zero-Order Hold:*

Theorem: The transfer function of zero-order hold is given by,

$$G_h = \frac{1-e^{-Ts}}{s}$$

Proof:

$G_h = \frac{1-e^{-\tau s}}{s}$ $\quad G_h = \frac{1-e^{-\tau s}}{s}$

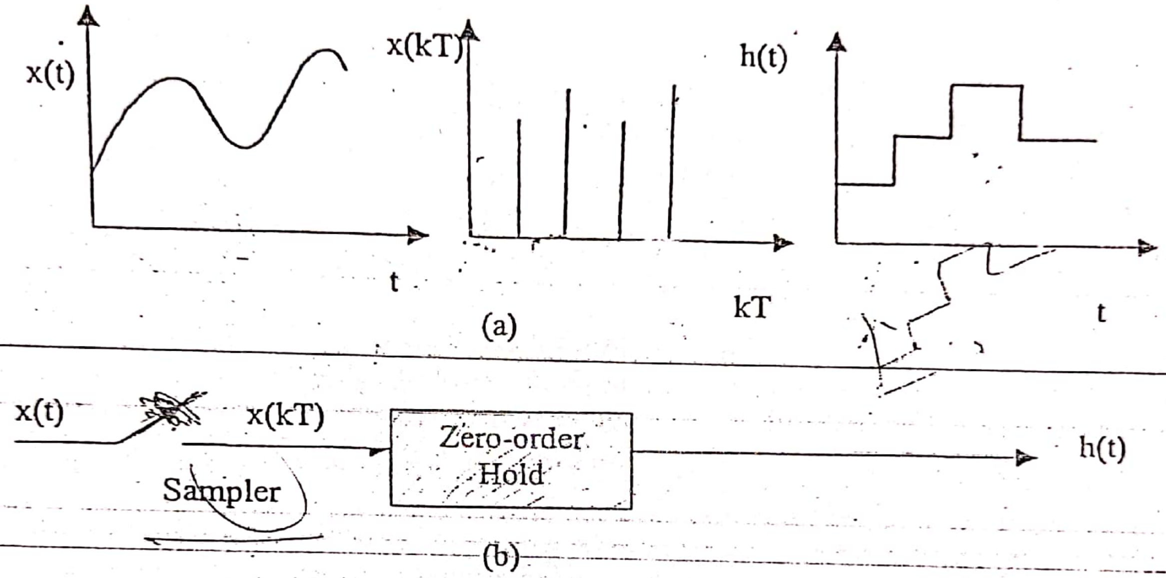
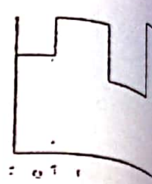


Fig 3.3 (Sampler and zero-order hold)

The figure above shows a sampler and a zero-order hold. The input signal x(t) is sampled at discrete instants and sampled signal is passed through the zero-order hold. The zero-order hold circuit smoothes the sampled signal to produce the signal h(t), which is constant from the last sampled value until the next sample is available, that is

$$h(kT+t) = x(kT), \qquad \text{for } 0 \le t < T \qquad \text{......................(3.8)}$$

The mathematical model for the combination of real sampler and a zero-order hold circuit is shown below in Fig. 3.4(a). As we know that the integral of an impulse function is a constant, we may assume that the zero-order hold is an integrator and the input to the zero-order hold circuit is a train of impulses. Then a mathematical model for a real sampler and zero-order may be constructed as shown in Fig. 3.4(b), where $Gh_o(s)$ is the transfer f... of the zero-order hold and $x^*(t)$ is the impulse sampled signal of $x(t)$.



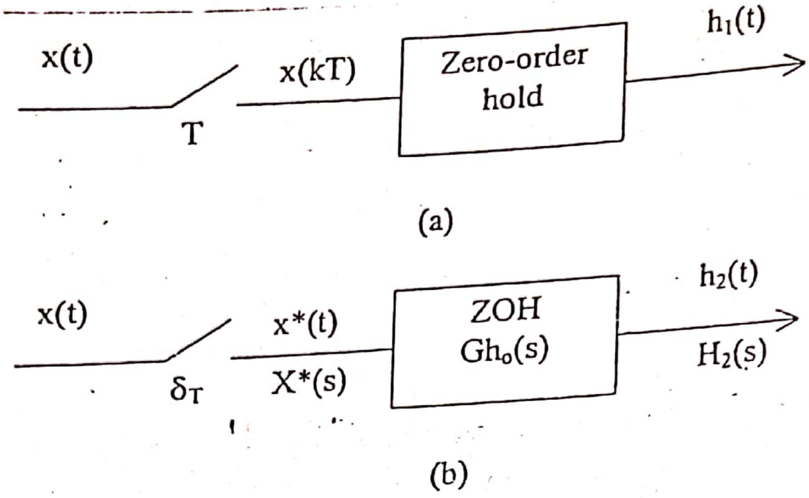


Fig 3.4 (A real sampler, zero order hold and mathematical model that consists of an impulse sampler and transfer function $G_{ho}(s)$)

Consider the sampler and zero-order hold shown in Fig 3.4 (a) Assume that the signal... zero for $t<0$. Then the output $h_1(t)$ is related to $x(t)$ as follows:

$$h_1(t) = x(0)[1(t) - 1(t-T)] + x(T)[1(t-T) - 1(t-2T)] + x(2T)[1(t-2T) - 1(t-3T)] + \ldots$$

$$= \sum_{k=0}^{\infty} x(kT)[1(t-kT) - 1(t-(k+1)T)] \qquad \ldots\ldots\ldots\ldots\ldots (3.9)$$

Since,

$$\boxed{\mathcal{L}[1(t-kT)] = \frac{e^{-kTs}}{s}}$$

The Laplace transform of Eq.(3.9) becomes,

$$\mathcal{L}[h_1(t)] = H_1(s) = \sum_{k=0}^{\infty} x(kT)\frac{e^{-kTs} - e^{-(k+1)Ts}}{s}$$

$$= \frac{1-e^{-Ts}}{s} \sum_{k=0}^{\infty} x(kT)e^{-kTs} \qquad \text{.............(3.10)}$$

Now, consider the mathematical model of Fig. 3.4(b). The output of this model must be same as that of real zero-order hold, or

$$\mathcal{L}[h_2(t)] = H_2(s) = H_1(s)$$

Thus, $$H_2(s) = \frac{1-e^{-Ts}}{s} \sum_{k=0}^{\infty} x(kT)e^{-kTs} \qquad \text{.......................(311)}$$

From Fig 3.4(b),

$$H_2(s) = Gh_0(s)X^*(s)$$

Since $$X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$\left(\frac{1-e^{-Ts}}{s}\right)^2 \cdot \frac{sT+1}{T}$

Hence from Eq. (3.11),

$$H_2(s) = \frac{1-e^{-Ts}}{s} X^*(s) \qquad \text{...................(3.12)}$$

Hence the transfer function of the zero-order hold may be given by,

$$\boxed{G_{ho}(s) = \frac{1-e^{-Ts}}{s}}$$

3.2.2 Transfer Function of First-Order Hold:

Theorem: The transfer function of the first-order hold may be given by

$$\boxed{G_{h1}(s) = \left(\frac{1-e^{-Ts}}{s}\right)^2 \frac{Ts+1}{T}} \qquad \text{..............(3.13)}$$

Proof: The output of the nth order hold circuit is given by

$$h(kT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \ldots\ldots + a_1\tau + x(kT) \qquad \text{....................(3.14)}$$

Let us now substitute $n = 1$ for the first order date-hold, we get

$$h(kT+\tau) = a_1\tau + x(kT) \qquad \text{....................(3.15)}$$

Where, $0 \le \tau < T$ and $k = 0, 1, 2, \ldots\ldots\ldots$

By applying the condition that

$$h((k-1)T) = x((k-1)T)$$

The constant a_1 can be determined as follows:

$$h((k-1)T) = -a_1 T + x(kT) = x((k-1)T).$$

or, $$a_1 = \frac{x(kT) - x((k-1)T)}{T}$$

Hence, Eq.(3.15) becomes

$$h(kT+\tau) = x(kT) + \frac{x(kT) - x((k-1)T)}{T}\tau \qquad \text{.................(3.1}$$

Where $0 \le \tau < T$. The extrapolation process of the first order hold is based on Eq.(3 continuous-time output signal h(t) obtained by use of the first-order hold is a piecewi signal, as shown in figure below.

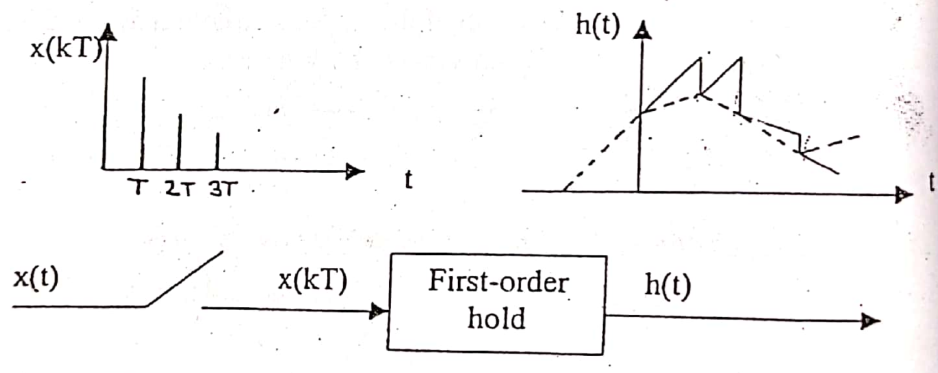


Fig 3.5 (Input and output of a first- order hold)

To derive the transfer function of the first-order hold circuit, it is convenient to as simple function for x(t). For example, a unit step-function, a unit-impulse function unit-ramp function would be the good choices for x(t).

Let us choose a unit-step function as x(t). Then, for the real sampler and first-order hold shown in Fig 3.6(a), the output h(t) of the first-order hold consists of straight lines that are extrapolations of the two preceding sampled values. The output h(t) is shown in the diagram.

The output curve h(t) may be written as follows:

$$h(t) = \left(1+\frac{t}{T}\right)1(t) - \frac{t-T}{T}1(t-T) - 1(t-T) \qquad \text{.................(3.17)}$$

The Laplace transform of this Eq.(3.17) becomes

$$H(s) = \left(\frac{1}{s}+\frac{1}{Ts^2}\right) - \frac{1}{Ts^2}e^{-Ts} - \frac{1}{s}e^{-Ts}$$

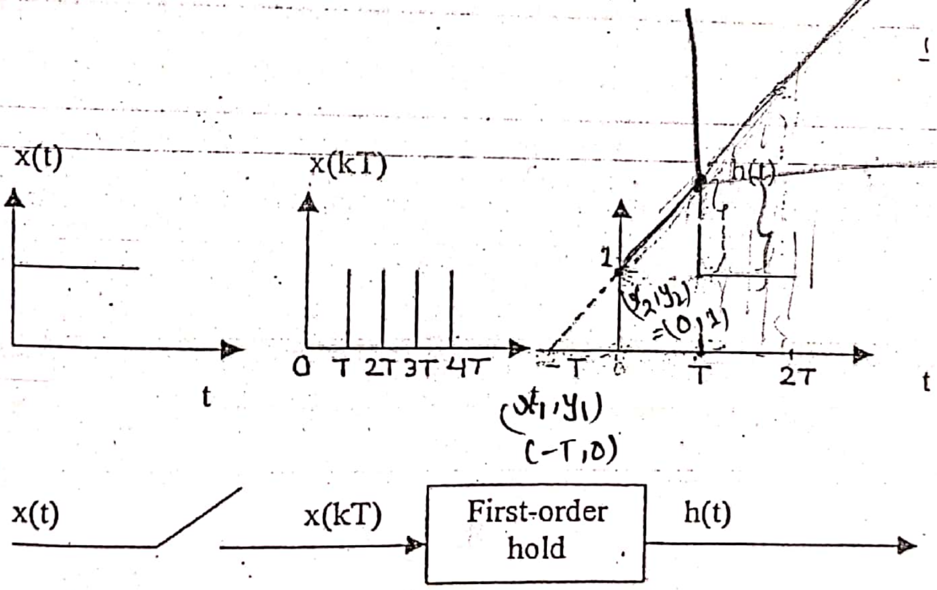
$y = mx + c$

$= \left(\frac{1}{T}t + 1\right)$

$1(t-T) : \frac{(t-T)}{T}$

$$= \frac{1-e^{-Ts}}{s} + \frac{1-e^{-Ts}}{Ts^2}$$

$$= (1-e^{-Ts})\frac{Ts+1}{Ts^2} \qquad \text{.................(3.18)}$$



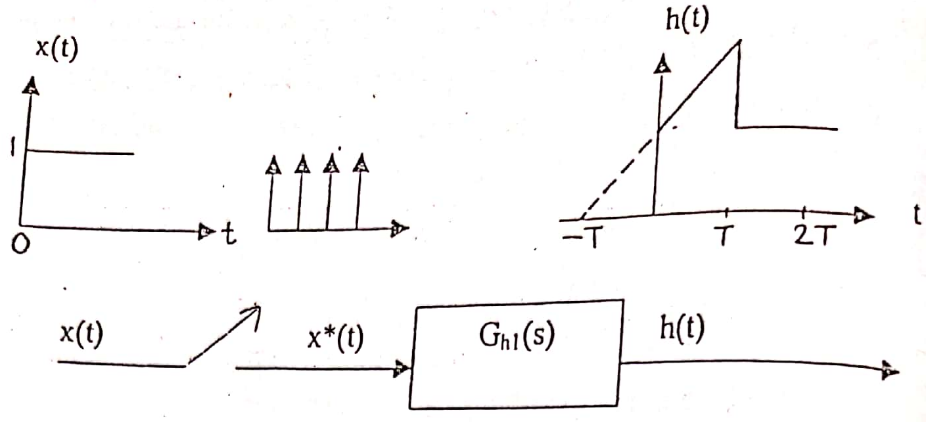
$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

(a)

$y - 0 = 1 - 0$

$\Rightarrow h(t) = \frac{1}{0-(-T)}(t-(-T))$

$= \frac{1}{T}(t+T) \Rightarrow \left(1+\frac{t}{T}\right)$



(b)

Fig 3.6(Real sampler cascaded with first-order hold and mathematical dodel consisting of impulse sampler and Gh1(s))

The Fig 3.6(b) shows a mathematical model of the real sampler cascaded with the fi hold shown in Fig 3.6(a). The model consists o the impulse sampler and $G_{h1}(s)$, The function of the first-order hold. The output signal of this model is the same as the o the real system. Hence, the output H(s) is also given by Eq.(3.18).

The Laplace transform of the input x*(t) to the first-order hold $G_{h1}(s)$ is

$$X^*(s)=\sum_{k=0}^{\infty}1(kT)e^{-kTs}=\frac{1}{1-e^{-Ts}}$$

z transform of unit step $= \frac{1}{1-z^{-1}}$ put $z = e^{Ts}$ laplace

Hence, the transfer function $G_{h1}(s)$ of the first-order hold is given by

$$Gh_1(s)=\frac{H(s)}{X^*(s)}=\left(1-e^{-Ts}\right)^2\frac{Ts+1}{Ts^2}$$

$$=\left(\frac{1-e^{-Ts}}{s}\right)^2\frac{Ts+1}{T}$$

Note that a real sampler combined with a first-order hold is equivalent to an impulse s combined with a transfer function

$$\frac{(1-e^{-Ts})^2(Ts+1)}{(Ts^2)}$$

3.3 Obtaining the Z-transform by the Convolution Integral

Let us consider the impulse sampler as shown in figure below:

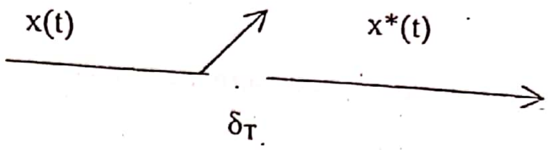


Fig 3.7(Impulse sampler)

The output of the impulse sampler is given by,

$$x^*(t) = \sum_{k=0}^{\infty} x(t)\delta(t-kT) = x(t)\sum_{k=0}^{\infty}\delta(t-kT) \qquad \text{.............(3.19)}$$

As we know that

$$\mathcal{L}[\delta(t-kT)] = e^{-kTs}$$

We have,

$$\mathcal{L}\left[\sum_{k=0}^{\infty}\delta(t-kT)\right] = 1 + e^{-Ts} + e^{-2Ts} + \ldots\ldots = \frac{1}{(1-e^{-Ts})}$$

$$X^*(s) = \mathcal{L}[x^*(t)] = \mathcal{L}\left[x(t)\sum_{k=0}^{\infty}\delta(t-kT)\right]$$

Here, $X^*(s)$ is the Laplace transform of the product of two time function $x(t)$ and $\sum_{k=0}^{\infty}\delta(t-kT)$. Note that it can't be equal to the product of the two corresponding Laplace transforms.

The Laplace transform of the product of two Laplace-transformable functions $f(t)$ and $g(t)$ can be given by,

$$\mathcal{L}[f(t)g(t)] = \int_0^{\infty} f(t)g(t)e^{-st}dt$$

$$= \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)G(s-p)dp \qquad \text{...........(3.20)}$$

Note that inversion integral is,

$$f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds, \qquad t>0$$

Where c is the abscissa of convergence for F(s). Thus,

$$\mathcal{L}\, f(t)g(t) = \frac{1}{2\pi j}\int_{0}^{\infty}\int_{c-j\infty}^{c+j\infty} F(p)e^{pt}dp\, g(t)e^{-st}dt$$

Because of the uniform convergence of the integrals considered we can invert the

integration.

$$\mathcal{L}\, f(t)g(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)dp\int_{0}^{\infty} g(t)e^{-(s-p)t}dt.$$

Noting that $\int_{0}^{\infty} g(t)e^{-(s-p)t}dt = G(s-p)$

We obtain

$$\mathcal{L}\, f(t)g(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)G(s-p)dp$$

Let us substitute x(t) and $\sum_{k=0}^{\infty}\delta(t-kT)$ for f(t) and g(t) respectively. Then the L

transform $X^*(s)$ is,

$$X^*(s) = \mathcal{L}\left[x(t)\sum_{k=0}^{\infty}\delta(t-kT)\right]$$

$$= \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(p)\frac{1}{1-e^{-T(s-p)}}dp \qquad \text{.................(3.21)}$$

Where the integration is along the line from c-j∞ to c+j∞ and this line is parallel

imaging axis in the p-plane and separates the poles of X(p) from those of $\frac{1}{1-e^{-T(s-p)}}$

Eq.(3.21) is known as convolution integral. Such an integral can be evaluated in terms of residues by forming a closed contour consisting of the line from c-j∞ to c+j∞ and a semicircle of infinite radius in the left or right half-plane.

3.3.1 *Convolution Integral in the Left Half of the p-Plane:*

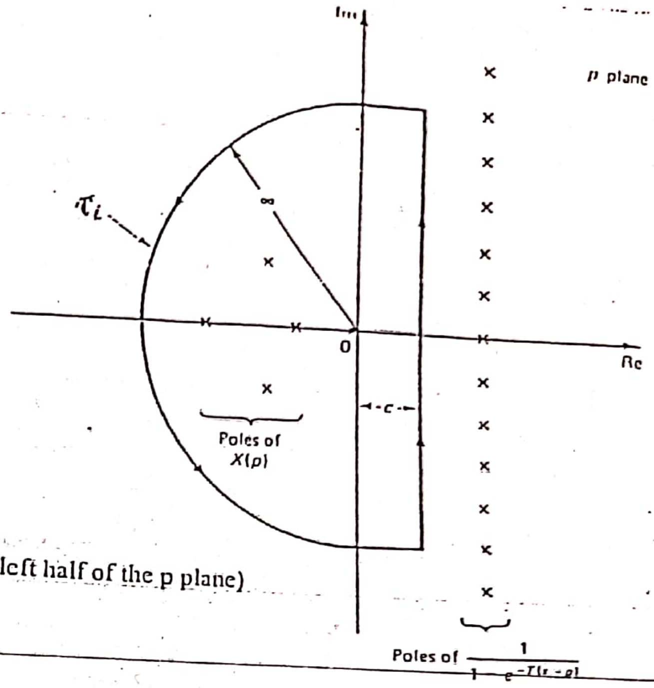


Fig 3.8 (Closed contour in the left half of the p plane)

As we have,

$$X^*(s) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(p)\frac{1}{1-e^{-T(s-p)}}dp$$

$$= \frac{1}{2\pi j}\oint x(p)\frac{1}{1-e^{-T(s-p)}}dp - \frac{1}{2\pi j}\int_{\tau_L}\frac{X(p)}{1-e^{-T(s-p)}}dp$$

Here we assume that $X(s)=\dfrac{q(s)}{p(s)}$ and p(s) is of higher degree in s than q(s), that is

$$\lim_{s\to\infty} X(s)=0$$

Using the contour shown above, along the semicircle, as $s\to\infty$, the integral $\to 0$, that is

$$X^*(s) = \frac{1}{2\pi j}\oint\frac{X(p)}{1-e^{-T(s-p)}}dp$$

$$=\sum\left[\text{Residues of } \frac{X(p)}{1-e^{-T(s-p)}} \text{ at the poles of } X(p)\right]$$

Letting $z=e^{Ts}$

$$\boxed{X(z)=\sum\left[\text{residues of } \frac{X(p)z}{z-e^{Tp}} \text{ at the poles of } X(p)\right]}$$

By changing the complex variable notation from p to s we obtain,

$$X(z) = \sum\left[\text{residues of } \frac{X(s)z}{z - e^{Ts}} \text{ at the poles of } X(s)\right]$$

Case (i) For a simple pole: If a pole at $s=s_j$ is a simple pole then the corresponding is

$$k_j = \lim_{s \to s_j}\left[(s-s_j)\frac{X(s)z}{z-e^{Ts}}\right]$$

Case (ii) For a multiple pole: If a pole at $s = s_j$ is a multiple pole of order n_i, then k_i is

$$k_i = \frac{1}{(n_i-1)!}\lim_{s\to s_i}\frac{d^{n_i-1}}{ds^{n_i-1}}\left[(s-s_i)^{n_i}\frac{X(s)z}{z-e^{Ts}}\right]$$

Example 1. Given

$$X(s) = \frac{1}{s^2(s+1)}$$

Obtain X(z) by use of the convolution integral in the left half plane.

Solution. The function X(s) has a double pole at s= 0 and a simple pole at s= -1

Hence,

$$\boxed{X(z) = \sum\left[\text{residue of } \frac{X(s)z}{z-e^{Ts}} \text{ at the poles of } X(s)\right]}$$

$$= \frac{1}{(2-1)!}\lim_{s\to 0}\left[\frac{d}{ds}\cancel{s^2}\frac{1}{\cancel{s^2}(s+1)}\frac{z}{z-e^{Ts}}\right] + \lim_{s\to -1}\left[\cancel{(s+1)}\frac{1}{s^2\cancel{(s+1)}}\frac{z}{z-e^{Ts}}\right]$$

$$= \lim_{s\to 0}\frac{-z\left[z-e^{Ts}+(s+1)(-T)e^{Ts}\right]}{(s+1)^2(z-e^{Ts})^2} + \frac{1}{(-1)^2}\frac{z}{z-e^{-T}}$$

$$= \frac{-z(z-1-T)}{(z-1)^2} + \frac{z}{z-e^{-T}} \qquad = \frac{z^2(T-1+e^{-T})+z(1-e^{-T}-Te^{-T})}{(z-1)^2(z-e^{-T})}$$

$$= \frac{(T-1+e^{-T})z^{-1}+z(1-e^{-T}-Te^{-T})z^{-2}}{(1-z^{-1})^2(1-e^{-T}z^{-1})}$$

Example 2. Given $X(s) = \dfrac{s+3}{(s+1)(s+2)}$

Obtain X(z) by the method of convolution integral in the left half plane.

Solution: The function X(s) has poles at s= -1 and s= -2

Now,

$$X(z) = \sum\left[\text{residues of } \frac{X(s)z}{z-e^{Ts}} \text{ at the poles of } X(s)\right]$$

$$= \lim_{s\to -1}\left[\frac{(s+1)(s+3)}{(s+1)(s+2)}\frac{z}{z-e^{Ts}}\right] + \lim_{s\to -2}\left[\frac{(s+3)(s+2)}{(s+1)(s+2)}\frac{z}{z-e^{Ts}}\right]$$

$$= \frac{2z}{z-e^{-T}} - \frac{z}{z-e^{-2T}}$$

$\frac{z}{z-e^{Ts}}$

3.3.2 *Integral in the Right Half Plane:*

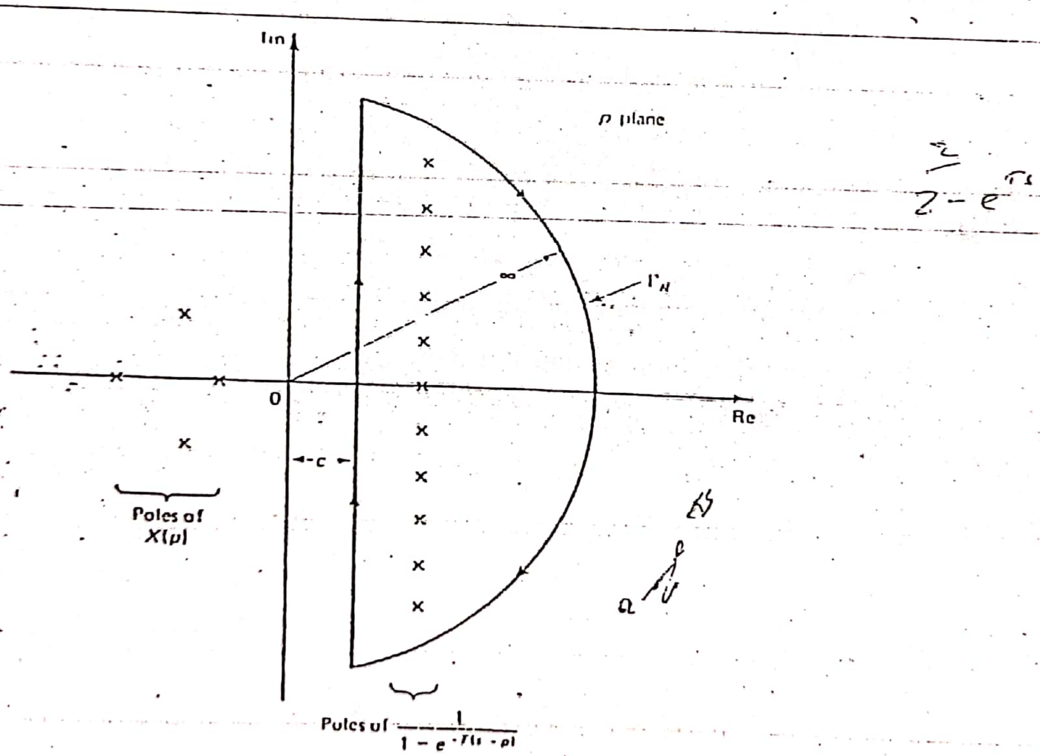


Fig 3.9 (Closed contour in the right half of the p plane)

The poles of $\dfrac{1}{1-e^{-T(s-p)}}$ may be obtained by solving $1-e^{-T(s-p)}=0$

$$e^{-T(s-p)} = 1$$

$$-T(s-p) = \pm 2\pi\, jk$$

$$s - p = \pm \frac{1}{T} 2\pi jk$$

$$p = s \pm j\frac{2\pi}{T}k, \quad (k = 0,1,2,3)$$

Thus there is an infinite number of poles.

In order to evaluate the given convolution integral, let us choose the contour which of the line from c-j∞ to c+j∞ and semicircle τ_R of an infinite radius in the right half plane as shown in Fig 3.9. The closed contour encloses all poles of $\frac{1}{1-e^{-T(s-p)}}$ but it enclose any poles of X(p).

Now, $X^*(s)$ can be written as,

$$X^*(s) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{X(p)}{1-e^{-T(s-p)}}\,dp$$

$$= \frac{1}{2\pi j}\oint \frac{X(p)}{1-e^{-T(s-p)}}\,dp - \frac{1}{2\pi j}\int_{\tau_R} \frac{X(p)}{1-e^{-T(s-p)}}\,dp$$

Let us consider the integral along the semicircle τ_R. Consider the degree of denomi X(s) is at least z greater than the degree of numerator, it can be shown that the integral is zero, or

$$\frac{1}{2\pi j}\int_{\tau_R} \frac{X(p)}{1-e^{-T(s-p)}} = 0$$

Therefore,

$$X^*(s) = \frac{1}{2\pi j}\oint \frac{X(p)}{1-e^{-T(s-p)}}\,dp$$

The integral along the closed contour can be obtained by evaluating residues. Hence,

$$X^*(s) = \left[\sum_{-\infty}^{\infty} \frac{X(p)}{\frac{d}{dp}\left(1 - e^{-T(s-p)}\right)} \right]_{p=s+j\left(\frac{2\pi}{T}\right)}$$

$$= \frac{1}{T} \sum_{k-\infty}^{\infty} X\left(s + j\frac{2\pi}{T}k\right)$$

$$X(z) = \left[\frac{1}{T} \sum_{k-\infty}^{\infty} X\left(s + j\frac{2\pi}{T}k\right) \right]_{s=\frac{1}{T}\ln z}$$

$\frac{d(1-e^{-T(s-p)})}{dp}$

$-\frac{d\,e^{-T(s-p)}}{d(}$

$-\frac{d\,e^{-Ts}\cdot e^{Tp}}{dp}$

$\Rightarrow e^{-Ts}\,T e^{Tp}$

3.3.3 *Obtaining z Transforms of Functions Involving the Term* $(1-e^{-Ts})/s$:

Suppose the transfer function G(s) follows the ZOH. Then the product of the transfer function of the ZOH and G(s) become

$$X(s) = \frac{1 - e^{-Ts}}{s} G(s)$$

Now, the function X(s) can be written as

$$X(s) = (1 - e^{-Ts}) \frac{G(s)}{s}$$

$$= (1 - e^{-Ts}) G_1(s) \qquad \text{.....................(3.22)}$$

Where

$$G_1(s) = \frac{G(s)}{s}$$

Consider the function,

$$X_1(s) = e^{-Ts} G_1(s) \qquad X_1(s) = \qquad \text{.....................(3.23)}$$

The inverse Laplace transform of Eq. (3.23) can be given by convolution integral,

$$x_1(t) = \int_0^t g_0(t - \tau) g_1(\tau) d\tau$$

Where $g_o(t) = \mathcal{L}^{-1} e^{-Ts} = \delta(t - T)$

$g_1(t) = \mathcal{L}^{-1} G_1(s)$

Thus,

$$x_1(t) = \int_0^t \delta(t - T - \tau) g_1(\tau) d\tau$$

$$= g_1(t - \tau)$$

By letting,

$$\mathcal{Z}[g_1(t)] = G_1(z)$$

$$\mathcal{Z}[x_1(t)] = \mathcal{Z}[g_1(t - \tau)] = z^{-1} G_1(z),$$

Referring to Eqs. (3.22) and (3.23), we get

$$X(z) = \mathcal{Z}[G_1(s) - X_1(s)]$$

$$= \mathcal{Z}[g_1(t)] - \mathcal{Z}[x_1(t)]$$

$$= G_1(z) - z^{-1} G_1(z)$$

$$= (1 - z^{-1}) G_1(z)$$

$$= (1 - z^{-1}) \mathcal{Z}[G_1(s)]$$

$$= (1 - z^{-1}) \mathcal{Z}\left[\frac{G(s)}{s}\right]$$

$$X(z) = (1 - z^{-1}) \mathcal{Z}\left[\frac{G(s)}{s}\right] \qquad \text{................ (3.24)}$$

Hence, we can conclude that if $X(s)$ involve a factor $(1-e^{-Ts})$ then, in finding the z tra of $X(s)$, the term $(1-e^{-Ts}) = (1-z^{-1})$ may be factored out so that $X(z)$ becomes the pro $(1-z^{-1})$ and the z transform of the remaining term.

Example 3. Obtain the z transform of   imp

$X(z) = (1-z^{-1}) z (G(s)/s)$

$$X(s) = \frac{1-e^{-Ts}}{s}\left(\frac{1}{s+1}\right)$$

$X(z) = (1-z^{-1}) z \left[\frac{G(s)}{s}\right]$

Solution. The z transform $X(z)$ is given by

$$X(z) = (1 - z^{-1}) \mathcal{Z}\left[\frac{G(s)}{s}\right]$$

$\frac{1}{s(s+1)}$

f sampling freq $\omega_s > 2\omega_m$ where $\omega_m \rightarrow$ maximum freq of the continuous time signal.

$$= (1 - z^{-1}) Z\left[\frac{1}{s(s+1)}\right]$$

$$= (1 - z^{-1}) Z\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

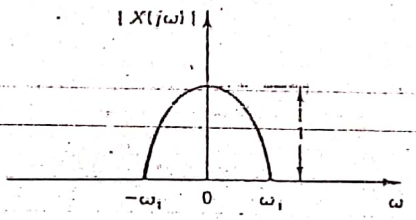
$$= (1 - z^{-1}) \cancel{Z}\left(\frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-T} z^{-1}}\right)$$

$\frac{1}{1 - z^{-1}} = \frac{1}{1 - e^{-T} z^{-1}}$

$$= \frac{(1 - e^{-T}) z^{-1}}{1 - e^{-T} z^{-1}}$$

3.4 Reconstructing Original Signals from Sampled Signals

Sampling theorem: Let us assume a continuous signal x(t) has the frequency spectrum as shown in figure below. This signal does not contain any frequency components above ω_1 rad/sec.



