Final

1. On my honor, I have neither received nor given any unauthorized assistance on this examination.

2.i. There are 13 spades in a deck of cards, we need 4 so $^{13}C_4$. We also need to make sure none of the other cards are spades, where there are 39 that aren't so $^{39}C_9$. Both must happen together, so:

$$^{13}C_4 = \frac{13!}{4!\,9!} = \frac{13\cdot12\cdot11\cdot10\cdot\cancel{9!}}{4\cdot3\cdot2\cdot1\cdot\cancel{9!}} = \frac{17{,}160}{24} = 715$$

$$^{39}C_9 = \frac{39!}{9!\,30!} = \frac{39\cdot38\cdot37\cdot36\cdot35\cdot34\cdot33\cdot32\cdot31\cdot\cancel{30!}}{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1\cdot\cancel{30!}} =$$

$= 211{,}915{,}132$

$715 \times 211{,}915{,}132 = 151{,}519{,}319{,}380$

Now we have to divide over total set of cards. There are 52 total and we are picking 13, so $^{52}C_{13}$.

$$\frac{52!}{13!\,39!} = \frac{52\cdot51\cdot50\cdot49\cdot48\cdot47\cdot46\cdot45\cdot44\cdot43\cdot42\cdot41\cdot40\cdot\cancel{39!}}{13\cdot12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1\cdot\cancel{39!}}$$

$= 635{,}013{,}559{,}600$

$$\frac{151{,}519{,}319{,}380}{635{,}013{,}559{,}600} = \underline{.239}$$

ii. If we already have a spade, we need 3 more out of 12 picks. There are 12 spades left, we need 3 so $^{12}C_3$ and the other 9 can't be so $^{39}C_9$.

We know $^{39}C_3 = 211{,}915{,}132$. For $^{12}C_3$:

$$\frac{12!}{3!\cdot9!} = \frac{12\cdot11\cdot10\cdot\cancel{9!}}{3\cdot2\cdot1\cdot\cancel{9!}} = \frac{1{,}320}{6} = 220$$

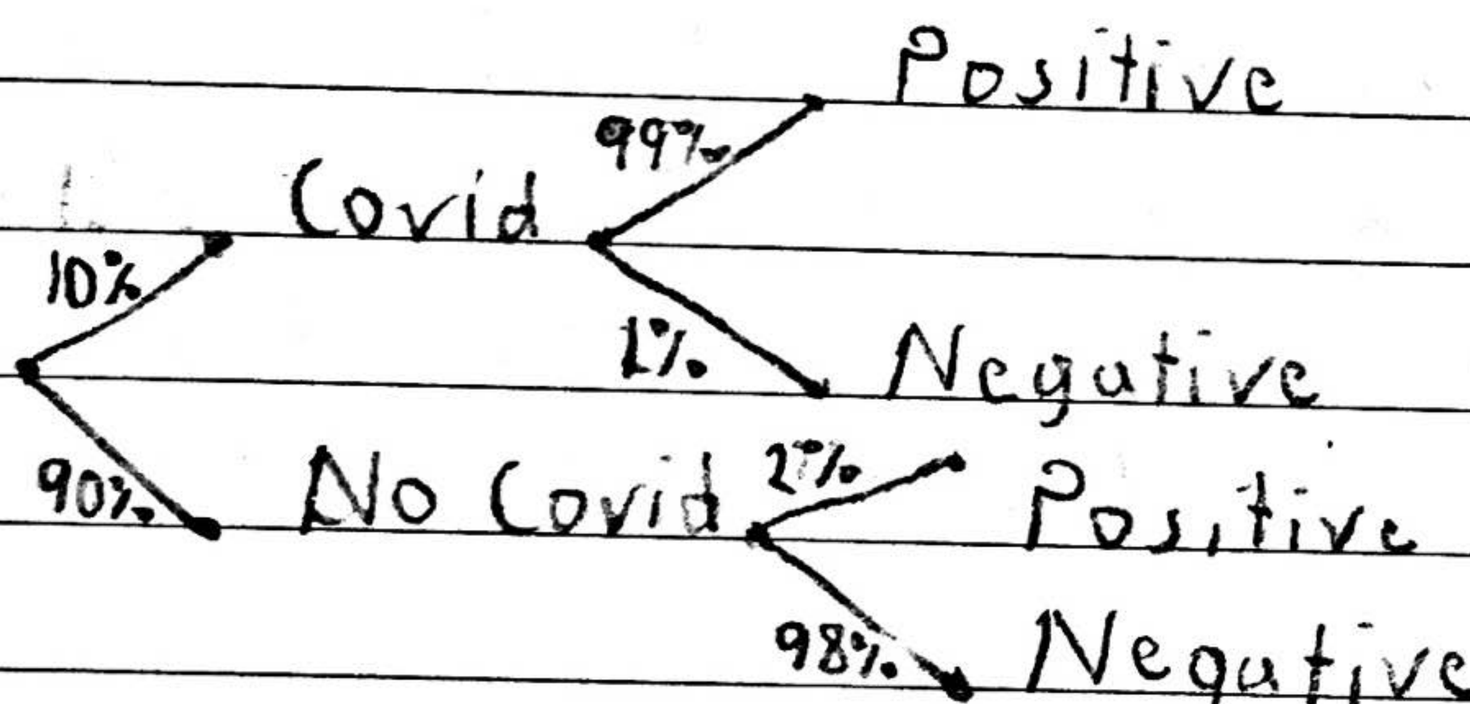
Now we take the combined chance of both happening over the entire deck of cards combos minus the one slot already taken, so $^{51}C_{12}$.

