Chapter 4
Markov Chains and its applications

a) Markov chains and Markov Process

Important classes of stochastic processes are Markov chains and Markov processes. A Markov chain is a discrete-time process for which the future behavior, given the past and the present, only depends on the present and not on the past. A Markov process is the continuous-time version of a Markov chain. Many queuing models are in fact Markov processes. This chapter gives a short introduction to Markov chains and Markov processes focusing on those characteristics that are needed for the modeling and analysis of queuing problems.

A Markov chain

A Markov chain, named after Andrey Markov, is a mathematical system that undergoes transitions from one state to another, between a finite or countable number of possible states. It is a random process characterized as memoryless: the next state depends only on the current state and not on the sequence of events that preceded it. This specific kind of "memoryless-ness" is called the Markov property. Markov chains have many applications as statistical models of real-world processes.

Formally Definition of Markov Chain

A Markov chain is a sequence of random variables X1, X2, X3, ... with the Markov property, namely that, given the present state, the future and past states are independent.

$$\Pr(X_{n+1}=x|X_1=x_1,X_2=x_2,\ldots,X_n=x_n)=\Pr(X_{n+1}=x|X_n=x_n)$$

Example; A simple whether model (Land of OZ Example)

The probabilities of weather conditions (modeled as either rainy or sunny), given the weather on the preceding day, can be represented by a transition matrix:

$$P=\begin{bmatrix}0.9 & 0.1\\ 0.5 & 0.5\end{bmatrix}$$

The matrix P represents the weather model in which a sunny day is 90% likely to be followed by another sunny day, and a rainy day is 50% likely to be followed by another rainy day. The columns can be labeled "sunny" and "rainy" respectively, and the rows can be labeled in the same order.

Notice that the rows of P sum to 1: This is because P is a stochastic matrix.

The weather on day 0 is known to be sunny. This is represented by a vector in which the "sunny" entry is 100%, and the "rainy" entry is 0%:

$$\mathbf{x}^{(0)} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

The weather on day 1 can be predicted by:

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} P = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \end{bmatrix}$$

The weather on day 2 can be predicted in the same way:

$$\mathbf{x}^{(2)} = \mathbf{x}^{(1)} P = \begin{bmatrix} 0.9 & 0.1 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.86 & 0.14 \end{bmatrix}$$

Or

$$\mathbf{x}^{(2)} = \mathbf{x}^{(1)} P = \mathbf{x}^{(0)} P^2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}^2 = \begin{bmatrix} 0.86 & 0.14 \end{bmatrix}$$

In general

$$\mathbf{x}^{(2)} = \mathbf{x}^{(1)} P = \begin{bmatrix} 0.9 & 0.1 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.86 & 0.14 \end{bmatrix}$$

General rules for day *n* are:

$$\mathbf{x}^{(n)} = \mathbf{x}^{(n-1)} P$$
$$\mathbf{x}^{(n)} = \mathbf{x}^{(0)} P^n$$

b) Markov chain or process Applications

Physics

Markovian systems appear extensively in thermodynamics and statistical mechanics, whenever probabilities are used to represent unknown or unmodelled details of the system, if it can be assumed that the dynamics are time-invariant, and that no relevant history need be considered which is not already included in the state description.

Queuing theory

Markov chains are the basis for the analytical treatment of queues (queuing theory). Agner Krarup Erlang initiated the subject in 1917. This makes them critical for optimizing the performance of telecommunications networks, where messages must often compete for limited resources (such as bandwidth).

Internet applications
The Page Rank of a webpage as used by Google is defined by a Markov chain. It is the probability to be at page i in the stationary distribution on the following Markov chain on all (known) web pages

Statistics
Markov chain methods have also become very important for generating sequences of random numbers to accurately reflect very complicated desired probability distributions, via a process called Markov chain Monte Carlo (MCMC) And many more.