$\omega_s = \frac{2\pi}{T}$

Fig 3.10 (A frequency spectrum)

The theorem states that, "If $\omega_s = \frac{2\pi}{T}$, where T is the sampling period, is greater than $2\omega_m$ or

$\omega_s > 2\omega_m$

where $2\omega_m$ corresponds to the frequency spectrum of the continuous signal x(t), then the signal x(t) can be reconstructed completely from the sampled signal $x^*(t)$".

As we know that the Laplace transform of the sampled signal x*(t) is given by,

$$X^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(s + j\omega_s k) \qquad \text{.................(3.25)}$$

By substituting $s = j\omega$, we will get the frequency spectrum as below

$$|X^*(j\omega)| = \frac{1}{T}\left|\sum_{k} X(j\omega + j\omega_s k)\right|$$

$$= \ldots\ldots + \frac{1}{T}\left|X[j(\omega-\omega_s)]\right| + \frac{1}{T}\left|X(j\omega)\right| + \frac{1}{T}\left|X[j(\omega+\omega_s)]\right| + \ldots\ldots$$

The following Fig 3.11 shows the plot of $|X^*(j\omega)|$ versus ω.

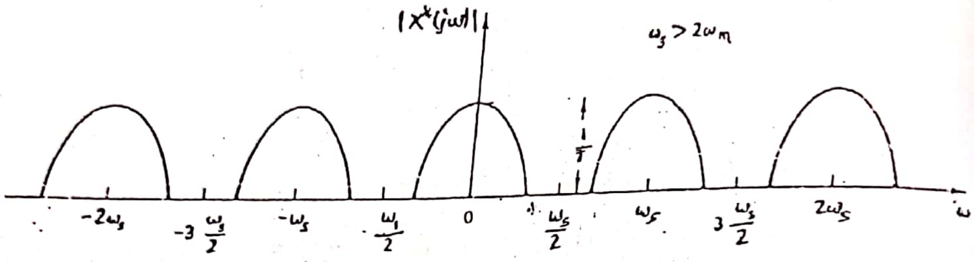


Fig 3.11 (Plot of the frequency spectrum $|X^*(j\omega)|$ versus ω for $\omega_s > 2\omega_1$)

Each plot of $|X^*(j\omega)|$ versus ω consists of $|X^*(j\omega)|$ repeated every $\omega_s = \frac{2\pi}{T}$ rad/sec. frequency spectrum of $|X^*(j\omega)|$, the component $|X(j\omega)|/T$ is called the primary component and the other component $|X[j(\omega \pm \omega_s k)]|/T$ is called complementary component.

If $\omega_s > 2\omega_1$, no two components of $|X^*(j\omega)|$ will overlap, and the sampled frequency spectrum will be repeated every ω_s rad/sec.

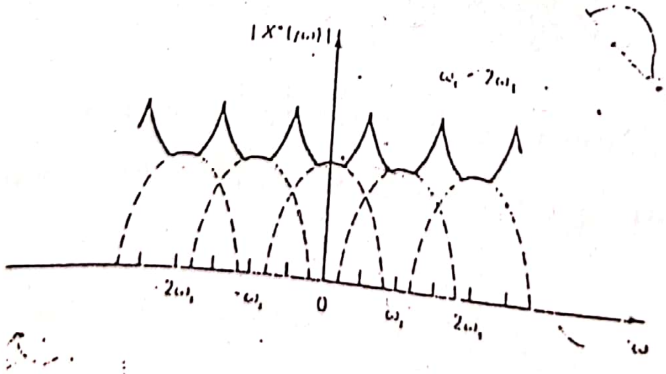


Fig 3.12 (Plot of the frequency spectrum $|X^*(j\omega)|$ versus ω for $\omega_s < 2\omega_1$)

If $\omega_s < 2\omega_1$, the original shape of $|X(j\omega)|$ no longer appears in the plot of $|X^*(j\omega)|$ versus ω because of the superposition of the spectrum. Hence, we see that the continuous time signal x(t) can be reconstructed from the impulse sampled signal x*(t) only if $\omega_s > 2\omega_1$.

3.5 The Pulse Transfer Function

The transfer function for the continuous-time system relates the Laplace transform of the continuous-time output to that of the continuous-time input, while the pulse transfer function relates the z transform of the output at the sampling instants to that of the sampled input.

Let us first discuss the convolution summation.

Convolution Summation: Let us consider a discrete-time system as shown in figure below.

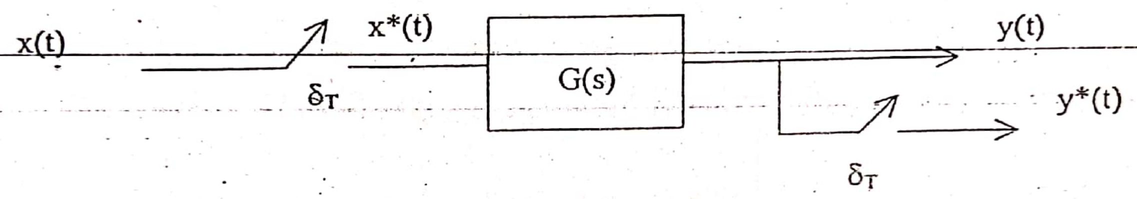


Fig 3.13 (Continuous – time system with an impulse sampler at the input)

Here, the sequence of impulses x*(t) is the input to the continuous-time plant whose transfer function is G(s). The output of the plant is a continuous signal y(t), If the output there is an another sampler, which is synchronized in phase with input sampler and operated at the same sampling period, then the output is a train of impulses.

If y(t) = 0 for t<0, then z transform of y(t) is

$$\mathcal{Z}[y(t)] = Y(z) = \sum_{k=0}^{\infty} y(kT)z^{-k} \qquad \text{.................(3.26)}$$

The output y(t) is related to the input x(t) by the convolution integral,

Or,

$$y(t)=\int_0^t g(t-\tau)x(\tau)d\tau \qquad \text{..........(3.27)}$$

$$=\int_0^t x(t-\tau)g(\tau)d\tau$$

Where g(t) is the impulse - response function of the system. For discrete-time sys... have a convolution summation, which is similar to the convolution integral. Since

$$x^*(t)=\sum_{k=0}^{\infty}x(t)\delta(t-kT)=\sum_{k=0}^{\infty}x(kT)\delta(t-kT) \qquad \text{..........(3.2...)}$$

is a train of impulses, the response y(t) of the system to the input x*(t) is the sum... individual impulse responses. Hence,

$$y(t)=\begin{cases} g(t)x(0), & 0\le t\le T \\ g(t)x(0)+g(t-T)x(T), & T\le t\le 2T \\ g(t)x(0)+g(t-T)x(T)+g(t-2T)x(2T), & 2T\le t\le 3T \\ \vdots & \\ g(t)x(0)+g(t-T)x(T)+\ldots\ldots\ldots+g(t-kT)x(kT), & kT\le t\le (k+1)T \end{cases}$$

Noting that for a physical system a response can not precede the input, we have g(t)=... t<0 or g(t-kT)=0 for t<kT. Hence the preceding equations may be combined i... equation:

$$y(t)=g(t)x(0)+g(t-T)x(T)+g(t-2T)x(2T)+\ldots\ldots\ldots\ldots+g(t-kT)x(kT)$$

$$=\sum_{h=0}^{k}g(t-hT)x(hT) \qquad 0\le t\le kT \qquad \text{..........(}$$

The values of the output y(t) at the sampling instants $t=kT$ $(k=0,1,2,3,\ldots\ldots\ldots)$ are given...

$$y(kT)=\sum_{h=0}^{k}g(kT-hT)x(hT) \qquad \text{..........(3.30)}$$

$$\sum_{h=0}^{k}x(kT-hT)g(hT) \qquad \text{..........(3.31)}$$

The summation in Eq.(3.30) or (3.31) is called the convolution summation. Note that the simplified notation

$$y(kT) = x(kT) * g(kT) \quad \text{.............. (3.32)}$$

is also used for the convolution summation.

Pulse Transfer function: Since we assumed that x(t)=0 for t<0, we have x(kT-hT)=0 for h>k. Also, since g(kT-hT)=0 for h>k, we may assume that the values of h in Eqs. (3.30) and (3.31) can be taken from 0 to ∞ rather than 0 to k without changing the value of the summation. Therefore, Eqs. (3.30) and (3.31) can be rewritten as,

$$y(kT) = \sum_{h=0}^{\infty} g(kT - hT)x(hT) \quad \text{...................... (3.33)}$$

$$= \sum_{h=0}^{\infty} x(kT - hT)g(hT) \quad \text{...................... (3.34)}$$

Where k=0, 1,2,..........

Hence, the z transform of y(kT) becomes

$$Y(z) = \sum_{k=0}^{\infty} y(kT)z^{-k}$$

$$= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} g(kT - hT)x(hT)z^{-k}$$

$$= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} g(mT)x(hT)z^{-(m+h)}$$

$$= \sum_{m=0}^{\infty} g(mT)z^{-m}\sum_{h=0}^{\infty} x(hT)z^{-h}$$

$$= G(z)X(z) \quad \text{.................. (3.35)}$$

Where m = k-h and

$$G(z) = \sum_{m=0}^{\infty} g(mT)z^{-m} = z \text{ transform of } g(t)$$

Eq. (3.35) relates the pulsed output Y(z) of the system to the pulse input X(z)
Eq.(3.35), we get

$$G(z) = \frac{Y(z)}{X(z)} \qquad \text{.........(}$$

Hence, the Eq. (3.36) gives the ratio of output Y(z) to the input X(z), called the pulse function of the discrete-time system. The figure below shows a block diagram for the transfer function G(z), together with the input X(z) and the output X(z).

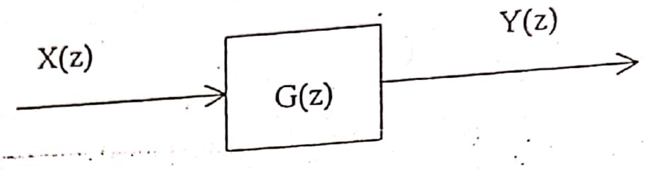


Fig 3.14 (Block diagram for a pulse-transfer-function system)

3.5.1 *Starred Laplace Transform of the Signal Involving Both Ordinary and Laplace Transforms:*

In analyzing discrete-time control systems, we often find that some signals in the sys starred (meaning that signals are impulse sampled) and others are not. To obtai transfer functions and to analyze discrete-time control systems, therefore, we must b obtain the transforms of output signals of systems that contain sampling operations in places in the loops. Let us consider an impulse-sampled system as shown in figure be

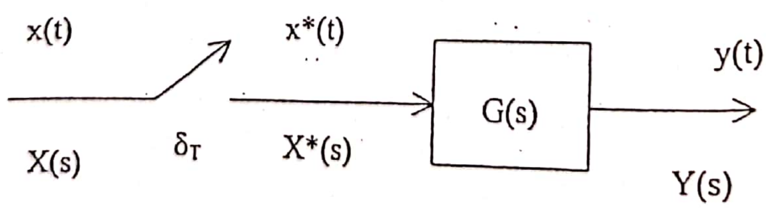


Fig 3.15 (Impulse-sampled system)

Then the output Y(s) is given by

$$Y(s) = G(s)X^*(s) \quad \text{.................(3.37)}$$

Here, Y(s) is the product of X*(s), which is periodic with period $\frac{2\pi}{\omega_s}$, and G(s), which is not periodic. The fact that the impulse-sampled signals are periodic can be seen from the fact that

$$X^*(s) = X^*(s \pm j\omega_s k), \qquad k = 0,1,2,....... \quad \text{.................(3.38)}$$

Let us now take the starred Laplace transform of Eq (3.37) we may factor out $X^*(s)$ so that,

$$Y^*(s) = [G(s)X^*(s)]^*$$
$$= [G(s)]^* X^*(s)$$
$$= G^*(s)X^*(s) \quad \text{............(3.39)}$$

This fact is very important in deriving the pulse transfer function and also in simplifying the block diagram of the discrete-time control system.

3.5.2 *General Procedures for Obtaining Pulse Transfer Functions:*

Let us consider a system that has an impulse sampler at the input as shown in figure (a). Let us obtain the pulse transfer function. The pulse transfer function G(z) of the system shown in figure (a) is

$$\frac{Y(z)}{X(z)} = G(z)$$

Next, consider the system as shown in figure (b). The transfer function G(s) is given by

$$\frac{Y(s)}{X(s)} = G(s)$$

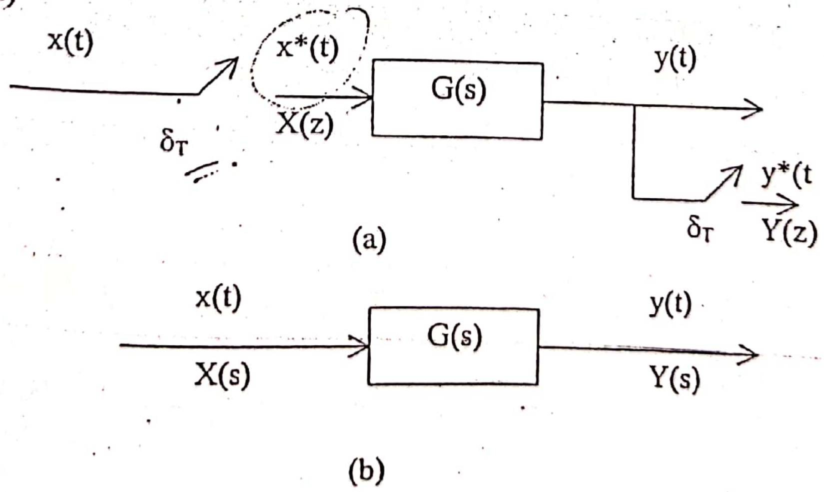


Fig 3.16 (Continuous-time system with an impulse sampler at the input and the continuous-time system)

The pulse transfer function of the system (b) is not equal to Z G(s) since there is no sampler. The presence or absence of the input sampler is crucial in determining the transfer function of a system.

Now, for system (a),

The Laplace transform of the output y(t) is

$$Y(s) = G(s)X^*(s)$$

Hence, by taking the starred Laplace transform of Y(s), we have

$$Y^*(s) = G^*(s)X^*(s)$$

or,

$$Y(z) = G(z)X(z)$$

For the system (b), the Laplace transform of the output y(t) is

$$Y(s) = G(s)X(s)$$

By taking starred Laplace transform, yields

$$Y^*(s) = [G(s)X(s)]^* = [GX(s)]^*$$

or,

$$Y(z) = GX(z) \neq G(z)X(z) \quad \} *$$

$y(z) = G\,x(z)$ --- I

$y(z) = G(z)\,X(z)$ --- II

$G\,x(z) \neq G(z)\,X(z)$

In discussing the pulse transfer function, we assume that there is a sampler at the input of the element in consideration. The presence or absence of a sampler at the output of the element (or the system) does not affect the pulse transfer function, because, if the sampler is not physically present at the output side of the system, it is always possible to assume that a fictitious sampler is present at the output. This means that, although the output signal is continuous, we can consider the values of the output only at $t = kT$ $(k = 0,1,2,\ldots\ldots)$ and get sequence y(kT).

Note that only for the case where the input to the system G(s) is an impulse sampled signal, the pulse transfer function given by

$$\boxed{G(z) = Z\,G(s)}$$

Example 4. Obtain the pulse transfer G(z) of the system where G(s) is given by

$$G(s) = \frac{1}{s+a}$$

Note that there is a sampler at the input of G(s).

Solution. Since there is a sampler at the input of G(s), the pulse transfer function is

$$G(z) = \mathcal{Z}[G(s)]$$

Here, $$ZG(s) = \mathcal{Z}\left[\frac{1}{s+a}\right]$$

$$= \frac{1}{1-e^{-aT}z^{-1}}$$

Hence, $$G(z) = \frac{1}{1-e^{-aT}z^{-1}}$$

Example 5. Obtain the pulse transfer function of the system where G(s) is given by

$$G(s) = \frac{1-e^{-Ts}}{s}\frac{1}{s(s+1)}$$

Note that there is a sampler at the input of G(s).

Solution. Since there is a sampler at the input of G(s), the pulse transfer function

$G(z) = \mathcal{Z}G(s)$.

Here, $$G(z) = \mathcal{Z}[G(s)] = \mathcal{Z}\left[(1-e^{-Ts})\frac{1}{s^2(s+1)}\right]$$

$$= (1-z^{-1})\mathcal{Z}\left[\frac{1}{s^2(s+1)}\right]$$

$$= (1-z^{-1})\mathcal{Z}\left[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right]$$

$$\frac{1}{1-z^{-1}}$$

$$\frac{Tz^{-1}}{(1-z^{-1})^2}$$

From the z transform table, we can write

$$G(z) = (1-z^{-1})\left[\frac{Tz^{-1}}{(1-z^{-1})^2} - \frac{1}{1-z^{-1}} + \frac{1}{1-e^{-T}z^{-1}}\right]$$

$$= \frac{(T-1+e^{-T})z^{-1} + (1-e^{-T}-Te^{-T})z^{-2}}{(1-z^{-1})(1-e^{-T}z^{-1})}$$

3.5.3 *Pulse Transfer Function of Cascaded Elements:*

Consider the systems shown in Fig 3.17 (a) and (b). Let us assume that the sam[e] synchronized and have the same sampling period. The pulse transfer function of Fig will be G(z) H(z) while the pulse transfer function of Fig 3.17 (b) will be GH(z), different from G(z) H(z).

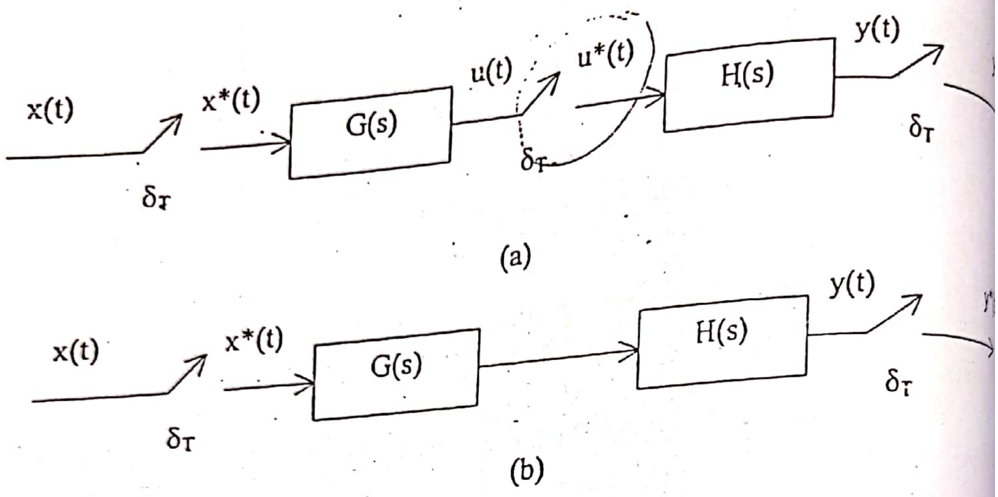


Fig 3.17 (Sampled system with a sampler between cascaded elements G(s) and H(s) and sampled system with no sampler between cascaded elements G(s) and H(s)

For Fig (a),

$$U(s) = G(s)X^*(s)$$

$$Y(s) = H(s)U^*(s)$$

Hence, by taking the starred Laplace transform of each of these two equations, we get

$$U^*(s) = G^*(s)X^*(s)$$

$$Y^*(s) = H^*(s)U^*(s)$$

Consequently,

$$Y^*(s) = H^*(s)U^*(s) = H^*(s)G^*(s)X^*(s)$$

or,

$$Y^*(s) = G^*(s)H^*(s)X^*(s)$$

In terms of z transform notation,

$$Y(z) = G(z)H(z)X(z)$$

Hence, the pulse transfer function is given by,

$$\frac{Y(z)}{X(z)} = G(z)H(z) \qquad \text{........(3.40)}$$

Let us now consider the system in figure (b). From the diagram we find

$$Y(s) = G(s)H(s)X^*(s) = GH(s)X^*(s)$$

Where,

$$GH(s) = G(s)H(s)$$

Taking the starred Laplace transform of Y(s), we have

$$Y^*(s) = [GH(s)]^* X^*(s)$$

In terms of z transform notation,

$$Y(z) = GH(z)X(z)$$

Hence, the pulse transfer function is given by,

$$\frac{Y(z)}{X(z)} = GH(z) \qquad \text{.......(3.41)}$$

Here, the pulse transfer functions for system (a) and (b) are different.

Example 6. Obtain the pulse transfer function for system (a) and (b) shown above where

$$G(s) = \frac{1}{s+a} \text{ and } H(s) = \frac{1}{s+b}$$

Solution. (G(S) H(S))

For system (a), the pulse transfer function is

$$\frac{Y(z)}{X(z)} = G(z)H(z)$$

Hence, $\frac{Y(z)}{X(z)} = G(z)H(z) = Z\left[\frac{1}{s+a}\right] Z\left[\frac{1}{s+b}\right]$

$$= \frac{1}{1-e^{-aT}z^{-1}} \frac{1}{1-e^{-bT}z^{-1}}$$

For system (b), the pulse transfer function is $\frac{Y(z)}{X(z)} = GH(z)$

Hence, $\frac{Y(z)}{X(z)} = \mathcal{Z}[G(s)H(s)] = \mathcal{Z}\left[\frac{1}{s+a}\frac{1}{s+b}\right]$

$$= \mathcal{Z}\left[\frac{1}{b-a}\left(\frac{1}{s+a} - \frac{1}{s+b}\right)\right]$$

$$= \frac{1}{b-a}\left(\frac{1}{1-e^{-aT}z^{-1}} - \frac{1}{1-e^{-bT}z^{-1}}\right)$$

$$= \frac{1}{b-a}\left[\frac{\left(e^{-aT} - e^{-bT}\right)z^{-1}}{\left(1-e^{-aT}z^{-1}\right)\left(1-e^{-bT}z^{-1}\right)}\right]$$

Clearly, we see that the pulse transfer function of two systems are different; that is,

$G(z)H(z) \neq GH(z)$.

3.5.4 *Pulse Transfer Function of Closed-loop Systems:*

Let us consider the closed-loop control system as shown in figure below.

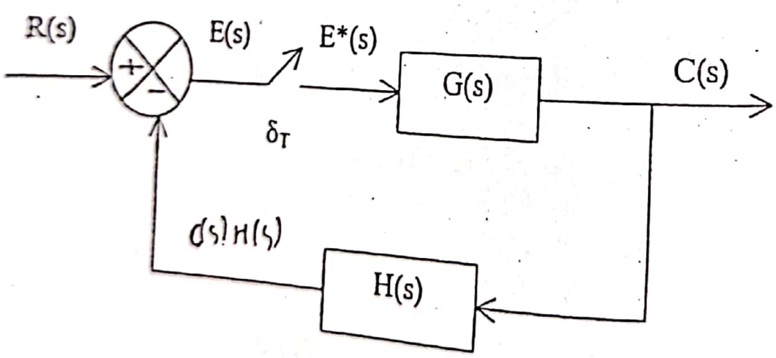


Fig 3.18 (Closed-loop control system)

From the diagram,

$$E(s) = R(s) - H(s)C(s)$$
$$C(s) = G(s)E^*(s)$$

Hence,

$$E(s) = R(s) - H(s)G(s)E^*(s)$$

Then, by taking the starred Laplace transform, we obtain

$$E^*(s) = R^*(s) - GH^*(s)E^*(s)$$

or,

$$E^*(s) = \frac{R^*(s)}{1 + GH^*(s)}$$

Since

$$C^*(s) = G^*(s)E^*(s)$$

We obtain

$$C^*(s) = \frac{G^*(s)R^*(s)}{1 + GH^*(s)}$$

In terms of the z transform notation,

$$C(z) = \frac{G(z)R(z)}{1 + GH(z)}$$

Hence, the pulse transfer function for the present closed-loop system is

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$$

3.5.5 Pulse Transfer Function of a Digital Controller:

The pulse transfer function of a digital controller may be obtained from the required input-output characteristics of the digital controller.

In general, the output m(k) may be given by the following type of difference equation:

$$m(k) + a_1 m(k-1) + a_2 m(k-2) + \dots\dots\dots + a_n m(k-n)$$

$$= b_0 e(k) + b_1 e(k-1) + \dots\dots\dots + b_n e(k-n) \qquad \dots\dots\dots(3.42)$$

Where e(k) = input to digital controller

Taking the z transform of Eq. (3.42), we get

$$M(z) + a_1 z^{-1} M(z) + a_2 z^{-2} M(z) + \dots\dots\dots + a_n z^{-n} M(z)$$

$$= b_0 E(z) + b_1 z^{-1} E(z) + \dots\dots\dots + b_n z^{-n} E(z)$$

or,

$$\left(1 + a_1 z^{-1} + a_2 z^{-2} + \dots\dots + a_n z^{-n}\right) M(z) = \left(b_0 + b_1 z^{-1} + \dots\dots + b_n z^{-n}\right) E(z)$$

The pulse transfer function $G_D(z)$ of the digital controller may then be given by

$$G_D(z) = \frac{M(z)}{E(z)} = \frac{b_0 + b_1 z^{-1} + \dots\dots + b_n z^{-n}}{1 + a_1 z^{-1} + \dots\dots + a_n z^{-n}} \quad \dots\dots (3.43)$$

3.5.6 *Closed-loop Pulse Transfer Function of a Digital Control System:*

The Fig. 3.19 (a) shows the block diagram of a digital control system. Here, the sampler converter, digital controller, zero-order hold, and D/A converter produce a continuous control signal u(t) to be fed to the plant. The Fig. 3.19(b) shows the transfer functions of blocks involved in the system.

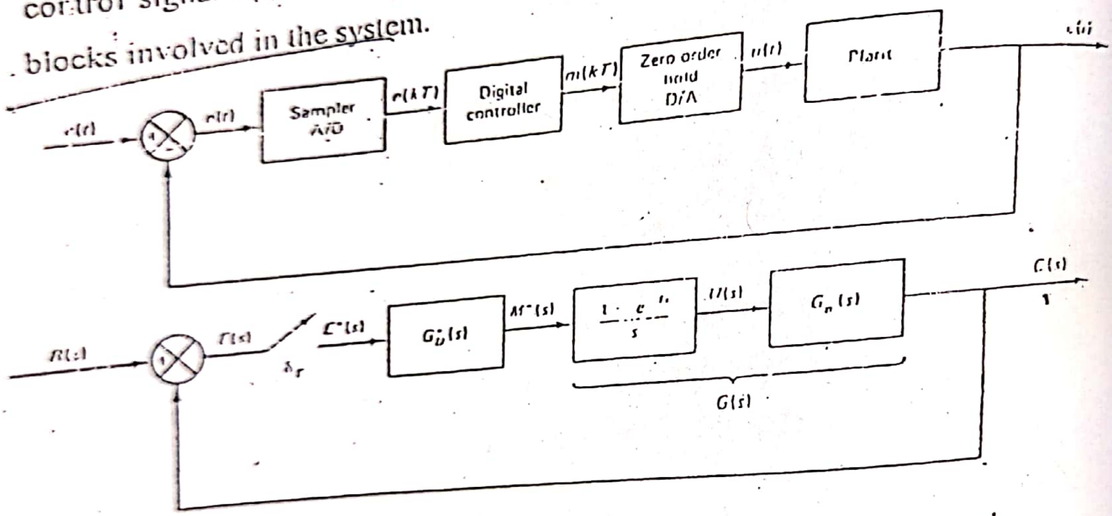


Fig 3.19 (Block diagram of a digital control system and equivalent block diagram showing transfer functions of blocks).

$G_D^*(s)$ represents the transfer function of digital controller.

Referring to the Fig 3.19 (b), let us define

$$\frac{1-e^{-Ts}}{s} G_p(s) = G(s)$$

Here,

$$C(s) = G(s)M^*(s)$$
$$= G(s)E^*(s)G_D^*(s)$$

or,

$$C^*(s) = G^*(s)E^*(s)G_D^*(s)$$

and.

$$E^*(s) = R^*(s) - C^*(s)$$

$$C^*(s) = G^*(s)G_D^*(s)[R^*(s) - C^*(s)]$$

or,

$$C^*(s)[1 + G^*(s)G_D^*(s)] = G^*(s)G_D^*(s)R^*(s)$$

or,

$$\frac{C^*(s)}{R^*(s)} = \frac{G^*(s)G_D^*(s)}{1 + G^*(s)G_D^*(s)}$$

$$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1 + G_D(z)G(z)}$$

This equation gives the closed-loop pulse transfer function of a digital controller.

3.5.1 *Transfer Function of a Digital PID Controller:*

Just as in continuous systems, there are three basic types of control: Proportional, Integral and Derivative, hence the name, PID.

Proportional Control: A discrete implementation of proportional control is identical to continuous, that is, where the continuous is

$$m(t) = Ke(t)$$

The discrete is

$$m(kt) = Ke(kt)$$

Taking the z transform, we get

$$M(z) = KE(z) \qquad \text{----- (3.44)}$$

Integral Control: In this, the control action is proportional to the integral of the actuating error. For continuous system, the output of the integral controller is given by,

$$m(t) = K\left[\frac{1}{T_i}\int_0^t e(t)dt\right]$$

The corresponding discrete equivalent is obtained by the trapezoidal summation as,

$$m(kt) = K\left\{\frac{T}{T_i}\left[\frac{e(0)+e(T)}{2}+\frac{e(T)+e(2T)}{2}+\ldots\ldots\ldots\ldots+\frac{e((k-1)T)+e(kT)}{2}\right]\right\}$$

or,

$$m(kT) = K\left[\frac{T}{T_i}\sum_{h=1}^{k}\frac{e((h-1)T)+e(hT)}{2}\right]$$

Define

$$\frac{e((h-1)T)+e(hT)}{2} = f(hT), \quad f(0)=0$$

Then

$$\sum_{h=1}^{k}\frac{e((h-1)T)+e(hT)}{2} = \sum_{h=1}^{k} f(hT)$$

Taking the z transform of this last equation, we get

$$\mathcal{Z}\left[\sum_{h=1}^{k}\frac{e((h-1)T)+e(hT)}{2}\right] = \mathcal{Z}\sum_{h=1}^{k} f(hT)$$

$$= \frac{1}{1-z^{-1}}\left[F(z)-f(0)\right]$$

$$= \frac{1}{1-z^{-1}}F(z)$$

$\sum_{h=1}^{k} f(hT)$

$= \sum_{h=0}^{\infty} f(hT) - f(0)$

$\frac{z^{-1}E(z)+E(z)}{2} =$

$F(z) = \frac{z^{-1}+1}{2} E(z)$

Notice that $F(z) = \mathcal{Z}[f(hT)] = \frac{1+z^{-1}}{2}E(z)$

Hence, $M(z) = K\left[\frac{T}{2T_i}\frac{1+z^{-1}}{1-z^{-1}}\right]E(z)$ (3.45)

Derivative Control: In this, the control action is proportional to the derivative actuating error signal. For the continuous system output of the derivative controller is by

$$m(t) = K\left[Td\frac{de(t)}{dt}\right]$$

We can obtain the discrete equivalent by approximating the derivative term by a two difference form.

$$m(kT) = K\left[Td\frac{e(kt)-e((k-1)T)}{T}\right]$$

e(kT)

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Taking the z transform we get,

$$M(z) = K\frac{Td}{T}(1-z^{-1})E(z)$$

----- (3.46)

Hence the output of the PID controller is obtained by adding the output due to proportional, integral and derivative controllers. Hence, we get

$$M(z) = K\left[1+\frac{T}{2T_i}\frac{1+z^{-1}}{1-z^{-1}}+\frac{T_d}{T}(1-z^{-1})\right]E(z)$$

----- (3.47)

The Eq (3.47) may be rewritten as follows:

$$M(z) = K\left[1-\frac{T}{2T_i}+\frac{T}{T_i}\frac{1}{1-z^{-1}}+\frac{T_d}{T}(1-z^{-1})\right]E(z)$$

$$=\left[K_p+\frac{K_I}{1-z^{-1}}+K_D(1-z^{-1})\right]E(z)$$

Where,

$$K_p = K-\frac{KT}{2T_i} = K-\frac{K_I}{2} = \text{Proportionl gain}$$

$$K_I = \frac{KT}{T_i} = \text{integral gain}$$

$$K_D = \frac{KT_d}{T} = \text{derivative gain}$$

Hence, the pulse transfer function for the PID controller becomes

$$\boxed{G_D(z) = \frac{M(z)}{E(z)} = K_p+\frac{K_I}{1-z^{-1}}+K_D(1-z^{-1})}$$

----- (3.48)

Example 7. Consider the control system with a digital PID controller shown in figure below. The transfer function of the plant is assumed to be

$$G_P(s) = \frac{1}{s(s+1)}$$ and the sampling period T is assumed to be 1 Sec.

