MASTER'S DEGREE IN AERONAUTICAL ENGINEERING

Navier Stokes resolution with the fractional step method (FSM)

COMPUTATIONAL ENGINEERING

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1 Introduction

The lid-driven cavity problem is a classic test case in CFD, characterized by a square cavity where the top lid moves with a constant horizontal velocity while the remaining walls remain stationary. This setup induces a shear-driven flow that develops primary and secondary vortices as the Reynolds number increases. The problem is widely used to evaluate the accuracy and stability of numerical algorithms for incompressible flow solutions.

Despite its simple geometry, the lid-driven cavity problem presents challenges such as corner singularities and complex flow structures at higher Reynolds numbers. These features make it an excellent benchmark for testing numerical solvers like the Fractional Step Method

2 Fractional Step Method Basics

2.1 Navier-Stokes

The two-dimensional incompressible flow is governed by the Navier-Stokes equations:

$$\nabla \cdot \mathbf{u} = 0 \quad \text{(Continuity equation)} \tag{1}$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla p + \mu \nabla^2 \mathbf{u} \quad \text{(Momentum equation)} \tag{2}$$

where $\mathbf{u} = (u, v)$ represents the velocity field, p is the pressure, ρ is the fluid density, and μ is the dynamic viscosity. The time integration of the equations yields:

$$\nabla \cdot \mathbf{u}^{n+1} = 0 \tag{3}$$

$$\rho \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} = \frac{3}{2}\mathbf{R}(\mathbf{u}^n) - \frac{1}{2}\mathbf{R}(\mathbf{u}^{n-1}) - \nabla p^{n+1} \tag{4}$$

where the term R has been defined as:

$$\mathbf{R}(\mathbf{u}) = -(\rho \mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \cdot (\mu \nabla \mathbf{u}) \tag{5}$$

From here on, the FSM method needs to be applied in order to obtain a solution to the equations.

2.2 Predictor velocity

Firstly, by applying the Helmholtz-Hodge theorem to Equation 4, the equation can be divided into a pure gradient field and a divergence-free vector parallel to the domain. For our case, the divergence free field is represented by a pressure gradient ∇p^{n+1}, and a predictor velocity $\mathbf{u}^p$ that is defined as:

$$\rho \frac{\mathbf{u}^p - \mathbf{u}^n}{\Delta t} = \frac{3}{2}\mathbf{R}(\mathbf{u}^n) - \frac{1}{2}\mathbf{R}(\mathbf{u}^{n-1}). \tag{6}$$

That can be further developed as:

$$\mathbf{u}^p = \mathbf{u}^n + \frac{\Delta t}{\rho}\left(\frac{3}{2}\mathbf{R}(\mathbf{u}^n) - \frac{1}{2}\mathbf{R}(\mathbf{u}^{n-1})\right). \tag{7}$$

However, this velocity field does not necessarily satisfy the incompressibility condition.

2.3 Velocity correction

To ensure $\nabla \cdot \mathbf{u}^{n+1} = 0$, and therefore a correct $\mathbf{u}^p$ value, the pressure Poisson equation is derived from the incompressibility condition:

$$\Delta p^{n+1} = \frac{\rho}{\Delta t}\nabla \cdot \mathbf{u}^p. \tag{8}$$

The final velocity field $\mathbf{u}^{n+1}$ is updated using the corrected pressure:

$$\mathbf{u}^{n+1} = \mathbf{u}^p - \frac{\Delta t}{\rho}\nabla p^{n+1}. \tag{9}$$

These steps are repeated iteratively until a steady-state solution is achieved.

3 Problem definition

The problem of the lid-driven cavity is defined by a square domain (See Figure 1) where the upper wall moves at a constant velocity, and it induces movement to the fluid due to shear stresses. The rest of the walls are static, wit no slip conditions.

