Assignment 8

1. c^k must have some value x. $c^k = x$ means
$$\sum_{k=0}^{\infty} x \cdot x^k = \frac{x}{1-x}$$
So if we make $c^k = \sqrt{x}$, we would have $\sqrt{x}$ on top and could square both the top and bottom so:
$$\sum_{k=0}^{\infty} \sqrt{x} \cdot x^k = \left(\frac{\sqrt{x}}{1-x}\right)^2 = \frac{x}{(1-x)^2}$$
So coefficient is $\sqrt{x}$.

2. If W_T, it means the gambler is starting at his target goal so there's no need to gamble because he already reaches it 100% of the time. If $W(x)$ is the probability of hitting the target, then we can say:
$$W_T = 1 = \frac{r^T - 1}{r-1} \cdot W_1, \text{ so}$$
$$W_1 = \frac{r-1}{r^T-1}$$

3. (21.2) states that
$$\Pr[\text{the gambler wins}] = \begin{cases} \frac{n}{T} & \text{for } p = \frac{1}{2} \\ \frac{r^n - 1}{r^T - 1} & \text{for } p \neq \frac{1}{2} \end{cases}$$
In this, n is the initial money, T is target goal to win, and r is $\frac{1-p}{p}$. For $p = \frac{1}{2}$, there's an equal chance of going up or down every bet, so the probability of winning is the money you start with over your target. For example is you initial is \$5 and target is \$7, you have a $\frac{5}{7}$ chance in winning cause you're already $\frac{5}{7}$ of the way to 7 and each turn is equal risk. For $p \neq \frac{1}{2}$, we keep everything the same except put the n and T

over the r. If $p = \frac{1}{4}$ instead, then $r = \frac{1-.25}{.25} = \frac{.75}{.25} = 3$ so the same $n = 5$, $T = 7$ example would be $\frac{3^5-1}{3^7-1}$ is much lower $\frac{242}{2{,}186}$. The probability value really alters the chance of success/failure because of the drift it can cause if p is too onesided, regardless of n's proximity to T.

If $p < \frac{1}{2} \rightarrow Pr[\text{winning}] = \frac{r^n-1}{r^T-1} < \frac{r^n}{r^T} = \left(\frac{1}{r}\right)^{T-n}$

4. For $X = 1$, $pr = \frac{1}{2}$:
   either heads $(\frac{1}{2})$ 1↑, or tails $(\frac{1}{2})$ 1↓

   For $X = 2$, $pr = 0$
   first must be tails, so now either heads $(\frac{1}{2})$ even or tails $(\frac{1}{2})$ 2↓

   For $X = 3$,
   even goes up to \$1, even goes down to 1↓, is tails 2↓ goes up to 1↓ or down to 3↓.
   1 out of 4 paths hit

   For $X = 4$, $pr = 0$
   Path 1 = 1↓ can go to even or 2↓
   Path 2 = 1↓ can also go even or 2↓
   Path 3 = 3↓ can go to 2↓ or 4↓,

   For $X = 5$,
   An even can go ↑1 (hit) or 1↓ (2x)
   A 2↓ can go ↓1 or 3↓ (3x)
   A 4↓ can go 3↓ or 5↓
   2 out of 12 paths hit

   For $X = 6$, $pr = 0$
   1↓ can go even or 2↓ (5x)
   3↓ can go 2↓ or 4↓ (4x)
   5↓ can go 3↓ or 6↓

For $X = 7$,

even can go 1↑ (hit) or 3↓ (5×)

2↓ can go 1↓ or 3↓ (9×)

4↓ can go 3↓ or 5↓ (4×)

3↓ can go 2↓ or 4↓

6↓ can go 5↓ or 7↓

5 out of 41 paths hit

$p_0 = 0$ $p_1 = p$ $p_2 = 0$ $p_3 = (1-p)p^2$ $p_4 = 0$

$p_5 = 2(1-p)^2(p^3)$ $p_6 = 0$ $p_7 = 6(1-p)^3(p^4) =$

$p_1 = .5$ $p_3 = (1-.5)(.5)^2 = .125$

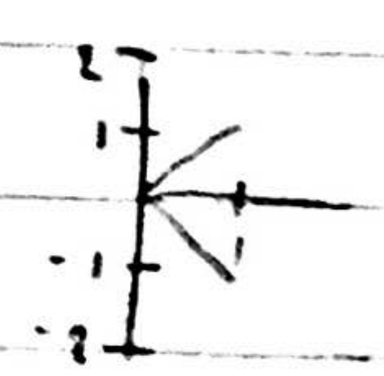
$p_5 = 2(1-.5)^2(.5)^3 = .0625$ $p_7 = 6(1-.5)^3(.5)^4 = .0469$