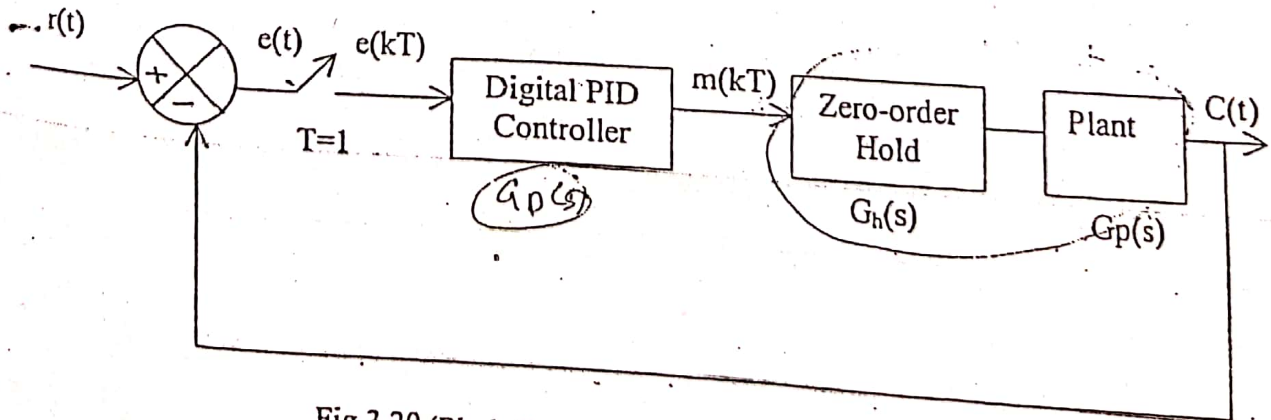

Fig 3.20 (Block diagram of a control system)

Also assume that $k_P = 1$, $k_I = 0.2$ and $k_D = 0.2$

Obtain the closed-loop pulse transfer function of the system.

Solution. The transfer function of the zero-order hold circuit is given by,

$$G_h(s) = \frac{1-e^{-s}}{s}$$

(Since the time period T = 1 Sec.)

Now, $$\mathcal{Z}\left[\frac{1-e^{-s}}{s}\frac{1}{s(s+1)}\right] = G(z) = \frac{0.3679z^{-1}+0.2642z^{-2}}{(1-0.3679z^{-1})(1-z^{-1})} \quad \text{---- (3.}$$

With $k_P = 1$, $k_I = 0.2$, and $k_D = 0.2$, the pulse transfer function of the digital becomes

$$G_D(z) = \frac{1.4-1.4z^{-1}+0.2z^{-2}}{1-z^{-1}} \quad \text{---- (3.5}$$

Hence, the closed loop pulse transfer function becomes

$$\boxed{\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1+G_D(z)G(z)}}$$ formula

$$= \frac{0.5151z^{-1}-0.1452z^{-2}-0.2963z^{-3}+0.0528z^{-4}}{1-1.8528z^{-1}+1.5906z^{-2}-0.6642z^{-3}+0.0528z^{-4}} \quad \text{---- (3.5}$$

3.6 Realization of Digital Controllers and Digital Filters

Realization of digital controllers and digital filters may involve either software or or both. The realization of pulse transfer function means determining the physical the appropriate combination of arithmetic and storage operations.

Computer programs for the digital computer will help to obtain the software whereas in a hardware realization we build a special-purpose processor using such as digital adders multipliers, and delay elements.

A digital filter processes a digital signal by passing desirable frequency component digital input signal and rejecting undesirable ones.

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In this section, we deal the block diagram realization of digital filters using delay elements adders and multipliers. This block diagram realization can be used as a basis for the design of software or hardware.

In the block diagram realization a pulse transfer function of z^{-1} represents a delay of one time unit.

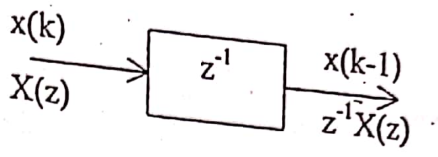


Fig 3.21 (Pulse transfer function showing a delay of one time unit)

The general form of a pulse transfer function between the output and the input is given by

$$G(z)=\frac{Y(z)}{X(z)}=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\dots\dots+b_mz^{-m}}{1+a_1z^{-1}+a_2z^{-2}+\dots\dots+a_nz^{-n}},\ n\geq m \qquad \text{-----(3.52)}$$

Where a_i's and b_i's are real coefficients (some of them may be zero). The pulse transfer function is in this form for many digital controllers.

As for example, the pulse transfer function for a PID controller is given by

$$G_D(z)=K_P+\frac{K_I}{1-z^{-1}}+K_D(1-z^{-1})$$

$$=\frac{\left[K_P(1-z^{-1})+K_I+K_D(1-z^{-1})^2\right]}{(1-z^{-1})}$$

$$=\frac{K_P-K_Pz^{-1}+K_I+K_D(1-2z^{-1}+z^{-2})}{(1-z^{-1})}$$

$$\text{or, } G_D(z)=\frac{(K_P+K_I+K_D)-(K_P+2K_D)z^{-1}+K_Dz^{-2}}{(1-z^{-1})}$$

$$=\frac{b_0+b_1z^{-1}+b_2z^{-2}}{1+a_1z^{-1}+a_2z^{-2}}$$

Where, $a_1=-1, \qquad a_2=0$

$b_0=K_P+K_I+K_D$

$b_1=-(K_P+2K_D)$

$b_2 = K_D$

3.6.1 *Direct Programming:*

In the direct programming method the coefficients a_i and b_i appear directly as multip... us consider the general form of the pulse transfer function,

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots\dots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots\dots + a_n z^{-n}}$$

or,

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \dots\dots\dots + a_n z^{-n} Y(z)$$
$$= b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \dots\dots\dots + b_m z^{-m} X(z)$$

or,

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \dots\dots\dots - a_n z^{-n} Y(z)$$
$$+ b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \dots\dots\dots + b_m z^{-m} X(z)$$

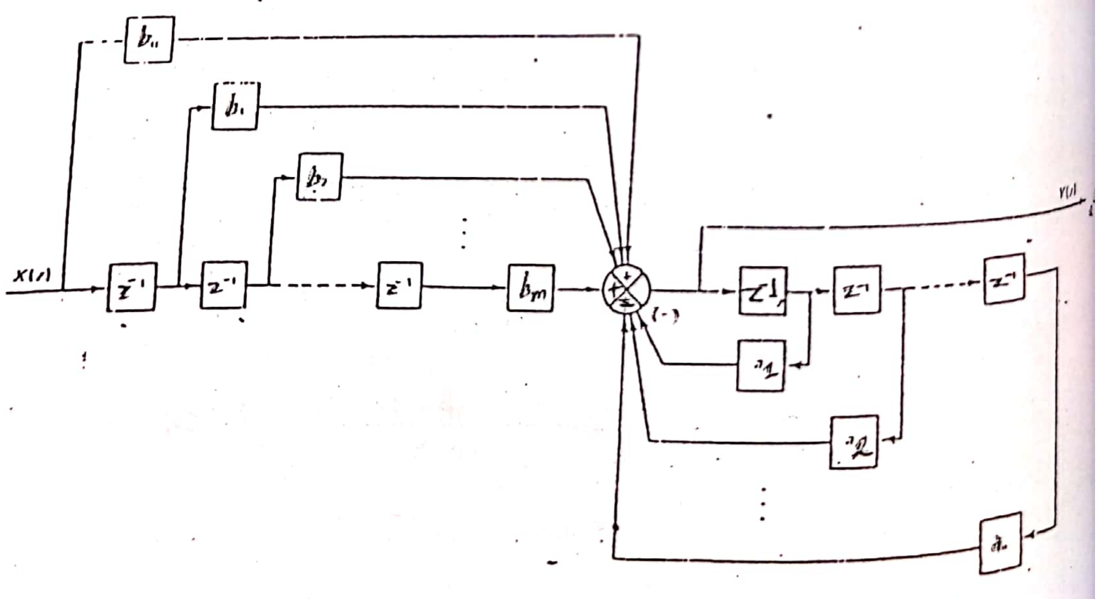


Fig 3.22 (Block diagram realization of a filter showing direct programming)

Hence, in direct programming realization, we realize the numerator and denominator of the pulse transfer function using separate sets of delay elements. The numerator uses a set of m delay elements and denominator uses a different set of n delay element. Thus the total number of delay elements used in direct programming is m+n.

3.6.2 *Standard Programming:*

The number of delay elements used in direct programming can be reduced from n+m to n (where $n \geq m$) with the help of standard programming method. This method uses a minimum possible number of delay elements. Let us consider the pulse transfer function as

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots\dots\dots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots\dots\dots + a_n z^{-n}} \quad \text{----- (3.53)}$$

Rearranging the pulse transfer function as follows:

$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{H(z)} \frac{H(z)}{X(z)}$$

$$= (b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots\dots\dots + b_m z^{-m}) \times \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots\dots + a_n z^{-n}} \quad \text{--- (3.54)}$$

Where,

$$\frac{Y(z)}{H(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots\dots\dots + b_m z^{-m} \quad \text{----- (3.55)}$$

and

$$\frac{H(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots\dots\dots + a_n z^{-n}} \quad \text{----- (3.56)}$$

Now, let us draw block diagrams for the systems given by Eqs (3.55) and (3.56).

Rewriting Eq (3.55),

$$Y(z) = b_0 H(z) + b_1 z^{-1} H(z) + b_2 z^{-2} H(z) + \dots\dots\dots + b_m z^{-m} H(z) \quad \text{----- (3.57)}$$

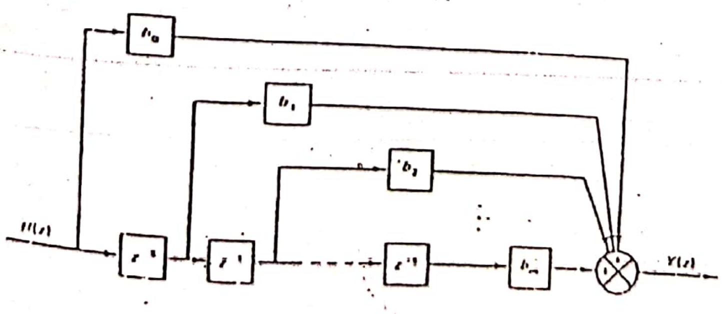


Fig 3.23 (Block diagram realization of Eq. 3.57)

Now, rewriting Eq (3.56),

$$H(z) = X(z) - a_1 z^{-1} H(z) - a_2 z^{-2} H(z) \; .... \; a_n z^{-n} H(z) \quad \text{-----(3.58)}$$

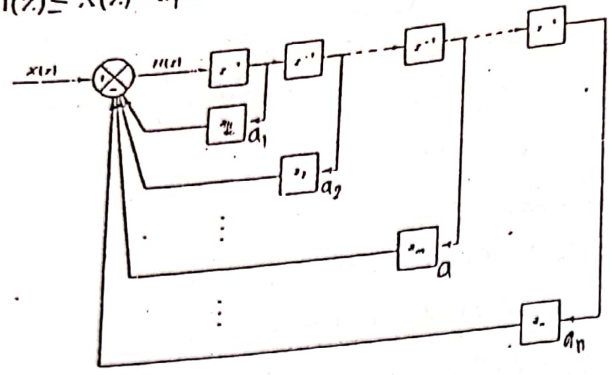


Fig 3.24 (Block diagram realization of Eq. 3.58)

Now, the combination of these two block diagrams will give the following block diagram

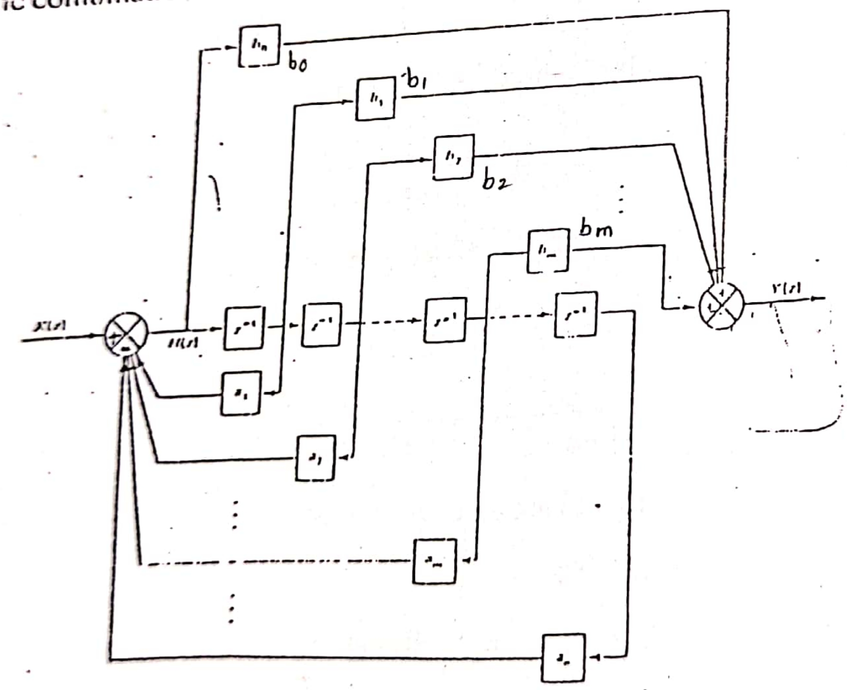


Fig 3.24 (Block diagram realization of Eq. 3.53) by standard programming)

Hence, in this method, we use only n delay elements. The coefficients a_1, a_2, ... a_n as feedback elements and the coefficients b_0, b_1, ... b_m appear as feed forward elements. Thus, the standard programming is preferred as it uses less number of delay elements.

3.6.3 *Series Programming:*

In the series programming method, we will implement the pulse transfer function $G(z)$ as a series connection of first-order and/or second-order pulse transfer functions. If $G(z)$ can be written as a product of pulse transfer functions $G_1(z)$, $G_2(z)$, ... $G_p(z)$, or

$$G(z) = G_1(z)G_2(z)..........G_p(z) \qquad \text{----- (3.59)}$$

Then the digital filter for $G(z)$ may be given as a series connection of the component digital filters $G_1(z)$, $G_2(z)$, ... $G_p(z)$, as shown in figure below:

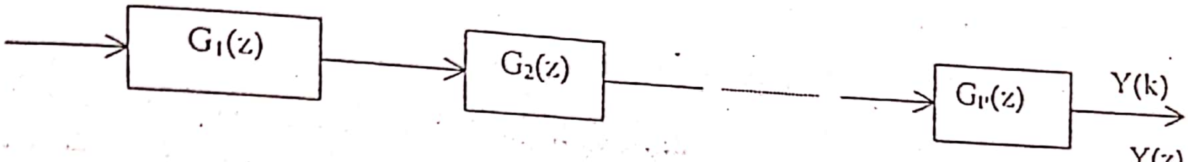


Fig 3.26 (Digital filter $G(z)$ decomposed into a series connection of $G_1(z)$, $G_2(z)$,......, $G_p(z)$)

Let us consider the pulse transfer function

$$G(z) = G_1(z)G_2(z)$$

$$= \frac{1+b_1 z^{-1}}{1+a_1 z^{-1}}\left(\frac{1+e_1 z^{-1}+f_1 z^{-2}}{1+c_1 z^{-1}+d_1 z^{-2}}\right) \qquad \text{----- (3.60)}$$

The block diagram representation of $\dfrac{Y_1(z)}{X_1(z)} = \dfrac{1+b_1 z^{-1}}{1+a_1 z^{-1}}$ is shown below.

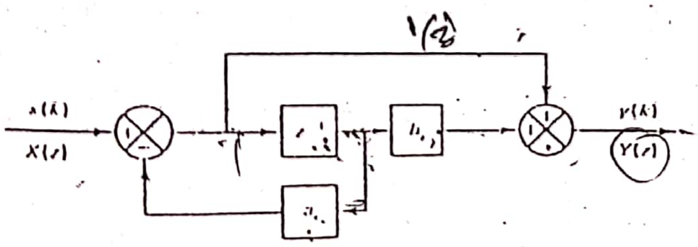


Fig 3.27

and that for $\dfrac{Y_2(z)}{X_2(z)} = \dfrac{1+e_1 z^{-1}+f_1 z^{-2}}{1+c_1 z^{-1}+d_1 z^{-2}}$ is shown below

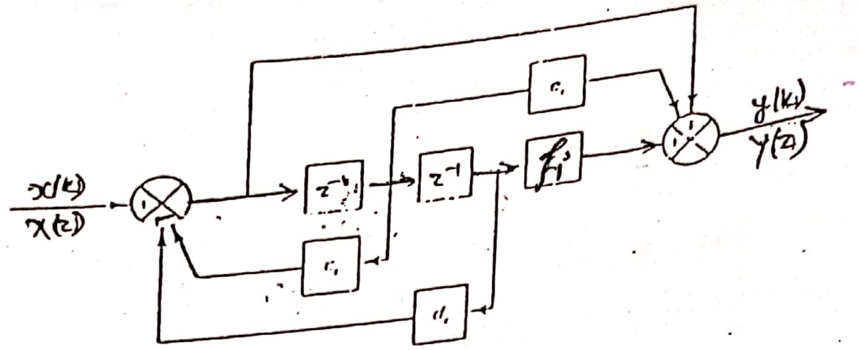


Fig 3.28

Thus by drawing these two diagrams in series, we get the block diagram representation series programming.

3.6.4 *Parallel Programming:*

In the parallel programming method the pulse transfer function is expanded into partial fractions. If G(z) is expanded on a sum of A, $G_1(z)$, $G_2(z)$,........$G_q(z)$ or,

$$G(z) = A + G_1(z) + G_2(z) + \dots\dots\dots + G_q(z) \qquad \text{----- (3.61)}$$

Where A is a constant then the block diagram for the digital filter G(z) can be obtained parallel connections of q+1 digital filters as shown in Fig 3.29.

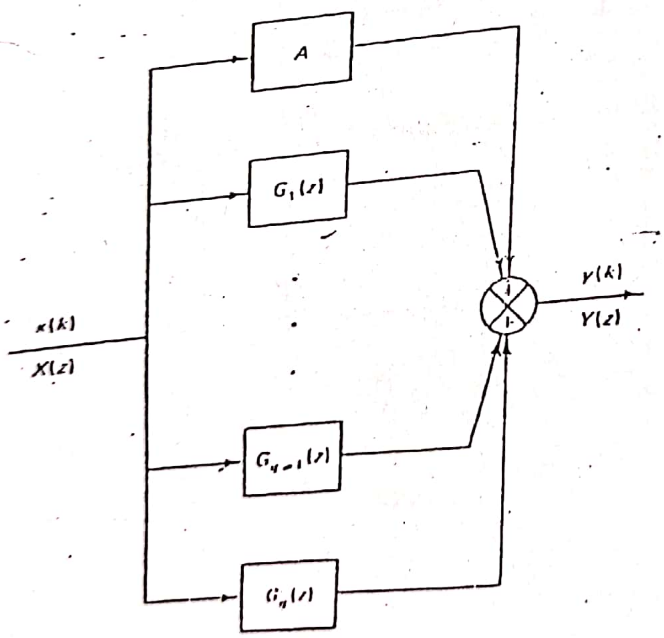


Fig 3.29 (Digital filter G(z) decomposed as a parallel connection of A, $G_1(z)$, $G_2(z)$...... $G_q(z)$)

Let us consider the pulse transfer function for the digital filter is

$$G(z) = \frac{b_i}{1 + a_i z^{-1}} + \frac{c_i + f_i z^{-1}}{1 + c_i z^{-1} + d_i z^{-2}}$$

Let us now draw the block diagram for

$$\frac{Y_i(z)}{X_i(z)} = \frac{b_i}{1+a_i z^{-1}}$$

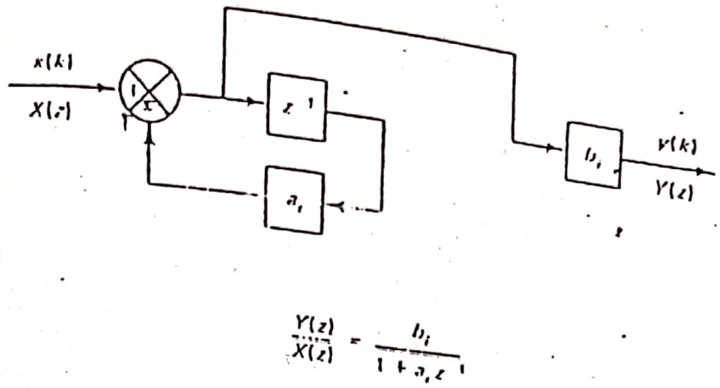


Fig 3.30

Again drawing the block diag

$$\frac{Y_2(z)}{X_2(z)} = \frac{e_i + f_i z^{-1}}{1 + c_i z^{-1} + d_i z^{-2}}$$

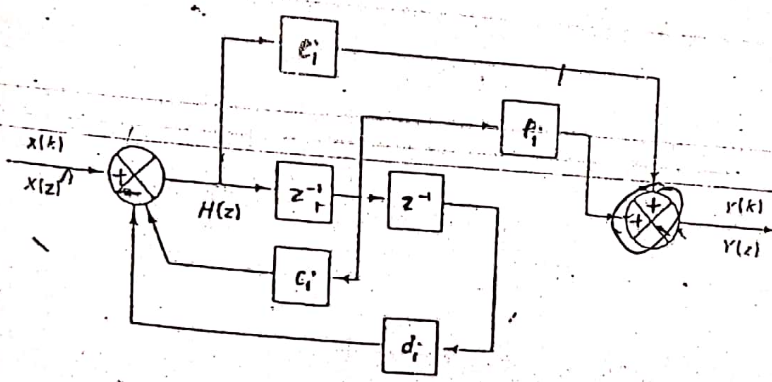


Fig 3.31

Now, by drawing these two diagrams in parallel, we get the block diagram representation for parallel programming.

3.6.5 *Ladder Programming:*

In this method, we expand the pulse transfer function G(z) into the following continued-fraction from.

$$G(z) = A_0 + \cfrac{1}{B_1 z + \cfrac{1}{A_1 + \cfrac{1}{B_2 z + \cfrac{1}{\ddots \; A_{n-1} + \cfrac{1}{B_n z + \cfrac{1}{A_n}}}}}} \qquad \text{----- (3.62)}$$

$A_0 + \frac{1}{B_1 z + \frac{1}{A_1 + \frac{1}{\ldots}}}$

The programming based on this scheme is known as ladder programming.

Let us define

$$G_i^{(B)}(z) = \frac{1}{B_i z + G_i^{(A)}(z)}, \qquad i = 1, 2, \ldots\ldots, n-1 \qquad \text{----- (3.63)}$$

$$G_i^{(A)}(z) = \frac{1}{A_i z + G_{i+1}^{(B)}(z)}, \qquad i = 1, 2, \ldots\ldots, n-1 \qquad \text{----- (3.64)}$$

$$G_n^{(B)}(z) = \cfrac{1}{B_n z + \cfrac{1}{A_n}}, \qquad \text{----- (3.65)}$$

Then G(z) may be written as

$$G(z) = A_0 + G_1^{(B)}(z) \qquad \text{----- (3.66)}$$

Let us now take a simple case when n = 2

$$G(z) = A_0 + \cfrac{1}{B_1 z + \cfrac{1}{A_1 + \cfrac{1}{B_2 z + \cfrac{1}{A_2}}}} \qquad \text{----- (3.67)}$$

Now, by the use of functions $G_1^{(A)}(z)$, $G_1^{(B)}(z)$ and $G_2^{(B)}(z)$, the transfer function G(z) written as follows:

$$G(z) = A_0 + \cfrac{1}{B_1 z + \cfrac{1}{A_1 + G_2^{(B)}(z)}}$$

$$= A_0 + \frac{1}{B_1 z + G_1^{(A)}(z)}$$

$$= A_0 + G_1^{(B)}(z)$$

----- (3.68)

Notice that $G_i^{(B)}(z)$ may be written as

$$G_i^{(B)}(z) = \frac{Y_i(z)}{X_i(z)} = \frac{1}{B_i z + G_i^{(A)}(z)} \quad \text{----- (3.69)}$$

or,

$$X_i(z) = Y_i(z)B_i z + Y_i(z)G_i^{(A)}(z)$$

or, $$Y_i(z) = \frac{1}{B_i} z^{-1} X_i(z) - \frac{1}{B_i} z^{-1} G_i^{(A)}(z) Y_i(z) \quad \text{----- (3.70)}$$

The block diagram for $G_i^{(B)}(z)$ given by Eq. (3.70) is shown below.

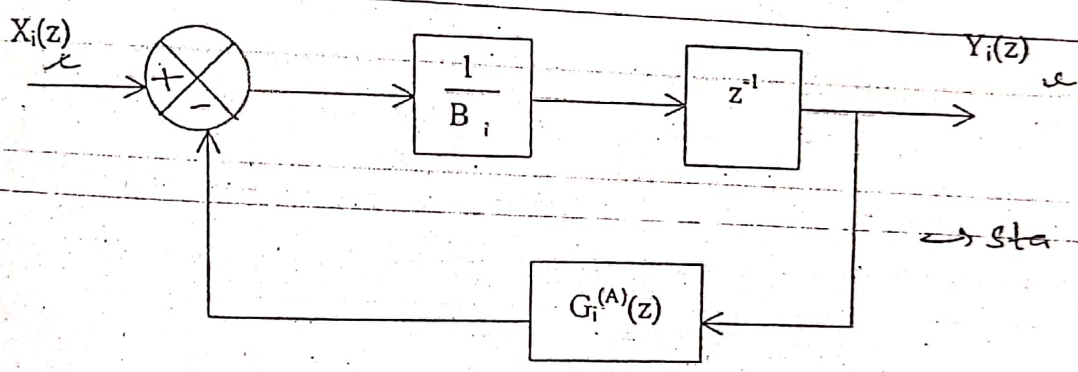


Fig 3.32 (Block diagram for $G_i^{(B)}(z)$ given by Eq. (3.70))

Now, $G_i^{(A)}(z)$ can be written as,

$$G_i^{(A)}(z) = \frac{1}{A_i + G_i + 1^{(B)}(z)} = \frac{Y_i(z)}{X_i(z)} \quad \text{----- (3.71)}$$

or,

$$Y_i(z)A_i + Y_i(z)G_{i+1}^{(B)}(z) = X_i(z)$$

or,

$$Y_i(z) = \frac{1}{A_i} X_i(z) - \frac{1}{A_i} Y_i(z) G_{i+1}^{(B)}(z) \quad \text{......(3.72)}$$

The block diagram for $G_i^{(A)}(z)$ given by Eq. (3.72) is shown below.

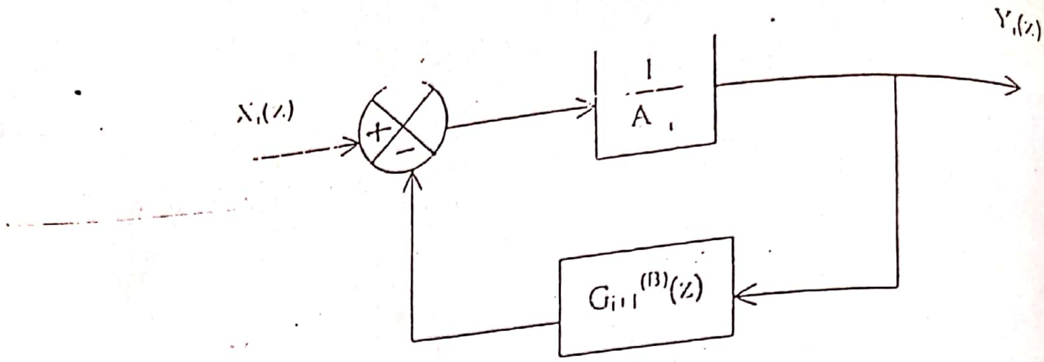
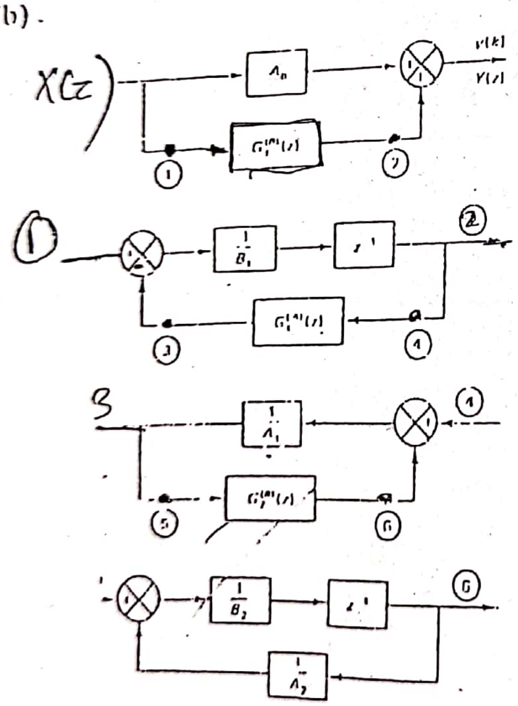
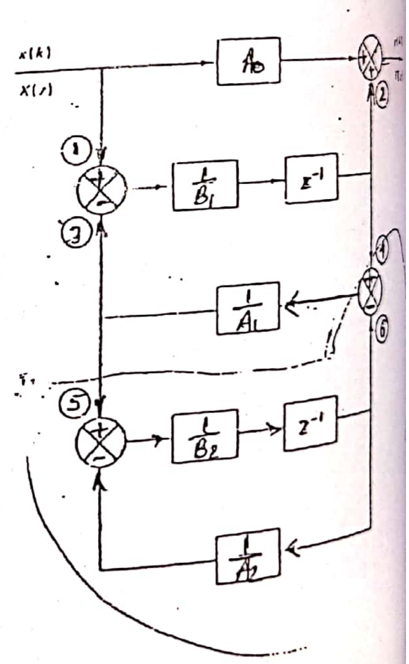


Fig 3.33 (Block diagram for $G_i^{(A)}(z)$ given by Eq. (3.72))

Hence for the case n = 2, by combining component digital filters as shown in figure below, it is possible to draw the block diagram of the digital filter G(z) as shown in Fig (b).



(a)

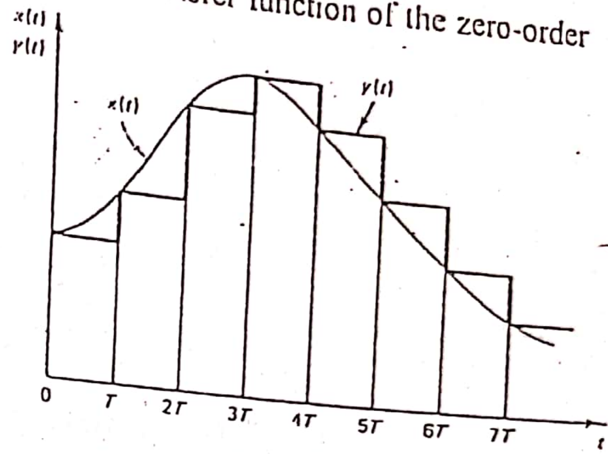


(b)

Fig 3.34 (Component block diagrams for ladder programming of G(z) given by Eq 3.62 when n=2 and combination of component block diagrams showing ladder programming of G(z)))

3.7. Problems

1. Consider a zero-order hold preceded by a sampler. The following figure shows the input $x(t)$ to the sampler and the output $y(t)$ of the zero-order hold. Obtain the expression for $y(t)$. Then find $Y(s)$ and obtain the transfer function of the zero-order hold.



Ans $Y(s) = \frac{1-e^{-Ts}}{s} X^*(s)$

$Gh_o(s) = \frac{1-e^{-Ts}}{s}$

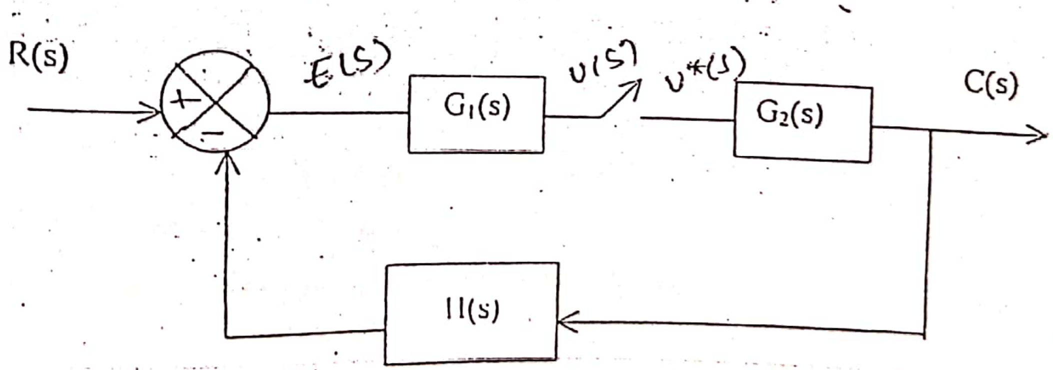
2. Consider the function $X(s) = \frac{1-e^{-Ts}}{s}$

show that $s = 0$ is not a pole of $X(s)$. Also show that $Y(s) = \frac{1-e^{-Ts}}{s^2}$ has a simple pole at $s = 0$.

