$\sigma_{eq} = \sqrt{\frac{2}{3}\sigma_{ij}'\sigma_{ij}'}$

1. ( )

$[a,b,c] = \vec{a}\cdot(\vec{b}\times\vec{c})$

$$\begin{vmatrix} \delta_{or} & \delta_{os} & \delta_{ot} \\ \delta_{pr} & \delta_{ps} & \delta_{pt} \\ \delta_{qr} & \delta_{qs} & \delta_{qt} \end{vmatrix} = e_{opq}e_{rst}$$

$e_{ijk}e_{ist} = \delta_{js}\delta_{kt} - \delta_{jt}\delta_{sk}$

$e_{ijk}e_{ijt} = 2\delta_{kt}$

$e_{ijk}e_{ijk} = 6$

$\beta_{i'k} = \frac{\partial x_{i'}}{\partial x_k}$

$T_{m'n'} = \beta_{m'i}\beta_{n'j}T_{ij}$

$T_{mn} = \beta_{i'm}\beta_{j'n}T_{i'j'}$

球形张量、偏斜张量

$|\lambda I - T| = 0$

$\Rightarrow \lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$

$I_1 = \mathrm{tr}\,T$

I_2：所有二阶主子式之和（顺序主子式）

$I_3 = \det T$

$\frac{1}{2}(T_{ii}T_{jj} - T_{ij}T_{ji})$

$\int_V \nabla * \tilde{T}\,dV = \int_S \vec{\nu} * \tilde{T}\,dS$

$\int_S \vec{\nu}\cdot(\nabla\times\tilde{T})\,dS = \oint_L d\vec{l}\cdot\tilde{T}$

名义应力：$\vec{\sigma}_0 = \frac{\vec{F}}{A_0}$（工程应力）

柯西应力：$\vec{\sigma} = \frac{\vec{F}}{A}$

正应力 ~ 切应力/剪应力

柯西公式/斜面应力公式：

$\vec{\sigma}_{(\nu)} = \vec{\nu}\cdot\tilde{\sigma}$

应力分量转换公式/转轴公式

第三主应力：$\sigma_1 \geq \sigma_2 \geq \sigma_3$

（重根、则重根对应作用平面任意正交方向作主方向）

正应力之最大、最小值：σ_1，σ_3

剪应力的六个极值：$\pm\frac{1}{2}(\sigma_1 - \sigma_2)$　方向分别为对应主应力

$\tau_{(\nu)}$　　$\pm\frac{1}{2}(\sigma_2 - \sigma_3)$　方向之对角线

$\pm\frac{1}{2}(\sigma_1 - \sigma_3)$

八面体正应力 $\sigma_8 = \frac{1}{3}I_1$，剪应力 $|\tau_8| = \frac{2}{3}\sqrt{\tau_{(1)}^2 + \tau_{(2)}^2 + \tau_{(3)}^2} = \frac{\sqrt{2}}{3}\sigma_{eq}$

动量定理：$\rho\frac{d\vec{v}}{dt} = \rho\vec{f} + \nabla\cdot\tilde{\sigma} \xrightarrow{\text{平衡}} \rho\vec{f} + \nabla\cdot\tilde{\sigma} = 0$

柱坐标系：
$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta r}}{\partial\theta} + \frac{\partial\sigma_{zr}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + f_r = 0$$
$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\sigma_{z\theta}}{\partial z} + 2\frac{\sigma_{r\theta}}{r} + f_\theta = 0$$
$$\frac{\partial\sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + f_z = 0$$

$\varepsilon_{ij} = \frac{1}{2}(g_{ij} - g_{ij}^{0})$　工程剪应变 γ_{ij}　$\tilde{F} = \frac{\partial x_i}{\partial a_j}\vec{e}_i\vec{e}_j$

格林应变张量 $E_{ij} = \frac{1}{2}\left(\frac{\partial x_m}{\partial a_i}\frac{\partial x_m}{\partial a_j} - \delta_{ij}\right)$

$\tilde{E} = \frac{1}{2}(\vec{u}\nabla + \nabla\vec{u} + \vec{u}\nabla\cdot\nabla\vec{u})$　$(\varepsilon_{ij}\vec{e}_i^{\,0}\vec{e}_j^{\,0})$