$$\frac{51!}{12!\,39!} = \frac{51\cdot 50\cdot 49\cdot 48\cdot 47\cdot 46\cdot 45\cdot 44\cdot 43\cdot 42\cdot 41\cdot 40\cdot 39!}{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 39!}$$

$= 158{,}753{,}389{,}900$

$$\frac{211{,}915{,}132\cdot 220}{158{,}753{,}389{,}900} = \underline{.294}$$

3.



A = has Covid

B = tests positive for Covid

$$Pr(B|A) = \frac{Pr(A|B)\cdot Pr(B)}{Pr(A|B)\cdot Pr(B) + Pr(A|\bar{B})\cdot Pr(\bar{A})}$$

$$\frac{(.99)(.10)}{(.99)(.10) + (.02)(.90)} = \frac{.099}{.117} = \underline{.846}$$

4.a. 26 reds, would need to pick 3 so $^{26}C_3$. We put this over the entire possibility of combos which is $^{52}C_3$. So:

$$\frac{26!}{3!\,23!} = \frac{26\cdot 25\cdot 24\cdot 23!}{3\cdot 2\cdot 1\cdot 23!} = \frac{15{,}600}{6} = 2{,}600$$

$$\frac{52!}{3!\,49!} = \frac{52\cdot 51\cdot 50\cdot 49!}{3\cdot 2\cdot 1\cdot 49!} = \frac{132{,}600}{6} = 22{,}100$$

$$\frac{2{,}600}{22{,}100} = \underline{.118}$$

b. $Prob(A_1) = \frac{26}{52} = \frac{1}{2}$
$Prob(A_2) = \frac{25}{51}$
$Prob(A_3) = \frac{24}{50} = \frac{12}{25}$

c. No because A_2 is dependent on A_1. If A_1 is red, $Pr(A_2) = \frac{25}{51}$. If A_1 isn't red, $Pr(A_2) = \frac{26}{51}$. A_1's outcome changes the probability of A_2's outcome so they are dependent.

5.a. ~~C~~ geometric sequence, because U is the variable that measures the number of trials before hitting 3 reds. We are accounting for the number of trials prior to its successful one.

b. We know from problem 4a., the probability of getting 3 reds is ${}^{26}C_3 / {}^{52}C_3 = .118 = \frac{2}{17}$. We can put this prob over 1 to get the expected number of trials this will take, So:
$E(U) = \frac{1}{p} = \frac{1}{\frac{2}{17}} = \frac{17}{2} = \underline{8.5}$

c. For $V(U)$, we know it's a geometric sequence so we want $\frac{1-p}{p^2}$ where $p = \frac{2}{17}$. So,
$\frac{1-\frac{2}{17}}{(\frac{2}{17})^2} = \frac{\frac{15}{17}}{\frac{4}{289}} = 63.75 / 8.5 = \underline{7.5}$

6.a. W is binomial, X is geometric, Y is negative binomial.

b. $E(W) = 60$

- $\frac{1}{6}$ chance for 3 each roll
360 rolls, $\frac{360}{60} = 60$ expected 3s

$V(W) = 50$

- Stand D is $\sqrt{np(1-p)}$, $n = 360$ $p = \frac{1}{6}$
so $\sqrt{(360 \cdot \frac{1}{6})(\frac{5}{6})} = 7.071$
7.071^2 for var is 50

c. $E(X) = 6$

Since 3 has a $\frac{1}{6}$ chance of appearing, it's expected to come every 6th turn.

