UNIT - III

LONG ANSWER QUESTIONS:

1. Test the convergency of series

(a) $\sum_{n=1}^{\infty} \frac{1}{(\sqrt{n}+\sqrt{n+1})}$ (b) $\sum_{n=1}^{\infty} \sqrt[3]{n^3+1} - n$ (c) $\sum_{n=1}^{\infty} \left(\frac{2^n+3}{3^n+1}\right)^{1/2}$

A(a) Given $\Sigma \sqrt[3]{n^3+1} - n$

Let $U_n = \sqrt[3]{n^3+1} - n$

$$(n^3)^{1/3}\left[\left(1+\frac{1}{n^3}\right)^{1/3}\right] - n$$

$$= n\left[\left(1+\frac{1}{n^3}\right)^{1/3} - 1\right]$$

using,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

We expand,

$$n\left[1 + \frac{1}{3}\frac{1}{n^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{n^3}\right)^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}\left(\frac{1}{n^3}\right)^3 + \cdots\right]$$

$$= n\left[\frac{1}{3n^3} - \frac{1}{9}\frac{1}{(n^3)^2} + \frac{5}{81}\frac{1}{(n^3)^3} + \cdots\right]$$

$$\frac{n}{n^3}\left[\frac{1}{3} - \frac{1}{9n^3} + \frac{5}{81(n^3)^2} + \cdots\right]$$

$$\therefore U_n = \frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{9n^3} + \frac{5}{81(n^3)^2} + \cdots\right]$$

consider,

$V_n = \frac{1}{n^2}$

$\Sigma V_n = \Sigma \frac{1}{n^2}$ which is in p-series where $p=2$

By p-test, $p>1$ so v_n is convergent

Consider, $\lim\limits_{n\to\infty} \frac{u_n}{v_n}$

$$\lim_{n\to\infty} \frac{\cancel{\frac{1}{n^2}}\left[\frac{1}{3} - \frac{1}{9n^3} + \frac{5}{81(n^3)^2} + \cdots\right]}{\cancel{\frac{1}{n^2}}}$$

$$\lim_{n\to\infty} \left[\frac{1}{3} - \frac{1}{9n^3} + \frac{5}{81(n^3)^2} + \cdots\right]$$

$$\lim_{n\to\infty} \left[\frac{1}{3} - 0 + 0\right]$$

$$= \frac{1}{3} \neq 0 \text{ (finite)}$$

By limit comparsion test, ΣU_n and ΣV_n either converge or diverge together.

Since ΣV_n is convergent.

Thus ΣU_n is also convergent.

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2. Test the convergency of series $\frac{1}{4\cdot 7\cdot 10} + \frac{1}{7\cdot 10\cdot 13} + \frac{1}{10\cdot 13\cdot 16} + \cdots$

sol Given series is,

$$\frac{1}{4\cdot 7\cdot 10} + \frac{1}{7\cdot 10\cdot 13} + \frac{1}{10\cdot 13\cdot 16} + \cdots\cdots$$

4, 7, 10 .... are in AP

$a = 4,\ d = 3$

$a_n = a + (n-1)d$

$= 4 + (n-1)3$

$= 3n+1$

7, 10, 13 ... are in AP

$a = 7,\ d = 3$

$a_n = a + (n-1)d$

$= 7 + (n-1)3$

$= 3n+4$

10, 13, 16 ... are in AP

$a = 10,\ d = 3$

$a_n = a + (n-1)d$

$= 10 + (n-1)3$

$= 3n+7$

Series is,

$$\frac{1}{4\cdot 7\cdot 10} + \frac{1}{7\cdot 10\cdot 13} + \frac{1}{10\cdot 13\cdot 16} + \cdots + \frac{1}{(3n+1)(3n+4)(3n+7)}$$

