Chapter 7

Analysis of Simulation Output

Nature of the problem

Once a stochastic variable has been introduced into simulation model, almost all the system variables describing the system behavior also become stochastic. Hence it needs some statistical method to analyze the simulation output. A large body of statistical methods has been developed over the years to analyze results in science, engineering and other fields. It seem natural to attempt applying these methods to analyze the simulation output but most of them pre-suppose that the results are mutually independent (IID) and the simulation process almost never produce raw output that is IID. For example: customer waiting times from queuing system are not IID. Thus it is difficult to apply classical statistical techniques to analysis of simulation model.

Some definitions

a) **Independently and identically distributed random variables**: Usually a random variable is drawn from an infinite population that has probability distribution with finite mean u and finite variance. This mean that the population distribution is not affected by the number of sample already made or does it change with time. Further the value of sample is not affected in anyway by value of another sample. Random variables that meet all these conditions are said to be independently and identically distributed.

b) **Central Limit theorem**: The theorem state that the sum of n IID variables drawn from population that has a mean of u and a variance of σ^2, is approximately distributed as a normal variable with a mean of μ and a variance of n.

1. Type of simulation on the basis of output

a) **Termination simulation or finite simulation**: the termination of a finite simulation takes place at a specified time or is caused by some specific events. For example: Banks opens at 8:30am with no customers present and 8 of 11 teller working and closed at 4:30 pm. The terminating simulation runs for some specified duration of time TE where E is a specified event that stops the simulation. It starts at time 0 under well specified initial condition and ends at the stooping time TE. For the banking system, the simulation analyst chooses to consider a terminating system because object of interest is one day's operations on the bank.

b) **Non terminating Simulation (Steady state simulation)**: The main propose of steady study simulation is the study of long run behavior of system. Performance measure is called a steady state parameter if it is a characteristic of the equilibrium distribution of an output stochastic process. Examples are: Continuously operating communication system where the objective of computation of mean delay of packet in the long run.

Here the non-terminating simulation has:

1. Runs continuously or at least over a very long period of time.

2. Initial conditions defined by analyst.

3. Runs for some analyst specified period of time T_E

4. Study the steady state (long run) properties of the system, properties that are not influenced by the initial condition of model.

(Note: whether a simulation is considered to be terminating or non-terminating depends on both the objective of study and nature of the system)

Estimation Methods

Any normal distribution can be transform into a standard distribution that has a mean of 0 and variance of 1. Let xi (i=1,2,3,……….n) be n IID random variables. Using central limit theorem, we have normal Variate

$$z = \frac{\sum_{i=1}^{n} x_i - n\mu}{6\sqrt{n}}$$

Dividing by n in both denominator and numerator, we get

$$z = \frac{\bar{x} - \mu}{6/\sqrt{n}}$$

Where

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

The variable $\bar{x}$ is the sample mean and it can be shown to be a consistent estimator for mean of the population form which sample is drawn.

a) Point estimation (For discrete data):

Let y_1, y_2 ...y_n be discrete time data with ordinary mean θ, then point estimator

$$\hat{\theta} = \frac{1}{n}\sum_{t=1}^{n} y_t$$

Is unbiased if its expected value is θ i.e. E($\hat{\theta}$)= θ and is biased if E($\hat{\theta}$)G θ and the difference E($\hat{\theta}$) - θ is called bias of $\hat{\theta}$.

As an example of a point estimate, assume you wanted to estimate the mean time it takes 12-year-olds to run 100 yards. The mean running time of a random sample of 12-year-olds would be an estimate of the mean running time for all 12-year-olds. Thus, the sample mean M, would be a point estimate of the Population mean, μ.

Example:

Following are the random sample of height of people of the town. If the population mean is 6.1 ft, find the bias of the point estimator.

5.5	6.1	5.7	6.6	5.2	6.0	5.6	6.3	5.9	5.8

θ^= (5.5 + 6.1 + 5.7 + 6.6 + 5.2 + 6.0 + 5.6 + 6.3 + 5.9 + 5.8)/10 = 5.87

Now, bias of estimator = 5.87 – 6.1 = - 0.23

For the continuous data {y(t): 0 <t <T} with mean Ø then the point estimator is given by

$$\hat{\phi} = \frac{1}{T_E}\int_0^{T_E} y(t)dt$$

b) Interval Estimation/Confidence Interval Estimation

Confidence interval is a measure used to analyze the correctness of the point estimator. In the above example we got point estimation of height of the people of a city but we don't know whether to accept or reject it.