$V(X) = 30$

Geometric so $\frac{1-p}{p^2}$, $p = \frac{1}{6}$
$\frac{1-\frac{1}{6}}{\frac{1}{6}^2} = \frac{\frac{5}{6}}{\frac{1}{36}} = 30$

d. $E(Y) = 24$

3 expected to come every 6th turn since $\frac{1}{6}$ chance, so 4th time is $6 \times 4 = 24$th roll.

$V(Y) = 120$

$\frac{r(1-p)}{p^2}$, $r = 4$ $p = \frac{1}{6}$

$\frac{4(1-\frac{1}{6})}{\frac{1}{6}^2} = \frac{4(\frac{5}{6})}{\frac{1}{36}} = \frac{\frac{20}{6}}{\frac{1}{36}} = 120$

7. $P[46 \leq X - \mu \leq 74]$ $np = 60$ (subtract 60 on both sides)
$P[-14 \leq X - \mu \leq 14] \rightarrow P[|X - \mu| \leq 14]$
As shown in prob. 6b., $np(1-p)$ where $n = 360$, $p = \frac{1}{6}$ means: $\sigma = \sqrt{50}$, $\sigma^2 = 50$. So:
$P[|X - \mu| < \frac{14\sigma}{\sqrt{50}}]$
So, by Tchebycheff's Inequility:

$P[|X-\mu| \geq \frac{140}{\sqrt{50}}] \leq \left(\frac{\sqrt{50}}{14}\right)^2 \rightarrow \frac{50}{196}$, So,
$P[46 \leq X-\mu \leq 74] = 1 - P[X \notin (46, 74)] \geq 1 - \frac{50}{196} = \frac{146}{196} \approx .749$

8. $Pr[W \geq c\,Ex[W]] \leq e^{-B(c)Ex[W]}$
From prob. 6b. we know $Ex[W] = 60$ because the 360 rolls with $\frac{1}{6}$ each roll for a 3 is 60 expected value.
Going from 60, want less than 75, so $c = 1.25$
$(60 \times 1.25 = 75)$
$B(c) = c\ln(c) - c + 1$
$B(1.25) = 1.25\ln(1.25) - 1.25 + 1$
$Pr[W \geq (1.25)(60)] \leq e^{-(.029)(60)}$; $B(c) = .029$
$Pr[W \geq 75] \leq e^{-1.77}$

9.i. We can use the equation $a_n = 2a_{n-1} + a_{n-2}$:
First, multiply both sides by x^n, so:
$a_n x^n = 2a_{n-1}x^n + a_{n-2}x^n$
For $n \geq 2$, we can put it in sums as:
$$\sum_{n=2}^{\infty} a_n x^n = 2\sum_{n=2}^{\infty} a_{n-1}x^n + \sum_{n=2}^{\infty} a_{n-2}x^n$$

1st term: $\sum_{n=2}^{\infty} a_n x^n = a_2 z^2 + a_3 z^3 + \ldots$
$\rightarrow f(x) - a_1 x - a_0$
2nd term: $2\sum_{n=2}^{\infty} a_{n-1}x^n = a_1 z^2 + a_2 z^3 + \ldots$
$\rightarrow 2x(f(x) - a_0)$
3rd term: $\sum_{n=2}^{\infty} a_{n-2}x^n = a_0 z^2 + a_1 z^3 + \ldots$
$\rightarrow (f(x))x^2$

$f(x) - a_1 x - a_0 = 2x(f(x) - a_0) - x^2 f(x)$
Knowing $a_0 = 5$ and $a_1 = 7$, we can rearrange the equation and solve.