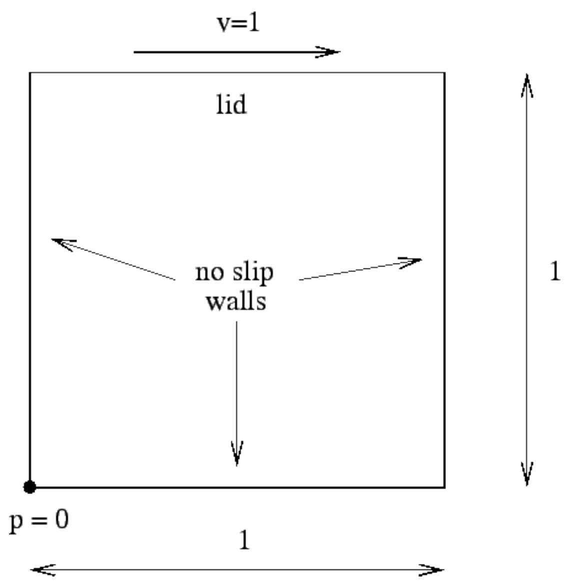

Figure 1. Lid driven cavity problem diagram. Extracted from [1]

3.1 Considerations

For this problem, a few assumptions will be made:

- Low Re number
- Incompressible flow and constant density ρ

3.2 Boundary conditions

Mathematically, the boundary conditions can be specified as:

- Top lid: $\mathbf{u} = (U, 0)$, where U is a constant.
- Bottom and side walls: $\mathbf{u} = (0, 0)$ (no-slip condition).

4 Numerical implementation

4.1 Staggered mesh arrangement

To avoid the checkerboard pattern problem in pressure, a staggered mesh arrangement is employed, where pressure points are located at cell centers, and velocity components are defined at the cell faces, and therefore, control volumes are centered around the faces instead of the nodes.

As for nomenclature, for any volume, either a cell or control volume, refers to the surrounding elements with the cardinal letters and based on the diagram seen in Figure 2

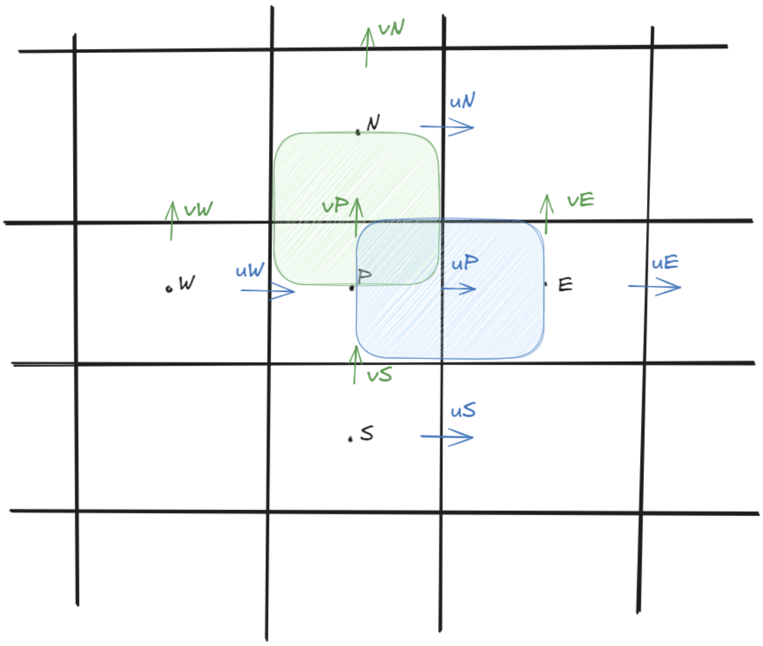

Figure 2. Staggered mesh diagram.

4.2 Convective-diffusion term (R)

Firstly, the term $\mathbf{R}(\mathbf{u^n})$ includes the convective term $-\rho\mathbf{u^n}\cdot\nabla\mathbf{u^n}$ and the diffusive term $\nabla\cdot(\mu\nabla\mathbf{u^n})$, needs to be integrated over control volumes. For structured meshes, the evaluation uses the Gauss theorem for surface integrals, allowing a transformation into surface fluxes around the control volume. Moreover, by considering staggered meshes, it is necessary to consider the calculation in both X and Y directions.

4.2.1 R in X direction

Applying the Gauss theorem, one can obtain the discretized form of the equations:

$$\mathbf{R}_{\mathbf{\Omega_{xP}}}(\mathbf{u^n}) = -\int_{\delta\Omega_x} \mathbf{n}\cdot(\rho\mathbf{u}u)dS + \int_{\delta\Omega_x} \mathbf{n}\cdot(\mu\nabla u)dS \tag{10}$$

$$\mathbf{R}_{\mathbf{\Omega_{xP}}} = -[\dot{m}_e u_e - \dot{m}_w u_w + \dot{m}_n u_n - \dot{m}_s u_s] + \left[\mu_e\frac{u_E - u_P}{d_{EP}}A_e - \mu_w\frac{u_P - u_W}{d_{WP}}A_w + \mu_n\frac{u_N - u_P}{d_{NP}}A_n - \mu_s\frac{u_P - u_S}{d_{SP}}A_s\right] \tag{11}$$

