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ECE-C

# M-II QB Key

## Unit V



LAQ's :

1. $\bar{r} = xi + yj + zk$

$d\bar{r} = dxi + dyj + dzk$

$\bar{f} \cdot d\bar{r} = (xyi - zj + x^2k)(dxi + dyj + dzk)$

$= xy\,dx - z\,dy + x^2 dz$

$x = t \qquad y = 2t \qquad z = t^3$

$dx = dt \qquad dy = 2dt \qquad dz = 3t^2 dt$

$\bar{f} \cdot d\bar{r} = t(2t)dt - t^3(2dt) + t^2(3t^2)dt$

$= 2t^2 dt - 2t^3 dt + 3t^4 dt$

$t : 0 \to 1$

$$\int_C \bar{f} \cdot d\bar{r} = \int_0^1 (2t^2 - 2t^3 + 3t^4)\,dt$$

$$= \left[\frac{2t^3}{3} - \frac{2t^4}{4} + \frac{3t^5}{5}\right]_0^1$$

$$= \frac{2}{3} - \frac{\cancel{2}}{\cancel{4}2} + \frac{3}{5}$$

$$= \frac{10 - 15 + 18}{30}$$

$$= \frac{13}{30}$$

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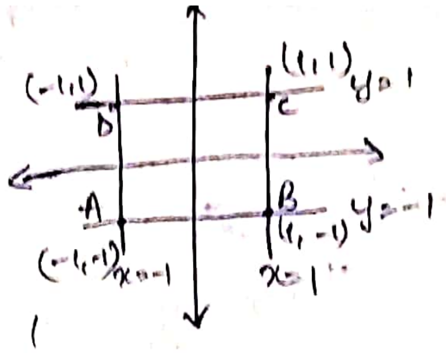
2. $A(-1,-1)\quad B(1,-1)\quad C(1,1)\quad D(-1,1)$




$\bar{F}\cdot d\bar{r} = (x^2+xy)\,dx + (x^2+y^2)\,dy$

$AB:\ x: -1 \to 1 \quad y=-1 \Rightarrow dy=0$

$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_{-1}^{1} (x^2+xy)\,dx = \left[\frac{x^3}{3}+\frac{x^2y}{2}\right]_{-1}^{1}$$

$$= \frac{1}{3} - \cancel{\frac{1}{2}} - \left(\frac{-1}{3}\right) - \left(\cancel{\frac{-1}{2}}\right) = \frac{2}{3}$$

$BC:\ y: -1 \to 1 \quad x=1 \Rightarrow dx=0$

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_{-1}^{1} (x^2+y^2)\,dy = \left[x^2y+\frac{y^3}{3}\right]_{-1}^{1}$$

$$= 1+\frac{1}{3} - \left(-1-\frac{1}{3}\right) = 2+\frac{2}{3} = \frac{8}{3}.$$

$CD:\ x: 1 \to -1 \quad y=1 \Rightarrow dy=0$

$$\int_{CD} \bar{F}\cdot d\bar{r} = \int_{1}^{-1} (x^2+xy)\,dx = \left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{1}^{-1}$$

$$= \frac{-1}{3}+\frac{1}{2} - \left(\frac{1}{3}+\frac{1}{2}\right) = -\frac{2}{3}.$$

$DA:\ y: 1 \to -1 \quad x=-1 \Rightarrow dx=0$

$$\int_{DA} \bar{F}\cdot d\bar{r} = \int_{y=1}^{-1} (x^2+y^2)\,dy = \left[x^2y+\frac{y^3}{3}\right]_{1}^{-1}$$

$$= \left[y+\frac{y^3}{3}\right]_{1}^{-1} = -1-\frac{1}{3}-1-\frac{1}{3} = \frac{-8}{3}.$$

$$\oint_C \bar{F}\cdot d\bar{r} = \int_{AB}\bar{F}\cdot d\bar{r} + \int_{BC}\bar{F}\cdot d\bar{r} + \int_{CD}\bar{F}\cdot d\bar{r} + \int_{DA}\bar{F}\cdot d\bar{r}$$

$$= \cancel{\frac{2}{3}} + \cancel{\frac{8}{3}} - \cancel{\frac{2}{3}} - \cancel{\frac{8}{3}} = \underline{\underline{0}}.$$

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3. $\bar{f} = (x^2 - y^2 + x)i - (2xy + y)j$

$\bar{r} = xi + yj + zk$

$d\bar{r} = dxi + dyj + dzk$

$\bar{f} \cdot d\bar{r} = (x^2 - y^2 + x)dx - (2xy + y)dy$

$y^2 = x$ $\qquad$ $x : 0 \to 1$

$2y\,dy = dx$ $\qquad$ $y : 0 \to 1$

$\bar{f} \cdot d\bar{r} = (y^4 \cancel{- y^2} + \cancel{y^2})2y\,dy - (2y^3 + y)dy$

$= (2y^5 - 2y^3 + y)\,dy$

Work done $= \int_C \bar{f} \cdot d\bar{r}$

$\int_C \bar{f} \cdot d\bar{r} = \int_0^1 (2y^5 - 2y^3 + y)dy$

$= \left[\frac{\cancel{2}y^6}{\cancel{6}3} - \frac{\cancel{2}y^4}{\cancel{4}2} + \frac{y^2}{2}\right]_0^1$

$= \frac{1}{3} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} = \frac{1}{3}$