$f(x)(x^2 - 2x + 1) = a_0 + a_1 x - 2a_0 x$

$5 + 7x - 2(5)x$

$5 + 7x - 10x$

$f(x)(x^2 - 2x + 1) = 5 - 3x$

$f(x) = \dfrac{5 - 3x}{(x^2 - 2x + 1)}$

ii. $a_0 = 5 \quad a_1 = 7 \qquad a_n = 2a_{n-1} - a_{n-2}$

$a_2 = 9 \qquad a_2 = 2a_1 - a_0 = 2(7) - 5 = 9$

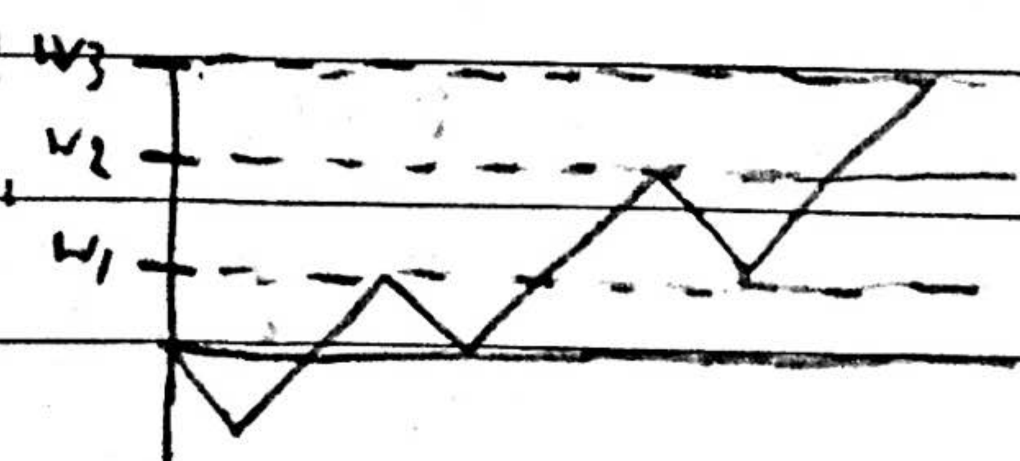
$a_3 = 11 \qquad a_3 = 2a_2 - a_1 = 2(9) - 7 = 11$

The pattern is 5, 7, 9, 11. Starts at 5 and goes up 2 each time. Therefore:

$\sum_{n=0}^{\infty} 2n + 5$

10.a. W_1, X_2, X_3: W_2 and W_3 need W_1 to happen in order for them to happen. W_1 can happen regardless if/when W_2 and W_3 happen. X_2 and X_3 are equations that happen at the end of the sequence regardless of their values.

b. W_2 is dependent on W_1, W_3 is dependent on W_1, W_3 is dependent on W_2.
For W_2 to hit, must hit W_1.
For W_3 to hit, must hit W_1 and W_2.



c. $W_1 = \dfrac{1p + 3p + 5p + \ldots}{p}$ W_1 can only happen on odd turns (1, 3, 5...)

$W_2 = \dfrac{2p + 4p + 6p + \ldots}{p}$ W_2 can only happen on even turns (2, 4, 6...)

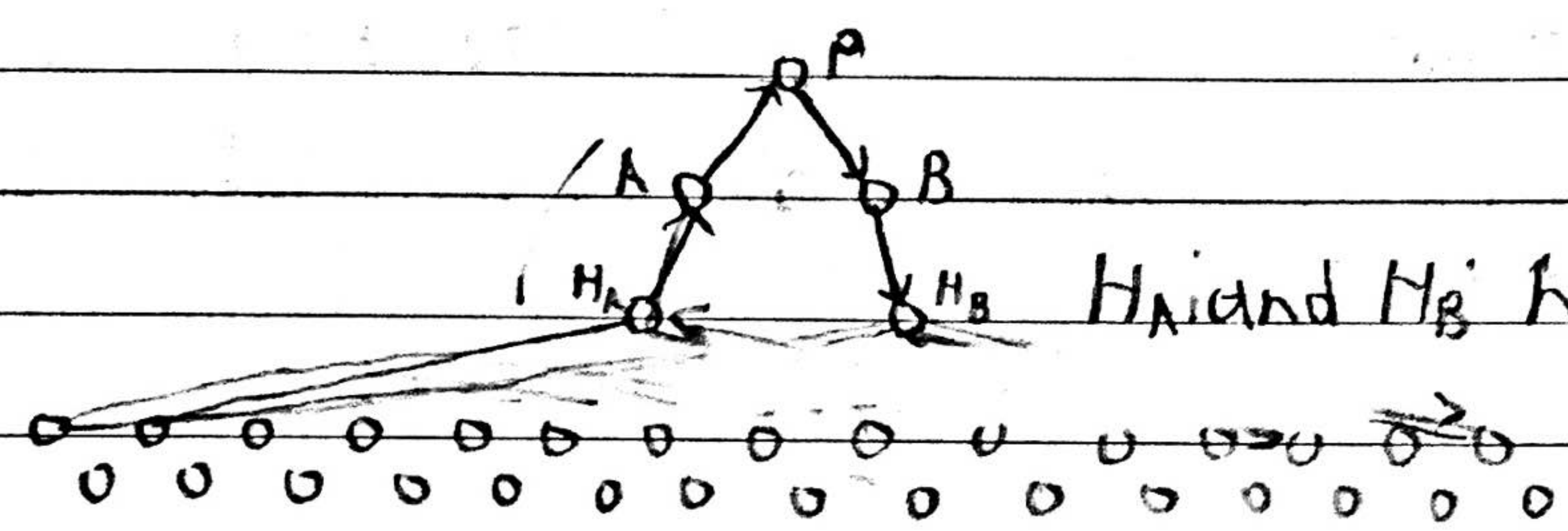
$W_3 = \frac{3p + 5p + 7p + \dots}{p}$ W_3 can only happen on odd turns starting at 3.