Then, by using appropriate numerical schemes, one can find the values of the velocities at the faces of the faces of the cells. If using a CDS, scheme, this would come out as:

$$u_e = CDS(u_P, u_E) \tag{12}$$
$$u_w = CDS(u_P, u_W) \tag{13}$$
$$u_n = CDS(u_P, u_N) \tag{14}$$
$$u_s = CDS(u_P, u_S) \tag{15}$$

And also the values of the mass fluxes, where x and y fluxes are accounted differently due to the mesh staggering. For the x components, the mass flow only depends on the value of the velocity at that face, and is found by averaging the mass flows:

$$\dot{m}_e = \rho \frac{u_E + u_P}{2} A_e \tag{16}$$
$$\dot{m}_w = \rho \frac{u_W + u_P}{2} A_w \tag{17}$$

And for the y mass flows, the staggered mesh (See Figure 3) makes the mass flow dependent on the velocities of two face velocities and their corresponding areas.

$$\dot{m}_n = \rho(v_P A_{Pn} + v_E A_{En}) \tag{18}$$
$$\dot{m}_s = \rho(v_S A_{Pn} + v_{ES} A_{En}) \tag{19}$$

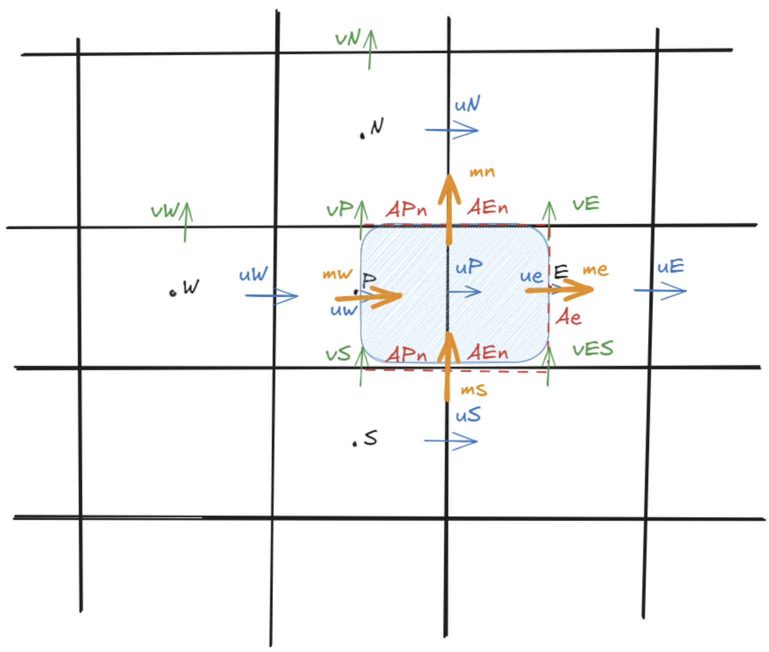

Figure 3. Velocities and mass flows for a staggered mesh in the X direction

4.2.2 R in Y direction

As for the y direction, the procedure would be very similar

$$\mathbf{R}_{\mathbf{\Omega_{yP}}} = -[\dot{m}_e v_e - \dot{m}_w v_w + \dot{m}_n v_n - \dot{m}_s v_s] + \left[\mu_e \frac{v_E - v_P}{d_{EP}} A_e - \mu_w \frac{v_P - v_W}{d_{WP}} A_w + \mu_n \frac{v_N - v_P}{d_{NP}} A_n - \mu_s \frac{v_P - v_S}{d_{SP}} A_s\right] \quad (20)$$

Yet again, if using a CDS, scheme, this would come out as:

$$v_e = CDS(v_P, v_E) \quad (21)$$

$$v_w = CDS(v_P, v_W) \quad (22)$$

$$v_n = CDS(v_P, v_N) \quad (23)$$

$$v_s = CDS(v_P, v_S) \quad (24)$$

And the mass fluxes can be obtained similarly:

$$\dot{m}_n = \rho \frac{v_N + v_P}{2} A_n \quad (25)$$

$$\dot{m}_s = \rho \frac{v_S + v_P}{2} A_s \quad (26)$$

$$\dot{m}_e = \rho(u_P A_{Pn} + u_N A_{Nn}) \quad (27)$$

$$\dot{m}_w = \rho(u_W A_{Pn} + u_{NW} A_{Nn}) \quad (28)$$

4.3 Predictor velocity

The intermediate or predictor velocity, denoted as $\mathbf{u}_p$, is computed by solving the momentum equations at each time step without ensuring incompressibility. It is derived as:

$$\mathbf{u}_p = \mathbf{u}^n + \frac{\Delta t}{\rho}\left[\frac{3}{2}R(\mathbf{u}^n) - \frac{1}{2}R(\mathbf{u}^{n-1})\right] \quad (29)$$

This step prepares the velocity field for the correction phase.

