Ryan Pollack

1. Claim: For $n \geq 12$, there's a way to purchase n cats by buying in only batches of 3 or 7.

BC: For $n = 12$, we can buy 4 batches of 3. (4×3 batch $= 12$ cats)
$\therefore$ So, for $n = 12$, the result holds true.
For $n = 13$, we can buy 2 batches of 3 and 1 batch of 7. (2×3 batch $+ 1 \times 7$ batch $= 13$ cats). So, for $n = 13$, the result holds true.
For $n = 14$, we can buy 2 batches of 7. (2×7 batch $= 14$ cats)
So, for $n = 14$, the result holds true.
So, by induction we assume that the result holds for some $n = k$. To show that the result holds for $n = k+1$, we rewrite as:

$$k+1 = k + 7 - 6$$
$$= k + (1 \times 7) - 2 \times 3$$

So we can have exactly $k+1$ cats. Therefore, the claim holds true, by the principle of induction.

2. Claim: The number of way to arrange $n \geq 1$ items in a row is $F(n) = n!$

BC: For $n = 1$, $F(n) = F(1) = 1$ $\rightarrow$ $(F(1) = 1!)$
For $n = 2$, $F(n) = F(2) = 2$ $\rightarrow$ $(F(2) = 2!)$
Let $n = k$, then by induction hypothesis, $F(k) = k!$
For $n = k+1$:

$$F(n) = F(k+1) = F(k)(k+1)$$
$$= k!\,(k+1) \qquad [F(k) = k!]$$
$$= (k+1)!$$

Therefore, $F(n) = n!$ and the claim holds true.

3. Claim: $1+x+x^2+\dots+x^{n-1}+x^n = \frac{x^{n+1}-1}{x-1}$

BC: $n=0$, where left side is 1 and right side is $\frac{x^{0+1}-1}{x-1} = 1$
so base case is true.
Inductive hypothesis: Let the claim be true for $0 \le n < n_0$.
Consider $n = n_0$
Rearrange sum from $1+x+x^2+\dots+x^{n_0-1}+x^{n_0}$ to be
$(1+x+x^2+\dots+x^{n_0-1})+x^{n_0}$. Use the inductive hypothesis so
$(1+x+x^2+\dots+x^{n_0-1}) = \frac{x^{(n_0-1)+1}-1}{x-1} = \frac{x^{n_0}-1}{x-1}$

Plug in to original to get:
$(1+x+x^2+\dots+x^{n_0-1})+x^{n_0} = \frac{x^{n_0}-1}{x-1}+x$

Cross multiply to get:
$(1+x+x^2+\dots+x^{n_0-1})+x^{n_0} = \frac{x^{n_0+1}-x+x-1}{x-1} = \frac{x^{n_0+1}-1}{x-1}$.

Therefore, the result holds true.

4. Case 1: n is odd. So,
$3 \cdot n = (2k+1)$, which equals the total squares on the board. When a knight makes a jump, he goes to one of the opposite colored square. The amount of white squares isn't equal to the amount of black squares. Suppose we have j black squares and $j+1$ white squares, where:
$j+(j+1) = 2j+1 = 3 \cdot n$
is total squares again, which contradicts the fact of a board. So, when n is odd, it's not possible for the knight to reach every square.

Case 2: n is even. So,
$3 \cdot n = 2k$, which is also total squares. Knights still jump from one square to one of opposite color. The number of white squares is equal to the number of black squares. Suppose there are j black squares and j white squares, where:
$3 \cdot n = 2j$
is total number of squares, which supports the fact of a board, so when n is even, it's possible for the knight to reach every square.

Bonus 1) It's a Frobenius Coin Problem, which says that for any 2 relatively prime integers m and n, the greatest integer that can't be written in $am + bn$ form for non negative numbers is $mn - m - n$.
So, $m = 3$ and $n = k$
From this, $n_0 = 3k - 3 - k + 1$ (+1 cause want bought)
For every $n \geq n_0$, it can be put in form $3a + kb$, where that number of cats can be bought.
Since cats are sold in batches of a and b, then:
$n_0 = ab - a - b + 1$

Bonus 2)