阿尔曼西张量 $e_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial a_m}{\partial x_i}\frac{\partial a_m}{\partial x_j}\right)$

$\tilde{e} = \frac{1}{2}(\vec{u}\nabla + \nabla\vec{u} - \vec{u}\nabla\cdot\nabla\vec{u})$　$(\varepsilon_{ij}\vec{e}_i\vec{e}_j)$

对数应变：$\epsilon_{zz} = \ln\frac{L}{L_0}$

变形前后、长度比 $\lambda_\nu = \frac{ds}{ds_0} = \sqrt{1 + 2\vec{\nu}\cdot\tilde{E}\cdot\vec{\nu}}$

单位向量 $\lambda_\nu\vec{\nu}' = (\tilde{I} + \vec{u}\nabla)\cdot\vec{\nu} = \tilde{F}\cdot\vec{\nu}$

夹角：$\cos(\vec{\nu}', \vec{t}') = \frac{\vec{\nu}\cdot\vec{t} + 2\vec{\nu}\cdot\tilde{E}\cdot\vec{t}}{\lambda_\nu\lambda_t}$

线应变分量 $E_\nu = \frac{1}{2}(\lambda_\nu^2 - 1)$　$(\varepsilon_{ii} = l_i)$

切应变分量 $E_{\nu t} = \frac{1}{2}(\lambda_\nu\vec{\nu}')\cdot(\lambda_t\vec{t}')$　$(\varepsilon_{ij} = \frac{\pi}{4} - \frac{\psi_{ij}}{2})$

$\boxed{\text{小变形}}$：$\lambda_\nu = \vec{\nu}\cdot\tilde{\varepsilon}\cdot\vec{\nu} =: \varepsilon_\nu$，工程正应变，

$\gamma_{\nu t} = 2\vec{\nu}\cdot\tilde{\varepsilon}\cdot\vec{t} = 2\varepsilon_{\nu t}$　工程剪应变

转动张量 $\tilde{\Omega} = \frac{1}{2}(\nabla\vec{u} - \vec{u}\nabla)$　$\Omega_{ij} = e_{ijk}\omega_k$

$(-\tilde{\Omega}\cdot d\vec{x} = \vec{\omega}\times d\vec{x})$

协调方程：$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$

$R_{mn} = e_{mil}e_{njk}\varepsilon_{ij,kl} = 0$　　$\nabla\times\tilde{\varepsilon}\times\nabla = 0$

二维：$\varepsilon_{\alpha\alpha,\beta\beta} - \varepsilon_{\alpha\beta,\alpha\beta} = 0$

单连通域上，位移单值 $\Leftrightarrow$ 满足协调方程

由应变求位移

~~直接积分~~: eg: $d\left(\frac{\partial u_1}{\partial x_2}\right) = \frac{\partial \varepsilon_{11}}{\partial x_2}dx_1 + \left(\frac{\partial \gamma_{12}}{\partial x_2} - \frac{\partial \varepsilon_{22}}{\partial x_1}\right)dx_2 + \frac{1}{2}\left(\frac{\partial\gamma_{12}}{\partial x_3}+\frac{\partial \gamma_{31}}{\partial x_2}-\frac{\partial\gamma_{23}}{\partial x_1}\right)dx_3$

线积分

有直接积分: eg: $\varepsilon_{ij}=0$

$\Rightarrow u_1=f_1(x_2,x_3)$ $u_2=f_2(x_1,x_3)$ $u_3=f_3(x_1,x_2)$

$\Rightarrow \gamma_{12}=\cdots=0$ $\gamma_{13}=\cdots=0$ $\gamma_{23}=\cdots=0$ ①

分离变量、再求导、得: $\frac{\partial^2 f_1}{\partial x_2^2}=0$、$\frac{\partial^2 f_1}{\partial x_3^2}=0$

$\Rightarrow f_1 = g_1(x_3)x_2+g_0(x_3) \Rightarrow g_1''\cdot x_2+g_0''=0$

$\Rightarrow g_1''=g_0''=0 \Rightarrow f_1=a_0+a_1x_2+a_2x_3+a_3x_2x_3$

同理可得 u_2、u_3、

代回①、进一步简化: $f_1=a_0-b_2x_2+a_2x_3,\dots$

柱坐标系: $\varepsilon_r=\frac{\partial u}{\partial r}$ $\varepsilon_\theta=\frac{\partial v}{r\partial\theta}+\frac{u}{r}$ $\varepsilon_z=\frac{\partial w}{\partial z}$