$.5 + .125 + .0625 + .0469 =$

Prob $X \leq 7$ is $.7344$

5. For $Y = 1$, pr $= 0$

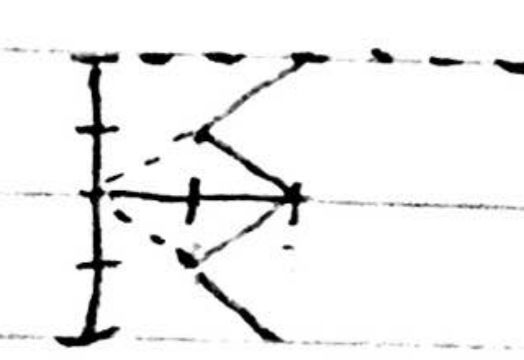
can only go to 1↑ or 1↓



For $X = 2$, pr $= \frac{1}{4}$

HH = hit, HT = even,

TH = even, TT = 2↓

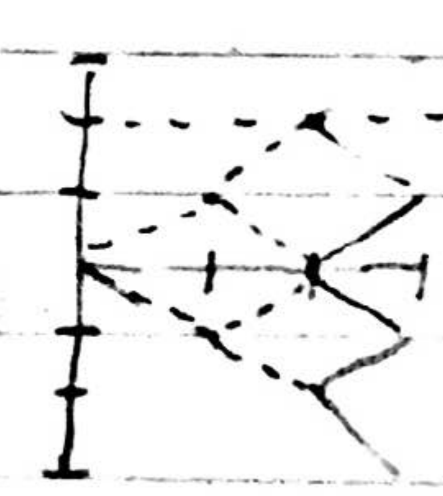


For $Y = 3$, pr $= 0$

HTH = 1↑ HTT = 1↓

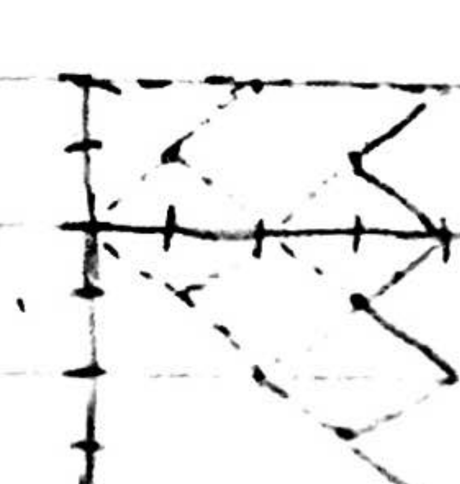
THH = 1↑ THT = 1↓

TTH = 1↓ TTT = 3↓



For $Y = 4$, pr $= \frac{1}{6}$

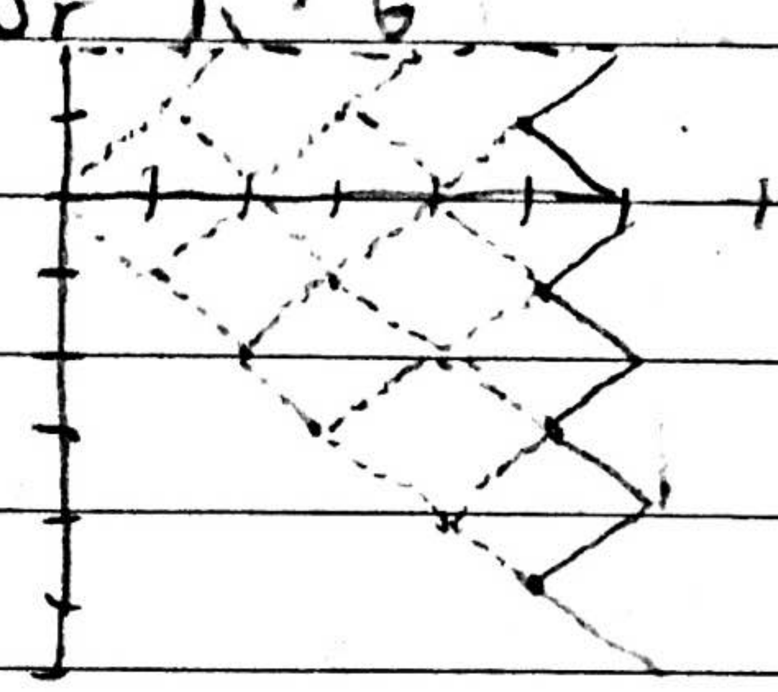
To hit need first 2 to break even then HH. 2 path out of 12 so $\frac{1}{6}$



