Midterm 1

1. On my honor, I have neither received nor given any unauthorized assistance on this examination.

2. $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \quad x = 2x \quad y = 3y \quad n = 18$. So:

$(2x+3y)^{18} = \sum_{k=0}^{n} \binom{18}{k} (2x)^{18-k} (3y)^k$

We need to have $x^{10} y^8$, so $k = 8$. Then:

$(2x+3y)^{18} = \sum_{k=0} \binom{18}{8} (2x)^{10} (3y)^8$

$= \binom{18}{8} 2^{10} \cdot x^{10} \cdot 3^8 \cdot y^8$

So, the coefficient of $x^{10} y^8$ in $(2x+3y)^{18}$ is $\binom{18}{8} 2^{10} 3^8$

3. $(x+y)^n \rightarrow$ total monomial is ${}^{(n+r-1)}C_n$ where $n = 18$ and $r = 2$ (for number of terms). So,

$n+r-1 = 18+2-1 = 19 \rightarrow {}^{19}C_{18} \rightarrow \binom{19}{18}$

$\frac{19!}{18!(19-18)!} = \frac{19!}{18! \cdot 1!} = \frac{19 \cdot \cancel{18!}}{\cancel{18!}} = 19$ monomials

4. $\left.\begin{array}{l} (x+y+z)^n \\ x^p y^q z^r \end{array}\right\} \rightarrow \frac{n!}{p! \cdot q! \cdot r!} \quad n = 25 \quad p = 10 \quad q = 8 \quad r = 7$

$\frac{25!}{10! \cdot 8! \cdot 7!} = 2.103 \times 10^{10}$

5. $(x+y+z)^n \rightarrow$ total monomial is ${}^{(n+r-1)}C_n$ where $n = 25$ and $r = 3$ (for number of terms). So

$n+r-1 = 25+3-1 = 20 \rightarrow {}^{27}C_{25} \rightarrow \binom{27}{25}$

$\frac{27!}{25!(27-25)!} = \frac{27!}{25! \cdot 2!} = \frac{27 \cdot 26 \cdot \cancel{25!}}{\cancel{25!} \cdot 2} = \frac{702}{2} =$

351 monomials

6.i. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = a + b - ab$

ii. $P(A \cap B^c) = P(A) \cdot (1 - P(B)) = a(1 - b) = a - ab$

iii. $P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \cdot P(B)}{P(B)} = \frac{ab}{b} = a$

$\uparrow$

(Independent, what happens to one event doesn't affect the other.)

7i. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - 0 = a + b$

ii. $P(A \cap B^c) = 1$ (A happens and B doesn't

iii. $P(A | B) = 0$ If B happens, as given, A can't happen.
(Mutually exclusive, if one event happens the other can't happens.)

8. 15 balls; 6 pockets

4 balls in pocket 1 $\rightarrow$ ${}^{15}C_4$

$\frac{15!}{4! \cdot 11!} = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot \cancel{11!}}{4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{11!}} = \frac{32{,}760}{24} = 1{,}365$

6 balls (out of 11 now) in pocket 2 $\rightarrow$ ${}^{11}C_6$

$\frac{11!}{6! \cdot 5!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot \cancel{6!}}{\cancel{6!} \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{55{,}440}{120} = 462$

3 balls (out of 5 now) in pocket 3 $\rightarrow$ ${}^{5}C_3$

$\frac{5!}{3! \cdot 2!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!} \cdot 2 \cdot 1} = \frac{20}{2} = 10$

2 balls (out of 2 now) in pocket 4 $\rightarrow$ ${}^{2}C_2$

$\frac{2!}{2!(2-2)!} = 1$

Multiply the 4 probabilities together so:

$1{,}365 \cdot 462 \cdot 10 \cdot 1 = 6{,}306{,}300$ ways

9. x_1, x_2, x_3 are balls in pockets 1, 2, and 3
We know there are 15 balls, so:
$x_1 + x_2 + x_3 = 15$
and we know x_1, x_2, and x_3 have to be ≥ 0:
This means:
$$^{(15+2)}P_2 = {}^{17}P_2 = \frac{17!}{(17-2)!} = \frac{17!}{15!} = \frac{17\cdot16\cdot15!}{15!} = 272 \text{ ways}$$

10. $x_1 + x_2 + x_3 = 15$, still, however this time x_1, x_2, and x_3 must be ≥ 1. So:
$(x_1 - 1) + (x_2 - 1) + (x_3 - 1) = 12 \leftarrow (15-3)$
If we make it so $y_1 = x_1 - 1$, $y_2 = x_2 - 1$ and $y_3 = x_3 - 1$, then can say:
$y_1 + y_2 + y_3 = 12$
So:
$$^{(12+2)}P_2 = {}^{14}P_2 = \frac{14!}{(14-2)!} = \frac{14!}{12!} = \frac{14\cdot13\cdot12!}{12!} = 182 \text{ ways}$$

11. Now for this one we have $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 15$
The number of ways to have 3 pockets be non-empty is $^6C_3 = \frac{6!}{3!\cdot3!} = \frac{6\cdot5\cdot4\cdot3!}{3!\;3\cdot2\cdot1} = \frac{120}{6} = 20$
3 pockets are ≥ 0, 3 others are ≥ 1
Lets say $x_1, x_2, x_3 \geq 0$ and $x_4, x_5, x_6 > 1$
Then, as we did in the problem before, say $y_4 = x_4 - 1$, $y_5 = x_5 - 1$, and $y_6 = x_6 - 1$. Now we have:
$x_1 + x_2 + x_3 + y_4 + y_5 + y_6 = 12$