$$U_n = \frac{1}{(3n+1)(3n+4)(3n+7)}$$

$$U_{n+1} = \frac{1}{[3(n+1)+1][3(n+1)+4][3(n+1)+7]}$$

$$= \frac{1}{(3n+4)(3n+7)(3n+10)}$$

Consider, $\frac{U_n}{U_{n+1}} = \frac{\frac{1}{(3n+1)(3n+4)(3n+7)}}{\frac{1}{(3n+4)(3n+7)(3n+10)}}$

$$= \frac{1}{(3n+1)\cancel{(3n+4)}\cancel{(3n+7)}} \times \frac{\cancel{(3n+4)}\cancel{(3n+7)}(3n+10)}{1}$$

$$= \frac{3n+10}{3n+1}$$

Consider,

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = l$$

$$\lim_{n\to\infty} \frac{3n+10}{3n+1}$$

$$= \frac{\cancel{n}\left(3+\frac{10}{n}\right)}{\cancel{n}\left(3+\frac{1}{n}\right)} = \frac{3+0}{3+0} = \frac{3}{3} = 1$$

Here $l = 1$ so, test fails.

Consider,

$$n\left[\frac{U_n}{U_{n+1}} - 1\right]$$

$$= n\left[\frac{3n+10}{3n+1} - 1\right]$$

$$= n\left[\frac{3n+10-3n-1}{3n+1}\right]$$

$$= n\left[\frac{9}{3n+1}\right] = \cancel{n}\left[\frac{9}{\cancel{n}\left(3+\frac{1}{n}\right)}\right] = \frac{9}{3+0} = \frac{\cancel{9}^{3}}{\cancel{3}} = 3$$

Co

Here $l = 3 > 1$

So By Raabe's test

The given series is convergent.

3. Test for the convergency of series.

(a) $x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot3}{2\cdot4}\frac{x^5}{5} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{x^7}{7} + \cdots$ (b) $\frac{x}{1\cdot2} + \frac{x^2}{2\cdot3} + \frac{x^3}{3\cdot4} + \cdots$

(c) $\sum_{n=1}^{\infty} \frac{x^{2n}}{(n+2)\sqrt{n+1}}$, $x > 0$

3(a) Given series is, $x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot3}{2\cdot4}\frac{x^5}{5} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{x^7}{7} + \cdots + \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)} \times \frac{x^{2n+1}}{2n+1}$

Let $U_n = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)} \times \frac{x^{2n+1}}{2n+1}$

$$U_{n+1} = \frac{1\cdot3\cdot5\cdots(2n-1)\,x^{2n+1}}{2\cdot4\cdot6\cdots(2n)(2n+1)} \times \frac{[2(n+1)-1]\,x^{2(n+1)+1}}{2(n+1)[2(n+1)+1]}$$

$$= \frac{1\cdot3\cdot5\cdots(2n-1)x^{2n+1}(2n+1)\,x^{2n+3}}{2\cdot4\cdot6\cdots(2n)(2n+1)(2n+2)(2n+3)}$$

Consider,

$$\frac{U_n}{U_{n+1}} = \frac{\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}\,\frac{x^{2n+1}}{2n+1}}{\frac{1\cdot3\cdots(2n)(2n+1)}{2\cdot4\cdot6\cdots(2n)(2n+2)}\,\frac{x^{2n+3}}{2n+3}}$$

$$= \frac{x^{2n+1}}{x^{2n+3}} \times \frac{2n+3}{2n+1}\left(\frac{2n+2}{2n+1}\right)$$

$$= \frac{1}{x^2}\,\frac{(2n+3)(2n+2)}{(2n+1)(2n+1)}$$

$$= \frac{1}{x^2}\,\frac{(2n+3)(2n+2)}{(2n+1)^2}$$

Consider,

$$\underset{n\to\infty}{Lt} \frac{u_n}{u_{n+1}}$$

$$\underset{n\to\infty}{Lt} \frac{1}{x^2} \frac{(2n+3)(2n+2)}{(2n+1)^2}$$

$$\frac{1}{x^2} \underset{n\to\infty}{Lt} \frac{\left(2+\frac{3}{n}\right)\left(2+\frac{2}{n}\right)}{\left(2+\frac{1}{n}\right)^2}$$