For $Y=5$, $pr=0$

At most, can only go ↑1

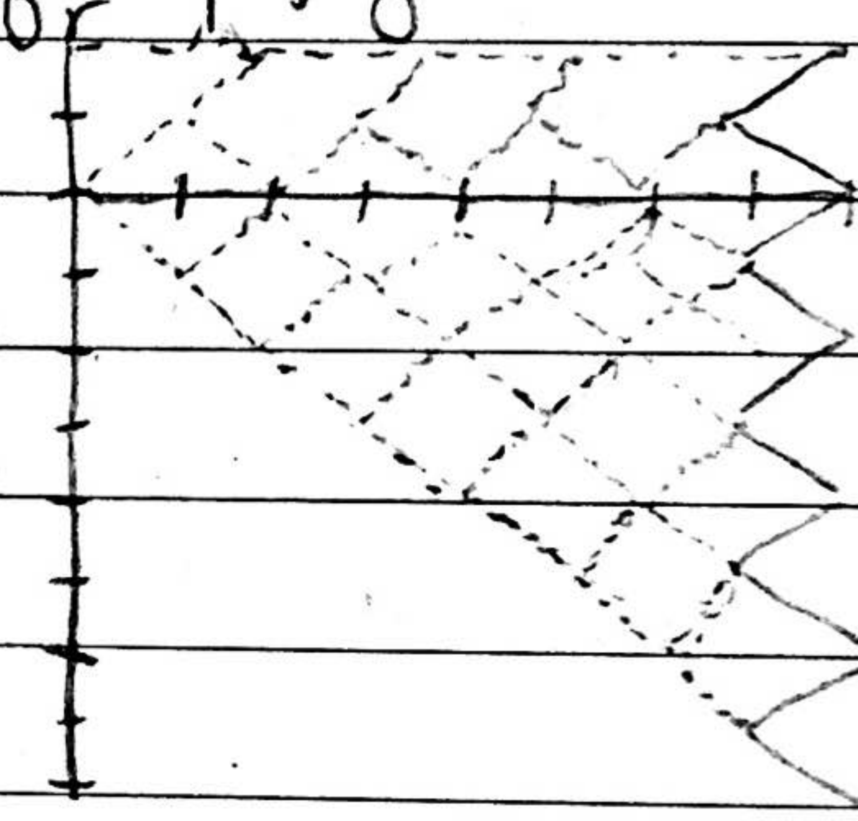
For $Y=6$, $pr=\frac{1}{8}$



For $Y=7$, $pr=0$

At most, can only go ↑1

For $Y=8$, $pr=\frac{1}{10}$



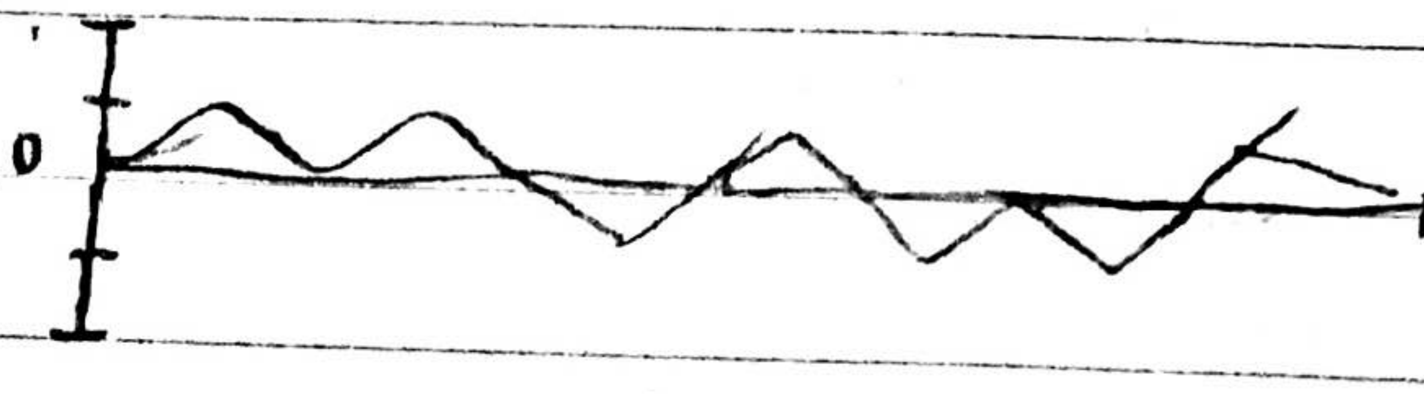
$\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}/2$

Prob $Y\leq 8$ is .3208

The probability that Y is odd is 0, Y must be even to hit the \$2 target.