$X_2 = 0$ is the product of $W_2 - W_1$

$X_2 = 1$ is the product of $W_3 - W_2$

d. Yes, it absolute was important. If $p = 1$, where the coin is always heads, then $W_1 = 1$, $W_2 = 2$, and $W_3 = 3$.

11.



H_A and H_B have 31 connections each

(2 connections each)

Yes, with 30 centers, you can go from $H_A \to A \to P \to B \to H_B$ then traverse through each center and end up back at H_B. (30 each, even num)

No, with 29 centers, you can go from $H_A \to A \to P \to B \to H_B$ and when you go through each center, you'll end up stuck at H_A. (29 each, odd num)

Yes, if you start at P and go $P \to B \to H_B$ and traverse through the centers, you end at the H_A. Then go $H_A \to A \to P$ for full one-time traversal.

12.

12. $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ $\quad x = 3x \quad y = y \quad n = 24$

$(3x + 2y)^{24} = \sum_{k=0}^{24} \binom{24}{k} (3x)^{24-k} (y)^k$

We need to have $x^9 y^{15}$ so $k = 15$. Then:

$(3x + 2y)^{24} = \sum_{k=0}^{n} \binom{24}{15} (3x)^9 (y)^{15}$

$= \binom{24}{15} \cdot 3^9 \cdot x^9 \cdot y^{15}$

So, the coefficient of $x^9 y^{15}$ in $(3x + 2y)^{24}$ is $\binom{24}{15} \cdot 3^9$.

13. $(x+y)^n \rightarrow$ total monomial is ${}^{(n+r-1)}C_n$ where $n=24$ and $r=2$ (for number of terms). So,

$n+r-1 \rightarrow 24+2-1=25 \rightarrow {}^{25}C_{24}$

$\frac{25!}{24!\,1!} = \frac{25 \cdot 24!}{24! \cdot 1} = 25$ monomials

14. There are 12 letters total and 6 unique letters. Technically, there are 12! ways to rearrange, however when similar letters change spots, the string doesn't change. So it's the total arrangement over the letter changes that makes the same string.

In FIDDLEFADDLE

$F=2 \quad D=4 \quad L=2 \quad E=2$

So,

$\frac{12!}{(2!)(4!)(2!)(2!)} = \frac{479{,}001{,}600}{2\cdot4\cdot3\cdot2\cdot2\cdot2} = \frac{479{,}001{,}600}{192} =$

2,494,800 different strings

15. With 3 people each with 6 choices, there's $6\cdot6\cdot6=216$ combos. N is total distinct floors chosen.

N=1 (all 3 go on 1 floor)

${}^6C_1 = 6$ (out of 216 combos)

N=2 (1 on own + 2 together)

${}^6C_2 \times [{}^3C_1 + {}^3C_1]$

$\downarrow \quad [3+3] = 6 \rightarrow 15 \times 6 = 90$

$\frac{6!}{2!\,4!} = \frac{6\cdot5\cdot4!}{2\cdot1\cdot4!} = \frac{30}{2} = 15$

N=3 (all on own floor)

${}^6C_3 \times 3! \quad \frac{6!}{3!\,3!} = \frac{6\cdot5\cdot4\cdot3!}{3\cdot2\cdot1\cdot3!} = \frac{120}{6} = 20 \times (3\cdot2\cdot1)$

$=120$

$E(N) = \sum xp$ where $x = N$. We put this over the entire possibility of combos (216). So:

$$\frac{(1)(6) + (2)(90) + 3(120)}{216} = \frac{6 + 180 + 360}{216} =$$

$$\frac{546}{216} = \underline{2.528}$$

E.C. Covariance $V(N) = \sum x^2 p - \mu$