$\gamma_{r\theta}=\frac{\partial u}{r\partial\theta}-\frac{v}{r}+\frac{\partial v}{\partial r}$ $\gamma_{\theta z}=\frac{\partial v}{\partial z}+\frac{\partial w}{r\partial\theta}$

$\gamma_{rz}=\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}$

协调方程: $\frac{\partial^2\varepsilon_r}{r^2\partial\theta^2}-\frac{\partial\varepsilon_r}{r\partial r}+\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\varepsilon_\theta)-\frac{1}{r^2}\frac{\partial^2}{\partial r\partial\theta}(r\gamma_{r\theta})=0$

$\varepsilon_x=\frac{1}{E}(\sigma_x-\nu\sigma_y-\nu\sigma_z)$ $\gamma_{ij}=\frac{1}{G}\tau_{ij}$ $G=\frac{E}{2(1+\nu)}$

$\varepsilon_{ij}=\frac{1+\nu}{E}\sigma_{ij}-\frac{\nu}{E}\sigma_{kk}\delta_{ij}$

令 $\theta=\varepsilon_{kk}$、$\Theta=\sigma_{kk}$、$K=\frac{E}{3(1-2\nu)}$、则:

$\theta=\frac{1}{3K}\Theta=\frac{1}{K}\sigma_0$ (体积应变 vs 静水应力)

令 $\lambda=\frac{\nu E}{(1+\nu)(1-2\nu)}$、则:

$\sigma_{ij}=2G\varepsilon_{ij}+\lambda\varepsilon_{kk}\delta_{ij}$ λ、G、第一二拉梅常数

$\Rightarrow \sigma_0=K\theta=3K\varepsilon_0$ $\sigma_{ij}'=2G\varepsilon_{ij}'$

一般形式: $\sigma_{ij}=C_{ijkl}\varepsilon_{kl}$ $W=\frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}=\frac{1}{2}\sigma_{kl}\varepsilon_{kl}$

应变能 $W=\int_0^{\tilde\varepsilon}\tilde\sigma:d\tilde\varepsilon$ $\frac{\partial W}{\partial\varepsilon_{ij}}=\sigma_{ij}$

$\Rightarrow$ 广义格林公式 $\frac{\partial\sigma_{kk}}{\partial\varepsilon}$ $\frac{\partial\sigma_{ij}}{\partial\varepsilon_{kl}}=\frac{\partial\sigma_{kl}}{\partial\varepsilon_{ij}}$

应变余能 $W_c=\int_0^{\tilde\sigma}\tilde\varepsilon:d\tilde\sigma$ $\frac{\partial W_c}{\partial\sigma_{ij}}=\varepsilon_{ij}\Rightarrow\frac{\partial\varepsilon_{kl}}{\partial\sigma_{ij}}=\frac{\partial\varepsilon_{ij}}{\partial\sigma_{kl}}$

且 $W+W_c=\sigma_{ij}\varepsilon_{ij}$

正定性: ~~$W=\frac{1}{2}K\varepsilon^2+G\varepsilon_{ij}\varepsilon_{ij}$~~

$W_c=\frac{1}{2K}$ $W=\frac{1}{2}K\varepsilon_{ii}^2+\frac{1}{2}\cdot2G\varepsilon_{ij}'\varepsilon_{ij}'$

$W_c=\frac{1}{2K}\sigma_0^2+\frac{1}{2\cdot2G}\sigma_{ij}'\sigma_{ij}'$

Lame-Navier 方程: $Gu_{i,jj}+(\lambda+G)u_{j,ji}+f_i=0(\rho\ddot{u}_i)$

Beltrami-Michell 方程:

$\nabla^2\sigma_{ij}+\frac{1}{1+\nu}\Theta_{,ij}=-(f_{i,j}+f_{j,i})-\frac{\nu}{1-\nu}f_{k,k}\delta_{ij}$

圣维南定理