$$\frac{1}{x^2} \frac{(2+0)(2+0)}{(2+0)^2} = \frac{1}{x^2}$$

If,

$\frac{1}{x^2} > 1$ (or) $x^2 < 1$, Σu_n converges

$\frac{1}{x^2} < 1$ (or) $x^2 > 1$, Σu_n diverges

$\frac{1}{x^2} = 1$ (or) $x^2 = 1$, test fails.

Let $\frac{1}{x^2} = 1$ (or) $x^2 = 1$

By Raabe's test,

$$\underset{n\to\infty}{Lt}\, n\left[\frac{u_n}{u_{n+1}} - 1\right]$$

$$\underset{n\to\infty}{Lt}\, n\left[\frac{(2n+3)(2n+2)}{(2n+1)^2} - 1\right]$$

$$= n\left[\frac{4n^2+4n+6n+6-4n^2+1+4n}{4n^2+1+4n}\right]$$

$$= \left[\frac{6n^2+5n}{4n^2+4n+1}\right]$$

$$= \frac{\cancel{n^2}\left[6+\frac{5}{n}\right]}{\cancel{n^2}\left[4+\frac{4}{n}+\frac{1}{n^2}\right]} \quad \underset{n\to\infty}{Lt} \frac{6+\frac{5}{n}}{4+\frac{4}{n}+\frac{1}{n^2}} = \frac{6+0}{4+0+0}$$

$$= \frac{\cancel{6}^{3}}{\cancel{4}_{2}} = \frac{3}{2} = 1.5$$

$$l = 1.5 > 1$$

By Raabe's test,

The given series is convergent.

b) Given series is, $\frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \cdots\cdots + \frac{x^n}{n(n+1)}$

Let $U_n = \frac{x^n}{n(n+1)}$

$U_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)}$

consider,

$$\frac{U_n}{U_{n+1}} = \frac{\frac{x^n}{n(n+1)}}{\frac{x^{n+1}}{(n+1)(n+2)}} = \frac{\cancel{x^n}}{n\cancel{(n+1)}} \times \frac{\cancel{(n+1)}(n+2)}{\cancel{x^n}\cdot x}$$

$$= \frac{\cancel{n}\left[1+\frac{2}{n}\right]}{\cancel{n}\, x} = \frac{\left(1+\frac{2}{n}\right)}{x}$$

$$\lim_{n\to\infty} \frac{1}{x}\left(1+\frac{2}{n}\right)$$

$$\frac{1}{x} \lim_{n\to\infty}\left(1+\frac{2}{n}\right)$$

$$\frac{1}{x}(1+0) = \frac{1}{x}$$

If $\frac{1}{x} > 1$ (or) $x < 1$ ΣU_n converges.

If $\frac{1}{x} < 1$ or $x > 1$ ΣU_n diverges

If $x = 1$, test fails.

By Raabe's test,

Consider,

$$n\left[\frac{U_n}{U_{n+1}} - 1\right]$$

$$= n\left[\frac{n+2}{n} - 1\right] = n\left[\frac{n+2-n}{n}\right] = 2 > 1$$ Here $l > 1$

The given series is convergent

(c) Given series $\sum \frac{x^{2n}}{(n+2)\sqrt{n+1}}$, $x > 0$

Let $u_n = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$

$$u_{n+1} = \frac{x^{2(n+1)}}{(n+1+2)(\sqrt{n+1+1})} = \frac{x^{2n+2}}{(n+3)\sqrt{n+2}}$$

