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1. $X_1 = 3$ $X_2 = 3^3$ $X_3 = 3^{27}$ $X_4 = 3^{7625\cdots}$

$X_2 = 27$ $X_3 = 7625\cdots$

exponents: $3 \rightarrow 1$ $3^3 \rightarrow 2$ $3^{3^3} \rightarrow 3$

(k-1) (k-1) (k-1)

$X_2 = 3^{X_1}$ $X_3 = 3^{X_2}$ $X_4 = 3^{X_3}$

$3^{3^{3^{\cdot^{\cdot^{\cdot}}}}}$ k-1 times $3^{X_{k-1}}$

$X_{k+1} = 3^{X_k}$

2. Base Case:

$k = 1$ $X_k \rightarrow X_1 \equiv 3 \pmod 4 = 3$

Inductive Hypothesis:

Let's assume $X_k \equiv 3 \pmod 4$ for all $k \geq 1$.

Inductive Step:

From the previous example, we know $X_{k+1} = 3^{X_k}$

With using Euler's theorem:

$GCD(3,4) = 1$, then $3^{\phi(4)} \equiv 1 \pmod 4$

Now we can solve $\phi(4)$ for a real value:

$\phi(4) = \phi(2^2) = 2^2(1-\frac{1}{2}) = 2 \longrightarrow 3^2 \equiv 1 \pmod 4$

So with the original $X_k \equiv 3 (\text{mod})$ we can find X_k:

$4 \text{ div } X_k - 3 \longrightarrow X_k \equiv 3 \pmod 2$

$2 \text{ div } X_k - 3 \longrightarrow X_k \equiv 1 \pmod 2$

From this, we get $X_k = 2a+1$ for some a. So plugging in, we have $X_{k+1} = 3^{X_k} = 3^{2a+1} = 3^{2a} \cdot 3 = (3^2)^a \cdot 3 = (1)^a \cdot 3 \pmod 4 =$

$(1)^a \cdot 3 \pmod 4 \equiv 3 \pmod 4$

Therefore, by induction, $X_k = 3 \pmod 4$ for all $k \geq 1$.

3. We can use Fermat's Little Theorem, which states that if p is a prime and $GCD(A, p) = 1$ (A not divisible by p), then $A^{p-1} \equiv 1 \pmod{p}$. With this example, $p = 5$, which is a prime and $GCD(A, 5) = 1$, so by Fermat's Little Theorem:

$A^{p-1} \equiv 1 \pmod{p} \to A^{(5)-1} \equiv 1 \pmod{5} \to A^4 \equiv 1 \pmod{5}$, so it checks out.

4. From problem 2, we gathered that $X_{k-1} \equiv 3 \pmod{4}$ With $k \geq 2$ and following a similar procedure as #2, we can see this will leave us $X_{k-1} = 4a+3$ for some a. Now, we can rewrite:

$X_{k+1} = 3^{X_k}$ to $X_k = 3^{X_{k-1}}$ and solve:

$3^{X_{k-1}} = 3^{4a+3} = (3^4)^a \cdot 3^3 \equiv (1)^a \cdot 3^3 \pmod{5} \equiv 2 \pmod{5}$

Therefore, $X_k \equiv 2 \pmod{5}$ for $k \geq 2$.

5. The first digit of X_k would need to be $X_k \pmod{10}$ We also know the only prime divisor of X_k is 3. So with $GCD(3, 10) = 1$, we can apply Euler's Theorem:

$3^{\phi(10)} \equiv 1 \pmod{10}$. Now we substitute $\phi(10)$ for real number.

$\phi(10) = 10(1-\frac{1}{2})(1-\frac{1}{5}) = 10(\frac{1}{2})(\frac{4}{5}) = 4$ so $3^4 \equiv 1 \pmod{10}$.

So from previous questions, we know $X_{k-1} \equiv 3 \pmod{4}$ goes to $X_{k-1} = 4a+3$ for some a. Once again we can rewrite:

$X_{k+1} = 3^{X_k}$ to $X_k = 3^{X_{k-1}}$ and solve:

$3^{X_{k-1}} = 3^{4a+3} = (3^4)^a \cdot 3^3 \equiv (1)^a \cdot 3^3 \pmod{10} = 27 \pmod{10}$

$27 \pmod{10}$ equivalent to $7 \pmod{10}$

$X_k \pmod{10} = 7$, and therefore proven.