4.4 Pressure correction

The pressure correction step ensures the incompressibility condition $\nabla \cdot \mathbf{u}^{n+1} = 0$ by solving a Poisson equation for the pressure correction p.

$$\Delta p^{n+1} = \frac{\rho}{\Delta t} \nabla \cdot \mathbf{u}_p \quad (30)$$

Then the equation can be integrated considering Gauss theorem:

$$\int_\Omega \Delta p^{n+1}\, d\Omega = \frac{\rho}{\Delta t} \int_\Omega \nabla \cdot \mathbf{u}^p\, d\Omega \quad (31)$$

$$\int_{\partial\Omega} \nabla p^{n+1} \cdot \mathbf{n}\, dS = \frac{\rho}{\Delta t} \int_{\partial\Omega} \mathbf{u}^p \cdot \mathbf{n}\, dS \tag{32}$$

Finally, resulting in a discretized equation:

$$\frac{p_E^{n+1} - p_P^{n+1}}{d_{EP}} A_e - \frac{p_P^{n+1} - p_W^{n+1}}{d_{WP}} A_w + \frac{p_N^{n+1} - p_P^{n+1}}{d_{NP}} A_n - \frac{p_P^{n+1} - p_S^{n+1}}{d_{SP}} A_s = \frac{\rho}{\Delta t} \left[u_e^p A_e - u_w^p A_w + v_n^p A_n - v_s^p A_s \right] \tag{33}$$

This equation can be reorganized in coefficients:

$$a_P p_P^{n+1} = a_E p_E^{n+1} + a_W p_W^{n+1} + a_N p_N^{n+1} + a_S p_S^{n+1} + b_P \tag{34}$$

$$a_E = \frac{A_e}{d_{EP}} \tag{35}$$

$$a_N = \frac{A_n}{d_{NP}} \tag{36}$$

$$a_W = \frac{A_w}{d_{WP}} \tag{37}$$

$$a_S = \frac{A_s}{d_{SP}} \tag{38}$$

$$b_P = -\frac{\rho}{\Delta t} \left[u_e^p A_e - u_w^p A_w + v_n^p A_n - v_s^p A_s \right] \tag{39}$$

Which can be solved by applying a linear solver, such as Gauss-Seidel

$$p_P^{n+1}{}_{i+1} = \frac{a_E p_E^{n+1}{}_i + a_W p_W^{n+1}{}_i + a_N p_N^{n+1}{}_i + a_S p_S^{n+1}{}_i + b_P}{a_P} \tag{40}$$

and iterating until the error between solutions is below a convergence tolerance (ϵ)

$$\epsilon = |p_P^{n+1}{}_{i+1} - p_P^{n+1}{}_i| \tag{41}$$

4.5 Next velocity calculation

After obtaining the pressure correction p^{n+1}, the final velocity $\mathbf{u}^{n+1}$ is computed:

$$\mathbf{u}^{n+1} = \mathbf{u}_p - \frac{\Delta t}{\rho} \nabla p^{n+1} \tag{42}$$

This calculated pressure gradient and its relationship with the cells and the control volumes can be observed in Figure 4

