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1. Let $N = 2^{10,000,000,000,000}$

$\log N = \log(2^{10,000,000,000,000})$

$= \log(2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2) \leftarrow$ [10,000,000,000,000 times

Since $\log(m \cdot n) = \log m + \log n$, then

$\log N = \log(2) + \log(2) + \log(2) + \ldots + \log(2)$

$= 10,000,000,000,000 \times \log(2)$

Since $\log(2) = .301$, then

$= 10,000,000,000,000 \times .301$

$= 3,010,000,000,000$

If $10^2 = 100 \rightarrow$ 3 digits (+1 from exponent)

$10^3 = 1000 \rightarrow$ 4 digits (+1 from exponent)

$10^4 = 10,000 \rightarrow$ 5 digits (+1 from exponent)

Then $10^n = n+1$ digits.

Therefore, $N+1$ from number for digits so

$2^{10,000,000,000,000}$ is 3,010,000,000,001 digits.

2. We can find the last 2 digits by understanding basis rules. We can rewrite $2^{10,000,000,000,000}$ as:

$(2^{10})^{1,000,000,000,000}$

$2^{10} = \underline{1,024}$ (take the last 2 digits and replug):

$(24)^{1,000,000,000,000}$

The basic rule is $(24)^{even}$ will always produce an answer where the last 2 digits are 76. To quickly prove:

If $(2)^{12}$ (last 2 digits of 2^{10})

$(2)^{10} = (24)^2 \rightarrow (24)(24) = 5\underline{76} \leftrightarrow$ 76 last 2 digits.

Also if $(2)^{18}$

$(2)^{16} = (24)^6 \rightarrow (24)(24)(24)(24)(24)(24) = 191,102,9\underline{76}$

As proven, any number put in the form $(24)^{even}$ will end with 76, so the last 2 digits of $2^{10,000,000,000,000}$ is 76.

3. We can find the last 2 digits the longer way by finding the last 2 digits of each 2^n until we see a pattern. So:

$2^2 = 4$	$\longrightarrow 04$	$2^{22} = 4,194,304$	$\longrightarrow 04$
$2^3 = 8$	$\longrightarrow 08$	$2^{23} = 8,388,608$	$\longrightarrow 08$
$2^4 = 16$	$\longrightarrow 16$	$2^{24} = 16,777,216$	$\longrightarrow 16$
$2^5 = 32$	$\longrightarrow 32$	$2^{25} = 33,554,432$	$\longrightarrow 32$
$2^6 = 64$	$\longrightarrow 64$	$2^{26} = 67,108,864$	$\longrightarrow 64$
$2^7 = 128$	$\longrightarrow 28$	$2^{27} = 134,217,728$	$\longrightarrow 28$
$2^8 = 256$	$\longrightarrow 56$	$2^{28} = 268,435,456$	$\longrightarrow 56$
$2^9 = 512$	$\longrightarrow 12$	$2^{29} = 538,870,912$	$\longrightarrow 12$
$2^{10} = 1,024$	$\longrightarrow 24$	$2^{30} = 1,073,741,824$	$\longrightarrow 24$
$2^{11} = 2,048$	$\longrightarrow 48$	$2^{31} = 2,147,483,648$	$\longrightarrow 48$
$2^{12} = 4,096$	$\longrightarrow 96$	$2^{32} = 4,294,967,296$	$\longrightarrow 96$
$\vdots$		$\vdots$	
$2^{20} = 1,048,576$	$\longrightarrow 76$	2^{40}	$\longrightarrow 76$

From this, we can see the cycle repeating every 20 numbers. So, every 2^{n+20} when $n>1$ will result in the last 2 digits being the same. In this case, $2^{20}, 2^{40}, 2^{60}, \ldots$ will all end in 76. Because $2^{10,000,000,000,000}$ is in this sequence, its last 2 digits must be 76.

Bonus. We can find the first digit by doing the same thing as problem 3, where we look for the pattern with the first digits. So,

$2^0 = 1 \rightarrow 1$ $\quad 2^{10} = 1{,}024 \rightarrow 1$ $\quad 2^{20} = 1{,}048{,}576 \rightarrow 1$
$2^1 = 2 \rightarrow 2$ $\quad 2^{11} = 2{,}048 \rightarrow 2$
$2^2 = 4 \rightarrow 4$ $\quad 2^{12} = 4{,}096 \rightarrow 4$
$2^3 = 8 \rightarrow 8$ $\quad 2^{13} = 8{,}198 \rightarrow 8$
$2^4 = 16 \rightarrow 1$ $\quad 2^{14} = 16{,}396 \rightarrow 1$
$2^5 = 32 \rightarrow 3$ $\quad 2^{15} = 32{,}796 \rightarrow 3$
$2^6 = 64 \rightarrow 6$ $\quad 2^{16} = 65{,}592 \rightarrow 6$
$2^7 = 128 \rightarrow 1$ $\quad 2^{17} = 131{,}184 \rightarrow 1$
$2^8 = 256 \rightarrow 2$ $\quad 2^{18} = 262{,}368 \rightarrow 2$
$2^9 = 512 \rightarrow 5$ $\quad 2^{19} = 524{,}736 \rightarrow 5$

We can see the pattern repeats every 10th number, where every 2^{n+10} will start with the same first digits. In this case, $2^0, 2^{10}, 2^{20}, 2^{30}, \ldots$ will all start with 1. Because $2^{10,000,000,000,000}$ is in this sequence, its first digits must be 1.