Confidence intervals are based on the premise that the data being produced by the simulation is represented well by a probability model. Suppose the model is the normally distributed with mean $\bar{x}$, Variance σ^2 and we have a sample of n size then the confidence interval is given by:

$$\bar{x} \pm \frac{\sigma}{\sqrt{n}} z_{\alpha/2}$$

In practice, the population variance is usually not known; in this case variance is replaced by the estimate calculated by the formula

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

This has a student t-distribution, with n – 1 degree of freedom. In terms of estimated variance s^2, the confidence interval for $\bar{x}$ is defined by

$$\bar{x} \pm \frac{S}{\sqrt{n}} t_{n-1,\ \alpha/2}$$

Here, the quantity $t_{n-1,\ \alpha/2}$ is found on the student t-distribution table.

Example

The daily production time of a product in a factory for 120 days is 5.8 hours and sample standard deviation (S) is 1.6. Calculate confidence interval for 95% confidence level.

Solution

Confidence Interval = $\bar{x} \pm \frac{S}{\sqrt{n}} t_{n-1, \alpha/2}$

$= 5.8 \pm (1.6/120)1.98$ ($t_{119, 0.025} = 1.98$)

$= 5.8 \pm 0.29$

Hence, the estimates between 5.8 ± 0.29 can be accepted for 95% confidence interval.

Simulation Run statistics

In the estimation method, it is assumed that the observations are mutually independent and the distribution from which they are draws is stationary. Unfortunately many statistics of interest in simulation do not meet these conditions.

For example: consider a single server system in which the arrival occurs with poison distribution and service time has an exponential and queue discipline is FIFO. Suppose the study objective is to measure the mean waiting rime. In simulation run, the simplest approach to estimate the mean waiting time by accumulating the waiting time of n successive entities and dividing by n. this is the sample mean, denoted by Waiting time measured in this way is not independent. Whenever a waiting line forms, the waiting time of each entity on the line clearly depends upon the waiting time of its predecessors. Such data are called auto-correlated. Another problem that must be faced is that distribution is not stationary. Te early arrivals get the service quickly, so a sample mean that include early arrivals is biased. The following figure show the mean waiting time for different sample sizes.

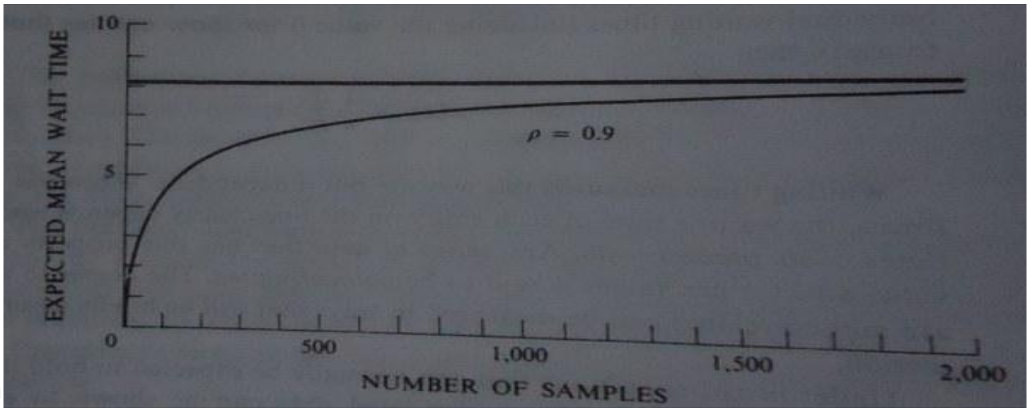

Fig: Mean wait time in M/M/1 system for different sample size

Replications of Runs

One way of obtaining independent result is to repeat simulation. Repeating the experiment with different random numbers for the sample size n gives a set of independent determination of sample mean$\bar{x}$. Suppose the experiment is repeated p times with independent random values of n sample sizes. Let x_{ij} be the i^th^ observation in j^th^ run and let the sample mean and the variance for the j^th^ run is denoted by $\bar{x}_j(n)$ and $s_j^2(n)$ respectively. Then for j^th^ run, the estimates are

$$\bar{x}_j(n) = \frac{1}{n}\sum_{i}^{n} x_{ij}$$

$$s_j^2(n) = \frac{1}{n-1}\sum_{i=1}^{n}\left[x_{ij} - \bar{x}_j(n)\right]^2$$

Combining the result of p independent measurement gives the following estimate for the mean $\bar{x}$ and variance s^2 of the populations as:

$$\bar{x} = \frac{1}{p}\sum_{j=1}^{p} \bar{x}_j$$

$$s^2 = \frac{1}{p}\sum_{j=1}^{p} s_j^2$$

Elimination of initial bias

The initial bias as shown in figure below needs to be removed. Two general approaches can be taken into remove the bias.

1) The system can be started in a more representative then an empty state.
2) The first part of the simulation can be removed.

