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1. I think my buddy's concern is valid because Big O's are based only as an upper bound and there can be one's much closer. If we were to continue the equation, we would have:

$X_0 = 1 \quad X_1 = 2 \quad X_2 = 3(2) + 2(1) = 8 \quad X_3 = 3(8) + 2(2) = 28$

$X_4 = 3(28) + 2(8) = 100$

To compare with 5^n

$5^0 = 1 \quad 5^1 = 5 \quad 5^2 = 25 \quad 5^3 = 125 \quad 5^4 = 625$

We can compare to something like 4^n:

$4^0 = 1 \quad 4^1 = 4 \quad 4^2 = 16 \quad 4^3 = 64 \quad 4^4 = 256$

From this, we can see how 4^n is growing quicker than the function as well but is much closer, showing that my buddy's concern of $O(5^n)$ being too big is valid.

2. Base Case:

$n = 7$

Inductive Hypothesis:

Let's assume $X_n \leq n!$ for all $n \geq 7$.

Inductive Step:

Try $n = 7, 8,$ and 9 for X_n and $n!$ So,

~X_n~	~$n!$~
$X_5 = 3(100) + 2(28) = 356$	$X_5 = 120$
$X_6 = 3(356) + 2(100) = 1{,}268$	$X_6 = 720$
$X_7 = 3(1{,}268) + 2(356) = 4{,}516$	$X_7 = 5{,}040$
$X_8 = 3(4{,}516) + 2(1{,}268) = 16{,}084$	$X_8 = 40{,}320$
$X_9 = 3(16{,}084) + 2(4{,}516) = 57{,}284$	$X_9 = 362{,}880$

After $n \geq 7$, $n!$ stays larger and grows at a exponentially quicker rate than X_n. Therefore, $X_n \leq n!$ for all $n \geq 7$.

3. From what we did in problem 1, β must be less than 4 because we proved 4^n grew at a higher rate than X_n. Of course, if we made α a fraction β could be much higher than 4. The largest α could be is 1. For example, if β was also 1, as X_n grows, $\alpha \cdot \beta^n$ would remain at 1 because an exponent to any n of 1 will remain 1. This satisfies $n=0$ where $X_0=1$ and $(1)\beta^0=1$ and all the others as n grows but $\alpha\beta^n$ stays constant. If α could be a super small fraction, β could be a super larger number. But, if α must be integer like 1, β could at its largest be 2 because when $n=1$, $X_1=2$ and $(1)\beta^1$ would take on the value of whatever β is, and that must ≥ 2.

4. This is kind of like problem 3 but reverse as we are coming from the top. The absolute smallest either γ or δ can be is $=1$. When $n=0$, $X_0=1$ so it's smallest we can get is 1. γ, much like α did, serves as the constant. We already know any δ of 4 will always be higher than X_n as it grows exponentially quicker so it doesn't need to serve as a multiplier. However, when $n=0$, δ^0 will always be 1 so γ has to be 1 so $X_0=1$ and $1\delta^0=1$ works. When γ wants to be a large constant, δ can be just above 1 when it still has slightly exponential growth and a large constant that gives it the cushion to grow rapidly.

5. Plug in: $n=1$ $\frac{\ln(2)}{1} \approx .693$ $n=2$ $\frac{\ln(8)}{2} \approx 1.03$ $n=3$ $\frac{\ln(28)}{3} \approx 1.11$

$n=4$ $\frac{\ln(100)}{4} \approx 1.15$

We can also use 3^n, which is smaller than X_n. $\lim_{n\to\infty} \frac{3^n}{n} = \infty$, since the smaller 3^n over n goes to ∞, $\lim_{n\to\infty} \ln(X_n)/n$ goes to ∞.

Bonus. Let's say $Y_n = x^n$. Then:

$x^n = (1-a)x^{n-1} + ax^{n-2}$

$x^2 - (1-a)x - a = 0$

Think this is some sort of recurrence relation, so should get $Y_n = \alpha_1(\text{something})^n + \alpha(\text{something})^n$ and then solve.

Bonus. Based on what it's asking, the function moves by taking the middle of the previous 2 values, so:

$$X_n = \frac{X_{n-1} + X_{n-2}}{2}$$

Following this function, we get

$X_0 = 0$ $X_1 = 1$ $X_2 = \frac{1+0}{2} = .5$ $X_3 = \frac{.5+1}{2} = .75$ $X_4 = \frac{.75+.5}{2} = .625$

$X_5 = \frac{.625+.75}{2} = .6875$ $X_6 = \frac{.6875+.625}{2} = .6719$ $X_7 = \frac{.6719+.6875}{2} = .6797$

As x goes to infinity, the function goes somewhere in the middle of .6797 and .6719.