Now, we can say:

$^{(12+5)}P_5 = {}^{17}P_5 = \frac{17!}{(17-5)!} = \frac{17!}{12!} = \frac{17\cdot16\cdot15\cdot14\cdot13\cdot\cancel{12!}}{\cancel{12!}} =$

742,560 × the 20 ways of 3 non-empty pockets so,
14,851,200

12. We can only move north and east. On any given path we will move north 4 blocks and east 3 blocks. The maximum movements we can have are 7, so $^7C_3$ (or $^7C_4$)

$\frac{7!}{3!(7-3)!} = \frac{7!}{3!\,4!} = \frac{7\cdot6\cdot5\cdot\cancel{4!}}{3\cdot2\cdot1\cdot\cancel{4!}} = \frac{210}{6} = 35$ paths.

13.i. First, there are 13 spades in a deck of cards, so $^{13}C_2$
We also need to make sure the other 3 cards aren't spades, where there are 39 that aren't, so $^{39}C_3$
Both need to happen simultaneously, so multiply:

$^{13}C_2 = \frac{13!}{2!\,11!} = \frac{13\cdot12\cdot\cancel{11!}}{2\cdot1\cdot\cancel{11!}} = \frac{156}{2} = 78$

$^{39}C_3 = \frac{39!}{3!\,36!} = \frac{39\cdot38\cdot37\cdot\cancel{36!}}{3\cdot2\cdot1\cdot\cancel{36!}} = \frac{54,834}{6} = 9,139$

78 × 9,139 = 712,842.

Now we have to divide over total set of cards and picking 5, so $^{52}C_5$.

$\frac{52!}{5!\,47!} = \frac{52\cdot51\cdot50\cdot49\cdot48\cdot47!}{5\cdot4\cdot3\cdot2\cdot1\cdot47!} = 2,598,960$

$\frac{712,842}{2,598,960} = .274$

ii. If we already have a spade, it means we need 1 more out of 4 picks. 12 spades left, need 1, so $^{12}C_1$ and the other 3 can't be spades so ${}^{39}C_3$ still. We know ${}^{39}C_3 = 9,139$. So, for ${}^{12}C_1$:

$$\frac{12!}{1!\,11!} = \frac{12 \cdot 11!}{1 \cdot 11!} = 12$$

Then we take the combined chance for both over entire deck of cards minus the 1 spade. ${}^{51}C_4$

$$\frac{51!}{4!\,47!} = \frac{51 \cdot 50 \cdot 49 \cdot 48 \cdot 47!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 47!} = 249,900$$

$$\frac{12 \cdot 9,139}{249,900} = .439$$

14.i. $P(\text{regular}) = \frac{2}{3}$ $P(\text{two-headed}) = \frac{1}{3}$

R TH head = H tail = T

Baye's Thereom

$$P(TH \mid H) = \frac{P(H \mid TH) \cdot P(TH)}{P(H \mid TH) \cdot P(TH) + P(H \mid R) \cdot P(R)}$$

$$\frac{1 \cdot \frac{1}{3}}{(1)(\frac{1}{3}) + (\frac{1}{2})(\frac{2}{3})} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

Probability of picking TH after 1 H is $\frac{1}{2}$ and odds are 1 in 3.

ii. For getting H twice, we still use Baye's Theorem.

$$P(TH \mid H \cap H) = \frac{P(H \cap H \mid TH) \cdot P(TH)}{P(H \cap H \mid TH) \cdot P(TH) + P(H \cap H \mid R) \cdot P(R)}$$

Since the coin flips are independent, we separate the probabilities.

$$= \frac{P(H \mid TH) \cdot P(H \mid TH) \cdot P(TH)}{P(H \mid TH) \cdot P(H \mid TH) \cdot P(TH) + P(H \mid R) \cdot P(H \mid R) \cdot P(R)}$$

$$= \frac{1 \cdot 1 \cdot \frac{1}{3}}{(1)(1)(\frac{1}{3}) + (\frac{1}{2})(\frac{1}{2})(\frac{2}{3})} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6}} = \frac{\frac{2}{6}}{\frac{2}{6} + \frac{1}{6}} = \frac{\frac{2}{6}}{\frac{3}{6}} = \frac{2}{3}$$

Probability of picking TH after 2 H in a row is $\frac{2}{3}$ and odds against are 2:1.

15. To prove by induction, we can start with base case by evaluating $k = 1$. So, the statement is:

$$\frac{(2n)!}{n!(n-k)!} > \frac{(2n)!}{(n-k)!(n-k)!} \text{ and } \geq \frac{(2n)!}{(n+k)(n-k)!}$$

If $k = 1$, then:

$$\frac{(2n)!}{1!(n-1)!} > \frac{(2n)!}{(n-1)!(n-1)!} \text{ and } \frac{(2n)!}{(n+1)!(n-1)!}$$

which is true. Now consider $n = k+1$, where

$$\frac{(2k+2)!}{(k+1)!((k+1)-k)!} > \frac{(2k+2)!}{((k+1)-1)!((k+1)-1)!} \text{ and } \frac{(2k+2)!}{((k+1)+1)!((k+1)-1)!}$$

$\downarrow$

$$\frac{(2k+2)!}{(k+1)!(1)!} > \frac{(2k+2)}{(k)!(k)!} \text{ and } \frac{(2k+2)!}{(k+2)!(k)!}$$

Still works, so this was proven by induction.