Consider,

$$\frac{u_n}{u_{n+1}} = \frac{\frac{x^{2n}}{(n+2)\sqrt{n+1}}}{\frac{x^{2n+2}}{(n+3)\sqrt{n+2}}} = \frac{\cancel{x^{2n}}}{(n+2)\sqrt{n+1}} \times \frac{\sqrt{n+2}\,(n+3)}{\cancel{x^{2n}}\cdot x^2}$$

$$= \frac{\sqrt{n+2}\;(n+3)}{\sqrt{n+1}\;(n+2)\,x^2}$$

Consider,

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}}$$

$$\lim_{n\to\infty} \frac{n\sqrt{n}\left(\sqrt{1+\frac{2}{n}}\;\left(1+\frac{3}{n}\right)\right)}{n\sqrt{n}\left[\sqrt{1+\frac{1}{n}}\left(1+\frac{2}{n}\right)x^2\right]}$$

$$= \frac{\sqrt{1+0}\;(1+0)}{\sqrt{1+0}\;(1+0)\,x^2} = \frac{1}{x^2}$$

By ratio test,

$\frac{1}{x^2} > 1$; $x^2 < 1$ $\sum u_n$ converges

$\frac{1}{x^2} < 1$; $x^2 > 1$ $\sum u_n$ diverges.

$\frac{1}{x^2} = 1$ i.e $x^2 = 1$, Test fails.

Consider,

$$= n\left[\frac{u_n}{u_{n+1}} - 1\right]$$

$$= n\left[\frac{\sqrt{n+2}\;(n+3)}{\sqrt{n+1}\;(n+2)} - 1\right]$$

$U_{n+1} = \frac{2\cdot5\cdot8\cdot11\cdots(3n-1)}{\ldots} \times \frac{3(n+1)-1}{\ldots}$

$$= n\left[\frac{\sqrt{n+2}\,(n+3) - \sqrt{n+1}\,(n+2)}{\sqrt{n+1}\,(n+2)}\right]$$

$$= \frac{n(n\sqrt{n})}{n\sqrt{n}}\left[\frac{\sqrt{1+\frac{2}{n}}\left(1+\frac{3}{n}\right) - \sqrt{1+\frac{1}{n}}\left(1+\frac{2}{n}\right)}{\sqrt{1+\frac{1}{n}}\left(1+\frac{2}{n}\right)}\right]$$

$$\lim_{n\to\infty} n\left[\frac{\sqrt{1+\frac{2}{n}}\left(1+\frac{3}{n}\right) - \sqrt{1+\frac{1}{n}}\left(1+\frac{2}{n}\right)}{\sqrt{1+\frac{1}{n}}\left(1+\frac{2}{n}\right)}\right]$$

$$= 0$$

Here $l < 0$

By Raabe's test

The given series is ~~convergent~~ divergent.

4. Test for convergence of series.

(a) $\frac{2}{1} + \frac{2\cdot5\cdot8}{1\cdot5\cdot9} + \frac{2\cdot5\cdot8\cdot11}{1\cdot5\cdot9\cdot13} + \cdots$ (b) $\sum_{n=1}^{\infty} \frac{1\cdot3\cdot5\cdots(2n+1)}{2\cdot5\cdot8\cdots(3n+2)}$

4(a) Sol: 2, 5, 8 ... are in AP

$a = 2,\ d = 3$

$a_n = a + (n-1)d$

$= 2 + (n-1)3$

$= 3n - 1$

4, 5, 9, 13 ... are in A.P

$a = 4,\ d = 4$

$a_n = a + (n-1)d$

$= 1 + (n-1)4$

$= 4n - 3$

Given series is,

$$\frac{2}{1} + \frac{2\cdot5\cdot8}{1\cdot5\cdot9} + \frac{2\cdot5\cdot8\cdot11}{1\cdot5\cdot9\cdot13} + \cdots + \frac{2\cdot5\cdot8\cdot11\cdots(3n-1)}{1\cdot5\cdot9\cdot13\cdots(4n-3)} + \cdots$$