3. Obtain the z transform of $X(s) = \frac{s}{(s+1)^2(s+2)}$ by using residue method.

Ans. $\frac{2-2e^{-T}z^{-1} - Te^{-T}z^{-1}}{(1-e^{-T}z^{-1})^2} - \frac{2}{1-e^{-2T}z^{-1}}$

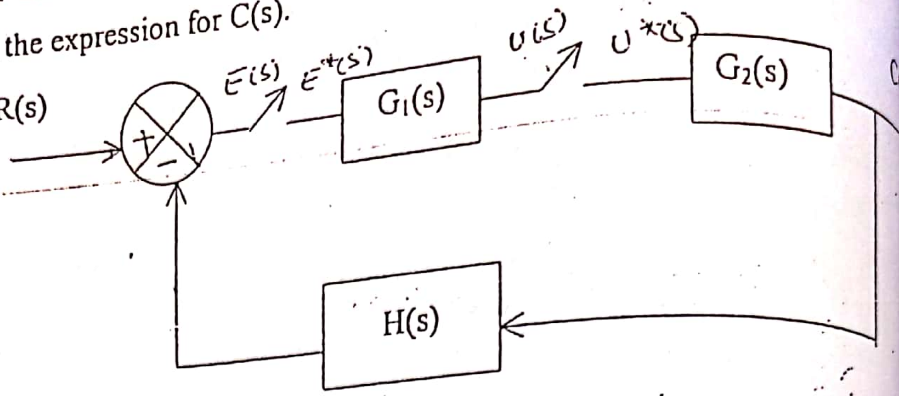
4. Obtain the discrete-time output $C(z)$ of the closed-loop control system shown in figure below. Also, obtain the continuous-time output $C(s)$.



Ans. $C(z) = \frac{G_2(z)G_1R(z)}{1+G_1G_2H(z)}$, $C(s) = G_2(s)\frac{[G_1R(s)]^*}{1+[G_1G_2H(s)]^*}$

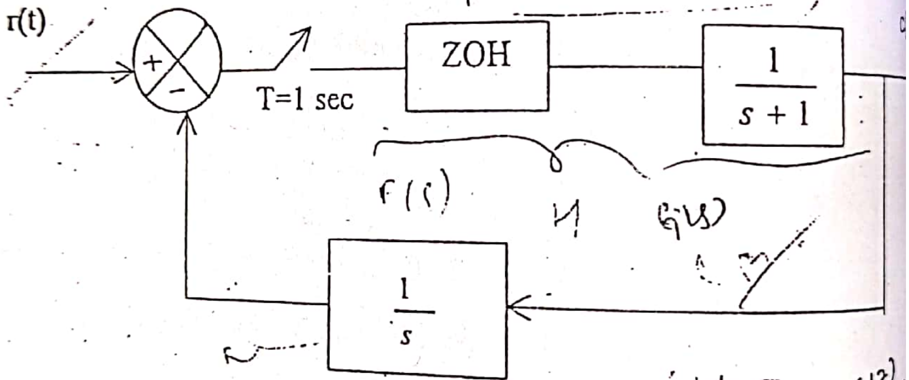
$C(z) = \frac{G_2(z)\, G_1R(z)}{1+G_1(z)\ldots}$

5. Obtain the closed-loop pulse transfer function of the system shown below. obtain the expression for C(s).



Ans. $\dfrac{C(z)}{R(z)} = \dfrac{G(z)G_2(z)}{1+G_1(z)G_2H(z)}$ $\quad C(s) = G_2(s)\dfrac{G_1^*(s)R^*(s)}{1+G_1^*(s)G_2H^*(s)}$

6. For the sampled data control system shown in figure below, find the output c(k) for r(t) = unit step.



Ans $c(k) = -j0.51(0.5+j0.62)^k + j0.51(0.5-j0.62)^k$

7. Consider the digital filter defined by

$$G(z) = \frac{Y(z)}{X(z)} = \frac{0.2+1.2z^{-1}+2.3z^{-2}+4z^{-3}}{1+5z^{-1}+6z^{-2}+7z^{-3}+8z^{-4}}$$

Draw the direct realization diagram and standard realization diagram.

8. Consider the digital filter defined by

$$G(z) = \frac{Y(z)}{X(z)} = \frac{3(z-1)(z^2+1.2z+1)}{(z+0.1)(z^2-0.3z+0.8)}$$

Draw a series realization diagram and a parallel realization diagram.

9\. Assume that a digital filter is given by the following difference equation:

$$y(k) + a_1 y(k-1) + a_2 y(k-2) = b_1 x(k) + b_2 x(k-1)$$

Draw block diagrams for the filter by using

(1) direct programming, (2) standard programming, and

(3) ladder programming.

10\. Consider the digital filter defined by

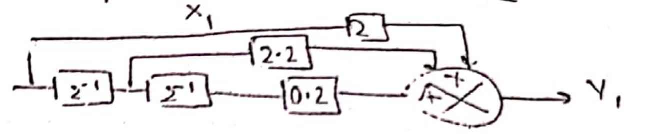
$$G(z) = \frac{2 + 2.2z^{-1} + 0.2z^{-2}}{1 + 0.4z^{-1} - 0.12z^{-2}} = \frac{Y(z)}{X(z)}$$ $\Rightarrow$ soln on next page

Realize this filter in the series scheme, the parallel scheme, and the ladder scheme.

(o) $\frac{Y(z)}{X(z)} = \frac{2 + 2.2z^{-1} + 0.2z^{-2}}{1 + 0.4z^{-1} - 0.12z^{-2}} = G_1(z) \cdot G_2(z)$

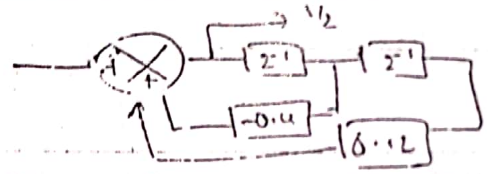
$2 + 2.2z^{-1} + 0.2z^{-2} \;\&\; \frac{1}{1 + 0.4z^{-1} - 0.12z^{-2}}$

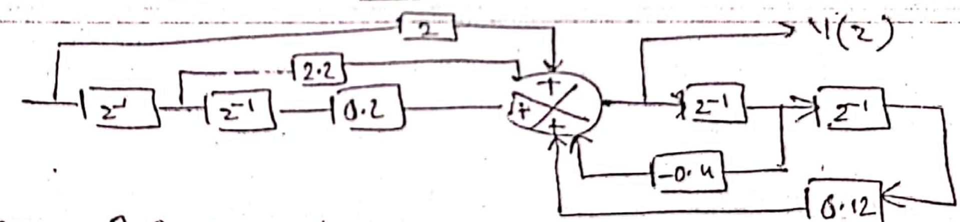
S: Series G_1: $\frac{Y_1}{X_1} = 2 + 2.2z^{-1} + 0.2z^{-2}$



$Y_1 = X_2$

$\frac{Y_2}{X_2} = \frac{1}{1 + 0.4z^{-1} - 0.12z^{-2}} \Rightarrow Y_2 = X_2 - 0.4z^{-1} + 0.12z^{-2}$



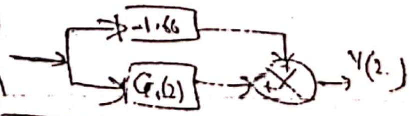


P: $\frac{G(z)}{z} = \frac{2z^2 + 2.2z + 0.2}{z(z^2 + 0.4z - 0.12)} = \frac{A}{z} + \frac{Bz + C}{z^2 + 0.4z - 0.12}$

$2z^2 + 2.2z + 0.2 = Az^2 + 0.4Az - 0.12A + Bz^2 + Cz$

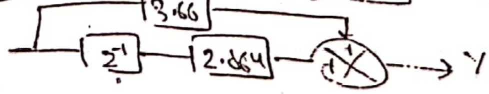
$2 = A + B$; $2.2 = 0.4A + C$; $A = -1.66$ $B = +3.66$, $C = 2.864$

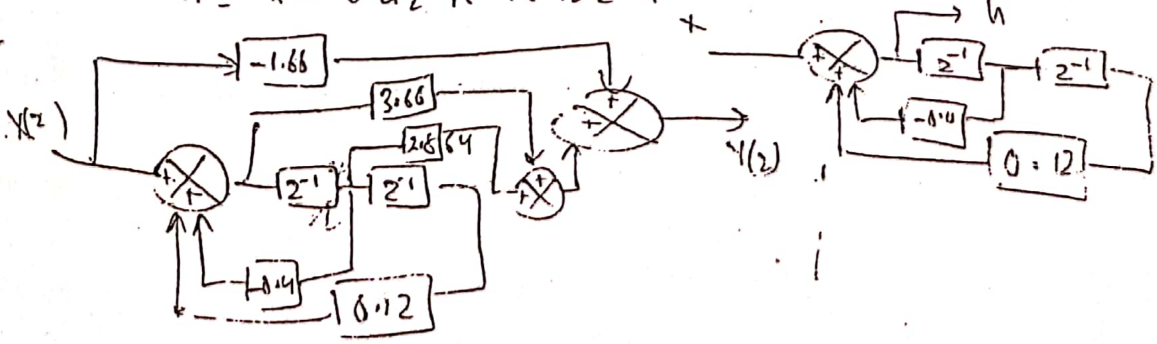
$G(z) = -1.66 + \left(\frac{3.66 + 2.864z^{-1}}{1 + 0.4z^{-1} - 0.12z^{-2}}\right) \rightarrow G_1(z)$



$G_1(z)$: $Y = H(3.66 + 2.864z^{-1})$

$H = X - 0.4z^{-1}H + 0.12z^{-2}H$





CHAPTER –IV

Design of Discrete-Time Control Systems by Conventi
Methods

4.1 Mapping of the Left Half of the s-Plane into the z-Plane

A linear dynamic system is stable if all the poles of the transfer function lie in the left the s-plane. In the z-plane the left-half s-plane corresponds to the unit circle centere origin or, the left-half s-plane maps into the inside of the unit in the z-plane.

Sincc, $z=e^{TS}$ and $s=\sigma+j\omega$

We can write,

$$z=e^{T(\sigma+j\omega)}$$

$$=e^{T\sigma}e^{Tj\omega}$$

$$\therefore |z|=e^{T\sigma}$$

and $\angle z=T\omega$.

Since σ is negative in the left-half of the s-plane, the left-half of the s-plane correspon

$$|z|=e^{T\sigma}<1.$$

The $j\omega$ axis in the s-plane where $\sigma=0$, corresponds to the unit circle in the z-plane (z= is, the imaginary axis in the s-plane corresponds to the unit circle in the z-plane, interior of the unit circle corresponds to the left-half of the s-plane.

Note that since $\angle z=\omega T$ the angle of z varies from $-\infty$ to ∞ as ω varies from – Consider a representative point on the $j\omega$ axis in the s-plane. As the point moves $-j\frac{1}{2}\omega_s$ to $j\frac{1}{2}\omega_s$ on the $j\omega$ axis, where ω_s is the sampling frequency, we have $|z|=1$, varies from $-\pi$ to π in the counterclockwise direction in the z-plane. As the point mo $j\frac{1}{2}\omega s$ to $j\frac{3}{2}\omega s$ on the $j\omega$ axis, the corresponding point in the z-plane traces out circle once in the counter clockwise direction. Thus, as the point in the s-plane moves from $-\infty$ to ∞, on the $j\omega$ axis, we trace the unit circle in the z-plane an infinite number of times. Hence, it is clear that each strip of width ω_s in the left-half of s-plane maps into the inside of the unit circle in the z-plane. This implies that the left-half of the s-plane may be divided into an infinite no. of periodic strip as shown below. The primary strip extends from $-j\frac{1}{2}\omega_s$ to $j\frac{1}{2}\omega_s$. The complementary strip extend from $j\frac{1}{2}\omega_s$ to $j\frac{3}{2}\omega_s$, $j\frac{3}{2}\omega_s$ to $j\frac{5}{2}\omega_s$ ...... and from $-j\frac{1}{2}\omega_s$ to $-j\frac{3}{2}\omega_s$, ...

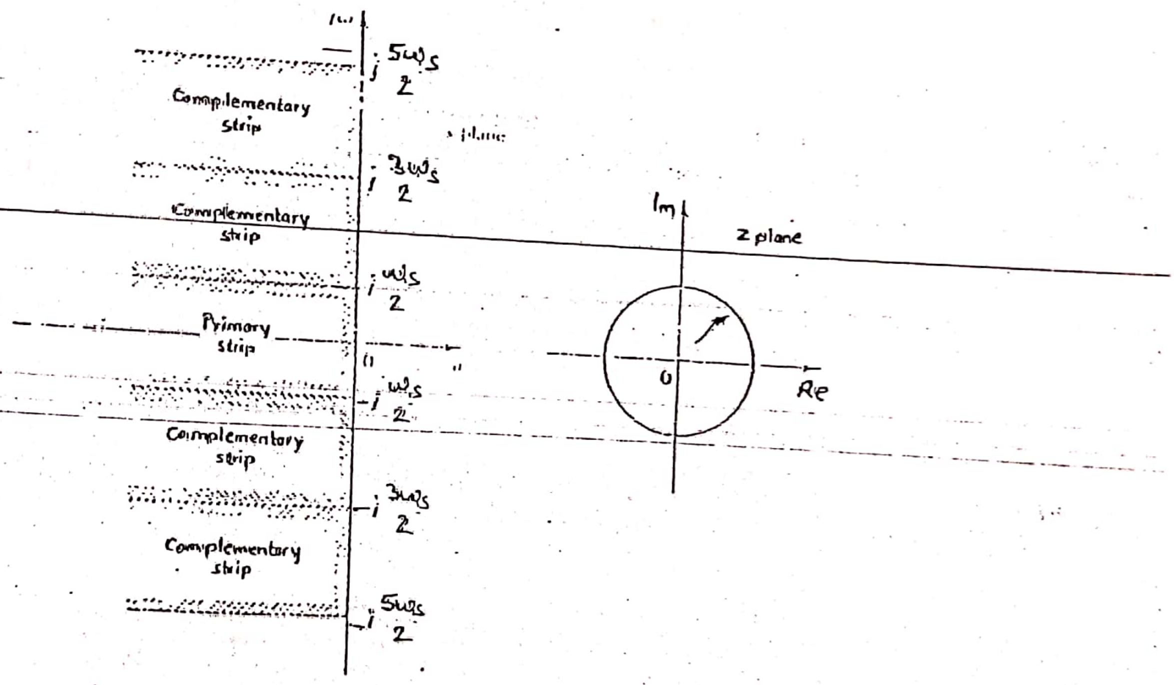


Fig 4.1 (Periodic strips in the s-plane and the corresponding region in the z-plane)

4.2 Stability Analysis of Closed-Loop System in the z-Plane

Stability analysis of a closed loop system :-

Let us consider a closed-loop discrete time system as shown below:-

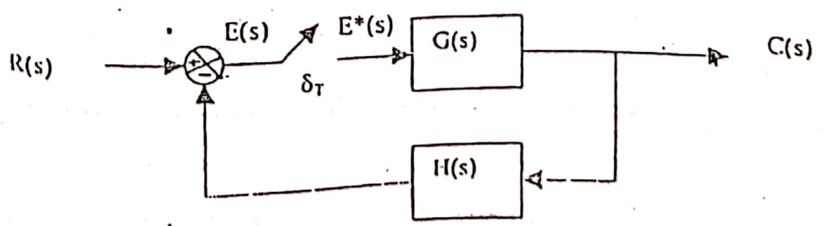


Fig 4.2 (Closed loop control system)

The pulse transfer function of this closed loop system is,

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)}$$

The stability of the system defined by Eq (4.1) as well as other types of discrete-time co
systems may be determined from the locations of the closed loop poles in the z-plane, or
roots of the characteristic equation.

$$P(z) = 1 + GH(z) = 0$$

as follows:

1. For the system to be stable, the closed-loop poles or the roots of the characte equation must lie within the unit circle in the z-plane. Any closed-loop pole ou the unit circle makes the system unstable.
2. If a simple pole lies at z=1, then the system becomes critically stable. Also system becomes critically stable if a single pair of conjugate complex poles lie o unit circle in the z-plane. Any multiple closed-loop pole on the unit circle make system unstable.
3. Closed-loop zeroes do not affect the absolute stability and therefore may be lo anywhere in the z-plane.

Example 1. Consider the closed-loop control system shown below. Determine the stabil the system when k=1.

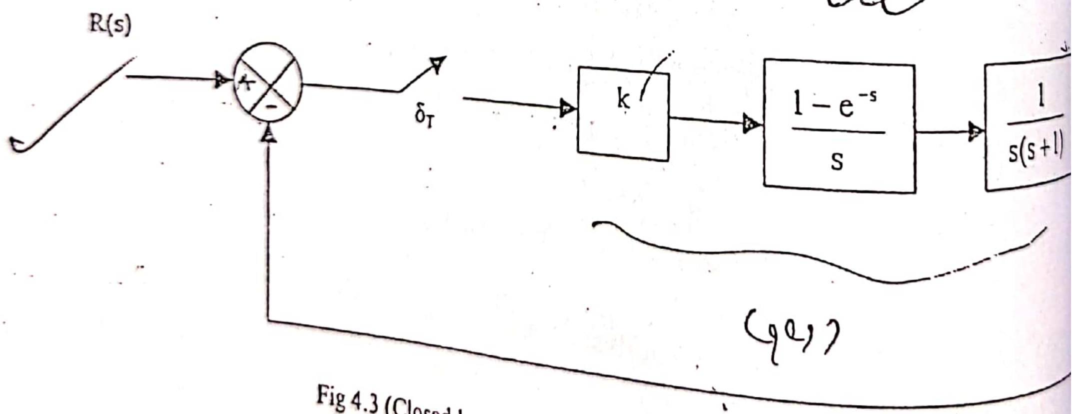


Fig 4.3 (Closed loop control system)

Solution.

$$G(s) = \frac{1-e^{-s}}{s}\,\frac{1}{s(s+1)}$$

$z = e^{TS}$

Now, $G(s) = \mathcal{Z}\left[\frac{1-e^{-s}}{s} \cdot \frac{1}{s(s+1)}\right]$

$$= \frac{0.3679z + 0.2642}{(z-0.3679)(z-1)}$$

The closed loop pulse transfer function of the given system is,

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+G(z)}$$

Hence, the characteristic equation is,

$$1+G(z)=0.$$

which becomes,

$$(z-0.3679)(z-1) + 0.3679z + 0.2642 = 0$$

Oor, $z^2 - z + 0.6321 = 0.$

The roots of the char equation are found to be

$z_1 = 0.5 + j\,0.6181$ and $z_2 = 0.5 - j\,0.6181$

Since, $|z_1| = |z_2| < 1.$

The system is stable.

4.3 Methods for Testing Absolute Stability

Jury Stability Test:- It is an algebraic criterion for determining whether or not the roots of the characteristic polynomial lie within a unit circle there by determining system stability. Consider that the characteristic polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_0, \qquad a_n > 0$$

The Jury's test consists of wo parts.

(1) A simple test for necessity.

(2) A second test for sufficiency.

The necessary condition for stability are

$P(1) > 0;$ $\quad P(-1) > 0$ for n even

$\quad < 0$ for n odd.

The sufficient condition for stability can be established through the following method. Prepare a table of coefficients of the characteristic polynomial as below:

Row	z^0	z^1	z^2	z^3	....	z^{n-2}	z^{n-1}
1	a_0	a_1	a_2	a_3	....	a_{n-2}	a_{n-1}
2	a_n	a_{n-1}	a_{n-2}	a_{n-3}	....	a_2	a_1
3	b_0	b_1	b_2	b_3	....	b_{n-2}	b_{n-1}
4	b_{n-1}	b_{n-2}	b_{n-3}	b_{n-4}	....	b_1	b_0
5	c_0	c_1	c_2	c_3	....	c_{n-2}	
6	c_{n-2}	c_{n-3}	c_{n-4}	c_{n-5}	....	c_0	
⋮	⋮	⋮	⋮	⋮			
2n-5	p_0	p_1	p_2	p_3			
2n-4	p_3	p_2	p_1	p_0			
2n-3	q_0	q_1	q_2				

where,

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}$$

$$b_0 = \begin{vmatrix} a_0 & a_n \\ a_n & a_0 \end{vmatrix}$$

$$c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix}$$

$$b_1 = \begin{vmatrix} a_0 & a_{n-1} \\ a_n & a_1 \end{vmatrix}$$

The sufficient condition for stability are.

$|a_0| < |a_n|$

$|b_0| > |b_{n-1}|$

$|c_0| > |c_{n-2}|$

$|q_0| > |q_2|$

$$c_1 = \begin{vmatrix} b_0 & b_{n-2} \\ b_{n-1} & b_1 \end{vmatrix}$$

Example2. Examine the stability of the following characteristic equation.

$$P(z) = 2z^4 + 7z^3 + 10z^2 + 4z + 1 = 0$$

Solution. For this characteristic equation.

$P(1) = 2+7+10+4+1 = 24 > 0$ satisfied

$P(-1) = 2-7+10-4+1 = 2 > 0$ satisfied

Row	Z^0	Z^1	Z^2	Z^3	Z^4
1	1	4	10	7	2
2	2	7	10	4	1
3	-3	-10	-10	-1	
4	-1	-10	-10	-3	
5	8	20	20		

The sufficient condition are,

$|a_0| < |a_4|$ i.e. $1 < 2$

$|-3| > |-1|$ satisfied

$|8| > |20|$ not satisfied

The system is therefore unstable.

Example 3. Examine the stability of the following characteristic equation :

$$P(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08 = 0$$

Solution . The necessary conditions are,

$P(1) = 1 - 1.2 + 0.07 + 0.3 - 0.3 - 0.08 = 0.09 > 0$

$P(-1) = 1 + 1.2 + 0.07 - 0.3 - 0.08$

$= 1.89 > 0,$

satisfied (for n even)

For the sufficient condition let us make a table as shown below.

Row	Z^0	Z^1	Z^2	Z^3	Z^4
1	-0.08	0.3	0.07	-1.2	1
2	1	-1.2	0.07	0.3	-0.08
3	-0.994	1.176	-0.0756	-0.204	
4	-0.204	-0.0756	1.176	-0.994	
5	0.946	-1.184	0.315		

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$a_0 = -0.08$, $a_4 = 1$

$|a_0| < |a_4|$, satisfied

b_0=- 0.994, b_3=-0.204

$|b_0| > |b_3|$, satisfied

c_0=0.946, c_2=0.315

$|c_0| > |c_2|$; satisfied

Hence, the system is satisfied i.e. all the roots lie inside the unit circle in the z-plane.

Example 4. Examine the stability of the following characteristic equation.

$$P(z)=z^3-1.1z^2-0.1z+0.2=0 \qquad \text{.........(4.2)}$$

Solution. $P(1)=1-1.1-0.1+0.2=0$

$P(-1)=-1.8<0$,n=3= odd, satisfied.

The Eq (4.2) indicates that at least one root is at z=1. Therefore the system can be critically stable. The remaining tests determine whether the system is critically stable or unstable.

Row	Z^0	Z^1	Z^2	Z^3
1	0.2	-0.1	-1.1	1
2	1	-1.1	-0.1	0.2
3	-0.96	-0.12	-0.12	

Here,

$|0.2|<1$ and

$|-0.9|>|-0.12|$ conditions are satisfied.

Hence, the system is critically stable.

Example 5. $G(z)=\dfrac{k(0.3769z+0.2642)}{(z-0.3679)(z-1)}$

Consider the discrete-time unity feedback control system (with sampling period T=1) Whose open loop pulse transfer function is given above. Determine the range of gain for the stability by use of the jury stability test and also find the frequency of sustained oscillation.

Solution .The closed loop transfer function is,

$$\frac{C(z)}{R(z)} = \frac{k(0.3679z + 0.2642)}{z^2 + (0.3679k - 1.3679)z + 0.3679 + 0.2642k}$$

Thus, the characteristic equation is,

$$P(z) = z^2 + (0.3679k - 1.3679)z + 0.3679 + 0.2642k = 0$$

Applying the condition,

$$|a_0| < |a_2|$$

or, $|(0.3679 + 0.2642\,k)| < 1$

or, $k < 2.3925$

The second condition is, ...(4.3)

$P(1) > 0$, $P(-1) > 0$, for $n=2$

So, $P(1) = 1 + 0.3679k - 1.3679 + 0.3679 + 0.2642k > 0$

or, $0.6321\,k > 0$

or, $k > 0$ ....(4.4)

~~Again, P(-1) = 1-0.3679 k + 1.3679 + .3679 + 0.2642 k~~

$= 2.7358 - 0.1037\,k > 0$

which fields,

$26.382 > k$. ....(4.5)

Hence, from the inequalities (4.3), (4.4) and (4.5) for the system to be stable,

$2.3925 > k > 0$.

Hence, the range of k for stability is between 0 and 2.3925.

If gain k=2.3925 is set then the system becomes critically stable that mean the sustained oscillations exist at the output.

with k=2.3925, the characteristic equation becomes,

$$z^2 - 0.4877z + 1 = 0$$

$$z = 0.2439 \pm j0.9698$$

Now, $\boxed{\omega_d = \frac{\omega_s}{2\pi}\angle z = \frac{2\pi}{2\pi}\angle z}$ $[T = 1\text{sec}]$

$$= \tan^{-1}\frac{0.9698}{0.2439} = 1.3244 \quad \text{rad/see}$$

Note : In s-plane, a constant damping-ratio can be given by,

$$s = -\delta\omega_n + j\omega_n\sqrt{1-\delta^2} = -\delta\omega_n + j\omega_d$$

$s = -\zeta\omega_n - j\omega_n\sqrt{1-}$

Where,

$$\omega_d = \omega_n\sqrt{1-\delta^2}$$

$\omega_d = \frac{\omega_s}{2\pi}$

Now, in z-plane,

$$z = e^{TS} = \exp(-\delta\omega_n T + j\omega_d T)$$

$$= \exp\left(-\frac{2\pi\delta}{\sqrt{1-\delta^2}}\frac{\omega_d}{\omega_s} + j2\pi\frac{\omega_d}{\omega_s}\right)$$

Hence, $$|z| = \exp\left(-\frac{2\pi\delta}{\sqrt{1-\delta^2}}\frac{\omega_d}{\omega_s}\right)$$

$$< z = 2\pi\frac{\omega_d}{\omega_s}$$

4.4 Stability Analysis by use of the Bilinear Transformation and Routh Stability Criterion

In the stability analysis of discrete time control system, another frequently used meth Bilinear transformation-coupled with R-H criterion. This, method requires transform from z-plane to ω-plane then apply R-H criterion.

The bilinear transformation is defined by

$$z = \frac{\omega+1}{\omega-1}$$

which gives, $$\omega = \frac{z+1}{z-1}$$

Example 6 . Use bilinear transformation and R-H stability criterion to determine the sta condition for the following characteristic equation.

$$P(z) = z^3 - 1.3z^2 - 0.08z + 0.24 = 0$$

Solution . Let us substitute, $z = \frac{\omega+1}{\omega-1}$ we get,

$$\left(\frac{\omega+1}{\omega-1}\right)^3 - 1.3\left(\frac{\omega+1}{\omega-1}\right)^2 - 0.08\left(\frac{\omega+1}{\omega-1}\right) + 0.24 = 0$$

or, $$-0.14\omega^3 + 1.06\omega^2 + 5.10\omega + 1.98 = 0$$

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or, $\omega^3 - 7.571\omega^2 - 36.43\omega - 14.14 = 0$

The Routh array :

ω^3	1	-36.42
ω^2	-7.571	-14.14
ω^1	-36.30	0
ω^0	-14.14	

-38 · 28 $\quad \dfrac{+36.30 \times 14.14}{-36.30}$

There is one sign change from 1 to -7.571. The system is unstable.

There is one sign change in first column, that means one root lie in the right half of the ω-plane or one root lie outside the unit circle in the z-plane.

4.5 Transient and Steady-State Response Analysis

The unit-step response curve showing the transient response specification t_d, t_r, t_p, M_p and t_s is shown here.

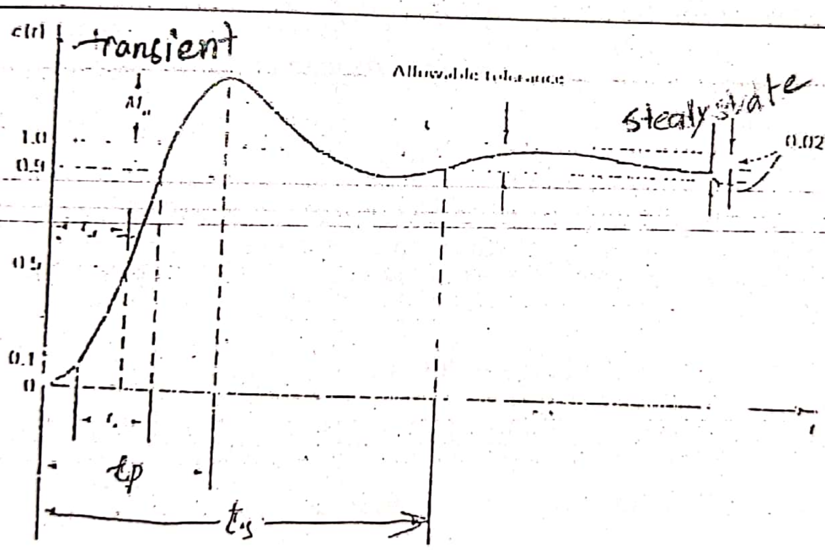

Fig 4.4 (Unit step response curve showing transient response specifications t_d, t_r, t_p, M_p, and t_s)

1. ***Delay time*** **:-** It is the time required to reach half the final value at the very first time. It is denoted by t_d.
2. ***Rise time*** **:-** For over damped system it is from 0% to 100%. For under damped system it is from 10% to 90% rise time is commonly used. It is denoted by t_r.
3. ***Peak time*** **:-** Time required for the response to reach the first peak of the overshoot. It is denoted by t_p.

4. *Maximum overshoot :-* It is the maximum peak value of the response c... measured from unity. If the final value of the response differs from unity the maximum overshoot is commonly used as below.

$$\%\ \text{Maximum overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

5. *Settling time :-* The settling time is the time required for the response curve to re... and stay within a range about the final value of a size specified as an abso... percentage of the final value, usually 2%.

4.6 Steady-State Error Analysis

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The Steady –State performance of a stable control system is generally judged by the stea... state error due to step, ramp and acceleration inputs. Any physical control system inhere... suffers steady-state error in response to certain types of inputs. That is, a system may hav... steady-state error with step inputs, but the same system may exhibit non zero steady-s... error in response to ramp input. Whether or not a given system will exhibit steady-state e... in response to a given type of input depends on the type of open-loop transfer function of system.

Consider the continuous –time control system whose open-loop transfer function G(s) H... given by,

$$G(s)H(s) = \frac{k(T_a s+1)(T_b s+1).....(T_n s+1)}{s^N (T_1 s+1)(T_2 s+1).....(T_p s+1)}$$

The term S^N is the denominator represents a pole of multiplicity N at the origin.

A system is said to be type 0, type1, type 2, if N=0, N=1, N=2, ... respectively.
Now, the discrete-time control system can be classified according to the number of ... loop poles at z=1. (an open loop pole at z=1 corresponds to an integration in the ... Suppose, the open-loop transfer function is given by the equation.

$$\text{Open-loop pulse transfer function} = \boxed{\frac{1}{(z-1)^N}\frac{B(z)}{A(z)}}$$

Where $\frac{B(z)}{A(z)}$ contains neither a pole nor a zero at z=1. Then the system can be classified as a type 0 system a type 1 system, or a type 2 system according to whether N=0, N=1, N=2 respectively.

Let us consider a discrete time control system as shown below :-

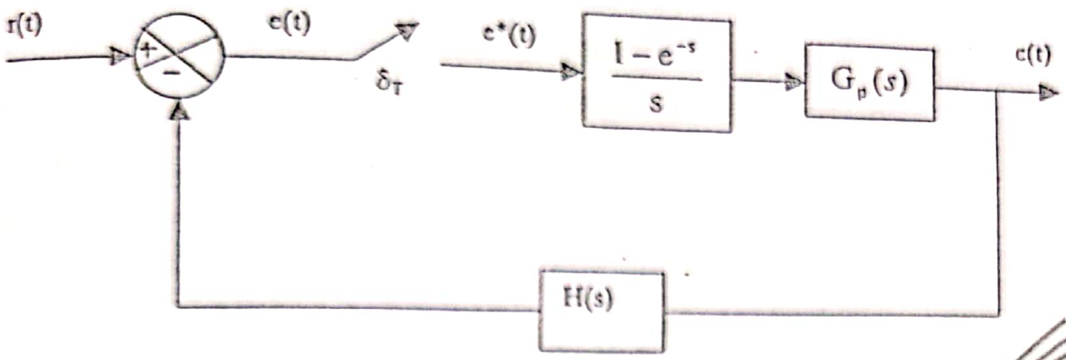


Fig 4.5 (Discrete- time control system)

Consider the system is stable so that we can apply final value theorem.

From the diagram

$$e(t) = r(t) - b(t)$$

Final value theorem,

$$\lim_{k \to \infty} e(kT) = \lim_{z \to 1} [(1-z^{-1})E(z)] \qquad \text{.............(4.6)}$$

from the figure define

$$G(z) = Z\left[\frac{1-e^{-Ts}}{s} G_p(s)\right]$$

$$= (1-z^{-1})\, Z\, \frac{G_p(s)}{s}$$

and $$GH(z) = (1-z^{-1})\, Z\, \frac{G_p(s)H(s)}{s}$$

Then we have

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)}$$

and $$E(z) = R(z) - B(z) = R(z) - GH(z)\,E(z)$$

..........(4.7)

or, $$E(z) = \frac{1}{1+GH(z)}R(z)$$

Substituting the value of E(z) to Eq (4.6), we get

$$e_{ss} = \lim_{z \to 1}\left[(1-z^{-1})\frac{1}{1+GH(z)}R(z)\right]$$

Let us consider three types of input : unit-step, unit ramp and unit-acceleration-input as in case of continuous-time control system.

