Midterm 2

1. On my honor, I have neither received nor given any unauthorized assistance on this examination.

2. .99 of all widgets have an expectation of working. In a sample of 1,000, 30 don't work.
Let total expected probability to work be $p = .99$
Let widgets in sample be $n = 1{,}000$
Let widgets that did work be $x = 970$ $(1{,}000 - 30)$
So, the working probability of the sample is $s = .97$ $\leftarrow \frac{970}{1{,}000}$

$$t = \frac{s - p}{\sqrt{\frac{p(1-p)}{n}}} = \frac{.97 - .99}{\sqrt{\frac{.99(1-.99)}{1000}}} = \frac{-.02}{\sqrt{\frac{(.99)(.01)}{1000}}} =$$

$$\frac{-.02}{.0031} = -6.4516$$

$2P(t > |t|)$
$2P(t > 6.4516)$
$2[1 - P(t < 6.4516)]$

$${}^{1000}C_{970}(.97)^{970}(.03)^{30}$$

3. Binary means either 0 or 1, two choices.
2 choices, 10 times, so $2^{10} = 1{,}024$ potential strings.
Since each individual value has a $\frac{1}{2}$ chance of being a 0 and $\frac{1}{2}$ chance of being a 1 then with a length of 10, the expected amount is 5 0s and 5 1s. So:

5 0's = 5   5 1's = 10   (All 0s = 10, All 1s = 20,
5 + 10 = 15   so 15 is the average)

Expected value of x is 15.

4. The value of X can range from 10 (All 0s) to 20 (All 1s) So the possible values are:
$X = 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20$
$\mu = 15 \quad n = 11$

$$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n} = \frac{\sum_{i=1}^{11}(x_i - 15)^2}{11} = \frac{110}{11} = 10$$

Diff. from mean

15   14/16   13/17   12/18   11/19   10/20

$0 + 1^2(2) + 2^2(2) + 3^2(2) + 4^2(2) + 5^2(2) = 110$

Variance is 10.

5. Markov's Inequality
$Pr[X \geq a] \leq \frac{Ex[X]}{a}$ $a = 18$ $Ex[X] = 15$ (mean)

$Pr[X \geq 18] \leq \frac{15}{18} = \frac{5}{6}$

6. Tchebyshev's Inequality

$Pr[|X| \geq 18] \leq \frac{Ex[|R|^2]}{x^2}$

$Pr[|X - \mu| \geq k] \leq \frac{\sigma^2}{k^2}$ $\mu = 15$ (mean) $k = 18$

$\sigma^2 = var = 10$

$Pr[|X - 15| \geq 18] \leq \frac{10}{324}$

7. Chernoff Bound

$Pr[T \geq c\,Ex[T]] \leq e^{-B(c)Ex[T]}$ $Ex(T) = 15$ (mean)

$c = 1.2$ (increase from 15 to 18)

$B(c) = c\ln(c) - c + 1$

$B(1.2) = 1.2\ln(1.2) - 1.2 + 1 = .0188$

$Pr[T \geq (1.2)(15)] \leq e^{-(.0188)(15)}$

$Pr[T \geq 18] \leq e^{-.282}$

8. Either 0 or 1 for each of the 100 values, so 2^{100} combos. The absolute lowest Y can be is 100, where all values are 0s. The absolute highest is 200, where all values are 1s. Y can be altered at a linear rate between 100 to 200 with each singular value change. So, the expected value falls in the middle at Y = 150.

9. Y can range from 100 to 200

$Ex(Y) = \mu = 150$ $n = 101 \leftarrow \{100, 101, 102, \ldots, 200\}$

$\sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \mu)^2}{n} = \frac{\sum_{i=1} (x_i - 150)^2}{101} = \frac{85{,}850}{101} = 850$

150 149/151 148/152 100/200

$0 + 1^2(2) + 2^2(2) + \ldots + 50^2(2) = 85{,}850$

Variance is 850.

10. Markov's Inequality

$Pr[Y \geq a] \leq \frac{Ex[Y]}{a}$ $a = 180$ $Ex[Y] = 150$ (mean)

$Pr[Y \geq 180] \leq \frac{150}{180} = \frac{5}{6}$

11. Tchebyshev's Inequality

$Pr[|Y| \geq 180] \leq \frac{Ex[|R|^2]}{k^2}$

$Pr[|Y - \mu| \geq k] \leq \frac{\sigma^2}{k^2}$ $k = 180$ $\mu = 150$ (mean)

$\sigma^2 = var = 850$

$Pr[|Y - 150| \geq 180] \leq \frac{850}{32,400} = \frac{85}{3,240}$

12. Chernoff Bound

$Pr[T \geq c\,Ex[T]] \leq e^{-B(c)Ex[T]}$

$Ex[T] = 150$ (mean)

$c = 1.2$ (increase from 150 to 180)

$B(c) = c\ln(c) - c + 1$

$B(1.2) = 1.2\ln(1.2) - 1.2 + 1 = .0188$

$Pr[T \geq (1.2)(150)] \leq e^{-(.0188)(150)}$

$Pr[T \geq 180] \leq e^{-2.82}$

13. True, Chernoff is exponential, and apply only distributions of sums of independent random variables in the real interval of [0,1] so they are sharper results.

False, Tchebyshev is useful since it can be applied to any probability distribution where mean and variance are defined but Chernoff is better.

14. for $n \geq 2$, $a_n = a_{n-1} + 6a_{n-2} - 3$, $a_0 = 1$, $a_1 = 2$

$r^2 - r - 6 = 0$

$(r-3)(r+2) = 0$

$r = 3, -2$ $\quad\quad r_1 = 3, r_2 = -2$

For $n = 0$:

$a_0 = a_1(r_1)^0 + a_2(r_2)^0 - 3$ $\quad\quad a_0 = 1$

$1 = a_1(3)^0 + a_2(-2)^0 - 3$

$1 = a_1 + a_2 - 3$

$4 = a_1 + a_2$

For $n = 1$:

$a_1 = a_1(r_1)^1 + a_2(r_2)^1 - 3$ $\quad\quad a_1 = 2$

$2 = a_1(3)^1 + a_2(-2)^1 - 3$

$5 = 3a_1 - 2a_2$

Multiply $4 = a_1 + a_2$ by 2 then add with other equation.

$(4 = a_1 + a_2)(2) \rightarrow 8 = 2a_1 + 2a_2$

$$\begin{aligned} 8 &= 2a_1 + 2a_2 \\ + \quad 5 &= 3a_1 - 2a_2 \\ \hline 13 &= 5a_1 \end{aligned}$$

$a_1 = \frac{5}{13}$ $\longrightarrow$ $4 = \frac{5}{13} + a_2 \rightarrow a_2 = \frac{48}{13}$

$a_n = a_1(r_1)^n + a_2(r_2)^n$

$= \left(\frac{5}{13}\right)(3)^n + \left(\frac{48}{13}\right)(-2)^n$