Assignment 5

1. The probability of x winning a game is $p$, so to win 4 consecutive games is $p \cdot p \cdot p \cdot p$, or $p^4$.

2. The prob of x winning series in at most 5 games is the prob of x winning in 4 games + x winning in 5 games. We know the first part is $p^4$, so for 5 games:

x must win game 5, choose 1 win for y in first 4 games $\binom{4}{1}$ multiplied by the probabity of that one win happening $(1-p)$ multiplied by the probability of the other 3 wins $(p^3)$ multiplied by the probability of x winning game 5 $(p)$. So:

$$p^4 + \binom{4}{1} p^3 (1-p) p \qquad \binom{4}{1} \rightarrow \frac{4!}{3!} \rightarrow \frac{4 \cdot \cancel{3!}}{\cancel{3!}} = 4$$

$$\downarrow$$

$$p^4 + 4p^3(1-p)p$$

3. Same thought process used in prob 2 with 5 games is used for 6 and 7 games. For 6 games, y must win 2 of first 5 and x wins the others and game 6. So:

$$\binom{5}{2} p^3 (1-p)^2 p \qquad \binom{5}{2} \rightarrow \frac{5!}{2!\,3!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{2 \cdot \cancel{3!}} = \frac{20}{2} = 10$$

$$\downarrow$$

$$10p^3(1-p)^2 p$$

For game 7, y must win 3 of first 6 and y wins the others and game 7. So:

$$\binom{6}{3} p^3 (1-p)^3 p \qquad \binom{6}{3} \rightarrow \frac{6!}{3!\,3!} = \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!} \cdot 3 \cdot 2} = 20$$

$$\downarrow$$

$$20p^3(1-p)^3 p$$

Now we put the prob of x winning the series in 4, 5, 6, and 7 games together. So:

$$p^4 + \binom{4}{1}p^3(1-p)p + \binom{5}{2}p^3(1-p)^2p + \binom{6}{3}p^3(1-p)^3p$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$p^4 + 4p^3(1-p)p + 10p^3(1-p)^2p + 20p^3(1-p)^3p$$

4. We can simply use our answer from prob 3. So:

$p = \frac{1}{2} \rightarrow \frac{1}{16} + 4(\frac{1}{8})(\frac{1}{2})(\frac{1}{2}) + 10(\frac{1}{8})(\frac{1}{4})(\frac{1}{2}) + 20(\frac{1}{8})(\frac{1}{8})(\frac{1}{2})$

$\frac{1}{16} + \frac{1}{8} + .15625 + .15625$

$\frac{1}{2}$ prob x wins series

$p = \frac{2}{3} \rightarrow \frac{16}{81} + 4(\frac{8}{27})(\frac{1}{3})(\frac{2}{3}) + 10(\frac{8}{27})(\frac{1}{9})(\frac{2}{3}) + 20(\frac{8}{27})(\frac{1}{27})(\frac{2}{3})$

$.1975 + .2634 + .2195 + .1463$

$.8267$ prob x wins series

5. Range of $X = \{4, 5, 6, 7\} \rightarrow 7 - 4 = 3$

6. For $X = 7$, we must have both teams to win 3 games in a pool of 6 then win game 7. So, each team having an equal chance of winning, we get:

$\binom{6}{3}(\frac{1}{2})^6(\frac{1}{2}) + \binom{6}{3}(\frac{1}{2})^6(\frac{1}{2})$ $\qquad \binom{6}{3} = \frac{6!}{3!3!} = \frac{6\cdot5\cdot4\cdot\cancel{3!}}{\cancel{3!}\cdot3\cdot2} = 20$

$20(\frac{1}{64})(\frac{1}{2}) + (20)(\frac{1}{64})(\frac{1}{2})$

$.15625 + .15625$

$.3125$ chance $X = 7$

7. $X \geq 6$ is $X = 6 + X = 7$, the latter we have. For $X = 6$, team must win 3 of first 5 then win 6th. So:

$\binom{5}{3}(\frac{1}{2})^5(\frac{1}{2}) + \binom{5}{3}(\frac{1}{2})^5(\frac{1}{2})$ $\qquad \binom{5}{3} = \frac{5!}{3!2!} = \frac{5\cdot4\cdot\cancel{3!}}{\cancel{3!}\cdot2} = 10$

$(10)(\frac{1}{32})(\frac{1}{2}) + (10)(\frac{1}{32})(\frac{1}{2})$

$.15625 + .15625 = .3125$ chance $X = 6$. So, for $Pr[X \geq 6]$, it's $.3125 + .3125 = .625$ chance $X \geq 6$

8. The total possible outcomes are $\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2)\ldots(6,5),(6,6)\}$ with a total of 36 different scenerios.