Let $U_n = \frac{2\cdot5\cdot8\cdot11\cdots(3n-1)}{1\cdot5\cdot9\cdot13\cdots(4n-3)}$

$$U_{n+1} = \frac{2\cdot5\cdot8\cdot11\cdots(3n-1)}{1\cdot5\cdot9\cdot13\cdots(4n-3)} \times \frac{3(n+1)-1}{4(n+1)-3}$$

$$= \frac{2\cdot5\cdot8\cdot11\cdots(3n-1)(3n+2)}{1\cdot5\cdot9\cdot13\cdots(4n-3)(4n+1)}$$

Consider,

$$\frac{U_n}{U_{n+1}} = \frac{\dfrac{2\cdot5\cdot8\cdot11\cdots(3n-1)}{1\cdot5\cdot9\cdot13\cdots(4n-3)}}{\dfrac{2\cdot5\cdot8\cdot11\cdots(3n-1)(3n+2)}{1\cdot5\cdot9\cdot13\cdots(4n-3)(4n+1)}}$$

$$= \frac{\cancel{2\cdot5\cdot8\cdot11\cdots(3n-1)}}{\cancel{1\cdot5\cdot9\cdot13\cdots(4n-3)}} \times \frac{\cancel{1\cdot5\cdot9\cdot13\cdots(4n-3)}(4n+1)}{\cancel{2\cdot5\cdot8\cdot11\cdots(3n-1)}(3n+1)}$$

$$\frac{U_n}{U_{n+1}} = \frac{4n+1}{3n+2}$$

$$= \frac{\cancel{n}\left[4+\frac{1}{n}\right]}{\cancel{n}\left[3+\frac{2}{n}\right]}$$

Consider,

$$\underset{n\to\infty}{Lt} \frac{4+\frac{1}{n}}{3+\frac{2}{n}} = \frac{4+0}{3+0} = \frac{4}{3} > 1$$

$l > 1$;

so By Ratio test,

The given series is convergent.

(b) sol Given,

$$U_n = \frac{1\cdot3\cdot5\cdots(2n+1)}{2\cdot5\cdot8\cdots(3n+2)}$$

$$U_{n+1} = \frac{1\cdot3\cdot5\cdots(2n+1)}{2\cdot5\cdot8\cdots(3n+2)} \times \frac{2n+1}{3n+5}$$

Consider,

$$\underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = l$$

Now,

$$\frac{U_n}{U_{n+1}} = \frac{1\cdot3\cdot5\cdots(2n+1)}{2\cdot5\cdot8\cdots(3n+2)} \times \frac{2\cdot5\cdot8\cdots(3n+2)}{1\cdot3\cdot5\cdots(2n+1)} \times \frac{3n+5}{2n+3}$$

$$\underset{n\to\infty}{Lt} \frac{3n+5}{2n+3}$$

$$\underset{n\to\infty}{Lt} \frac{\cancel{n}\left[3+\frac{5}{n}\right]}{\cancel{n}\left[2+\frac{3}{n}\right]} = \frac{3+0}{2+0} = \frac{3}{2} = 1.5 > 1$$

By Ratio test,

The given series is convergent.

5. Test the convergence of series $\frac{\sqrt{2}-1}{3^2-1} + \frac{\sqrt{3}-1}{4^2-1} + \frac{\sqrt{4}-1}{5^2-1} + \cdots$

Sol Given series is,

$$\frac{\sqrt{2}-1}{3^2-1} + \frac{\sqrt{3}-1}{4^2-1} + \frac{\sqrt{4}-1}{5^2-1} + \cdots \frac{\sqrt{n+1}-1}{(n+2)^2-1}$$