Static Position Error Constant :-

For a unit step input $r(t) = 1(t)$,

$$R(z) = \frac{1}{(1-z^{-1})}$$

So, $$e_{ss} = \lim_{z \to 1}\left[(1-z^{-1})\frac{1}{1+GH(z)}\frac{1}{(1-z^{-1})}\right]$$

$$= \lim_{z \to 1}\frac{1}{1+GH(z)}$$

Let us define state position error constant k_p as follows :-

$$k_p = \lim_{z \to 1} GH(z)$$

Then we can write, $$e_{ss} = \frac{1}{1+k_p}$$

The steady-state actuating error in response to a unit step input becomes zero if $k_p = \infty$, which requires that GH(z) have at least one pole at z=1

Static Velocity Error Constant :- For a unit ramp input,

$r(t) = t\,1(t)$,

$$R(z) = \frac{Tz^{-1}}{(1-z^{-1})^2}$$

So, $$e_{ss} = \lim_{z \to 1}\left[(1-z^{-1})\frac{1}{1+GH(z)}\frac{Tz^{-1}}{(1-z^{-1})^2}\right] = \lim_{z \to 1}\left[\frac{T}{(1-z^{-1})GH(z)}\right]$$

Let us define the static velocity error constant k_v as

$$k_v = \lim_{z \to 1}\left[\frac{(1-z^{-1})GH(z)}{T}\right]$$

Then, $$e_{ss} = \frac{1}{k_v}$$

If $k_v = \infty$, then the steady-state actuating error in response to a unit-ramp input is zero. This requires GH(z) to possess a double pole at z=1.

Static Acceleration Error Constant :- For a unit acceleration input,

$$r(t) = \frac{1}{2}t^2 1(t)$$

$$\therefore R(z) = \frac{T^2(1+z^{-1})z^{-1}}{2(1-z^{-1})^3}$$

Then, $$e_{ss} = \lim_{z \to 1}\left[(1-z^{-1})\frac{1}{1+GH(z)}\frac{T^2(1+z^{-1})z^{-1}}{2(1-z^{-1})^3}\right]$$

$$= \lim_{z \to 1}\left[\frac{T^2}{(1-z^{-1})^2 GH(z)}\right]$$

Now, define static acceleration error constant k_a as

$$k_a = \lim_{z \to 1}\left[\frac{(1-z^{-1})^2 GH(z)}{T^2}\right]$$

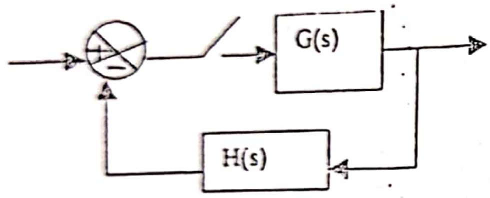
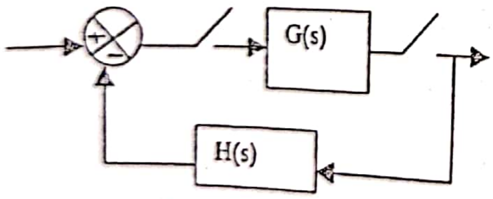
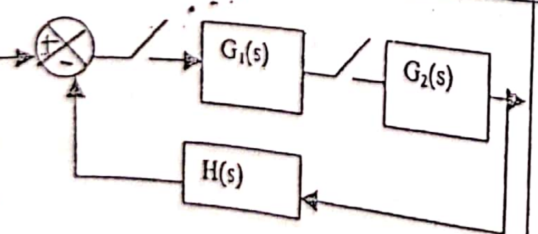
Then, $e_{ss} = \dfrac{1}{k_a}$

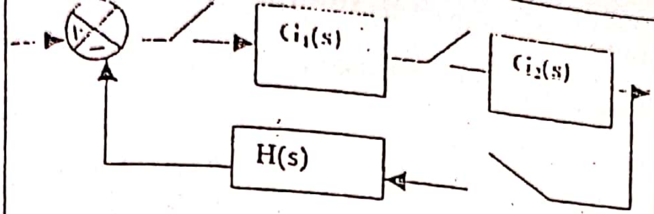
The steady-state actuating error in response to a unit acceleration input becomes zero if $k_a = \infty$. This requires GH(z) to possess a triple pole at z=1.

The following table shows the system types and the corresponding steady-state errors in response to step, ramp and acceleration inputs for the discrete-time control system.

System	Steady State Error in Response to		
	Step input r (t) =1	Ramp input r (t) =t	Acceleration inp $r(t)=\frac{1}{2}t^2$
Type 0	$1/(1+k_p)$	∞	∞
Type 1	0	$1/k_v$	∞
Type 2	0	0	$1/k_a$

The table below shows the static error constants for typical closed-loop configurations of discrete-time control system.

Closed loop configuration	Values of k_p, k_v, and k_a
	$k_p = \lim_{z\to 1} GH(z)$ $k_v = \lim_{z\to 1} \frac{(1-z^{-1})GH(z)}{T}$ $k_a = \lim_{z\to 1} \frac{(1-z^{-1})^2 GH(z)}{T^2}$
	$k_p = \lim_{z\to 1} G(z)H(z)$ $k_v = \lim_{z\to 1} \frac{(1-z^{-1})G(z)H(z)}{T}$ $k_a = \lim_{z\to 1} \frac{(1-z^{-1})^2 G(z)H(z)}{T^2}$
	$k_p = \lim_{z\to 1} G_1(z)G_2H(z)$ $k_v = \lim_{z\to 1} \frac{(1-z^{-1})G_1(z)G_2H(z)}{T}$ $k_a = \lim_{z\to 1} \frac{(1-z^{-1})^2 G_1(z)G_2H(z)}{T^2}$



$$k_p = \lim_{z \to 1} G_1(z)G_2(z)H(z)$$

$$k_v = \lim_{z \to 1} \frac{(1-z^{-1})G_1(z)G_2(z)H(z)}{T}$$

$$k_a = \lim_{z \to 1} \frac{(1-z^{-1})G_1(z)G_2(z)H(z)}{T^2}$$

4.7 Design Based on the Root-Locus Methods

General Rules for Constructing Root Loci :-

1. Obtain the characteristic equation

   $1+F(z)=0$ and rearrange as follows:

$$1+\frac{k(z+z_1)(z+z_2)\ldots\ldots(z+z_m)}{(z+p_1)(z+p_2)\ldots\ldots(z+p_n)}=0$$

Where k is the gain and $k > 0$.

From the factored form of the open-loop pule transfer function, locate the open loop poles and zeroes in z-plane.

2. Find the originating points and terminating points of the root loci. The points on the root loci corresponding to $k=0$ are open loop poles and those corresponding to $k=\infty$ are open loop zeroes.

   $(z+p_1)(z+p_2)\ldots\ldots\ldots\ldots(z+p_n)+k(z+z_1)(z+z_2)\ldots\ldots\ldots(z+z_m)=0$

or, $$\sum_{j=1}^{n}(z+p_j)+k\sum_{i=1}^{m}(z+z_i)=0$$

When $k=0$, this equation has roots at $-p_j$ $(j=1,2,\ldots.n)$ which are open loop poles. The root locus branches therefore start at open loop poles.

when $k=\infty$, $$\frac{1}{k}\sum_{j=1}^{n}(z+p_j)+\sum_{i=1}^{m}(z+z_i)=0$$

This equation has roots at $-z_i$ $(i=1,2\ldots m)$ which are open loop zeroes. The root locus branches therefore terminate at open loop zeroes.

In case, $m < n$, the open loop transfer function has (n-m) zeroes at infinity. Therefore (n-
branches of root locus terminate at infinity along the asymptotes.

3. A point on the real axis lies on the root locus if the number of open-loop poles
zeroes on the real axis to the right of this point is odd.

4. The (n-m) branches of root locus which tend to infinity, do so along straight
asymptotes whose angles are given by,

$$\phi_A = \frac{(2q+1)180}{(n-m)}, q = 0,1,2,\ldots\ldots,(n-m-1)$$

where, n = no, of finite poles of F(z)

m = no, of finite zeroes of F(z)

5. The asymptotes cross the real axis at a point known as centroid, determined by
relation.

$$-\sigma_A = \frac{\sum \text{real parts of poles} - \sum \text{reals parts of zeroes}}{\text{no.of poles} - \text{no.of zeroes}}$$

6. The breakaway points (points at which multiple roots of the characteristic eq
occur) of the root locus are the solution of $\frac{dk}{dz} = 0$

If the characteristic equation

1+F(z) =0 is written as

$$1+k\frac{B(z)}{A(z)}=0$$

then,

$$k = \frac{-A(z)}{B(z)}$$

So,

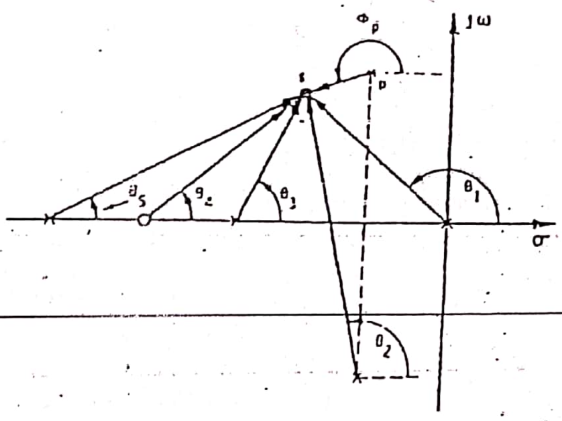
$$\frac{dk}{dz} = -\frac{A'(z)B(z) - A(z)B'(z)}{B^2(z)} = 0$$

If the value of k corresponding to a root $z = z_0$ of $\frac{dk}{dz} = 0$ is positive, point $z = z_0$ is the break way point. Since k is assumed to be non-negative, if the value of k thus obtained is negative then point $z = z_0$ is not a break way point.

7. The angle of departure from an open-loop pole is given by,

$$\phi p = \pm 180(2q + 1) + \phi, q = 0,1,2,.........$$

ϕ is the net angle contribution, at this pole of all other open loop poles and zeroes.



$\boxed{1 + A - B = 40^\circ}$

Fig 4.6 (Diagram showing the angle of departure)

$$\phi = \theta_4 - (\theta_1 + \theta_2 + \theta_3 + \theta_5)$$

$$\therefore \quad \phi_p = 180 + \theta_4 - (\theta_1 + \theta_2 + \theta_3 + \theta_5)$$

Similarly, the angle of arrival at an open loop zero is given by,

$$\phi_z = \pm \; 180(2q+1) - \phi$$

8. Find the points where the root loci cross the imaginary axis. The points where the root loci intersect the imaginary axis can be found by setting $z = jv$ in the characteristic equation, equating both the real part and the imaginary part to zero, and solving for v and k. The values of v and k thus found give the location at which the root loci cross the imaginary axis and the value of the corresponding gain k respectively.

9. The open loop gain k in pole-zero form at any point z_0 on the root locus is given by

$$k = \frac{\text{product of phasor length from } z_0 \text{ to open loop poles}}{\text{product of phasor length from } z_0 \text{ to open loop zeroes}}$$

Example 7. Draw root locus diagrams in the z-plane for the system in following figure for the following three sampling periods. T= 1 sec, T= 2 sec, T= 4 sec

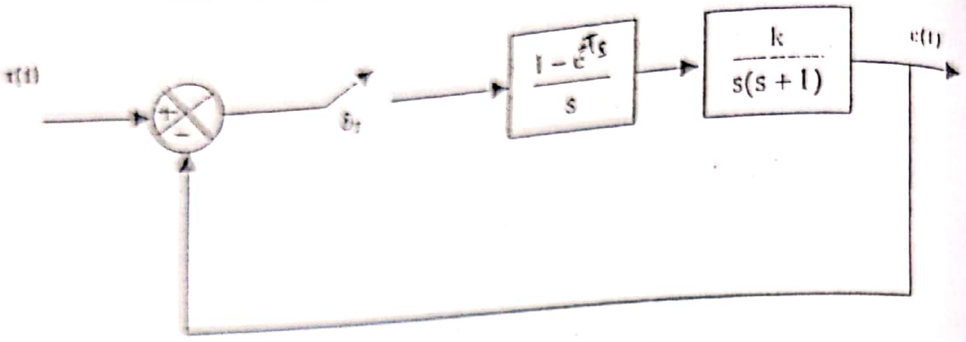


Fig 4.7 (Closed loop control system)

Solution.

$$G(z) = Z\left[\frac{1-e^{-Ts}}{s}\,\frac{k}{s(s+1)}\right]$$

$$= (1-z^{-1})Z\left[\frac{k}{s^2(s+1)}\right]$$

$$= \frac{k\left[(T-1+e^{-T})z^{-1} + (1-e^{-T}-Te^{-T})z^{-2}\right]}{(1-z^{-1})(1-e^{-T}z^{-1})}$$

1. Sampling period T=1 : For T=1,

$$G(z) = \frac{0.3679\,k(z+0.7181)}{(z-1)(z-0.3679)}$$

$$\frac{k(0.3679z+0.2642)}{(z-1)(z-0.3679)}$$

zeroes at z=-0.7181

and poles at z=1, z=0.3679.

To find break away points :-

$$k = -\frac{(z-1)(z-0.3679)}{0.3679(z+0.7181)} = -\frac{z^2 - 1.3679z + 0.3679}{0.3679(z+0.7181)}$$

$$\frac{dk}{dz} = 0$$

Or.

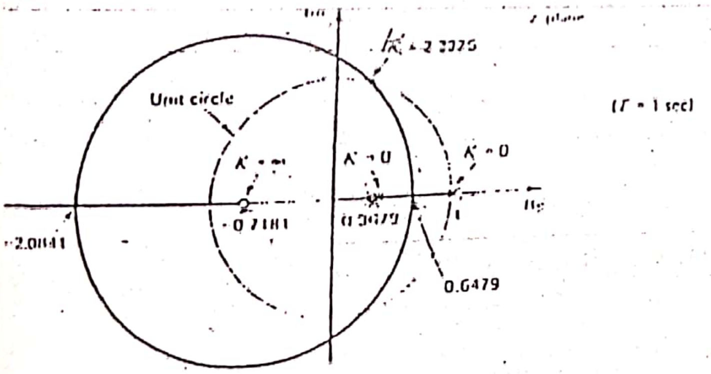
$$\frac{0.3679(z+0.7181)(2z-1.3679) - (z^2 - 1.3679z + 0.3679)0.3679}{[0.3679(z+0.7181)]^2} = 0$$

Which gives, z 0.6479, 2.841

So, draw a circle of radius at centre at -0.7181.

$$k = \left| \frac{(z-1)(z-.3679)}{0.3679(z+0.7181)} \right| = 2.3925$$

Hence, the critical gain is found to be k= 2.3925 (for stability).



1.2 × 1.4
2

Fig 4.8 (Root locus diagram for the system when T 1 sec)

2. Sampling period T 2, with T 2 G(z) becomes

$$G(z) = \frac{1.1353k(z+0.5232)}{(z-1)(z-0.1353)}$$

The break way point are at z= 0.4783 and z=-1.5247.

CamScanner

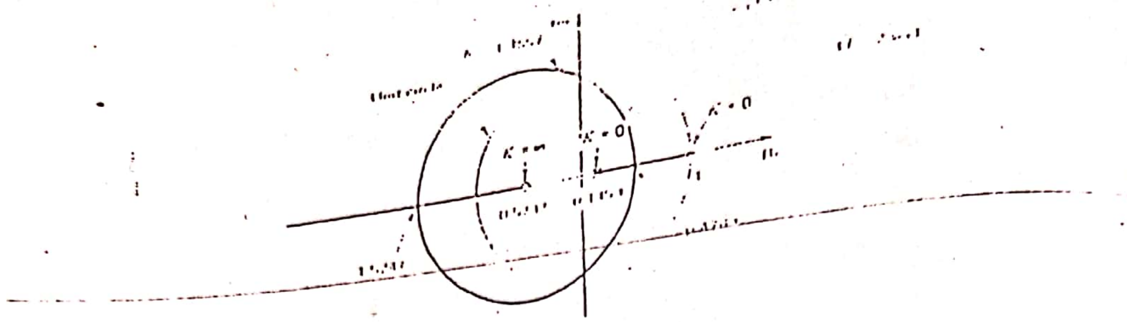


Fig 4.9 (Root locus diagram for the system when T = 2 sec)

3. With Sampling period T = 4 sec.

$$G(z) = \frac{3.0183\,(z + 0.3010)}{(z - 1)(z - 0.0183)}$$

The break way points are at z= 0.3435, and z=-0.9455

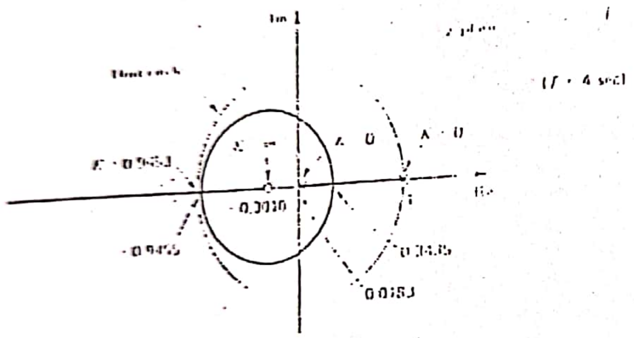


Fig 4.10 (Root locus diagram for the system when T= 4 sec)

The critical gain k for stability is k=0.9653. From these three cases considered notice that smaller the sampling period is, the larger the critical gain k for stability.

4.8 Design Based on the Frequency Response Method

The frequency response concept plays the same role in digital control systems as it does continuous-time control systems. The familiarity with Bode diagrams (logarithmic plots necessary in the extension of the conventional frequency response techniques to the analysis and design of discrete-time control systems.

Frequency response methods have very frequently been used in the compensator design. The reason for this is the simplicity of the methods. While performing frequency response tests on a discrete-time system, it is important that the system have a low-pass filter before the sampler so that sidebands are filtered out. Then the response of the linear time invariant system to a sinusoidal input preserves the frequency and modifies only the amplitude and phase of the input signal. Thus, the amplitude and the phase are the only two quantities that must be dealt with.

4.8.1 *Response of a Linear Time-Invariant Discrete-Time System to a Sinusoidal Input :-*

Consider the stable linear time-invariant discrete-time system shown in Figure below. The input to be system G(z) before sampling is .

$$u(t) = \sin \omega t$$

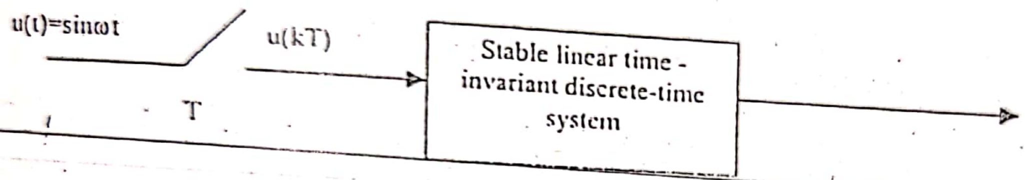


Fig 4.11 (Stable linear time-invariant discrete – time system)

The sampled signal u(kT) is

$$u(kT) = \sin k\omega T$$

The z transform of the sampled input is

$$U(Z) = Z\,[\sin k\omega T] = \frac{z \sin \omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})}$$

The response of the system is given by

$$X(z) = G(z)U(z) = G(z)\frac{z \sin \omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})}$$

$$= \frac{az}{z - e^{j\omega T}} + \frac{\bar{a}z}{z - e^{-j\omega T}} + [\text{terms due to poles of } G(z)] \qquad \text{..........(4.8)}$$

Multiplying both sides of Eq (4.8) by $(z - e^{j\omega T})/z$, we obtain

$$G(z)\frac{\sin \omega T}{z - e^{-j\omega T}} = a + \frac{\bar{a}(z - e^{j\omega T})}{z - e^{-j\omega T}} + \frac{z - e^{j\omega T}}{z}\,[\text{terms due to poles of } G(z)]$$

The second term on the right-hand side of this last equation approaches zero as z approac[hes] $e^{j\omega T}$. Since the system considered here is stable, the third term on the right-hand side approaches zero as z approaches $e^{j\omega T}$. Hence, by letting z approach $e^{j\omega T}$, we have

$$a = G(z)\frac{\sin\omega T}{z - e^{-j\omega T}}\bigg|_{z=e^{j\omega T}} = \frac{G(e^{j\omega T})}{2j}$$

The coefficient $\bar{a}$ the complex conjugate of a, is then obtain as follows:

$$\bar{a} = -\frac{G(e^{-j\omega T})}{2j}$$

Let us define $G(e^{j\omega T}) = Me^{-j\theta}$

Then $G(e^{-j\omega T}) = Me^{-j\theta}$

Eq (4.8) can now be written as

$$X(z) = \frac{Me^{j\theta}}{2j}\frac{z}{z - e^{j\omega T}} - \frac{Me^{-j\theta}}{2j}\frac{z}{z - e^{-j\omega T}} + [\text{terms due to poles of } G(z)]$$

or, $$X(z) = \frac{M}{2j}\left(\frac{e^{j\theta}z}{z - e^{j\omega T}} - \frac{e^{-j\theta}z}{z - e^{-j\omega T}}\right) + [\text{terms due to poles of } G(z)]$$

The inverse z transform of this last equation is

$$x(kT) = \frac{M}{2j}(e^{jk\omega T}e^{j\theta} - e^{-jk\omega T}e^{-j\theta}) + Z^{-1}[\text{terms due to poles of } G(z)] \quad \text{......}$$

The last term on the right-hand side of Eq (4.9) represents the transient response. Sin[ce] system G(z) has been assumed to be stable, all transient response terms will disapp[ear] steady state and we will get the following steady-state response $x_{ss}(kT)$:

$$x_{ss}(kT) = \frac{M}{2j}[e^{j(k\omega T+\theta)} - e^{-j(k\omega T+\theta)}] = M\sin(k\omega T + \theta) \quad \text{......(}$$

where M, the gain of the discrete-time system when subjected to a sinusoidal input, is [given] by

$$M = M(\omega) = |G(e^{j\omega T})|$$

and θ, the phase angle, is given by

$$\theta = \theta(\omega) = \angle G(e^{j\omega T})$$

In terms of $G(e^{j\omega T})$ Eq (4.10) can be written as follows:

$$\boxed{x_{ss}(kT) = |G(e^{j\omega T})|\sin(k\omega T + \angle G(e^{j\omega T}))}$$

We have shown that $G(e^{j\omega T})$ indeed gives the magnitude and phase of the frequency response of G(z). Thus, to obtain the frequency response of G(z), we need only to substitute $e^{j\omega T}$ for z in G(z). The function $G(e^{j\omega T})$ is commonly called the sinusoidal pulse transfer function. Nothing that

$$e^{j(\omega+(2\pi/T))T} = e^{j\omega T} e^{j2\pi} = e^{j\omega T}$$

We find that the sinusoidal pulse transfer function $G(e^{j\omega T})$ is periodic, with the period equal to T.

Example 8 .Consider the system defined by

$$x(kT) = u(kT) + ax((k-1)T), \qquad 0 < a < 1$$

Where u(kT) is the input and x(kT) the output. Obtain the steady-state output x(kT) when the input u(kT) is the sampled sinusoid, or $u(kT) = A \sin k\omega T$.

Solution . The z transform of the system equation is

$$X(z) = U(z) + az^{-1}X(z)$$

By defining G(z) =X(z)/U(z), we have

$$G(z) = \frac{X(Z)}{U(z)} = \frac{1}{1-az^{-1}}$$

Let us substitute $e^{j\omega T}$ for z in G(Z). Then sinusoidal pulse transfer function $G(e^{j\omega T})$ can be obtained as

$$G(e^{j\omega T}) = \frac{1}{1-ae^{j\omega T}} = \frac{1}{1-a\cos\omega T + ja\sin\omega T}$$

The amplitude of $G(e^{j\omega T})$ is

$$|G(e^{j\omega T})| = M = \frac{1}{\sqrt{1+a^2-2a\cos\omega T}}$$

and the phase angle of $G(e^{j\omega T})$ is

$$\angle G(e^{j\omega T}) = \theta = -\tan^{-1}\frac{a\sin\omega T}{1-a\cos\omega T}$$

Then the steady-state output $x_{ss}(kT)$ can be written as follows:

$$x_{ss}(kT) = AM\sin(k\omega T+\theta)$$

$$= \frac{A}{\sqrt{1+a^2-2a\cos\omega T}}\sin\left(k\omega T - \tan^{-1}\frac{a\sin\omega T}{1-a\cos\omega T}\right)$$

4.8.2 *Bilinear Transformation and the w Plane:-* Before we can advantageously apply, well-developed frequency-response methods to the analysis and design of discrete-time control systems, certain modifications in the z plane approach are necessary. Since in the z plane the frequency appears as $z = e^{j\omega T}$, if we treat frequency response in the z plane, the simplicity of the logarithmic plots will be completely lost. Thus, the direct application of the frequency-response methods is not worthy of consideration. In fact, since the transformation maps the primary and complementary strips of the left half of the s plane into the unit circle in the z plane, conventional frequency-response methods, which deal with the entire left half plane, do not apply to the z plane.

The difficulty, however, can be overcome by transformation the pulse transfer function in the z plane into that in the w plane. The transformation, commonly called the w transformation, a bilinear transformation, is defined by

$$z = \frac{1+(T/2)w}{1-(T/2)w} \qquad \text{..........(4.11)}$$

$z = \frac{1+(T/2)w}{1-(T/2)w}$

Where T is sampling period involved in the discrete-time control system under consideration. By converting a given pulse transfer function in the z plane into a rational function of w, the frequency-response methods can be extended to discrete-time control systems. By solving Eq (4.11) for w, we obtain the inverse relationship.

$$w = \frac{z}{T}\,\frac{z-1}{z+1} \qquad \text{..........(4.12)}$$

$w = 2/T \; \frac{z-1}{z+1}$

Through the z transformation and the w transformation, the primary strip of the left half of the s plane is first mapped into the inside of the unit circle in the z plane and then mapped into the entire left half of the w plane. The two mapping processes are depicted in figure below. (Note that in the s plane we consider only the primary strip). Notice that the origin of the z plane is mapped into the point w = -2/T in the w plane. Notice also that a varies from 0 to $j\omega_s/2$ along the $j\omega$ axis in the s plane, z varies from 1 to −1 along the unit circle in the z plane and w varies from 0 to ∞ along the imaginary axis in the w plane.

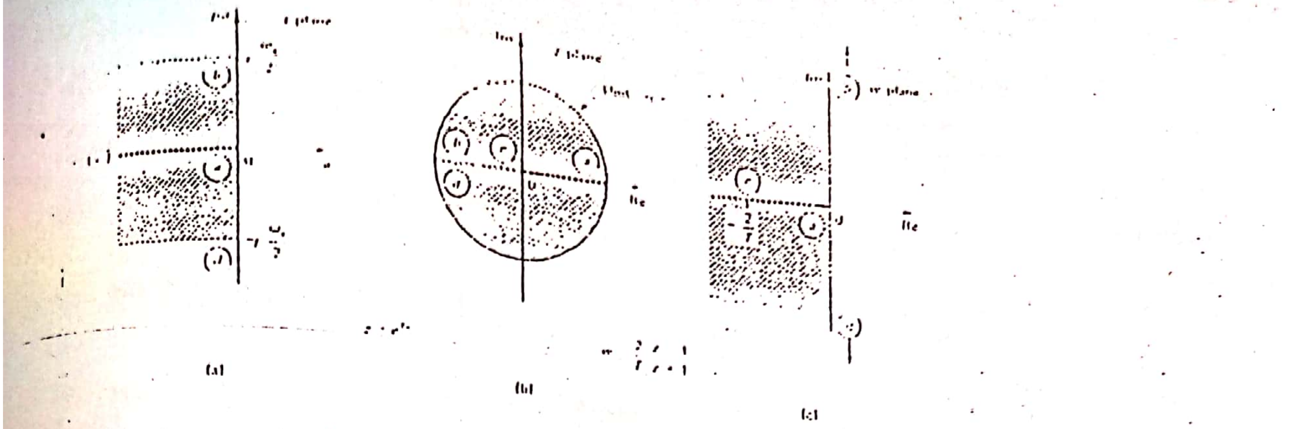


Fig 4.12 (Diagrams showing mappings from the s plane to the z-plane and from the z-plane to the w-plane)

Although the left half of the w plane corresponds to the imaginary axis of the s plane, there are differences between the two planes. The chief difference is that the behavior in the s plane over the frequency range $-\frac{1}{2}\omega_s \le \omega \le \frac{1}{2}\omega_s$ maps to the range $-\infty < v < \infty$, where v is the fictitious frequency in w plane. This means that, although the frequency response characteristics of the analog filter will be reproduced in the discrete or digital filter, the frequency scale on which the response occurs will be compressed from an infinite interval in the analog filter to a finite interval in the digital filter.

Once the pulse transfer function G(z) is transformed into G(w) by means of the w transformation, it may be treated as a conventional transfer function in w. Conventional frequency response techniques can then be used in the w plane, and so the well-established frequency-response design techniques can be applied to the design of discrete-time control systems.

As noted earlier, v represents the fictitious frequency. By replacing w by jv, conventional frequency-response techniques may be used to draw the Bode diagram for the transfer function in w.

Although the w plane resembles the s plane geometrically, the frequency axis in the w plane is distorted. The fictitious frequency v and the actual frequency ω are related as follows:

$$w\Big|_{w=jv} = jv = \frac{2}{T}\frac{z-1}{z+1}\Bigg|_{z=e^{j\omega T}} = \frac{2}{T}\frac{e^{j\omega T}-1}{e^{j\omega T}+1}$$

$$= \frac{2}{T}\frac{e^{j(1/2)\omega T}-e^{-j(1/2)\omega T}}{e^{j(1/2)\omega T}+e^{-j(1/2)\omega T}} = \frac{2}{T}j\tan\frac{\omega T}{2}$$

$$v = \frac{2}{T}\tan\frac{\omega T}{2} \qquad \text{..........(4.13)}$$

Eq (4.13) gives the relationship between the actual frequency ω and the fictitious freque

v. Note that as the actual frequency ω moves from $-\frac{1}{2}\omega_s$ to 0 the fictitious frequenc

moves from $-\infty$ to o, and as ω moves from 0 to $\frac{1}{2}\omega_s$, v moves from 0 to ∞.

Referring to Eq (4.13) the actual frequency ω can be translated into the fictitious frequency

Example 9 . Consider the transfer-function system shown is figure below. The sampl period T is assumed to be 0.1 sec. Obtain G(w).

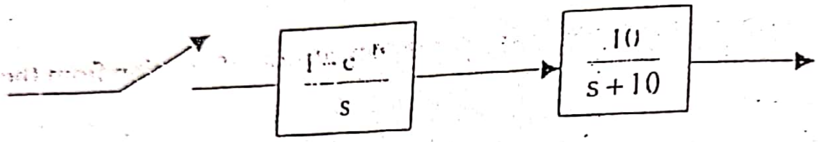


Fig 4.13 (Transfer function system)

Solution . The z transform of G(s) is.

$$G(z) = \mathcal{Z}\left[\frac{1-e^{-Ts}}{s}\,\frac{10}{s+10}\right]$$

$$= (1-z^{-1})\mathcal{Z}\left[\frac{10}{s(s+10)}\right]$$

$$= \frac{0.6321}{z-0.3679}$$

By use of the bilinear transformation we get,

$$z = \frac{1+(T/2)w}{1-(T/2)w} = \frac{1+0.05w}{1-0.05w}$$

G(z) can be transformed into G(w) as follows:

$$G(w) = \frac{0.6321}{\frac{1+0.05w}{1-0.05w}-0.3679} = \frac{0.6321(1-0.05w)}{0.6321+0.06840w}$$

$$= 9.241\frac{1-0.05w}{w+9.241}$$

Notice that the location of the pole of the plant is s=-10 and that of the pole in the w plane is w =-9.241. The gain value in the s plane is 10 and that in the w plane is 9.241. (Thus, both the pole locations and the gain values are similar in the plant and the w plane). However, G(w) has a zero at w = 2/T = 20, although the plant does not have any zero.

Note that we have

$$\lim_{w \to 0} G(w) = \lim_{s \to 0} \frac{10}{s+10}$$

This fact is very useful in checking the numerical calculations in transforming G(s) into G(w).

To summarize, the w transformation, a bilinear transformation, maps the inside of the unit circle of the z plane into the left half of the w plane. The overall result due to the transformations from the s plane into the z plane and from the z plane into the w plane is that the w plane and the s plane are similar over the region of interest in the s plane. This is because some of the distortions caused by the transformation from the s plane into the z plane are partly compensated for by the transformation from the z plane into the w plane.