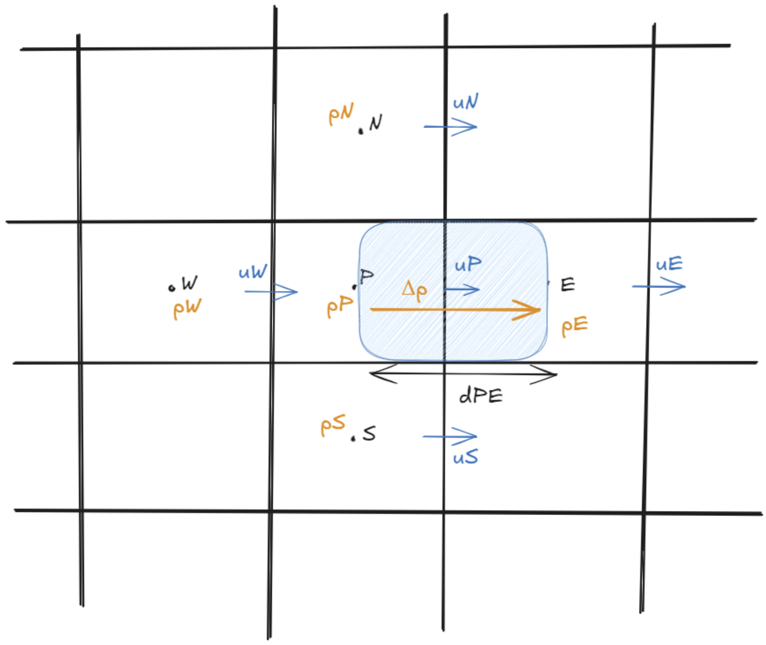

Figure 4. Relationship of the pressures with the staggered mesh in the x direction.

So that it can be divided in the x and y directions, yielding:

$$u_P^{n+1} = u_P^p - \frac{\Delta t}{\rho} \cdot \frac{p_E^{n+1} - p_P^{n+1}}{d_{EP}} \tag{43}$$

$$v_P^{n+1} = v_P^p - \frac{\Delta t}{\rho} \cdot \frac{p_N^{n+1} - p_P^{n+1}}{d_{NP}} \tag{44}$$

4.6 CFL condition

The CFL (Courant-Friedrichs-Lewy) condition is used to determine the time step size for stability for the next time:

$$\Delta t_c = \min\left(0.35 \frac{\Delta x}{|\mathbf{u^{n+1}}|}\right)$$

$$\Delta t_d = \min\left(0.20 \frac{\Delta x^2}{\mu/\rho}\right)$$

The time step is chosen as $\Delta t = \min(\Delta t_c, \Delta t_d)$.

4.7 Resolution algorithm

The FSM algorithm proceeds as follows:

1. Parameters input:

- Physical parameters
- Numerical parameters

2. While $t^n < t_{max}$

   (a) Iterate all nodes and compute $R(\mathbf{u}^n)$ in the staggered mesh velocity field. Check for boundary conditions.

   (b) Calculate predictor velocity: $\mathbf{u}_p = \mathbf{u}^n + \frac{\Delta t}{\rho}\left[\frac{3}{2}R(\mathbf{u}^n) - \frac{1}{2}R(\mathbf{u}^{n-1})\right]$.

   (c) Solve the Poisson equation for pressure correction: $\Delta p^{n+1} = \frac{\rho}{\Delta t}\nabla \cdot \mathbf{u}_p$.

   - Iterate all nodes and calculate next pressure $p_P^{n+1}{}_{i+1}$. Check for boundary conditions.
   - If error $\epsilon = |p^{n+1}{}_{i+1} - p^{n+1}{}_i| < tolerance \rightarrow$ Stop iterating
   - Update pressure field: $p_i^{n+1} = p_P^{n+1}{}_{i+1}$

   (d) Correct the velocity: $\mathbf{u}^{n+1} = \mathbf{u}_p - \frac{\Delta t}{\rho}\nabla p^{n+1}$.

   (e) Calculate time step: $\Delta t = \min(\Delta t_c, \Delta t_d)$.

   (f) Update variables : $\mathbf{u}^n = \mathbf{u}^{n+1} \quad \mathbf{p}^n = \mathbf{p}^{n+1} \quad \mathbf{t}^n = \Delta t + \mathbf{t}^n$

3. Final calculations: Any post-processing can now be made, as the velocity and pressure fields have been resolved.

4. Output and plotting: Output of the desired values and plotting of the data and fields.

References

[1] MIT. (n.d.). *Lid-driven cavity* [Accessed: 2024-11-18]. MIT CalculiX Documentation. https://web.mit.edu/calculix_v2.7/CalculiX/ccx_2.7/doc/ccx/node14.html

[2] Escola Superior d'Enginyeries Industrial, Aeronàutica i Audiovisual de Terrassa (ESEIAAT), C. (n.d.). Numerical methods in heat transfer and fluid dynamics [Lecture slides covering topics such as the Fractional Step Method, staggered and collocated meshes, and related computational exercises.]. *Universitat Politècnica de Catalunya (UPC)*.