Let $U_n = \frac{\sqrt{n+1}-1}{(n+2)^2-1}$

$$= \frac{\sqrt{n\left(1+\frac{1}{n}\right)}-1}{\left(n\left[1+\frac{2}{n}\right]\right)^2-1}$$

$$= \frac{\sqrt{n}\left[\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}\right]}{n^2\left[\left(1+\frac{2}{n}\right)-\frac{1}{n^2}\right]} = \frac{\left[\sqrt{1+\frac{1}{\sqrt{n}}}-\frac{1}{\sqrt{n}}\right]}{n^2 \cdot n^{-1/2}\left[\left(1+\frac{2}{n}\right)^2-\frac{1}{n^2}\right]}$$

$$U_n = \frac{\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}}{n^{3/2}\left[\left(1+\frac{2}{n}\right)^2 - \frac{1}{n^2}\right]}$$

$$V_n = \frac{1}{n^{3/2}}$$

$\Sigma U_n = \Sigma \frac{1}{n^{3/2}}$ is in p- series.

$p = \frac{3}{2} > 1$

By p-test, ΣV_n is convergent.

$$\lim_{n\to\infty} \frac{U_n}{V_n}$$

$$= \frac{\frac{\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}}{\cancel{n^{3/2}}\left[\left(1+\frac{2}{n}\right)^2 - \frac{1}{n^2}\right]}}{1/\cancel{n^{3/2}}}$$

$$\lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}}{\left[\left(1+\frac{2}{n}\right)^2 - \frac{1}{n^2}\right]} = \frac{\sqrt{1+0} - 0}{(1+0)^2 - 0} = \frac{1}{1} = 1 \neq 0 \text{ finite}$$

Therefore, By limit comparsion test, ΣU_n and ΣV_n either converge or diverge together.

since ΣV_n is convergent

ΣU_n is convergent

6. Test for absolute or conditional convergence of series.

$$\frac{1}{5\sqrt{2}} - \frac{1}{5\sqrt{3}} + \frac{1}{5\sqrt{4}} \cdots\cdots (-1)^n \frac{1}{5\sqrt{n}} \cdots$$

sol The given series,

$$\sum_{n=2}^{\infty} (-1)^n \cdot \frac{1}{5\sqrt{n}}$$ which is an alternating series

7. Test the convergence...

Let,

$U_n = \frac{1}{5\sqrt{n}}$, $U_{n+1} = \frac{1}{5\sqrt{n+1}}$

(i) Consider, $U_n - U_{n+1}$

$$\frac{1}{5\sqrt{n}} - \frac{1}{5\sqrt{n+1}}$$

$$= \frac{1}{5}\left[\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}\sqrt{n+1}}\right] > 0$$

since

$U_n - U_{n+1} > 0$

$U_n > U_{n+1}$

(ii) Consider,

$$\lim_{n\to\infty} U_n$$

$$\lim_{n\to\infty} \frac{1}{5\sqrt{n}} = 0$$

By Leibnitz's test,

Given series is convergent.

II) $\sum_{n=2}^{\infty} |U_n|$

$$\sum_{n=2}^{\infty} \frac{1}{5\sqrt{n}}$$

$U_n = \frac{1}{5\sqrt{n}}$

$V_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$

$\sum V_n = \sum \frac{1}{n^{1/2}}$

It is in p-series

$p = \frac{1}{2} < 1$

$\sum V_n$ diverges

$$\lim_{n\to\infty} \frac{U_n}{V_n}$$

$$= \frac{\frac{1}{5\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \frac{1}{5} \neq 0 \text{ (finite)}$$

By limit comparsion test,

$|U_n|$ is ~~convergent~~ divergent

U_n is convergent, $|U_n|$ is divergent

series is conditionally convergent.