Note that if

$$G(z) = \frac{b_0 z^m + b_1 z^{m-1} + \ldots\ldots + b_m}{z^n + a_1 z^{n-1} + \ldots\ldots + a_n}, \qquad m \le n$$

Where the a_i's and b_i's are constants, is transformed into the w plane by the transformation:

$$z = \frac{1+(T/2)w}{1-(T/2)w}$$

then G(w) takes the form

$$G(w) = \frac{\beta_0 w^n + \beta_0 w^{n-1} + \ldots\ldots + \beta_n}{\alpha_0 w^n + \alpha_1 w^{n-1} + \ldots\ldots + \alpha_n}$$

Where the α_i's and the β_i's are constants (some of them may by zero). Thus, G(w) is a ratio of polynomials in w, where the degrees of the numerator and denominator may or may not be equal.

4.8.3 ***Bode Diagrams:-*** Design by means of Bode diagrams has been widely used in dealing with single-input-single-output continuous-time control systems. In particular, if the transfer function is in a factored form, the simplicity and ease with which the asymptotic Bode diagram can be drawn and reshaped are well known.

As stated earlier, the conventional frequency-response methods apply to the tran functions in the w plane. Recall that the Bode diagram consists of two separate plots, logarithmic magnitude $|G(jv)|$ versus log v and the phase angle $\angle G(jv)$ versus log Sketching of the logarithmic magnitude is based on the factoring of G(jv), so that it works the principle of adding the individual factored terms instead of multiplying individual term Familiar asymptotic plotting techniques can be applied, and therefore the magnitude cu can be quickly drawn by using straight-line asymptotes. Using the Bode diagram, a dig compensatory or digital controller may be designed with conventional design techniques. It is important to note that there may be a difference in the high-frequency magnitudes G(jω) curve and G(jv). The high-frequency asymptote of the logarithmic magnitude curve G(jv) may be a constant - decibel line (that is, a horizontal line).

On the other hand, if $\lim_{s\to\infty} G(s) = 0$, then the magnitude of G(jω) always approaches z (...) as ω approaches infinity. For example, referring to the previous example we obtain G(w) for G(s) as follows:

$$G(w) = 9.241\left(\frac{1-0.05w}{w+9.241}\right)$$

The high-frequency magnitude of G(jv) is

$$\lim_{v\to\infty}|G(jv)| = \lim_{v\to\infty}\left|9.241\left(\frac{1-0.05jv}{jv+9.241}\right)\right| = 0.4621$$

While the high-frequency magnitude of the plant is

$$\lim_{\omega\to\infty}\left|\frac{10}{j\omega+10}\right| = 0$$

The difference in the Bode diagrams at the high-frequency end can be explained as follo First, recall that we are interested only in the frequency range $0 \le \omega \le \frac{1}{2}\omega_s$, wh corresponds $0 \le v \le \infty$. Then, noting that $v = \infty$ in the w plane corresponds to $\omega = \frac{1}{2}\omega_s$ in s plane, it can be said that $\lim_{v\to\infty}|G(jv)|$ corresponds to $\lim_{\omega\to\frac{\omega_s}{2}}\left|\frac{10}{(j\omega+10)}\right|$, which is constant. (It is important to note that these two values are generally not equal to each other

From the pole-zero point of view , it can be said that when $|G(jv)|$ is a nonzero constant at $v = \infty$ it is implied that G(w) constants the same number of poles and zeros.

In general, one or more zeros of G(w) lie in the right half of the w plane. The presence of a zero in the right half of the w plane means that G(w) is a nonminimum phase transfer function. Therefore, we must be careful in drawing the phase angle curve in the Bode diagram.

Procedures to draw Bode diagrams :-

Let us consider a digital control system as shown in figure below.

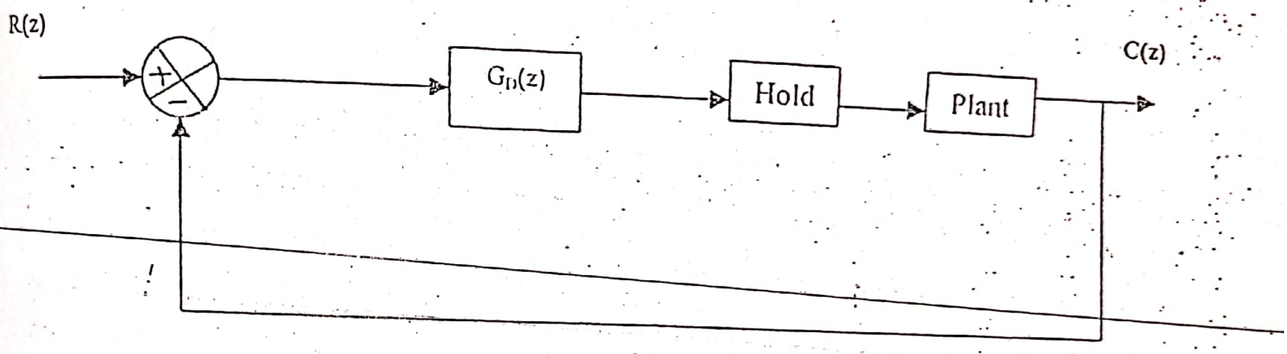


Fig 4.14 (Digital control system)

1. First, obtain G(z), the z transform of the plant preceded by a hold. Then transform G(z) into a transfer function G(w) through the bilinear transformation.

$$z = \frac{1+(T/2)w}{1-(T/2)w}$$

2. Substitute $\omega = jv$ into $G(\omega)$ and plot the Bode diagram for G(jv).

Example 10 . Draw a Bode diagram for the control system shown below. Assume that the sampling period is 0.2 sec, or T=0.2 and the value of k=2.

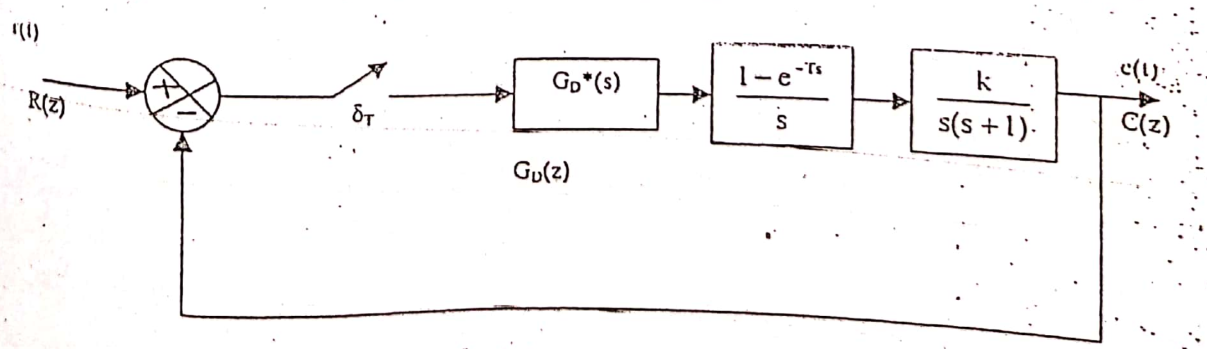


Fig 4.15 (Digital control system)

Solution . First, we obtain the pulse transfer function $G(z)$ to the plant that is preceded by the zero-order hold.

$$G(z) = \mathcal{Z}\left[\frac{1-e^{-Ts}}{s}\,\frac{k}{s(s+1)}\right]$$

$$= (1-z^{-1})\mathcal{Z}\left[\frac{k}{s^2(s+1)}\right]$$

$$= 0.01873\left[\frac{k(z+0.9356)}{(z-1)(z-0.8187)}\right]$$

$$= \frac{k(0.01873z+0.01752)}{z^2-1.8187z+0.8187}$$

Next, we transform the pulse transfer function $G(z)$ into a transfer function $G(w)$ by means of the bilinear transformation.

$$z = \frac{1+(T/2)w}{1-(T/2)w} = \frac{1+0.1w}{1-0.1w}$$

Thus.

$$G(w) = \frac{k\left[0.01873\left(\frac{1+0.1w}{1-0.1w}\right)+0.01752\right]}{\left(\frac{1+0.1w}{1-0.1w}\right)^2-1.8187\left(\frac{1+0.1w}{1-0.1w}\right)+0.8187}$$

$$= \frac{k(-0.000333w^2-0.09633w+0.9966)}{w^2+0.9969w}$$

$$\cong \frac{k\left(1+\frac{w}{300}\right)\left(1-\frac{w}{10}\right)}{w(w+1)}$$

By setting K=2, we plot the Bode diagram of $G(w)$.

$$G(w) = \frac{2(-0.000333w^2-0.09633w+0.9966)}{w^2+0.9969w}$$

$$\cong \frac{2\left(1+\frac{w}{300}\right)\left(1-\frac{w}{10}\right)}{w(w+1)}$$

Figure 4.16 shows the bode diagram for the system. For the magnitude curve we have used straight line asymptotes. The magnitude and the phase angle of $G(jv)$ are shown by dashed curves. The phase margin can be read from the bode diagram (dashed curves) as 30° and the gain margin as 14.5 dB.

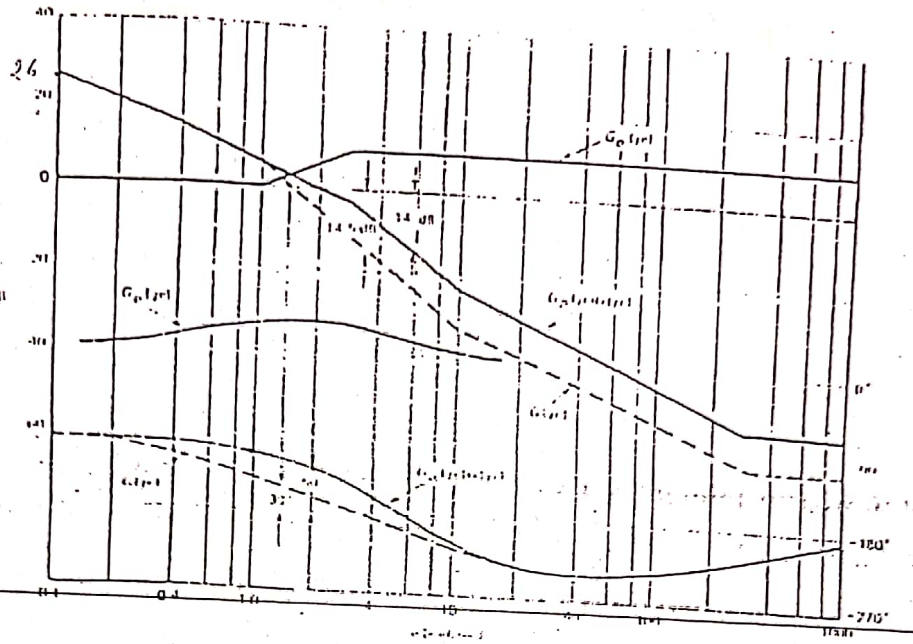


Fig 4.16(Bode diagram for the example 10)

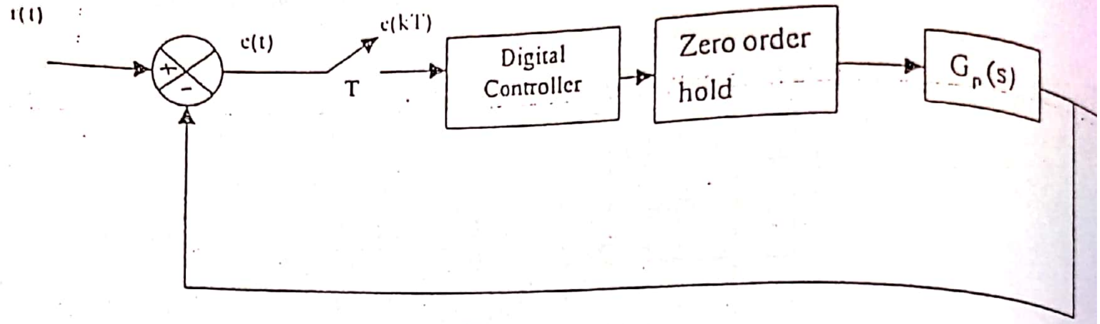
4.9 Analytical Design Method

In this section we specifically present an analytical design method for digital controllers that will force the error sequence, when subjected to a specific type of time-domain input, to become zero after a finite number of sampling periods and, in fact, to become zero and stay zero after the minimum possible number of sampling periods.

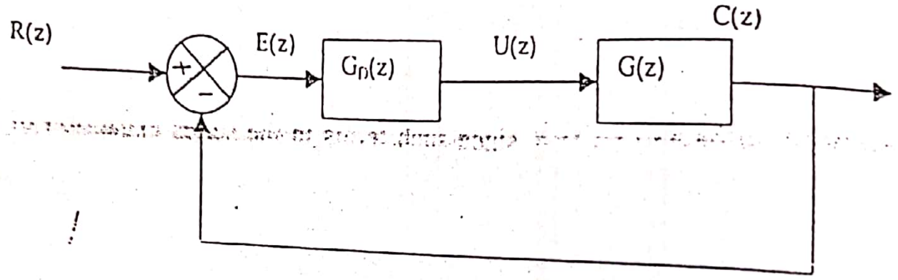
If the response of a closed-loop control system to a step input exhibits the minimum possible settling time (that is, the output reaches the final value in the minimum time and stay there), no steady -state error, and no ripples between the sampling instants, then this type of response is commonly called a deadbeat response.

Design of Digital Controllers for Minimum Settling Time with Zero Steady-State Err...

Consider the digital control system shown in figure below. The error signal e(t), which is difference between the input r(t) and the output c(t), is sampling every time interval T.



Fig(a)



Fig(b)

Fig 4.18 (Digital control system and the diagram showing equivalent digital control system

The input to the digital controller is the error signal e(kT). The output of the digital contr... is the control signal u(kT). The control signal u(kT) is fed to the zero-order hold, and output of the hold, u(t), which is a piecewise continuous-time signal, is fed to the plant. desired to design a digital controller $G_D(z)$ such that the closed-loop control system exhibit the minimum possible setting time with zero steady-state error in response to a st... ramp, or an acceleration input. It is required that the output not exhibit inter sampling rip... after the steady state is reached. The system must satisfy any other specifications, if requ... such as a specification for the static velocity error constant.

Let us define the z transform of the plant that is preceded by the zero-order hold as G(z),

$$G(z) = Z\left[\frac{1-e^{-Ts}}{s} G_p(s)\right]$$

Then the open-loop pulse transfer function becomes $G_D(z)\,G(z)$, as shown in figure (b).

Next, define the desired closed-loop pulse transfer function as F(z).

$$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1+G_D(z)G(z)} = F(z) \qquad \text{..........(4.14)}$$

Since it is required that the system exhibit a finite settling time with zero steady –state error, the system must exhibit a finite impulse response. Hence, the desired closed-loop pulse transfer function must be of the following form.

$$F(z) = \frac{a_0 z^N + a_1 z^{N-1} + \text{.........} + a_N}{z^N}$$

or,

$$F(z) = a_0 + a_1 z^{-1} + \text{.........} + a_N z^{-N} \qquad \text{..........(4.15)}$$

where $N \geq n$ and n is the order of the system. [Note that F (z) must not contain any terms with positive powers in z, since such terms in the series expansion of F (z) imply that the output precedes the input, which is not possible for a physically realizable system.] In our design approach, we solve the closed-loop pulse transfer function for the digital controller $G_D(z)$. That is, we find the pulse transfer function $G_D(z)$ that will satisfy Eq (1.14). Solving Eq (4.14) for $G_D(z)$, we obtain .

$$G_D(z) = \frac{F(z)}{G(z)[1-F(z)]} \qquad \text{..........(4.16)}$$

The designed system must be physically realizable. The conditions for physical realizability place certain constraints on the closed-loop pulse transfer function F(z) and the digital controller plus transfer function $G_D(z)$. The condition for physical realizability may be stated as follows:

1. The order of the numerator of $G_D(z)$ must be equal to or lower than the order of the denominator. (Otherwise, the controller requires future input data to produce the current output).
2. If the plant $G_p(s)$ involves a transportation lag e^{-Ls}, then the designed closed-loop system must involve at least the same magnitude of the transportation lag. (Otherwise, the closed-loop system would have to respond before an input was given, which is impossible for a physically realizable system).

3. If $G(z)$ is expanded into a series in z^{-1}, the lowest-power term of the series expans... $F(z)$ in z^{-1} must be at least as large as that of $G(Z)$. For example, if an expansion o... into a series in z^{-1} begins with the z^{-1} term, then the first term of $F(z)$ given by Eq ... must be zero, or a_0 must equal 0; that is, the expansion has to be of the form.

$F(z) = a_1 z^{-1} + a_2 z^{-2} + \ldots\ldots\ldots + a_N z^{-N}$

where $N \geq n$ and n is the order of the system. This means that the plant cannot re... instantaneously when a control signal of finite magnitude is applied: the respond come... delay of at least one sampling period if the series expansion of $G(z)$ begins with a term i...

In addition to the physical realizabilty conditions, we must pay attention to the sta... aspects of the system. Specifically, we must avoid canceling an unstable pole of the... by a zero of the digital controller. If such a cancellation is attempted, any error in the... zero cancellation will diverge as time elapses and the system will become uns... Similarly, the digital controller pulse transfer function should not involve unstable po... cancel plant zeros that lie outside the unit circle.

Next, let us investigation what will happen to the closed-loop pulse transfer function F... $G(z)$ involves an unstable (or critically stable) pole, that is, a pole $z = \alpha$ outside (or on... unit circle. [Note that the following argument applies equally, if $G(z)$ involves two or... unstable –or critically stable –poles.] Let us define

$$G(z) = \frac{G_1(z)}{z-\alpha}$$

where $G_1(z)$ does not include a term that cancels with $z - \alpha$. Then the closed – loop transfer function becomes

$$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1+G_D(z)G(z)} = \frac{G_D(z)\frac{G_1(z)}{z-\alpha}}{1+G_D(z)\frac{G_1(z)}{z-\alpha}} = F(z) \qquad \ldots\ldots\ldots(4.17)$$

Since we require that no-zero of $G_D(z)$ cancel the unstable pole of $G(z)$ at $z = \alpha$, we have.

$$1 - F(z) = \frac{1}{1+G_D(z)\frac{G_1(z)}{z-\alpha}} = \frac{z-\alpha}{z-\alpha+G_D(z)G_1(z)}$$

That is, $1-F(z)$ must have $z = \alpha$ as a zero. Also notice that from equation (4) if zeros of $G(z)$ do not cancel poles of $G_D(z)$, the zeros of $G(z)$ become zeros of $F(z)$. [$F(z)$ may involve additional zeros.]

Let us summarize what we have stated concerning stability.

1. Since the digital controller $G_D(z)$ should not cancel unstable (or critically stable) poles of $G(z)$, all unstable (or critically stable) poles of $G(z)$ must be included in $1-F(z)$ as zeros.

2. Zeros of $G(z)$ that lie inside the unit circle may be canceled with poles of $G_D(z)$. However, zeros of $G(z)$ that lie on or outside the unit circle must not be canceled with poles of $G_D(z)$. Hence, all zeros of $G(z)$ that lie on or outside the unit circle must be included in $F(z)$ as zeros.

Now we shall proceed with the design. Since $e(kT) = r(kT) - c(kT)$, referring to Eq (4.14) we have,

$$E(z) = R(z) - C(z) = R(z)\,[1 - F(z)] \qquad \text{..........(4.18)}$$

Note that for a unit-step input $r(t) = 1(t)$

$$R(z) = \frac{1}{1 - z^{-1}}$$

For a unit-ramp input $r(t) = t1(t)$,

$$R(z) = \frac{Tz^{-1}}{(1 - z^{-1})^2}$$

And for a unit-acceleration input $r(t) = \frac{1}{2}t^2 1(t)$,

$$R(z) = \frac{T^2 z^{-1}(1 + z^{-1})}{2(1 - z^{-1})^3}$$

Thus, in general, z transforms of such time-domain polynomial inputs may be written as

$$R(z) = \frac{P(z)}{(1 - z^{-1})^{q+1}} \qquad \text{..........(4.19)}$$

where $P(z)$ is a polynomial in z^{-1}. Notice that for a unit-step input $P(z) = 1$ and $q=0$; for a unit-ramp input, $P(z) = Tz^{-1}$ and $q=1$; and for a unit-acceleration input, $P(z) = \frac{1}{2}T^2z^{-1}(1+z^{-1})$ and $q=2$.

By substituting Eq (4.19) into Eq (4.18) we obtain

$$E(z) = \frac{P(z)[1 - F(z)]}{(1 - z^{-1})^{q+1}} \quad \text{.........(4.20)}$$

To ensure that the system reaches steady state in a finite number of sampling periods and maintains zero steady-state error, E(z) must be a polynomial in z^{-1} with a finite number of terms. The, by referring to Eq (4.20), we choose the function 1-F(z) to be of the form.

$$1-F(z)=(1-z^{-1})^{q+1}N(z) \quad \text{.........(4.21)}$$

where N(z) is a polynomial in z^{-1} with a finite number of terms. Then,

$$E(z) = P(z)\,N(z) \quad \text{.........(4.22)}$$

which is a polynomial in z^{-1} with a finite number of terms. This means that the error signal becomes zero in a finite number of sampling periods.

From the preceding analysis, the pulse transfer function of the digital controller can be determined as follows. By first letting F(z) satisfy the physical realizability and stability conditions and then substituting Eq (4.21) into Eq (4.16) we obtain.

$$G_D(z) = \frac{F(z)}{G(z)(1 - z^{-1})^{q+1} N(z)} \quad \text{.........(4.23)}$$

Eq (4.23) gives the pulse transfer function of the digital controller that will produce zero steady-state error after a finite number of sampling periods.

For a stable $G_p(s)$, the condition that the output not exhibit intersampling ripples after the settling time is reached may be written as follows:

$c(t \geq nT) = \text{constant}$, for step inputs

$\dot{c}(t \geq nT) = \text{constant}$, for ramp inputs

$\ddot{c}(t \geq nT) = \text{constant}$, for acceleration inputs

The applicable condition must be satisfied when the system is designed. In designing the system, the condition on c(t), $\dot{c}(t)$, or $\ddot{c}(t)$ must be interpreted in terms of u(t). Note that the plant is continuous time and the input to the plant is u(t), a continuous –time function therefore, to have no ripples in the output c(t), the control signal u(t) at steady-state must be either constant or monotonically increasing (or monotonically decreasing) for step, ramp, or acceleration inputs.

...digital control system in figure below, where the plant transfer function $G_p(s)$ is given by $G_p(s) = \frac{1}{s(s+1)}$

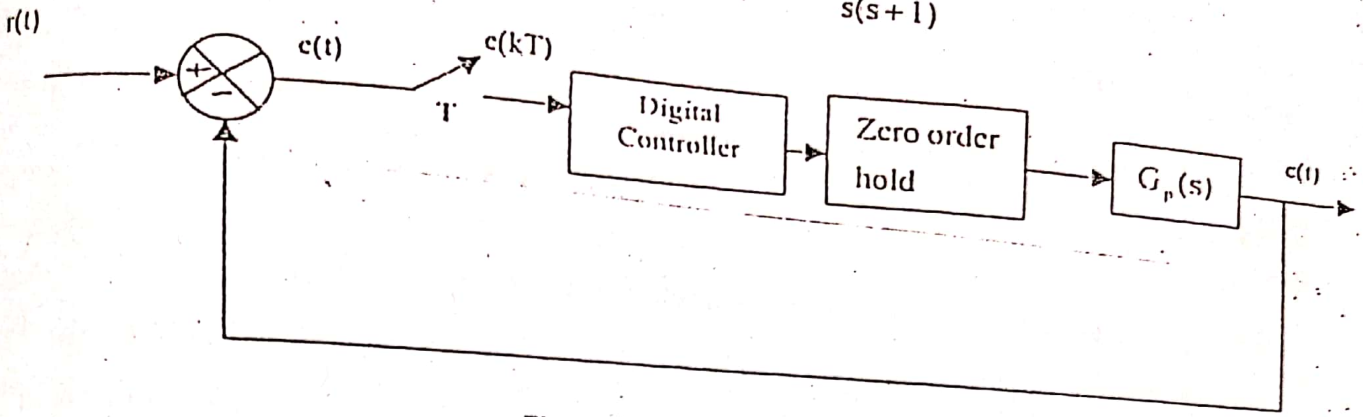


Fig 4.19 (A Digital control system)

Design a digital controller $G_D(z)$ such that the closed-loop system will exhibit a deadbeat response to a unit-step input. (In a deadbeat response the system should not exhibit inter sampling ripples in the output after the settling time is reached.) The sampling period T is assumed to be 1 sec. Then, using the digital controller $G_D(z)$ so designed, investigate the response of this system to a unit-ramp input.

Solution . The first step in the design is to determine the z transform of the plant that is preceded by the zero-order hold.

$$G(z) = \mathcal{Z}\left[\frac{1-e^{-Ts}}{s}\frac{1}{s(s+1)}\right]$$

$$= (1-z^{-1})\mathcal{Z}\left[\frac{1}{s^2(s+1)}\right]$$

$$= (1-z^{-1})\left[\frac{z^{-1}}{(1-z^{-1})^2} - \frac{1}{(1-z^{-1})} + \frac{1}{1-0.3679z^{-1}}\right]$$

$$= \frac{0.3679(1+0.7181z^{-1})z^{-1}}{(1-z^{-1})(1-0.3679z^{-1})} \quad \text{.........(4.24)}$$

$z = \frac{1+(T/2)\omega}{1-(T/2)\omega}$

Now redraw the block diagram of the system as shown in figure below.

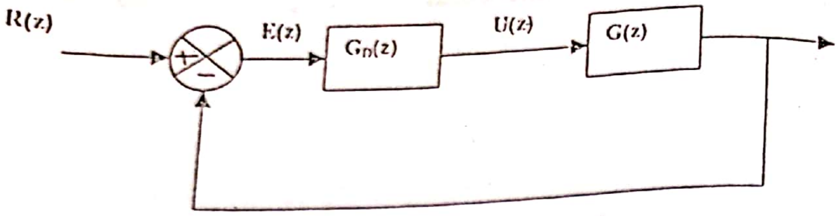


Fig 4.20 (Diagram showing equivalent digital control system)

Define the closed-loop pulse transfer function as F(z), or

$$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1+G_D(z)G(Z)} = F(z)$$

Notice that if G(z) is expanded into a series in z^{-1} then the first term will be $0.3679z^{-1}$. He
F(z) must begin with a term in z^{-1}.

Hence, we can assume F(z) to be of the following form.

$$F(z) = a_1z^{-1}+a_2z^{-2} \quad \text{.........(4.25)}$$

Since the input is a step function, we can write $1-F(z)=(1-z^{-1})\,N(z)$ ....(4.26)

Since G(z) has a critically stable pole at z=1, the stability requirement states that 1-F(z) have a zero at z=1. However, the function 1-F(z) already has a term $1-z^{-1}$ and there satisfies the requirement.

Since the system should not exhibit inter sampling ripples and the input is a step function require $c(t \geq 2T)$ to be constant. Noting that u(t), the output of the zero-order hold, continuous-time function, a constant $c(t \geq 2T)$ requires that u(t) also be constant for $t \geq 2T$ terms of the z transform, U(z) must be to the following type of series in z^{-1}.

$$U(z) = b_0+b_1z^{-1}+b(z^{-2}+z^{-3}+z^{-4}+.......)$$

Where b is constant. Because the plant transfer function $G_p(s)$ involves an integrator, b be zero. (Otherwise, the output cannot stay constant). Consequently, we have

$$U(z) = b_0+b_1z^{-1}$$

From Fig (b), U(z) can be given as follows:

$$U(z)=\frac{C(z)}{G(z)}=\frac{C(z)}{R(z)}\frac{R(z)}{G(z)}=F(z)\frac{R(z)}{G(z)}$$

$$=F(z)\frac{1}{1-z^{-1}}\frac{(1-z^{-1})(1-0.3679z^{-1})}{0.3679(1+0.7181z^{-1})z^{-1}}=F(z)\frac{1-0.3679z^{-1}}{0.3679(1+0.7181z^{-1})z^{-1}}$$

For U(z) to be a series in z^{-1} with only two terms, F(z) must be of the following form:

$$F(z)=(1+0.7181z^{-1})\,z^{-1}F_1 \qquad \text{..........(4.27)}$$

where F_1 is a constant. Then U(z) can be written as follows:

$$U(z)=2.7181(1-0.3679z^{-1})F_1 \qquad \text{..........(4.28)}$$

Eq (4.28) gives U(z) in terms of F_1. Once constant F_1 is determined, U(z) can be given as a series in z^{-1} with only two terms.

Now we shall determined N(z), F(z), and F_1. By substituting Eq (4.25) into Eq (4.26) we obtain.

$$1-a_1z^{-1}-a_2z^{-2}=(1-z^{-1})N(z)$$

The left-hand side of its last equation must be divisible by $1-z^{-1}$. If we divide the left-hand side by $1-z^{-1}$, the quotient is $1+(1-a_1)z^{-1}$ and the remainder is $(1-a_1-a_2)z^{-2}$. Hence, N(z) is determined as.

$$N(z)=1+(1-a_1)z^{-1} \qquad \text{..........(4.29)}$$

and the remainder must be zero. This requires that

$$1-a_1-a_2=0 \qquad \text{..........(4.30)}$$

Also, from Eqs (4.25) and (4.27) we have

$$F(z)=a_1z^{-1}+a_2z^{-2}=(1+0.7181z^{-1})z^{-1}F_1$$

Hence,

$$a_1+a_2z^{-1}=(1+0.7181z^{-1})\,F_1$$

Division of the left-hand side of this last equation by $1+0.7181z^{-1}$ yields the quotient a_1 and the remainder $(a_2-0.7181a_1)z^{-1}$. By equating the quotient with F_1 and the remainder with zero, we obtain.

$$F_1=a_1$$

and

$$a_2-0.7181a_1=0 \qquad \text{..........(4.31)}$$

Solving Eqs (4.30) and (4.31) for a_1 and a_2 gives

$a_1=0.5820, \qquad a_2=0.4180$

Thus, $F(z)$ is determined as

$$F(z) = 0.5820z^{-1} + 0.4180z^{-2} \qquad \text{.........(4.32)}$$

and

$$F_1 = 0.5820$$

Eq (4.29) gives

$$N(z) = 1 + 0.4180z^{-1} \qquad \text{........(4.33)}$$

The digital controller pulse transfer function $G_D(z)$ is then determined as follows.

$$G_D(z) = \frac{F(z)}{G(z)(1-z^{-1})N(z)}$$

$$= \frac{(1+0.7181z^{-1})z^{-1}(0.5820)}{\dfrac{0.3679(1+0.7181z^{-1})z^{-1}}{(1-z^{-1})(1-0.3679z^{-1})}(1-z^{-1})(1+0.4180z^{-1})}$$

$$= \frac{1.5820 - 0.5820z^{-1}}{1+0.4180z^{-1}}$$

With the digital controller thus designed, the closed-loop pulse transfer function becomes follows.

$$\frac{C(z)}{R(z)} = F(z) = 0.5820z^{-1} + 0.4180z^{-2}$$

$$= \frac{0.5820(z+0.7181)}{z^2}$$

The system output in response to a unit-step input $r(t) = 1$ can be obtained as follows.

$$C(z) = F(z)\,R(z)$$

$$= (0.5820z^{-1} + 0.4180z^{-2})\frac{1}{1-z^{-1}}$$

$$= 0.5820z^{-1} + z^{-2} + z^{-3} + z^{-4} + \dots\dots\dots$$

Hence,

$c(0) = 0$

$c(1) = 0.5820$

$c(k) = 1, \qquad k = 2, 3, 4, \dots\dots$

Notice that substitution of 0.5820 for F_1 in Eq (5.28) yields

$$U(z) = 2.7181(1-0.3679z^{-1})(0.5820)$$
$$= 1.5820 - 0.5820\, z^{-1}$$

Thus, the control signal u(k) becomes zero for $k \geq 2$, as required. There is no inter sampling ripple in the output after the setting time is reached.

Next, let us investigate the response of this system to a unit-ramp input.

$$C(z) = F(z)\,R(z)$$

$$= (0.5820\, z^{-1} + 0.4180\, z^{-2})\frac{z^{-1}}{(1-z^{-1})^2}$$

$$= 0.5820\, z^{-2} + 1.5820\, z^{-3} + 2.5820\, z^{-4} + 3.5820\, z^{-5} + \dots\dots\dots$$

For the unit-ramp response, the control signal U(z) is obtained as follows. referring to Eqs (4.24) and (4.32),

$$U(z) = \frac{C(z)}{G(z)} = \frac{F(z)}{G(z)}R(z) = \frac{F(z)}{G(z)}\frac{z^{-1}}{(1-z^{-1})^2}$$

$$= (1.5820 - 0.5820z^{-1})\frac{z^{-1}}{1-z^{-1}}$$

$$= 1.5820\, z^{-1} + z^{-2} + z^{-3} + z^{-4} + \dots\dots$$

The signal u(k) becomes constant (b=1) for $k \geq 2$. Hence, the system output will not exhibit inter sampling ripples.