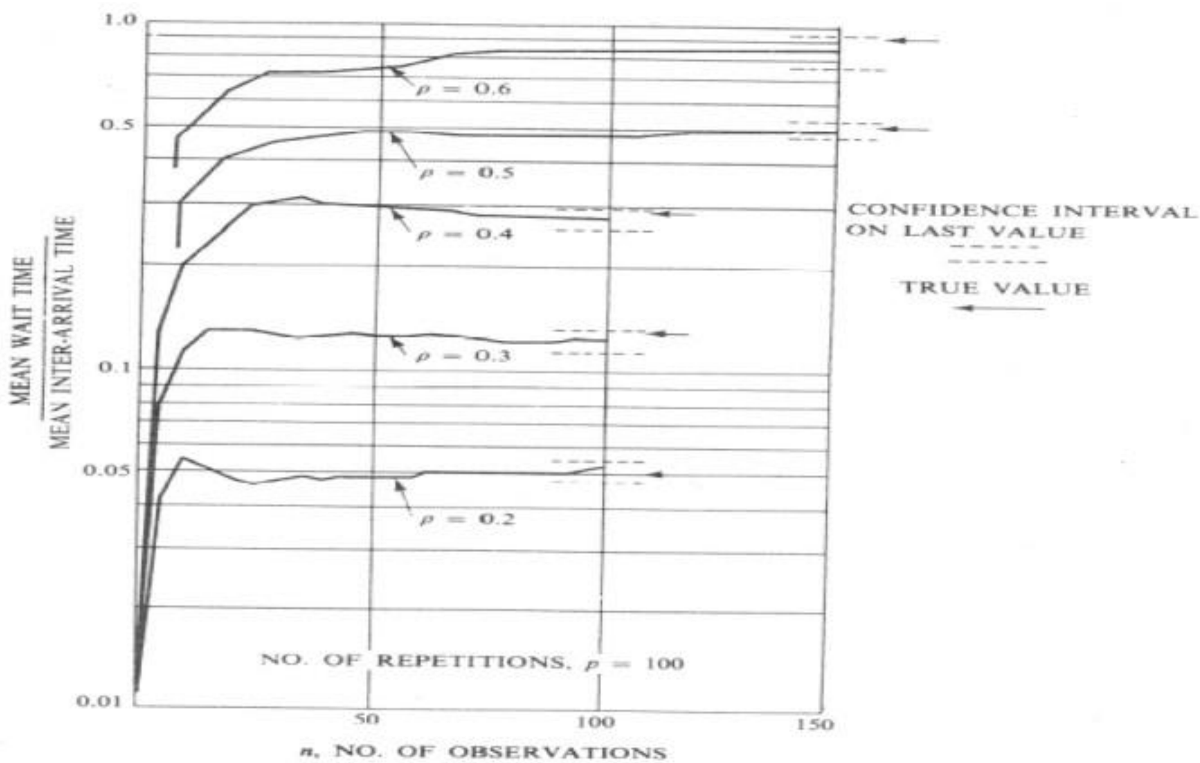

Fig: Experimentally measured wait time in M/M/1 system for different sample size

The some simulation studies where the information about expected value is available, it is feasible to select better initial conditions. The ideal solution is to know the steady state distribution for the system and select the initial conditions from that distribution. The more common approach to remove initial bias is to illuminate an initial section of run.

The run is started form an idle state and stopped after a certain period of time. The entities existing in that system at that time are left as they are. The run is then restarted with the statistics being gathered from the point of restart. No simple rules can be given to decide how long an interval would be eliminated. It is advisable to use some pilot runs starting form idle state to judge how long the initial bias remains.

Student-t Distribution

	Confidence Probability			
v	.80	.90	.96	.98
1	3.078	6.314	15.895	31.821
2	1.886	2.920	4.849	6.965
3	1.638	2.353	3.482	4.541
4	1.533	2.132	2.999	3.747
5	1.476	2.015	2.757	3.365
6	1.440	1.943	2.612	3.143
7	1.415	1.895	2.517	2.998
8	1.397	1.860	2.449	2.896
9	1.383	1.833	2.398	2.821
10	1.372	1.812	2.359	2.764
11	1.363	1.796	2.328	2.718
12	1.356	1.782	2.303	2.681
13	1.350	1.771	2.282	2.650
14	1.345	1.761	2.264	2.624
15	1.341	1.753	2.249	2.602
16	1.337	1.746	2.235	2.583
17	1.333	1.740	2.224	2.567
18	1.330	1.734	2.214	2.552
19	1.328	1.729	2.205	2.539
20	1.325	1.725	2.197	2.528
25	1.316	1.708	2.167	2.485
30	1.310	1.697	2.147	2.457
40	1.303	1.684	2.123	2.423
50	1.299	1.676	2.109	2.403
75	1.293	1.665	2.090	2.377
100	1.290	1.660	2.081	2.364
∞	1.282	1.645	2.054	2.326