6. $n_1=1$. On the first turn, somebody will always be up \$1 because it has to be H or T, meaning $Z=1$.
Prob$(Z\leq 1)>\frac{1}{2}$ → Prob$(1\leq 1)=1$ and $1>\frac{1}{2}$.
$n_2=4$, the first two turns will be HH, TT, HT, or TH, meaning you either breakeven $(\frac{1}{2})$ or someone is up \$2 $(\frac{1}{2})$. The third turn has $pr=0$ of getting someone up \$2 from the even, but the 4th time does which puts it above $\frac{1}{2}$ chance.
Prob$(W\leq 4)>\frac{1}{2}$
$E(Z)=1$ because someone will always have a \$1 lead after the 1st turn.

7. Prob(W = 2n)



The value can ever only be 1↑, 0, or 1↓. To go the length of 2n it need to stay between -1 and 1. The pattern is that it starts by going to -1 or 1, then next flip must be opposite of previous to bring it back to even. Next flip can be either, but flip after must be opposite to that now to bring it back to even and the pattern repeats. If right before 2n, the value is 0, neither H or T makes it hit \$2. If value is -1 or 1, both have $\frac{1}{2}$ chance of reach \$2.

$\frac{1}{2} + \frac{1}{2} + \frac{0}{2} = \frac{2}{6} = \frac{1}{3}$ Prob(W = 2n) = $\frac{1}{3}$

Prob(W > 2n) is $\frac{1}{2}$ because from even, it has to go HH or TT and not HT or TH to get to \$2, and $\frac{2}{4} = \frac{1}{2}$

Given that W > 2n, we want to know the probability it hits it's \$2 only 2 turns later, as 2n+2 indicates. The next 2 turns can only be HH, HT, TH, TT at even, or $\frac{1}{2}$ to hit \$2. At -1 or 1, hitting the \$2 on first would be 2n+1 and going back to even with 1 turn left wouldn't be enough. So, with the $\frac{1}{3}$ chance we are on even at 2n's times the prob $\frac{1}{2}$ of getting HH or TT, Prob(W = 2n+2 | W > 2n) = $\frac{1}{6}$.

8. All odd num. W have 0 prob of hitting \$2.
At $W=2$, $pr=\frac{1}{2}$ because HH, TT ✓, HT, TH X $\frac{2}{4}$
At $W=4$, $pr=\frac{1}{2}$ because HTHH, HTTT ✓, HTHT, HTTH X
$\frac{4}{8}$ THHH, THTT ✓, THHT, THTH X
Will have $\frac{1}{2}$ chance of happening each even num if it's still going so $\frac{1}{2}$ @ 2, $\frac{1}{4}$ @ 4, $\frac{1}{8}$ @ 8, $\frac{1}{16}$ @ 10. So,
$E(W) \approx 8$

9. $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\mu)^2 = \mu = 8 \quad n = \frac{\infty}{2} = 100$ (only even nums)

This calculate dif from the mean, so

$$\begin{array}{ccccccccc} & & & & 8 & & & \\ 2 & 4 & 6 & & & 10 & 12 & 14 & 16 \\ 6^2 + & 4^2 + & 2^2 + & & & 2^2 + & 4^2 + & 6^2 + & 8^2 \ldots \end{array}$$

However weighted less and less as num continues because W stops once value hits \$2, so
$2 \rightarrow \frac{1}{2}$ prob $\quad 4 \rightarrow \frac{1}{4}$ prob, etc.
$V(W) \approx 4 \qquad Var(W) = E((W-\mu)^2) = \sigma^2$

10. $Cov(Z,W) = \frac{\sum_{i=1}^{n}(z_i-\bar{z})(w_i-\bar{w})}{n} = 0 \quad \bar{z}=1$ because always has \$1 lead after 1 turn
$\bar{w}=8$

$Cov(X,Y) = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{n} = 0$

11. Z is independent because it will always equal 1 since first turn must always give someone \$1 lead. Since Z is a constant always recording 1, W also is independent. X is also independent because it's target of \$1 always hits before Y's of \$2 and Y must have X hit first so Y is dependent.