Note that the static velocity error constant k_v for the present system is

$$K_V = \lim_{z \to 1}\left[\frac{1-z^{-1}}{T}G_D(z)G(z)\right]$$

$$= \lim_{z \to 1}\left[(1-z^{-1})\frac{F(z)}{(1-z^{-1})N(z)}\right]$$

$$= \lim_{z \to 1}\frac{0.5820z^{-1} + 0.4180z^{-2}}{1+0.4180z^{-1}} = 0.7052$$

Thus, the steady-state error in the unit-ramp response is

$$e_{ss} = \frac{1}{k_V} = 1.4180$$

Example 12 . Consider a design problem the same as that of previous example except the static velocity error constant k_v is specified. (Because of this addition constraint settling time will be longer than 2 sec.) The block diagram of the digital control system under consideration is shown in figure below. The plant transfer function $G_p(s)$ under consideration is.

$$G_p(s) = \frac{1}{s(s+1)}$$

The design specifications are (1) that the closed-loop system is to exhibit a finite settling with zero steady-state error in the unit-step response, (2) that the output is not to exhibit sampling ripple after the settling time is reached, (3) that the static velocity error consta is to be 4 sec^{-1}, and (4) that the settling time is to be the minimum possible that will satisf these specifications. The sampling period T is assumed to be 1 sec. Design a di controller $G_D(z)$ that satisfies the given specifications. After the controller is desig investigate the response of the system to a unit-ramp input.

Solution . The z transform of the plant that is preceded by the zero-order hold was obta in the previous example as.

$$G(z) = \mathcal{Z}\left[\frac{1-e^{-Ts}}{s}\frac{1}{s(s+1)}\right]$$

$$= \frac{0.3679(1+0.7181z^{-1})z^{-1}}{(1-z^{-1})(1-0.3679z^{-1})}$$

Define the closed-loop pulse transfer function as F(z).

$$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1+G_D(z)G(z)} = F(z)$$

Since the first term in the expansion of G(z) is $0.3679z^{-1}$, F(z) must begin with a term in z

$$F(z) = a_1 z^{-1} + a_2 z^{-2} + \dots\dots + a_N z^{-N}$$

where $N \geq n$ and n is the order of the system (that is, n =2 in the present case). Because o added constraint, we may assume N > 2. We shall try N =3. Thus, we assume.

$$F(z) = a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} \qquad \dots\dots\dots(4.34)$$

(If a satisfactory result is not obtained, we must assume N > 3.) Since the input is a function we require that.

$$1-F(z) = (1-z^{-1})\,N(z) \qquad \dots\dots\dots(4.35)$$

Note that the present of a critically stable pole at $z = 1$ in the plant pulse transfer function $G(z)$ requires $1-F(z)$ to have a zero at $z = 1$. However, the function $1-F(z)$ already has a term $1-z^{-1}$ and therefore satisfies the stability requirement.

The requirement that the static velocity error constant be 4 sec^{-1} can be written as follows.

$$k_v = \lim_{z \to 1}\left[\frac{1-z^{-1}}{T} G_D(z)G(z)\right]$$

$$= \lim_{z \to 1}\left[(1-z^{-1})\frac{F(z)}{(1-z^{-1})N(z)}\right]$$

$$= \frac{F(1)}{N(1)} = 4$$

Where we used q=0. Notice that Eq (4.35) we have $F(1) = 1$. Hence, k_v can be written as follows.

$$k_v = \frac{1}{N(1)} = 4 \qquad \text{.........(4.36)}$$

~~Since the system output~~ should not exhibit inter sampling ripples after the setting time is reached, we require $U(z)$ to be of the following form.

$$U(z) = b_0+b_1z^{-1}+ b_2z^{-2}+b(z^{-3}+z^{-4}+z^{-5}+.........)$$

Because the plant transfer function $G_p(s)$ involves an integrator, b must be zero. Consequently, we have.

$$U(z) = b_0+b_1z^{-1}+ b_2z^{-2}$$

Also, from figure (b), U(z) can be given by

$$U(z) = \frac{C(z)}{G(z)} = \frac{C(z)}{R(z)}\frac{R(z)}{G(z)} = F(z)\frac{R(z)}{G(z)}$$

$$= F(z)\frac{1-0.3679z^{-1}}{0.3679(1+0.7181z^{-1})z^{-1}}$$

For U(z) to be a series in z^{-1} with three terms, F(z) must be of the following form.

$$F(z) = (1+0.7181z^{-1})z^{-1}F_1(z) \qquad \text{.........(4.37)}$$

Where $F_1(z)$ is a first-degree polynomial in z^{-1}. Then U(z) can be written as follows.

$$U(z) = 2.7181(1-0.3679z^{-1})F_1(z) \qquad \text{.........(4.38)}$$

From Eqs (4.34) and (4.35), we have

$$1-F(z) = 1-a_1z^{-1}-a_2z^{-2}-a_3z^{-3} = (1-z^{-1})N(z)$$

If we divide $1-a_1z^{-1}-a_2z^{-2}-a_3z^{-3}$ by $1-z^{-1}$, the quotient is $1+(1-a_1)z^{-1}+(1-a_1-a_2)z^{-2}+$ and remainder is $(1-a_1-a_2-a_3)z^{-3}$. Hence, N(z) is determined as .

$$N(z) = 1+(1-a_1)z^{-1}+(1-a_1-a_2)z^{-2} \qquad \text{.........(4.39)}$$

and the remainder must be zero, so that

$$1-a_1-a_2-a_3=0 \qquad \text{.........(4.40)}$$

Note that from Eq (4.36) we require $N(1) = \frac{1}{4}$. Therefore, by substituting $z^{-1}=1$ into (4.39) we obtain.

$$2a_1+a_2=2.75 \qquad \text{.........(4.41)}$$

Also, Eq (4.37) can be rewritten as

$$F(z) = a_1z^{-1}-a_2z^{-2}-a_3z^{-3} = (1+0.7181z^{-1})z^{-1}F_1(z)$$

Hence,

$$a_1+a_2z^{-1}+a_3z^{-2} = (1+0.7181z^{-1})F_1(z)$$

Division of the left-hand side of this last equation by $1+0.7181z^{-1}$ yields the quo $[a_1+(a_2-0.7181a_1)z^{-1}]$ and the remainder $[a_3-0.7181(a_2-0.7181a_1)]z^{-2}$. By equating the quo with $F_1(z)$ and the remainder with zero, we obtain.

$$F_1(z) = a_1+(a_2-0.7181a_1)z^{-1}$$

and

$$a_3-0.7181(a_2-0.7181a_1)=0 \qquad \text{.........(4.42)}$$

Solving Eq (4.40), (4.41) and (4.42) for a_1, a_2 and a_3 gives

$a_1 = 1.26184,\qquad a_2 = 0.22633,\qquad a_3 = -0.48816$

Thus, F(z) is determined as

$$F(z) = 1.26184z^{-1} + 0.22633z^{-2} - 0.48816z^{-3}$$

and

$$F_1(z) = 1.26184 - 0.67979z^{-1}$$

Eq (4.39) gives

$$N(z) = 1 - 0.26184z^{-1} - 0.48817z^{-2}$$

The digital controller pulse transfer function $G_D(z)$ is then determined as follows:

$$G_D(z) = \frac{F(z)}{G(z)(1-z^{-1})N(z)}$$

$$= \frac{0.3679(1+0.7181z^{-1})z^{-1}}{(1-z^{-1})(1-0.3679z^{-1})z^{-1}} \cdot \frac{(1+0.7181z^{-1})z^{-1}(1.26184-0.67980z^{-1})}{(1-z^{-1})(1-0.26184z^{-1}-0.48817z^{-2})}$$

$$= 3.4298\frac{(1-0.5387z^{-1})(1-0.3679z^{-1})}{(1-0.8418z^{-1})(1+0.5799z^{-1})}$$

With the digital controller thus designed, the system output in response to a unit-step input $r(t)=1$ is obtained as follows.

$C(z) = F(z)\,R(z)$

$$= (1.26184z^{-1}+0.22633\,z^{-2}-0.48816\,z^{-3})\frac{1}{1-z^{-1}}$$

$$= 1.2618\,z^{-1}+1.4882\,z^{-2}+z^{-3}+z^{-4}+\ldots\ldots\ldots$$

Hence,

$c(0) = 0$

$c(1) = 1.2618$

$c(2) = 1.4882$

$c(k) = 1, \qquad k = 3,4,5,\ldots\ldots\ldots\ldots$

The unit-step response sequence has a maximum overshoot of approximately 50%. The settling time is 3 sec.

Notice that from Eq (4.38) we have

$U(z) = 2.7181(1-0.3679\,z^{-1})(1.26184-0.67979\,z^{-1})$

$= 3.4298-3.1096\,z^{-1}+.6798\,z^{-2}$

Thus, the control signal $u(k)$ becomes zero for $k \geq 3$. Consequently, there are no intersampling ripples in the response. Notice that the assumption of N=3, that is, the assumption of $F(z)$ as given by equation 1, is satisfactory.

Next, let us investigate the response of this system to a unit-ramp input.

$C(z) = F(z)\,R(z)$

$$= (1.26184\,z^{-1}+0.22633\,z^{-2}-0.48816\,z^{-3})\frac{z^{-1}}{(1-z^{-1})^2}$$

$$= 1.2618\,z^{-2}+2.7500\,z^{-3}+3.7500\,z^{-4}+\ldots\ldots\ldots\ldots$$

In the unit-ramp response, the control signal $U(z)$ is obtained as follows.

$$U(z) = \frac{C(z)}{G(z)} = \frac{F(z)R(z)}{G(z)} = \frac{F(z)}{G(z)}\frac{1}{1-z^{-1}}\frac{z^{-1}}{1-z^{-1}}$$

$$= (3.4298 - 3.1096z^{-1} + 0.6798z^{-2})\frac{z^{-1}}{1-z^{-1}}$$

$$= 3.4298\, z^{-1} + 0.3202\, z^{-2} + z^{-3} + z^{-4} + z^{-5} + \ldots\ldots\ldots$$

The signal u(k) becomes constant (b=1) for $k \geq 3$. Hence, the system output will not ex

intersampling ripples.

4.10 Problems :

1. Consider the system described by

   $y(k) - 0.6\, y(k-1) - 0.81\, y(k-2) + 0.67\, y(k-3) - 0.12\, y(k-4) = x(k)$

   Where x (k) is the input and y(k) is the output of the system. Determine the stab

   of the system.

Ans. Critical stable.

2. Consider the characteristic equation of the second-order system.

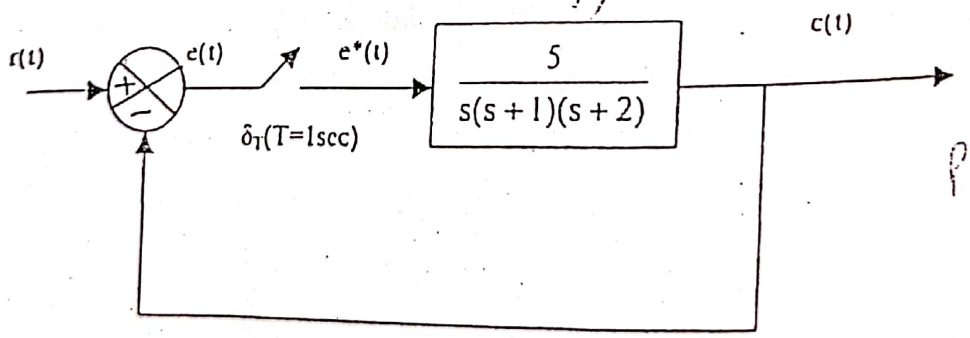
   $F(z) = a_2z^2 + a_1z + a_0 = 0$; $a_2 > 0$

   Determine the stability of the system.

Ans. Unstable.

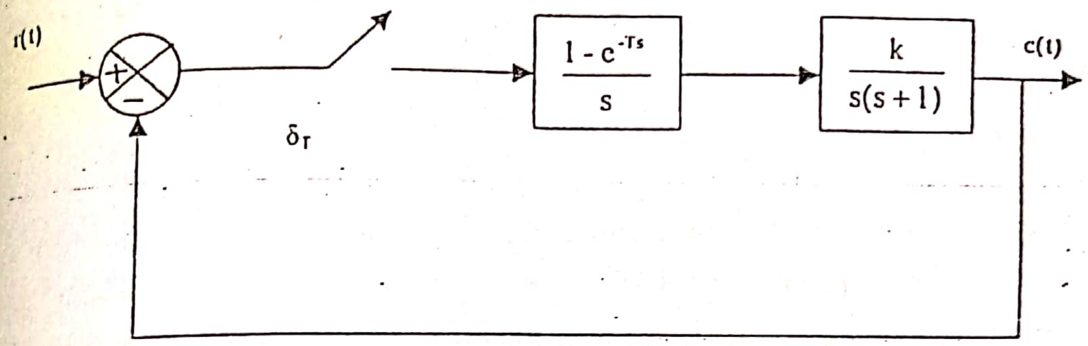
3. Consider the sampled data system as shown in figure below. Determine characteristic equation in the z-domain and ascertain its stability via the bili transformation.

G(s)



Ans. Stable

4. Draw root locus diagrams in the z plane for the system shown in figure below for the following three sampling periods. T=1 sec, T=2 sec, and T= 4 sec.



5. Draw the Bode diagram in the w-plane for the transfer function of the plant

$G(s) = \frac{1}{s(s+2)}$ which is preceded by a zero-order hold.

CHAPTER- V

State-Space Analysis

5.1 Introduction

Conventional methods such as the root-locus and frequency-response methods are usef dealing with single-input- single -output systems. These methods are simple and require a reasonable number of computations, but they are applicable only to linear time-inv systems having a single input and single output. They are based on input-output relatio of the systems, that is, the transfer function or the pulse transfer function. Conven methods do not apply to nonlinear systems except in simple cases. Also, the conven methods do not apply to the design of optimal and adaptive control systems, whic mostly time varying and/ or nonlinear.

A modern control system may have many inputs and many outputs. For multiple input multiple outputs, we need state space methods

5.2 Concept of the State-Space Method

The state-space method is based on the description of system equations in terms of n order difference equation. Which may be combined into a first-order vector-matrix diffe equation. The use of the vector-matrix notation greatly simplifies the mathem representation of the equations of the systems.

The use of the state-space concept enables us to design control systems with respect to performance indexes. In the state-space the design can be carried out for a class of in instead of a specific input function such as the impulse function, step function, or sinus function. The state-space methods also enables us to include initial conditions in the de which is not possible in the conventional methods.

Definitions

State : The state of a dynamic system is the smallest set of variables (called state varia such that the knowledge of these variables at $t=t_0$, completely determines the behavior o system for any time $t \geq t_0$. It is to be noted that the concept of a state is not only limited physical systems, but also applicable to biological systems, economic system, social syste and others.

State Variables: The state variables of a dynamic system are the variables making up the smallest set of variables that determine the state of the dynamic system. If at least n variables $x_1, x_2, \ldots, x_n$ are needed to describe completely the behavior of the dynamic system (so that once the input is given for $t \geq t_0$ and the initial state at $t=t_0$ is specified, the future state of the system is completely determined), then such n variables are a set of state variables.

State Vector : It n state variables are needed to describe completely the behavior of a given system, then these n state variables can be considered the n components of a vector X. Such a vector is thus a vector that determines uniquely the system state x(t) for any time $t \geq t_0$, once the state at $t=t_0$ is given and the input u(t) for $t \geq t_0$ is specified.

State Space : The n-dimensional space whose coordinate axes consist of the x_1 axis, x_2 axis, ..., x_n axis is called a state space. Any state can be represented by a point in the state space.

State- Space Equations : In state-space analysis we need to concern with three types of variables that are involved in the modeling of dynamic systems : input variables, output variable and the state variables.

For time-varying (linear or nonlinear) discrete –time systems, the state equation may be written as .

$$x(k+1) = f[x(k), u(k), k] \quad \text{..........(5.1)}$$

and the output equation as

$$y(k) = g[x(k), u(k), k] \quad \text{..........(5.2)}$$

For linear time-varying discrete-time systems, the state equation and output equation may be simplified to

$$x(k+1) = G(k) + H(k)\,u(k) \quad \text{............(5.3)}$$

$$y(k) = C(k)x(k) + D(k)u(k) \quad \text{............(5.4)}$$

Where,

$x(k)$	= n- vector	(state vector)
$y(k)$	= m- vector	(output vector)
$u(k)$	= r - vector	(input vector)
$G(k)$	= n x n matrix	(state matrix)
$H(k)$	= n x r matrix	(input matrix)
$C(k)$	= m x n matrix	(output matrix)
$D(k)$	= m x r matrix	(direct transmission matrix)

For the time-invariant system the state equation and output equation can be simplified to

$$x(k+1) = Gx(k) + Hu(k) \qquad \text{.........(5.5)}$$

$$y(k) = Cx(k) + Du(k) \qquad \text{.........(5.6)}$$

The Fig. 5.1 shows the block diagram representation of the discrete-time control system defined by Eqs (5.5) and (5.6)

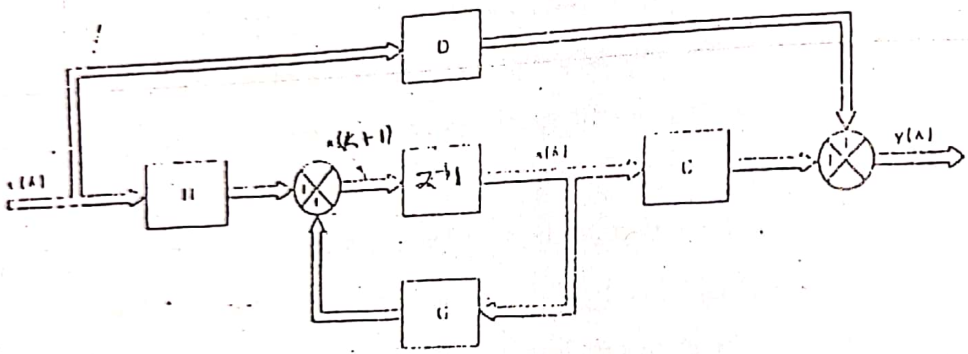


Fig 5.1 (Block diagram of a linear time invariant discrete-time control system represented in the state space)

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5.3 State-Space Representation of Discrete-Time System

Let us consider the discrete-time system described by

$$y(k)+a_1y(k-1) + a_2y(k-2)+ \ldots + a_ny(k-n)$$

$$= b_0u(k) + b_1u(k-1) + \ldots\ldots + b_nu(k-n) \qquad \text{.........(5.7)}$$

Where $u(k)$ is the input and $y(k)$ is the output of the system at the kth sampling instant. Note that some of the coefficients a_i (i=1,2, ...n) and bj (j=0,1,2,....., n) may be zero. The Eq (5.7) may be written in the form of pulse transfer function as

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \dots\dots + b_n z^{-n}}{1 + a_1 z^{-1} + \dots\dots + a_n z^{-n}}$$

$$\dots\dots\dots(5.8)$$

or. $$\frac{Y(z)}{U(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \dots\dots + b_n}{z^n + a_1 z^{n-1} + \dots\dots + a_n}$$

$$\dots\dots\dots(5.9)$$

As there are many ways to realize state-space representations for the discrete time system described by Eqs (5.7), (5.8), or (5.9), we will present the following representations:

(i) Controllable canonical form

(ii) Observable canonical form

(iii) Diagonal canonical form

(iv) Jordan canonical form

(i) ***Controllable canonical form*** :A control system is controllable if every state variables can be controlled in a finite time period by some unconstrained control signal.

The Eqs (5.8) or (5.9) may be put in controllable canonical form as shown below:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots\dots & 0 \\ 0 & 0 & 1 & \dots\dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots\dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots\dots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$\dots\dots(5.10)$$

$$y(k) = \left[b_n - a_n b_0 \vdots b_{n-1} - a_{n-1} b_0 \vdots \cdots \vdots b_1 - a_1 b_0 \right] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ \vdots \\ x_n(k) \end{bmatrix} + b_0 u(k) \qquad \dots\dots\dots(5.11)$$

Eqs (5.10) and (5.11) are the state equation and output equation respectively.

Example 1. Obtain the state-space representation in the controllable canonical form following discrete-time system.

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$$\frac{Y(z)}{U(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \dots + b_n}{z^n + a_1 z^{n-1} + \dots + a_n} \qquad \text{.........(5.}$$

Solution . The given system can be modified to

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_n z^{-n}} \qquad \text{.........(5.}$$

$$= b_0 + \frac{(b_1 - a_1 b_0) z^{-1} + (b_2 - a_2 b_0) z^{-2} + \dots + (b_n - a_n b_0) z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_n z^{-n}} \qquad \text{.........(5}$$

or,

$$Y(z) = b_0 U(z) + \frac{(b_1 - a_1 b_0) z^{-1} + (b_2 - a_2 b_0) z^{-2} + \dots + (b_n - a_n b_0) z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_n z^{-n}}$$

......(5

Let us define,

$$Y_1(z) = \frac{(b_1 - a_1 b_0) z^{-1} + (b_2 - a_2 b_0) z^{-2} + \dots + (b_n - a_n b_0) z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_n z^{-n}} U(z) \qquad \text{.........(}$$

Then, the Eq (5.15) becomes,

$$Y(z) = b_0 U(z) + Y_1(z) \qquad \text{.........(5}$$

Eq (5.16) can be rewritten as,

$$\frac{Y_1(z)}{(b_1 - a_1 b_0) z^{-1} + (b_2 - a_2 b_0) z^{-2} + \dots + (b_n - a_n b_0) z^{-n}}$$

$$= \frac{U(z)}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_n z^{-n}} = Q(z) \qquad \text{.........(5.}$$

So.

$$Q(z) = -a_1 z^{-1} Q(z) - a_2 z^{-2} Q(z) + \dots + a_n z^{-n} Q(z) + U(z) \qquad \text{.........(5.}$$

and,

$$Y_1(z) = (b_1 - a_1 b_0) z^{-1} Q(z) + (b_2 - a_2 b_0) z^{-2} Q(z) + \ldots\ldots + (b_n - a_n b_0) z^{-n} Q(z) \quad \ldots\ldots(5.20)$$

Let us now define the state variables as follows:

$$\left.\begin{aligned} &\boxed{X_1(z) = z^{-n} Q(z)} \\ &X_2(z) = z^{-n+1} Q(z) \\ &\vdots \\ &X_{n-1}(z) = z^{-2} Q(z) \\ &X_n(z) = z^{-1} Q(z) \end{aligned}\right\} \quad \ldots\ldots(5.21)$$

$X_1(z) = z^{-n} Q(z)$

$X_2(z) = z^{-(n-1)} Q(z)$

$\Rightarrow X_2 = z X_1(z)$

$\Rightarrow z X_n(z) = Q(z)$

Then we can write

$$\begin{aligned} &z X_1(z) = X_2(z) \\ &z X_2(z) = X_3(z) \\ &\vdots \\ &z X_{n-1}(z) = X_n(z) \end{aligned}$$

In terms of difference equations, the preceding n-1 equations become

$$\begin{aligned} &x_1(k+1) = x_2(k) \\ &x_2(k+1) = x_3(k) \\ &\vdots \\ &x_{n-1}(k+1) = x_n(k) \end{aligned} \quad \ldots\ldots(5.22)$$

Now, by substituting Eq (5.21) into the Eq (5.22), we can write

$$z X_n(z) = -a_1 X_n(z) - a_2 X_{n-1}(z) - \ldots\ldots - a_n X_1(z) + U(z)$$

or,

$$x_n(k+1) = -a_n x_1(k) - a_{n-1} x_2(k) - \ldots\ldots - a_1 x_n(k) + u(k) \quad \ldots\ldots(5.23)$$

Also, Eq (5.20) can be written as,

$$Y_1(z) = (b_1 - a_1 b_0) X_n(z) + (b_2 - a_2 b_0) X_{n-1}(z) + \ldots + (b_n - a_n b_0) X_1(z)$$

or,

$$y_1(k) = (b_n - a_n b_0) x_1(k) + (b_{n-1} - a_{n-1} b_0) x_2(k) + \ldots\ldots + (b_1 - a_1 b_0) x_n(k) \quad \ldots\ldots(5.24)$$

From (5.17) and (5.24), we get

$$y(k) = b_0 u(k) + (b_n - a_n b_0) x_1(k) + (b_{n-1} - a_{n-1} b_0) x_2(k) + \ldots\ldots + (b_1 - a_1 b_0) x_n(k) \quad \ldots\ldots(5.25)$$

Combining Eqs (5.22) and (5.23), we get the state equation as below:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(k) \quad \text{..(5}$$

and from Eq (5.25), we get the output equation as below:

$$y(k) = \left[b_n - a_n b_0 \vdots b_{n-1} - a_{n-1} b_0 \vdots \cdots \vdots b_1 - a_1 b_0\right] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + b_0 u(k) \quad \text{......(5.}$$

Example 2. Consider the discrete-time system defined by

$$\frac{Y(z)}{U(z)} = \frac{0.368\ (z+0.717)}{(z-1)\ (z-0.368)}$$

Obtain the state-space representation of this system

Solution . The given system can be modified as

$$\frac{Y(z)}{U(z)} = \frac{0.368z^{-1} + 0.264z^{-2}}{1-1.368z^{-1} + 0.368z^{-2}}$$

or, $$\frac{Y(z)}{0.368z^{-1} + 0.264z^{-2}} = \frac{U(z)}{1-1.368z^{-1} + 0.368z^{-2}} = Q(z)$$

or, $$Q(z) = U(z) + 1.368z^{-1}Q(z) - 0.368z^{-2}\,Q(z) \quad \text{.........(5.}$$

and $$Y(z) = 0.368z^{-1}Q(z) + 0.264z^{-2}Q(z) \quad \text{.........(5.}$$

Let us define

$$z^{-2}Q(z) = X_1(z)$$

$$z^{-1}Q(z) = X_2(z)$$

So, $$z X_1(z) = X_2(z)$$ $\Rightarrow Q(z) = z X_2(z)$

or, $$x_1(k+1) = x_2(k)$$

Eq (5.28) can be rewritten as,

$$z X_2(z) = U(z) + 1.368 X_2(z) - 0.368 X_1(z) \quad \text{..........(5.30)}$$

or,

$$x_2(k+1) = u(k) + 1.368\, x_2(k) - 0.368\, x_1(k)$$ ....(5.31)

Now, from Eq (5.29), we get

$$y(k) = 0.368\, x_2(k) + 0.264\, x_1(k)$$ ....(5.32)

Hence, the state-space model of the system is,

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.368 & 1.368 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$ ....(5.33)

and $$y(k) = \begin{bmatrix} 0.264 & 0.368 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$ ....(5.34)

(ii) ***Observable canonical form*** : Consider the pulse transfer function system defined by

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \dots\dots + b_n z^{-n}}{1 + a_1 z^{-1} + \dots\dots + a_n z^{-n}}$$ ....(5.35)

The state-space representation of the discrete-time system given by equation may be put in the following form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & -a_n \\ 1 & 0 & \cdots & \cdots & 0 & 0 & -a_{n-1} \\ \vdots & \vdots & & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & -a_2 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} u(k)$$ ....(5.36)

$$y(k) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + b_0 u(k)$$ ....(5.37)

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The state-space representation given by Eqs (5.36) and (5.37) is called an observa-

canonical form.

Proof : Rewriting the pulse transfer function as follows :

$$Y(z) + a_1 z^{-1}Y(z) + a_2 z^{-2}Y(z) + \ldots\ldots + a_n z^{-n}Y(z)$$

$$= b_0 U(z) + b_1 z^{-1}U(z) + b_2 z^{-2}U(z) + \ldots\ldots + b_n z^{-n}U(z)$$

or,

$$Y(z) + z^{-1}\left[a_1 Y(z) - b_1 U(z)\right] + z^{-2}\left[a_2 Y(z) - b_2 U(z)\right] + \ldots\ldots + z^{-n}\left[a_n Y(z) - b_n U(z)\right] - b_0 U$$

or,

$$Y(z) = b_0 U(z) + z^{-1}\Big(b_1 U(z) - a_1 Y(z) + z^{-1}\Big\{b_2 U(z) - a_2 Y(z) + z^{-1}\big[b_3 U(z) - a_3 Y(z) + \ldots\big]$$

........(5.38)

Let us define the state variables as follows:

$$\begin{aligned} X_n(z) &= z^{-1}\left[b_1 U(z) - a_1 Y(z) + X_{n-1}(z)\right] \\ X_{n-1}(z) &= z^{-1}\left[b_2 U(z) - a_2 Y(z) + X_{n-2}(z)\right] \\ &\vdots \\ X_2(z) &= z^{-1}\left[b_{n-1} U(z) - a_{n-1} Y(z) + X_1(z)\right] \\ X_1(z) &= z^{-1}\left[b_n U(z) - a_n Y(z)\right] \end{aligned}$$

.......(5.39)

Hence from Eq (5.38), we get

$$Y(z) = b_0 U(z) + X_n(z)$$

.......(5.40)

From (5.39),

$$z X_1(z) = b_n U(z) \quad a_n Y(z)$$

or, $$z X_1(z) = b_n U(z) - a_n\left[b_0 U(z) + X_n(z)\right]$$

$$= b_n U(z) - a_n b_0 U(z) - a_n X_n(z)$$

$$= (b_n - a_n b_0)\, U(z) - a_n X_n(z)$$

$$\therefore\ x_1(k+1) = -a_n x_n(k) + (b_n - a_n b_0)\, u(k)$$

Similarly,

$$x_2(k+1) = x_1(k) - a_{n-1}x_n(k) + (b_{n-1} - a_{n-1}b_0)\, u(k)$$

$$\vdots$$

$$x_{n-1}(k+1) = x_{n-2}(k) - a_2x_n(k) + (b_2 - a_2b_0)\, u(k)$$

$$x_n(k+1) = x_{n-1}(k) - a_1x_n(k) + (b_1 - a_1b_0)\, u(k)$$

and, $\quad y(k) = x_n(k) + b_0u(k)$

Hence, the state-space representation can be given as shown in Eqs (5.36) and (5.37).

Example 3. Obtain the state-space representation of the following pulse transfer function in observable canonical form.

$$\frac{Y(z)}{U(z)} = \frac{0.368z^{-1} + 0.264z^{-2}}{1 - 1.368z^{-1} + 0.368z^{-2}}$$

Solution . The given pulse transfer function can be rewritten as,

$$Y(z) - 1.368z^{-1}Y(z) + 0.368z^{-2}Y(z)$$

$$= 0.368z^{-1}U(z) + 0.264z^{-2}U(z)$$

or, $$Y(z) = 1.368z^{-1}Y(z) - 0.368z^{-2}Y(z)$$

$$+ 0.368z^{-1}U(z) + 0.264z^{-2}U(z)$$

or, $$Y(z) = z^{-1}\left[1.368Y(z) + 0.368U(z)\right]$$

$$+ z^{-2}\left[0.264U(z) - 0.368Y(z)\right]$$

or, $$Y(z) = z^{-1}\Big[1.368\; Y(z) + 0.368\; U(z)$$

$$+ z^{-1}\left\{0.264\; U(z) - 0.368\; Y(z)\right\}\Big]$$

Let us define

$$X_2(z) = z^{-1}\left[1.368\; Y(z) + 0.368\; U(z) + X_1(z)\right]$$

$$X_1(z) = z^{-1}\left[0.264\; U(z) - 0.368\; Y(z)\right]$$

$$\therefore Y(z) = X_2(z)$$

So, $x_1(k+1) = 0.264\,u(k) - 0.368x_2(k)$

$x_2(k+1) = x_1(k) + 1.368x_2(k) + 0.368\,u(k)$

Hence, the state-space representation of the given pulse transfer function is given as foll

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & -0.368 \\ 1 & 1.368 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.264 \\ 0.368 \end{bmatrix} u(k)$$

and $$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

(iii) ***Diagonal canonical form:*** Consider the pulse transfer function of the system is given by

$$\frac{Y(z)}{U(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \dots + b_n}{z^n + a_1 z^{n-1} + \dots + a_n} \quad \text{.........(5.41)}$$

If the poles of the pulse transfer function are all distinct, then the state-space representation may be put in diagonal canonical form as follows:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & p_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(k) \quad \text{.........(5.42)}$$

and

$$y(k) = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + b_0\,u(k) \quad \text{.........(5.43)}$$

Proof : The given pulse transfer function can be modified as follows:

$$\frac{Y(z)}{U(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \dots + b_n}{z^n + a_1 z^{n-1} + \dots + a_n}$$

$$= b_0 + \frac{(b_1 - a_1 b_0)z^{n-1} + (b_2 - a_2 b_0)z^{n-2} + \ldots\ldots + (b_n - a_n b_0)}{(z-p_1)(z-p_2)\ldots\ldots\ldots(z-p_n)} \qquad \ldots(5.44)$$

Since, all the poles of the pulse transfer function $\frac{Y(z)}{U(z)}$ are distinct, $\frac{Y(z)}{U(z)}$ can be expanded into the following form:

$$\frac{Y(z)}{U(z)} = b_0 + \frac{c_1}{z-p_1} + \frac{c_2}{z-p_2} + \ldots\ldots\ldots + \frac{c_n}{z-p_n} \qquad \ldots\ldots(5.45)$$

where,

$$c_i = \lim_{z \to p_i}\left[\frac{Y(z)}{U(z)}(z-p_i)\right]$$

or,
$$Y(z) = b_0\, U(z) + \frac{c_1}{z-p_1}U(z) + \frac{c_2}{z-p_2}U(z) + \ldots\ldots + \frac{c_n}{z-p_n}U(z) \qquad \ldots\ldots(5.46)$$

Let us define the state variables as follows:

$$X_1(z) = \frac{1}{z-p_1}U(z)$$

$$X_2(z) = \frac{1}{z-p_2}U(z) \qquad \ldots\ldots(5.47)$$

$$\vdots$$

$$X_n(z) = \frac{1}{z-p_n}U(z)$$

or,

$$z\,X_1(z) = p_1 X_1(z) + U(z)$$

$$z\,X_2(z) = p_2 X_2(z) + U(z) \qquad \ldots\ldots(5.48)$$

$$\vdots$$

$$z\,X_n(z) = p_n X_n(z) + U(z)$$

and

$$Y(z) = b_0 U(z) + c_1\, X_1(z) + c_2\, X_2(z) + \ldots\ldots + c_n\, X_n(z) \qquad \ldots\ldots(5.49)$$

Now,

$$x_1(k+1)= p_1x_1(k)+ u(k) \qquad \text{.........(5.50)}$$
$$x_2(k+1)= p_2x_2(k)+ u(k)$$
$$\vdots$$
$$x_n(k+1)= p_nx_n(k)+ u(k)$$

and

$$y(k)= c_1\, x_1\,(k)+ c_2\, x_2\,(k)+......+ c_n\, x_n\,(k) + b_0\, u(k) \qquad \text{........(5.51)}$$

Hence, the state-space representation can be given as Eqs (5.42) and (5.43).