7. Test the convergency of

(a) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}$ (b) $\frac{2}{1^2}x+\frac{3^2}{2^3}x^2+\cdots\frac{(n+1)^n}{n^{n+1}}x^n+\cdots\ (x>0)$

(a) Sol Given that,

$$\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}$$

$$U_n=\left(1+\frac{1}{n}\right)^{-n^2}$$

$$=\left(\frac{n+1}{n}\right)^{-n^2}$$

$$\therefore U_n=\left(\frac{n}{n+1}\right)^{n^2}$$

$$(U_n)^{1/n}=\left(\left(\frac{n}{n+1}\right)^{n^2}\right)^{1/n}$$

$$=\left(\frac{n}{n+1}\right)^{n}$$

$$=\frac{\cancel{n^n}(1)}{\cancel{n^n}\left(1+\frac{1}{n}\right)^n}=\frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty}(U_n)^{1/n}$$

$$\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}$$

Here $l<1$,

so By n^{th} root test

The given series is convergent.

9. Test the series for absolute / conditional convergence

(a) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^2}$

(c) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}$

9(a) Given series is,

I) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^2}$ which is alternating series

$U_n = \frac{1}{n(\log n)^2}$; $U_{n+1} = \frac{1}{(n+1)[\log(n+1)]^2}$

(i) Consider,

$U_n - U_{n+1}$

$\frac{1}{n(\log n)^2} - \frac{1}{(n+1)[\log(n+1)]^2}$

$= \frac{(n+1)[\log(n+1)]^2 - n(\log n)^2}{n(\log n)^2 (n+1)(\log(n+1))^2} > 0$

since $U_n - U_{n+1} > 0$

$U_n > U_{n+1}$

(ii) Consider,

$\lim_{n\to\infty} U_n$

$\lim_{n\to\infty} \frac{1}{n(\log n)^2} = 0$

By leibnitz's test alternating series is convergent.

II) $\sum_{n=2}^{\infty} |U_n|$

$\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^2}$

$= \frac{1}{n(\log n)^2}$

$f(n) = \frac{1}{n(\log n)^2}$ which is non-negative and decreasing function

consider,

$$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{1}{x(\log x)^2}\,dx$$

$$\int_2^\infty \left(\frac{1}{x}\right) \log x^{-2}\,dx$$

$$= \left[\frac{(\log x)^{-2+1}}{-2+1}\right]_2^\infty$$

$$= \left[(\log x)^{-1}\right]_2^\infty$$

$$= \left[\frac{1}{\infty} - \frac{1}{\log 2}\right] = \frac{1}{\log 2} = \text{finite}$$

By integral test,

$\Sigma|U_n|$ is also convergent

so,

The given series is absolutely convergent.

(b) sol Given,

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1} \qquad \cos n\pi = (-1)^n$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$$

$$U_n = \frac{1}{n^2+1} \qquad U_{n+1} = \frac{1}{(n+1)^2+1} = \frac{1}{n^2+2n+2}$$

Consider,

$$U_n - U_{n+1} = \frac{1}{n^2+1} - \frac{1}{n^2+2n+2}$$

$$\frac{n^2+2n+2-n^2-1}{(n^2+1)(n^2+2n+2)}$$

$$= \frac{(2n+1)}{(n^2+1)(n^2+2n+2)} > 0$$

$U_n - U_{n+1} > 0$

$U_n > U_{n+1}$

Consider,

$$\lim_{n\to\infty} U_n$$

$$\lim_{n\to\infty} \frac{1}{n^2+1}$$

$= 0$

By leibnitz's test

series is convergent

II) Consider,

$$\Sigma|U_n| = \frac{1}{n^2+1}$$

$$= \frac{1}{n^2\left(1+\frac{1}{n^2}\right)}$$

consider $v_n = \frac{1}{n^2}$ which is in p-series

ΣV_n is convergent

Consider,

$$\lim_{n\to\infty} \frac{U_n}{V_n}$$

$$= \frac{\frac{1}{n^2\left(1+\frac{1}{n^2}\right)}}{\frac{1}{n^2}} = \cancel{1 \neq 0}$$

$= 1 \neq 0$ (finite) By limit comparsion test,

$\Sigma|U_n|$ is convergent

Given series is absolutely convergent