Assignment 6

1. Variance → $Var[R] = Ex[R^2] - Ex^2[R]$

Indepence means the trial of X_1 doesn't affect the trial of X_2 and vice versa. So:

$Ex[(X_1 - X_2)^2]$

$Ex[(X_1 - X_2)(X_1 - X_2)]$

$Ex[X_1^2 - 2X_1X_2 + X_2^2]$

$Ex[X_1^2] + Ex[X_2^2] - 2Ex[X_1X_2]$

If $Var[R] = Ex[R^2] - Ex^2[R]$, then $Ex[R^2] = Var[R] + Ex^2[R]$ so:

$(Var[X_1] + Ex^2[X_1]) + (Var[X_2] + Ex^2[X_2]) - 2 \cdot Ex[X_1] \cdot Ex[X_2]$

$Ex = \mu \quad Var = \sigma^2$

$(\sigma^2 + \mu^2) + (\sigma^2 + \mu^2) - 2 \cdot \mu \cdot \mu$

$2\sigma^2 + 2\mu^2 - 2\mu^2 = 2\sigma^2$

This answers makes sense because each trial is independent to each other so the answer is the variances multiplied together.

2. $cov[X, Y]$ D_1 = dice 1 roll D_2 = dice 2 roll

$cov[D_1 + D_2, D_1 - D_2]$

$Var[D_1] + Var[D_2]$ doesn't equal 0 meaning they aren't fully independent of each other. Therefore, they have some correlation so X and Y are corelated.

$V(X) = \sum x^2 p - E(X)$

3.a. $\frac{1}{2}$ prob for each flip it lands on heads.

Chebychev's Inequality from textbook:

$$Pr[|R| \geq x] \leq \frac{Ex[|R|^2]}{x^2}$$

With prob of .5 for each attempt and c = # of H:

$$Pr[|c-\mu| \geq k\sigma] \leq \frac{1}{k^2}$$

$\mu \rightarrow$ attempts $\times$ probability $\rightarrow 100 \times .5 = 50$

$\sigma^2 \rightarrow$ attempts $\times$ prob(1 - prob) $= 100 \times .5 \times .5 = 25$

If $\sigma^2 = 25$, then $\sigma = 5$

$k\sigma$ is the requirement the number of head flips must hit, which is 75.

$75 = k\sigma \rightarrow 75 = 5k \rightarrow k = 15$. So:

$$Pr[|c - 50| \geq 75] \leq \frac{1}{15^2}$$

b. Markov's Inequality from textbook:

$$Pr[R \geq x] \leq \frac{Ex[R]}{x}$$

Let's say R is c for # of heads and x is 75. So:

$$Pr[c \geq 75] \leq \frac{Ex[|c|]}{75}$$

c. Coin is fair so should be heads, we want to see 50% increase minimum to 75. So:

$$Pr[c > 1.5(50)] \leq e^{-\beta(c) \cdot 50}$$

$c\ln(c) - c + 1 = .0187$

$$Pr[c > 75] \leq e^{(.0187)(50)}$$

4.a. Range(1, 2, 3, 4, 5, 6, 7, 8, 9, 10) = 10 - 1 = 9

The expected value is going to be the average so

$\frac{10+1}{2} = 5.5$ $E(W) = 5.5$

$V(W) = \sum x^2 p - \mu^2$

Each key has a .1 chance of working so

1	2	3	4	5	6	7	8	9	10

$\mu \rightarrow .1 + .2 + .3 + .4 + .5 + .6 + .7 + .8 + .9 + 1 = 5.5$

$\sum x^2 p$ $.1 + .4 + .9 + 1.6 + 2.5 + 3.6 + 4.9 + 6.4 + 8.1 + 10 = 38.5$

$38.5 - 5.5^2 \rightarrow 38.5 - 30.25 = 8.5$

b. Range(1, 2, 3, 4, ... can go on forever) = ∞

W is equal to failing over amongst the 10 keys until it hits success. If n is num of attempts, then:

$P(success) = \frac{1}{n}$ $P(fail) = \frac{n-1}{n}$

Geometric distribution where $p = P(success) = \frac{1}{n}$, so:

$E(W) = \frac{1}{p} = \frac{1}{\frac{1}{n}} = n$ n is 10 keys.

V(W) doesn't change, so 8.5.

5. Find expected value by finding each chance of person going on floor over all the possibilities. With 4 people each 7 choices, it's $7 \cdot 7 \cdot 7 \cdot 7 = 2,401$.
N is "total" distinct floors chosen.

$N = 1$ (all 4 go to 1 floor)

$^7C_1 = 7$

$N = 2$ (1 on own and 3 together or 2 on each)

$^7C_2 \times (^4C_1 + ^4C_1 + ^4C_2)$

$21 \times (4 + 4 + 6) = 21 \times 14 = 294$

$\frac{7!}{2!\,5!} = \frac{7 \cdot 6 \cdot 5!}{2 \cdot 1 \cdot 5!} = \frac{42}{2} = 21$

$N = 3$ (2 on own and 2 together)

$^7C_3 \times (^4C_2 \times 2! \times 3)$

$35 \times (6 \times 2 \times 3) = 35 \times 36 = 1,260$

$\frac{7!}{3! \cdot 4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{3 \cdot 2 \cdot 1 \cdot 4!} = \frac{210}{6} = 35$

$N = 4$ (all to their own floor)

$^7C_4 \times 4!$

$35 \times (4 \cdot 3 \cdot 2) = 35 \times 24 = 840$

$\frac{7!}{4! \cdot 3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4! \cdot 3 \cdot 2} = \frac{210}{6} = 35$

$E(N) = \sum xp$; where $x = N$. So:

$\frac{(1)(7) + (2)(294) + (3)(1,260) + (4)(840)}{2,401}$

$\frac{7 + 588 + 3,780 + 3,360}{2,401} = \frac{7,735}{2,401} = 3.22$ $E(N) = 3.2216$

$V(N) = \sum x^2 p - \mu$ $\mu = E^2(N)$

$x^2 \longrightarrow \frac{25,963}{2,401} - 3.2216 = 10.8134 - 10.3787$

$V(N) = .4347$