Example 4. Obtain a state-space representation of the following pulse-transfer func[tion] system in the diagonal canonical form

$$\frac{Y(z)}{U(z)} = \frac{1+6z^{-1}+8z^{-2}}{1+4z^{-1}+3z^{-2}}$$

$$= 1 + \frac{(6-4)z^{-1} + (8-3)\ldots}{1+4z^{-1}+3z^{\ldots}}$$

Solution. The given pulse transfer function can be rewritten as

$$\frac{Y(z)}{U(z)} = \frac{z^2+6z+8}{z^2+4z+3}$$

$$= 1+\frac{2z+5}{z^2+4z+3}$$

$$= 1+\frac{2z+5}{(z+1)(z+3)}$$

$$= 1 + \frac{2z^{-1}+5z^{-2}}{1+4z^{-1}+3z^{-2}}$$

$$= 1 + \frac{2z+5}{z^2+4z+3}$$

$$= 1 + \frac{2z+5}{(z+1)(z+3)}$$

or,

$$\frac{Y(z)}{U(z)} = 1+\frac{c_1}{z+1}+\frac{c_2}{z+3}$$

Where,

$$c_i = \lim_{z\to p_i}\left[\frac{Y(z)}{U(z)}(z-p_i)\right]$$

Residue

$$c_1 = \left[\frac{2z+5}{(z+1)(z+3)}(z+1)\right]_{z=-1}$$

$$= \frac{-2+5}{2} = \frac{3}{2}$$

$$c_2 = \left[\frac{2z+5}{(z+1)}\right]_{z=-3}$$

$$= \frac{-6+5}{-3+1} = \frac{1}{2}$$

or, $$\frac{Y(z)}{U(z)} = 1 + \frac{3/2}{z+1} + \frac{1/2}{z+3}$$

or, $$Y(z) = U(z) + \frac{3/2}{z+1}U(z) + \frac{1/2}{z+3}U(z)$$

Let us define the state variables as follows:

$$X_1(z) = \frac{1}{z+1}U(z)$$

$$X_2(z) = \frac{1}{z+3}U(z)$$

or, $$zX_1(z) = U(z) - X_1(z)$$ $\Rightarrow zX_1(z) = P_L X_1(z) + U(z)$

$$zX_2(z) = U(z) - 3X_2(z)$$

and $$Y(z) = U(z) + \frac{3}{2}X_1(z) + \frac{1}{2}X_2(z)$$

Hence,

$$x_1(k+1) = u(k) - x_1(k)$$ $x_1(k+1) = u(k) - x_1$

$$x_2(k+1) = u(k) - 3x_2(k)$$

and $$y(k) = u(k) + \frac{3}{2}x_1(k) + \frac{1}{2}x_2(k)$$

Hence, the state –space representation is given by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u(k)$$

and $$y(k) = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + u(k)$$

(iv) ***Jordon Canonical Form :*** Consider the pulse transfer function system defined

$$\frac{Y(z)}{U(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \ldots + b_n}{z^n + a_1 z^{n-1} + \ldots + a_n} \qquad \ldots\ldots(5.$$

If the system involves a multiple pole of order m at $z=p_i$ and all other poles are distinc

the state-space representation is given by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+2) \\ \vdots \\ x_m(k+1) \\ \hline x_{m+1}(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \left[\begin{array}{cccccccc} p_1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & p_1 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & p_i & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & p_{m+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & p_n \end{array}\right]$$

$$+ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \hline 1 \\ \vdots \\ 1 \end{bmatrix} u(k) \qquad \ldots\ldots\ldots(5.$$

$$y(k) = \begin{bmatrix} c_1 & c_2 & \ldots\ldots & c_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + b_0 u(k) \qquad \ldots\ldots\ldots(5.$$

Proof : Since the system pulse transfer function can be written in the form

$$\frac{Y(z)}{U(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \ldots + b_{n-1} z + b_n}{(z-p_1)^m (z-p_{m+1})(z-p_{m+2})\ldots\ldots(z-p_n)}$$

$$= b_0 + \frac{(b_1 - a_1 b_0) z^{n-1} + (b_2 - a_2 b_0) z^{n-2} + \ldots\ldots + (b_n - a_n b_0)}{(z-p_1)^m (z-p_{m+1})\ldots\ldots\ldots(z-p_n)}$$

$$= b_0 + \frac{c_1}{(z-p_1)^m} + \frac{c_2}{(z-p_1)^{m-1}} + \dots\dots + \frac{c_m}{z-p_1}$$

$$+ \frac{c_{m+1}}{z-p_{m+1}} + \frac{c_{m+2}}{z-p_{m+2}} + \dots\dots + \frac{c_n}{z-p_n} \qquad \dots\dots\dots(5.55)$$

Then, we obtain

$$Y(z) = b_0U(z) + \frac{c_1}{(z-p_1)^m}U(z) + \frac{c_2}{(z-p_1)^{m-1}}U(z) + \dots\dots + \frac{c_m}{z-p_1}U(z) + \frac{c_{m+1}}{z-p_{m+1}}U(z)$$

$$+ \frac{c_{m+2}}{z-p_{m+2}}U(z) + \dots\dots + \frac{c_n}{z-p_n}U(z) \qquad \dots\dots(5.56)$$

Let us define the first m state variables $X_1(z), X_2(z), \dots\dots\dots X_m(z)$ by the equations

$$X_1(z) = \frac{1}{(z-p_1)^m}U(z)$$

$$X_2(z) = \frac{1}{(z-p_1)^{m-1}}U(z)$$

$$X_m(z) = \frac{1}{(z-p_1)}U(z)$$

and the remaining (n-m) state variables $X_{m+1}(z), X_{m+2}(z), \dots\dots, X_n(z)$ by the equation

$$X_{m+1}(z) = \frac{1}{z-p_{m+1}}U(z)$$

$$X_{m+2}(z) = \frac{1}{z-p_{m+2}}U(z) \qquad \dots\dots\dots(5.57)$$

$$\vdots$$

$$X_n(z) = \frac{1}{z-p_n}U(z)$$

Hence,

$$\frac{X_1(z)}{X_2(z)} = \frac{1}{z-p_1}$$

$$\frac{X_2(z)}{X_3(z)} = \frac{1}{z-p_1}$$

$$\vdots$$

$$\frac{X_{m-1}(z)}{X_m(z)} = \frac{1}{z-p_1} \qquad \text{.........(5.5}$$

Hence, by taking inverse z transform, we obtain

$$\begin{aligned}
x_1(k+1) &= p_1 x_1(k) + x_2(k) \\
x_2(k+1) &= p_1 x_2(k) + x_3(k) \\
&\vdots \\
x_{m-1}(k+1) &= p_1 x_{m-1}(k) + x_m(k) \\
x_m(k+1) &= p_1 x_m(k) + u(k) \\
&\vdots \\
x_{m+1}(k+1) &= p_{m+1} x_{m+1}(k) + u(k) \\
&\vdots \\
x_n(k+1) &= p_n x_n(k) + u(k)
\end{aligned} \qquad \text{.........(5.5}$$

and the output equation is,

$$Y(z) = c_1 X_1(z) + c_2 X_2(z) + \ldots\ldots + c_m X_m(z) + c_{m+1} X_{m+1}(z)$$

$$+ c_{m+2} X_{m+2}(z) + \ldots\ldots + c_n X_n(z) + b_0 U(z) \qquad \text{.........(5.6}$$

By taking the inverse z transform, we obtain

$$y(k) = c_1 x_1(k) + c_2 x_2(k) + \ldots\ldots + c_m x_m(k) + c_{m+1} x_{m+1}(k)$$

$$+ c_{m+2} x_{m+2}(k) + \ldots\ldots + c_n x_n(k) + b_0 u(k)$$

Hence, from Eqs (5.59) and (5.60), we obtain the state-space representation given (5.53) and (5.54).

5.4 Solving Discrete –Time State-Space Equation

5.4.1 Solution of the Linear Time-Invariant Discrete-Time State-Space Equation : In general, the discrete-time equations are solved by means of a recursion procedure which is quite simple and convenient for digital computation.

Let us consider ~~the state equation~~ the state equation and output equation as follows:

$$x(k+1) = Gx(k) + Hu(k) \qquad \text{.........(5.61)}$$

$$y(k) = Cx(k) + Du(k) \qquad \text{.........(5.62)}$$

The Eq (5.61) can be solved by the method of recursion as follows:

$$x(1)=Gx(0)+Hu(0)$$

$$x(2)=Gx(1)+Hu(1)$$

$$= G^2x(0) + GHu(0)+Hu(1)$$

$$x(3) = Gx(2)+Hu(2)=G^3x(0)+G^2Hu(0)+GHu(1)+Hu(2)$$

By repeating this procedure, we get

$$x(k) = G^k x(0) + \sum_{j=0}^{k-1} G^{k-j-1} Hu(j) \qquad k=1,2,3 \quad \text{.........} \qquad \text{......(5.63)}$$

Hence, it is clear that Eq (5.63) consists of two parts, one representing the contribution of the initial state x(0) and the other the contribution of the input u(j), where j=0,1,2,......,k-1. The output y(k) is given by

$$y(k) = CG^k X(0) + C\sum_{j=0}^{k-1} G^{k-j-1} Hu(j) + Du(k) \qquad \text{........(5.64)}$$

5.4.2 State Transition Matrix : It is possible to write the solution of the homogeneous state equation

$$x(k+1) = Gx(k)$$

as

$$\text{..........(5.65)}$$

$$x(k) = G\Psi(k),$$

Where $\Psi(k)$ is a unique n x n matrix satisfying the condition

$$\text{..........(5.66)}$$

$$\Psi(k+1) = G\Psi(k), \quad \Psi(0) = I$$

$$\text{........(5.67)}$$

Clearly, $\Psi(k)$ can be written as

$$\boxed{\Psi(k)=G^k} \qquad \text{.........(5.68)}$$

From Eq (5.66), it is clear that the solution of Eq (5.65) is simply a transformation of initial state. Therefore, the unique matrix $\Psi(k)$ is called the *State Transition Matrix*.

As we know that the solution of the state equation is,

$$x(k) = G^k x(0) + \sum_{j=0}^{k-1} G^{k-j-1} Hu(j) \qquad \text{.........(5.69)}$$

Hence, in terms of the state transition matrix $\Psi(k)$, Eq (5.69) can be written as

$$x(k) = \Psi(k)x(0) + \sum_{j=0}^{k-1} \Psi(k-j-1)Hu(j) \qquad \text{.........(5.70)}$$

$$= \Psi(k)x(0) + \sum_{j=0}^{k-1} \Psi(j)Hu(k-j-1) \qquad \text{.........(5.71)}$$

$\Psi(k)$

Again the output equation

$$y(k) = CG^k x(0) + C\sum_{j=0}^{k-1} G^{k-j-1} Hu(j) + Du(k) \qquad \text{.........(5.72)}$$

In terms of state transition matrix, Eq (5.72) can be written as

$$y(k) = C\Psi(k)x(0) + C\sum_{j=0}^{k-1} \Psi(k-j-1)Hu(j) + Du(k) \qquad \text{.........(5.73)}$$

$$= C\Psi(k)x(0) + C\sum_{j=0}^{k-1} \Psi(j)Hu(k-j-1) + Du(k) \qquad \text{.........(5.74)}$$

5.4.3 z - Transform Approach to the Solution of Discrete –Time State Equation :-

Let us consider the discrete-time system describe by the following equation,

$$x(k+1)=Gx(k)+Hu(k) \qquad \text{.........(5.75)}$$

Taking the z transform of both sides of Eq (5.75), we get

$$zX(z)-zx(0)=GX(z)+HU(z)$$

where $X(z)=\mathcal{Z}[x(k)]$ and $U(z)=\mathcal{Z}[u(k)]$

Then

$$(zI-G)X(z)=zx(0)+HU(z)$$

Premultipying both sides of this last equation by $(zI-G)^{-1}$, we obtain

$$X(z)=(zI-G)^{-1}zx(0)+(zI-G)^{-1}HU(z) \qquad \text{........(5.76)}$$

Taking the inverse z transform of both sides of Eq (5.76), we get

$$x(k)=\mathcal{Z}^{-1}\left[(zI-G)^{-1}z\right]x(0)+\mathcal{Z}^{-1}\left[(zI-G)^{-1}HU(z)\right] \qquad \text{........(5.77)}$$

Now, as we know the solution of discrete time equation $x(k+1)=Gx(k)+Hu(k)$ is

$$x(k)=G^{k}x(0)+\sum_{j=0}^{k-1}G^{k-j-1}Hu(j) \qquad \text{.........(5.78)}$$

Comparing Eq (5.77) with (5.78), we get

$$G^{k}=\mathcal{Z}^{-1}\left[(zI-G)^{-1}z\right] \qquad \text{.........(5.79)}$$

and

$$\sum_{j=0}^{k-1}G^{k-j-1}Hu(j)=\mathcal{Z}^{-1}\left[(zI-G)^{-1}HU(z)\right] \qquad \text{.........(5.80)}$$

Where $k=1,2,3,\ldots\ldots$

Example 5. Obtain the state transition matrix of the following discrete-time system:

$$x(k+1)=Gx(k)+Hu(k)$$

$$y(k)=Cx(k)$$

where $G=\begin{bmatrix}0 & 1\\ -0.16 & -1\end{bmatrix}, H=\begin{bmatrix}1\\1\end{bmatrix}, C=\begin{bmatrix}1 & 0\end{bmatrix}$

Then obtain the state $x(k)$ and the output $y(k)$ when the input $u(k)=1$ for $k=0,1,2,\ldots\ldots$ Assume that the initial state is given by,

$$x(0)=\begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}$$

Solution . As we know that the state transition matrix $\Psi(k)$ is

$(z^2+z)+0.16$

$$\Psi(k) = G^k = \mathcal{Z}^{-1}[(zI-G)^{-1}z]$$

Therefore, we first obtain $(zI-G)^{-1}$

$$(zI-G)^{-1} = \begin{bmatrix} z & -1 \\ 0.16 & z+1 \end{bmatrix}^{-1}$$

$$\begin{vmatrix} z & -1 \\ 0.16 & z+1 \end{vmatrix}^{-1} = \frac{1}{(z^2+z)+0.16}\begin{bmatrix} z+1 & -0.16 \\ 1 & z \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{z+1}{(z+0.2)(z+0.8)} & \dfrac{1}{(z+0.2)(z+0.8)} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\frac{4}{3}}{z+0.2} + \dfrac{-\frac{1}{3}}{z+0.8} & \dfrac{\frac{5}{3}}{z+0.2} + \dfrac{-\frac{5}{3}}{z+0.8} \\ \dfrac{-\frac{0.8}{3}}{z+0.2} + \dfrac{\frac{0.8}{3}}{z+0.8} & \dfrac{-\frac{1}{3}}{z+0.2} + \dfrac{-\frac{4}{3}}{z+0.8} \end{bmatrix}$$

The state transition matrix $\Psi(k)$ is now obtained as follows:

$$\Psi(k) = G^k = \mathcal{Z}^{-1}[(zI-G)^{-1}z]$$

$$= \mathcal{Z}^{-1}\begin{bmatrix} \frac{4}{3}\frac{z}{z+0.2} - \frac{1}{3}\frac{z}{z+0.8} & \frac{5}{3}\frac{z}{z+0.2} - \frac{5}{3}\frac{z}{z+0.8} \\ -\frac{0.8}{3}\frac{z}{z+0.2} + \frac{0.8}{3}\frac{z}{z+0.8} & -\frac{1}{3}\frac{z}{z+0.2} + \frac{4}{3}\frac{z}{z+0.8} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{3}(-0.2)^k - \frac{1}{3}(-0.8)^k & \frac{5}{3}(-0.2)^k - \frac{5}{3}(-0.8)^k \\ -\frac{0.8}{3}(-0.2)^k + \frac{0.8}{3}(-0.8)^k & -\frac{1}{3}(-0.2)^k + \frac{4}{3}(-0.8)^k \end{bmatrix} \qquad \text{.........(5.81)}$$

Eq (5.81) gives the state transition matrix.

Next, compute x(k). The z transform of x(k) is given by

$$Z\;[x(k)] = X(z) = (zI-G)^{-1}zx(0) + (zI-G)^{-1}HU(z)$$

$$= (zI-G)^{-1}[zx(0) + HU(z)]$$

Since $U(z) = \dfrac{1}{1-z^{-1}} = \dfrac{z}{z-1}$

We obtain $\quad zx(0) + HU(z) = \begin{bmatrix} z \\ -z \end{bmatrix} + \begin{bmatrix} \dfrac{z}{z-1} \\ \dfrac{z}{z-1} \end{bmatrix} = \begin{bmatrix} \dfrac{z^2}{z-1} \\ \dfrac{-z^2+2z}{z-1} \end{bmatrix}$

Hence $X(z)=(zI-G)^{-1}[zx(0)+HU(z)]$

$$=\begin{bmatrix} \dfrac{(z^2+2)z}{(z+0.2)(z+0.8)(z-1)} \\ \dfrac{(-z^2+1.84z)z}{(z+0.2)(z+0.8)(z-1)} \end{bmatrix}$$

$$=\begin{bmatrix} \dfrac{\frac{-17}{6}z}{z+0.2}+\dfrac{\frac{22}{9}z}{z+0.8}+\dfrac{\frac{25}{18}z}{z-1} \\ \dfrac{\frac{3.4}{6}z}{z+0.2}+\dfrac{\frac{-17.6}{9}z}{z+0.8}+\dfrac{\frac{7}{18}z}{z-1} \end{bmatrix}$$

Thus, the state vector x(k) is given by

$$x(k)=Z^{-1}[X(z)]=\begin{bmatrix} -\frac{17}{6}(-0.2)^k+\frac{22}{9}(-0.8)^k+\frac{25}{18} \\ \frac{3.4}{6}(-0.2)^k-\frac{17.6}{9}(-0.8)^k+\frac{7}{18} \end{bmatrix}$$

Finally, the output y(k) is obtained as follows:

$$y(k)=Cx(k)=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix} \frac{-17}{6}(-0.2)^k+\frac{22}{9}(-0.8)^k+\frac{25}{18} \\ \frac{3.4}{6}(-0.2)^k-\frac{17.6}{9}(-0.8)^k+\frac{7}{18} \end{bmatrix}$$

$$=-\frac{17}{6}(-0.2)^k+\frac{22}{9}(-0.8)^k+\frac{25}{18}$$

5.4.4. *Solution of Linear Time-Varying Discrete-Time State Equations* : Consider the following linear time-varying discrete-time state equation and output equation:

$$x(k+1)=G(k)\,x(k)+H(k)\,u(k) \qquad \text{.........(5.82)}$$

$$y(k)=C(k)\,x(k)+D(k)\,u(k) \qquad \text{.........(5.83)}$$

The solution of Eq (5.82) may be found easily by recursion, as follows:

$$x(h+1)=G(h)\,x(h)+H(h)u(h)$$

$$x(h+2)=G(h+1)\,x(h+1)+H(h+1)\,u(h+1)$$

$$=G(h+1)\,G(h)x(h)+G(h+1)\,H(h)\,u(h)+H(h+1)u(h+1)$$

$$\vdots$$

Let us define the state transition matrix (fundamental matrix) for the system defined Eq (5.82) as $\Psi(k,h)$. It is a unique matrix satisfying the conditions

$$\Psi(k+1,h)=G(k)\,\Psi(k,h), \qquad \Psi(h,h)=I$$

where $k=h,h+1,h+2,\ldots$ It can be seen that the state transition matrix $\Psi(k,h)$ is given by equation.

$$\Psi(k,h)=G(k-1)\,G(k-2)\ldots G(h), \qquad k>h \qquad \ldots\ldots(5.8$$

Using $\Psi(k,h)$, the solution of Eq (5.82) becomes

$$x(k)=\Psi(k,h)x(h)+\sum_{j=h}^{k-1}\Psi(k,j+1)\,H(j)\,u(j), \qquad k>h \qquad \ldots\ldots(5.8$$

Notice that the first term on the right-hand side of Eq (5.85) is the contribution of the ini state $X(h)$ to the current state $X(k)$ and that the second term is the contribution of the in $u(h), u(h+1),\ldots\ldots u(k-1)$.

Eq (5.85) can be verified easily. Referring to Equation (5.84), we have

$$\Psi(k+1,h)=G(k)G(k-1)\ldots G(h)=G(k)\,\Psi(k,h) \qquad \ldots\ldots(5.86)$$

If we substitute Eq (5.86) into

$$x(k+1)=\Psi(k+1,h)\,x(h)+\sum_{j=h}^{k}\Psi(k+1,j+1)\,H(j)u(j)$$

we obtain

$$x(k+1)=G(k)\,\Psi(k,h)x(h)+\sum_{j=h}^{k-1}\Psi(k+1,j+1)\,H(j)\,u(j)+\Psi(k+1,k+1)\,H(k)\,u(k)$$

$$=G(k)\left[\Psi(k,h)x(h)+\sum_{j=h}^{k-1}\Psi(k,j+1)H(j)u(j)\right]+H(k)u(k)$$

$$=G(k)\,x(k)+H(k)\,u(k)$$

Thus, we have shown that Eq (5.85) is the solution of Eq (5.82).
Once we get the solution $x(k)$, the output equation, Eq (5.83), becomes as follows:

$$y(k)=C(k)\,\Psi(k,h)\,x(h)+\sum_{j=h}^{k-1}C(k)\,\Psi(k,j+1)\,H(j)u(j)+D(k)\,u(k), \qquad k>h$$

If $G(k)$ is nonsingular for all k values considered, so that the inverse of $\Psi(k,h)$ exists, then the inverse of $\Psi(k,h)$, denoted by $\Psi(h,k)$, is given as follows:

$$\Psi^{-1}(k,h) = \Psi(h,k)$$

$$= [G(k-1)\, G(k-2) \ldots G(h)]^{-1}$$

$$= G^{-1}(h) G^{-1}(h+1) \ldots G^{-1}(k-1) \qquad \text{........(5.87)}$$

Summary on $\Psi(k,h)$. A summary on the state transition matrix $\Psi(k,h)$ gives the following:

1. $\Psi(k,k) = I$
2. $\Psi(k,h) = G(k-1)G(k-2)\ldots G(h), \qquad k > h$
3. If the inverse of $\Psi(k,h)$ exists, then

   $\Psi^{-1}(k,h) = \Psi(h,k)$
4. If $G(k)$ is nonsingular for all k values considered, then

   $\Psi(k,i) = \Psi(k,j)\, \Psi(j,i)$, for any i,j,k

If $G(k)$ is singular for any value of k, then

$\Psi(k,i) = \Psi(k,j)\, \Psi(j,i)$; for $k > j > i$

5.5 Pulse Transfer Function Matrix

Theory 4.

A single-input-single-output discrete-time system may be modeled by a pulse transfer function. Extension of the pulse-transfer-function concept to a multiple-input-multiple-output discrete-time system gives us the pulse-transfer-function matrix. In this section we shall investigate the relationship between state-space representation and representation by the pulse-transfer-function matrix.

Pulse Transfer Function Matrix : The state-space representation of an n th-order linear time-invariant discrete-time system with r inputs and m outputs can be given by

$$x(k+1) = Gx(k) + Hu(k) \qquad \text{........(5.88)}$$

$$y(k) = Cx(k) + Du(k) \qquad \text{........(5.89)}$$

Where $x(k)$ is an n-vector, $u(k)$ is an r-vector, $y(k)$ is an m-vector, G is an $n \times n$ matrix, H is an $n \times r$ matrix, C is an $m \times n$ matrix, and D is an $m \times r$ matrix. Taking the z transform of Eqs (5.88) and (5.89), we obtain

$$zX(z) - zx(0) = GX(z) + HU(z)$$

$$Y(z) = CX(z) + DU(z)$$

Noting that the definition of the pulse transfer function calls for the assumption of zero in conditions, here we also assume that the initial state x(0) is zero. then we obtain

$$X(z) = (zI-G)^{-1} HU(z)$$

and

$$Y(z)=[C(zI-G)^{-1}H+D]\ U(z) =F(z)U(z)$$

Where .... (5.90)

$$F(z) =C(zI-G)^{-1}H+D$$

F(z) is called the pulse transfer function matrix. It is an m x r matrix. The pulse tran function matrix F(z) characterizes the input-output dynamics of the given discrete-t system.

Since the inverse of matrix (zI-G) can be written as

$$(zI-G)^{-1} = \frac{adj(zI-G)}{|zI-G|}.$$

the pulse transfer function matrix F(z) can be given by the equation

$$F(z)=\frac{C\ adj\ (zI-G)H}{|zI-G|}+D$$

Clearly, the poles of F(z) are the zeros of $|zI-G| = 0$. This means that the characte equation of the discrete-time system is given by

$$|zI-G| = 0$$

or, $z^n+a_1z^{n-1}+a_2z^{n-2}+......+a_{n-1}z+a_n=0$

Where the coefficients a_i depend on the elements of G.

5.6 Discretization of Continuous-Time State-Space Equations

In digital control of continuous-time plants, we need to convert continuous-time state-s equations into discrete-time state-space equations. Such conversion can be don introducing fictitious samplers and fictitious holding devices into continuous-time syst The error introduced by discretization may be made negligible by using a sufficiently s sampling period compared with the significant time constant of the system.

5.6.1 Review of Solution of Continuous-Time State Equations: We shall first review the matrix exponential, e^{At}. The matrix exponential is defined by

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \ldots + \frac{1}{k!}A^kt^k + \ldots = \sum_{k=0}^{\infty} \frac{A^kt^k}{k!}$$

Because of the convergence of the infinite series $\sum_{K=0}^{\infty} A^kt^k/k!$, the series can be differentiated term by term to give

$$\frac{d}{dt}e^{At} = A + A^2t + \frac{A^3t^2}{2!} + \ldots + \frac{A^kt^{k-1}}{(k-1)!} + \ldots$$

$$= A\left[I + At + \frac{A^2t^2}{2!} + \ldots + \frac{A^{k-1}t^{k-1}}{(k-1)!} + \ldots\right] = Ae^{At}$$

$$= \left[I + At + \frac{A^2t^2}{2!} + \ldots + \frac{A^{k-1}t^{k-1}}{(k-1)!} + \ldots\right]A = e^{At}A$$

The matrix exponential has the property that

$$e^{A(t+s)} = e^{At}e^{As}$$

This can be proved as follows:

$$e^{At}e^{As} = \left(\sum_{k=0}^{\infty}\frac{A^kt^k}{k!}\right)\left(\sum_{k=0}^{\infty}\frac{A^ks^k}{k!}\right) = \sum_{k=0}^{\infty} A^k\left[\sum_{i=0}^{k}\frac{t^is^{k-i}}{i!(k-i)!}\right]$$

$$= \sum_{k=0}^{\infty} A^k\frac{(t+s)^k}{k!} = e^{A(t+s)}$$

In particular, if $s=-t$, then

$$e^{At}e^{-At} = e^{-At}e^{At} = e^{A(t-t)} = I$$

Thus, the inverse of e^{At} is e^{-At}. Since the inverse of e^{At} always exists, e^{At} is nonsingular. It is important to point out that

$$e^{(A+B)t} = e^{At}e^{Bt}, \qquad \text{if } AB = BA.$$

$$e^{(A+B)t} \neq e^{At}e^{Bt}, \qquad \text{if } AB \neq BA$$

We shall next obtain the solution of the continuous-time state equation

$$\dot{x} = Ax + Bu \qquad \text{.........(5.91)}$$

Where x is the state vector (n-vector), u the input vector (r-vector), A an n x n cons matrix, and B an n x r constant matrix.

By writing Eq (5.91) as

$$\dot{x}(t) - Ax(t) = Bu(t)$$

and premultiplying both sides of this last equation by e^{-At}, we obtain

$$e^{-At}\left[\dot{x}(t) - Ax(t)\right] = \frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At}Bu(t)$$

Integrating the preceding equation between 0 and t gives

$$e^{-At}x(t) = x(0) + \int_0^t e^{-A\tau} Bu(\tau)\, d\tau$$

or,

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau \qquad \text{........(5.92)}$$

Eq (5.92) is the solution of Eq (5.91). Note that the solution of the state equation starting w the initial state $x(t_0)$ is

$$X(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)} BU(\tau)\, d\tau \qquad \text{.........(5.93)}$$

5.6.2 *Discretization of Continuous-Time State-Space Equations* Theory + **:** In what follows shall present a procedure for discretizing continuos-time state-space equations. We assu that the input vector u(t) changes only at equally spaced sampling instants. Note that sampling operation here is fictitious. We shall derive the discrete-time state equation a output equation that yield the exact values at t=kT, where k=0,1,2,.....

Consider the continuous-time state equation and output equation

$$\dot{x} = Ax + Bu \qquad \text{.........(5.94)}$$

$$y = Cx + Du \qquad \text{.........(5.95)}$$

In the following analysis, to clarify the presentation, we use the notation kT and (k+1)T instead of k and k+1. The discrete-time representation of Eq (5.94) will take the form

$$x((k+1)T) = G(T)\, x(kT) + H(T)\, u(kT) \qquad \text{........(5.96)}$$

Note that the matrices G and H depend on the sampling period T. Once the sampling period T is fixed, G and H are constant matrices.

We assume that the input u(t) is sampled and fed to a zero-order hold so that all the components of u(t) are constant over the interval between any two consecutive sampling instants, or

$$u(t) = u(kT), \qquad \text{for } kT \le t < kT + T \qquad \text{..........(5.97)}$$

Since

$$x((k+1)T) = e^{A(k+1)T}\, x(0) + e^{A(k+1)T} \int_0^{(k+1)t} e^{-A\tau} Bu(\tau)\, d\tau \qquad \text{........(5.98)}$$

and

$$x(kT) = e^{AkT} x(0) + e^{AkT} \int_0^{kT} e^{-A\tau} Bu(\tau)\, d\tau \qquad \text{.........(5.99)}$$

Multiplying Eq (5.99) by e^{AT} and subtracting it from Eq (5.98) gives us

$$x((k+1)T) = e^{AT} x(kT) + e^{A(k+1)T} \int_{kT}^{(k+1)T} e^{-A\tau} Bu(\tau)\, d\tau$$

Since from Eq (5.97) u(t) =u(kT) for kT ≤ t < kT +T, we may substitute u(τ) =u(kT)= constant in this last equation

$$x((k+1)T) = e^{AT} x(kT) + e^{AT} \int_0^{T} e^{-At} Bu(kT)\, dt$$

$$= e^{AT} x(kT) + \int_0^{T} e^{A\lambda} Bu(kT)\, d\lambda \qquad \text{.........(5.100)}$$

Where λ =T –t. If we define

$$G(T) = e^{AT} \qquad \text{........(5.101)}$$

$$H(T) = \left(\int_0^T e^{A\lambda} d\lambda\right) B \quad \text{.......(5.10}$$

then Eq (5.100) becomes

$$x((k+1)T) = G(T)\, x(kT) + H(T)u(kT) \quad \text{.......(5.103)}$$

Which is Eq (5.96). Thus, Eqs (5.101) and (5.102) give the desired matrices G(T) and H(

Note that G(T) and H(T) depend on the sampling period T. Referring to Eq (5.95), the ou equation becomes,

$$y(kT) = Cx(kT) + Du(kT) \quad \text{........(5.104)}$$

Where matrices C and D are constant matrices and do not depend on the sampling period. If matrix A is nonsingular, then H(T) given by equation (9) can be simplified to

$$H(T) = \left(\int_0^T e^{A\lambda} d\lambda\right) B = A^{-1}(e^{AT} - I)B = (e^{AT} - I)A^{-1}B$$

Example 6. Consider the continuous-time system given by

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+a}$$

Obtain the continuous-time state-space representation of the system. Then discretize the s equation and output equation and obtain the discrete-time state-space representation of system. Also, obtain the pulse transfer function for the system.

Solution : The continuous-time state-space representation of the system is simply,

$$\dot{x} = -ax + u$$

$$y = x$$

Now we discretize the state equation and the output equation.

$$G(T) = e^{-aT}$$

$$H(T) = \int_0^T e^{-a\lambda} d\lambda = \frac{1 - e^{-aT}}{a}$$

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