a. $X = \{2,3,4,5,6,7,8,9,10,11,12\} \Rightarrow 12 - 2 = 10$
$Y = \{1,2,3,4,5,6\} \rightarrow 6 - 1 = 5$
$Z \rightarrow \{(1-1),(2-1),(3-1),(4-1),(5-1),(6-1)\}$
$Z = \{0,1,2,3,4,5\} \rightarrow 5 - 0 = 5$
$W \rightarrow$ Multiply each element of X with each element of Z.
$W = \{0,2,3,4,5,6,7,8,9,10,11,12,14,16,18,20,21,24,15,18,21,27,30,33,36,28,32,40,44,48,25,35,45,50,55,60\} \rightarrow 60 - 0 = 60$

b. Portition of $X = (2,3,4,5,6,7,8,9,10,11,12)$
Portion of $Z = (0,1)$

c. $X = 7$ when $(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$, which is $\frac{6}{36}$ or $\frac{1}{6}$ prob.
$Z = 1$ when $(2-1),(3-2),(4-3),(5-4),(6-5)$ and there reverse since absolutes, so $\frac{10}{36} = \frac{5}{18}$

Check $P(A \cap B) = P(A)P(B)$, to check independence.
$P(A \cap B) \rightarrow (3,4),(4,3) = \frac{2}{36} = \frac{1}{18}$
$\frac{1}{18} = \left(\frac{1}{6}\right)\left(\frac{5}{18}\right)$
$\frac{1}{18} \neq \frac{5}{108}$
Therefore, they are not independent.

9.a. The sample spaces of 3 hats for 2 people is:
$\{(H_1, H_2), (H_1, H_3), (H_2, H_1), (H_2, H_3), (H_3, H_1), (H_3, H_2)\}$
Since X is indicator for neither man getting their own hat, it can only be 0 or 1 so:
$Range(X) = \{0, 1\} = 1 - 0 = 1$
Y is simply didn't get their own hat, so it's either neither, one did, or both, meaning 0, 1, 2 so:
$Range(Y) = \{0, 1, 2\} = 2 - 0 = 2$

b. Partition of X for $X = 0 \rightarrow \{(H_2, H_3), (H_1, H_2), (H_1, H_3)\}$
$X = 1 \rightarrow \{(H_2, H_1), (H_2, H_3), (H_3, H_1)\}$
Partion of Y for $Y = 0 \rightarrow \{(H_1, H_2)\}$
$Y = 1 \rightarrow \{(H_1, H_3), (H_3, H_2)\}$
$Y = 2 \rightarrow \{(H_2, H_1), (H_3, H_1), (H_2, H_3)\}$

c. We can check independence with $Pr(X \cap Y) = P(X)P(Y)$ when $X = 0$ and $Y = 0$.
Using the partitions, $X = 0$ has 3 sample spaces so $\frac{3}{6} = \frac{1}{2}$.
$Y = 0$ has 1 sample space so $\frac{1}{6}$
For $P(X \cap Y)$, only $(H_1, H_2)$ applies so $\frac{1}{6}$. So:
$\frac{1}{6} = (\frac{1}{2})(\frac{1}{6})$
$\frac{1}{6} \neq \frac{1}{12}$
Therefore, X and Y are not independent.

10. 2 tails has a $\frac{1}{4}$ prob $(\frac{1}{2} \cdot \frac{1}{2})$. Since S is the number of successes, N is the faction of S over 12 and the prob of each. S is $\frac{1}{4}$, we can say
12 tries $\times$ $\frac{1}{4}$ will be 2 tails $= 3S$, so:
$$E(N) = \frac{E(S)}{12} = \frac{3}{12} = \frac{1}{4}$$

EC. Prob of 2 heads in a row is $\frac{1}{4}$, prob of not getting 2 heads on turn is $\frac{3}{4}$. So:

Player A:

$$P(A) = \frac{1}{4} + \left(\frac{3}{4}\right)^3 \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^6 \cdot \frac{1}{4} + \dots$$
$$= \frac{1}{4}\left[1 + \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^6 + \dots\right]$$
$$= \frac{1}{4}\left[1 + \left(\frac{3}{4}\right)^3 + \left(\left(\frac{3}{4}\right)^3\right)^2 + \dots\right]$$
$$= \frac{1}{4}\left[1 - \left(\frac{3}{4}\right)^3\right]^{-1} = \frac{1}{4}\left(\frac{4^3 - 3^3}{4^3}\right)^{-1} = \frac{1}{4}\left(\frac{64}{64-27}\right) = \frac{16}{37}$$

Player B:

$$P(B) = \left(\frac{3}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^7\left(\frac{1}{4}\right) + \dots$$
$$= \left(\frac{3}{4} \cdot \frac{1}{4}\left[1 + \left(\frac{3}{4}\right)^3 + \left(\left(\frac{3}{4}\right)^3\right)^2 + \dots\right]\right.$$
$$= \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{64}{64-27} = \frac{12}{37}$$

Player C:

Probs must add up to be 1 so:

$$1 - \underset{P(A)}{\frac{16}{37}} - \underset{P(B)}{\frac{12}{37}} = \frac{9}{27}$$