4. $\bar{f} \cdot d\bar{r} = (2y+3)dx + xz\,dy + (yz - x)dz$

$x = 2t^2$ $\qquad$ $y = t$ $\qquad$ $z = t^3$

$dx = 4t\,dt$ $\qquad$ $dy = dt$ $\qquad$ $dz = 3t^2dt$

$\bar{f} \cdot d\bar{r} = (2t+3)4t\,dt + 2t^2(t^3)dt + (t(t^3) - 2t^2)3t^2dt$

$= (8t^2 + 12t)dt + 2t^5dt + (3t^6 - 6t^4)dt$

$= (3t^6 + 2t^5 - 6t^4 + 8t^2 + 12t)\,dt$

$y: 0 \to 1 \Rightarrow t: 0 \to 1$

Work done $= \int_C \bar{F} \cdot d\bar{r} = \int_0^1 (3t^6 + 2t^5 - 6t^4 + 8t^2 + 12t)\, dt$

$$= \left[ \frac{3t^7}{7} + \frac{\cancel{2}t^6}{\cancel{6}3} - \frac{6t^5}{5} + \frac{8t^3}{3} + \frac{\cancel{12}^{6}t^2}{\cancel{2}} \right]_0^1$$

$$= \frac{3}{7} + \frac{1}{3} - \frac{6}{5} + \frac{8}{3} + 6$$

$$= \frac{45 + 35 - 126 + 280 + 1056}{105}$$

$$= \frac{\cancel{1290}^{\cancel{258}^{86}}}{\cancel{105}_{\cancel{21}_{7}}} = \frac{86}{7}$$

5. curl $\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix}$

$= i(3x^2z^2 - 3x^2z^2) - j(6xyz^2 - 6xyz^2) + k(2xz^3 - 2xz^3)$

$= \underline{0}$

$\therefore$ Conservative.

$\bar{F}$ is conservative. Find $\phi$ which is its scalar potential.

$$f = \nabla \phi$$

$$(2xyz^3)i + (x^2z^3)j + (3x^2yz^2)k = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$$

$$2xyz^3 = \frac{\partial\phi}{\partial x} \qquad x^2z^3 = \frac{\partial\phi}{\partial y} \qquad 3x^2yz^2 = \frac{\partial\phi}{\partial z}$$

$$\phi = \int 2xyz^3\,dx \qquad \phi = \int x^2z^3\,dy \qquad \phi = \int 3x^2yz^2\,dz$$

$$\phi = x^2yz^3.$$

Work done $= \left[x^2yz^3\right]_{(1,-1,2)}^{(3,2,-1)}$

$$= \left[(3^2(2)(-1)^3) - (1(-1)(2^3))\right]$$

$$= -18+8 = \underline{\underline{-10}}$$

6. $\bar{f} = zi + xj - 3y^2zk.$

$\phi = x^2+y^2 = 16$

$\nabla\phi = 2xi + 2yj.$

$|\nabla\phi| = \sqrt{(2x)^2+(2y)^2} = \sqrt{4(x^2+y^2)} = \sqrt{4(16)} = \underline{\underline{8}}.$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2xi+2yj}{8}$$

$$\bar{f}\cdot\hat{n} = \frac{(zi+xj-3y^2zk)\,(2xi+2yj)}{8} = \frac{2xz+2xy}{8}$$

$$= \frac{x(y+z)}{4}.$$

Let $R$ be the projection of $S$ on $yz$ plane, then

$$\int_S \bar{f}\cdot\hat{n}\,ds = \iint_R \bar{f}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{n}\cdot i|}$$

$|\hat{n}\cdot i| = \frac{x}{4}$.

$$\int_S \bar{F}\cdot\hat{n}\,ds = \iint_R \frac{\cancel{x}(y+z)}{\cancel{4}}\,\frac{dy\,dz}{\cancel{x}/\cancel{4}}$$

$x^2+y^2=16$.

At $x=0$ $y^2=16$ $\Rightarrow y: 0\to 4$.
$y=4$.

$$\int_S \bar{F}\cdot\hat{n}\,ds = \int_{z=0}^{5}\left(\int_{y=0}^{4}(y+z)\,dy\right)dz$$

$$= \int_{z=0}^{5}\left[\frac{y^2}{2}+yz\right]_0^4 dz.$$

$$= \int_{z=0}^{5}\left(\frac{\cancel{16}^{8}}{\cancel{2}}+4z\right)dz = \left[8z+\frac{\cancel{4}^{2}z^2}{\cancel{2}}\right]_0^5$$

$$= 40+2(25) = 40+50 = \underline{\underline{90}}$$

7. $\bar{F} = (x+y^2)i - 2xj + 2yzk$

$\phi = 2x+y+2z-6$.

$\nabla\phi = 2i+j+2k$.

$|\nabla\phi| = \sqrt{2^2+1^2+2^2} = \sqrt{9} = 3$.

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2i+j+2k}{3}$$

$$\bar{F}\cdot\hat{n} = \frac{((x+y^2)i - 2xj + 2yzk)(2i+j+2k)}{3}$$

$$= \frac{2(x+y^2) - 2x + 4yz}{3}$$

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$$\bar{f}\cdot\hat{n} = \frac{(\cancel{2x}+2y^2-\cancel{2x}+4yz)}{3} = \frac{2}{3}(y^2+2yz)$$

$$= \frac{2y}{3}(y+2z).$$

Let R be the projection of S on yz plane then

$$\int_S \bar{f}\cdot\hat{n}\,ds = \iint_R \bar{f}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{n}\cdot i|}$$

$$|\hat{n}\cdot i| = \frac{2}{3}.$$

$$\int_S \bar{f}\cdot\hat{n}\,ds = \iint_R \frac{\cancel{2}y}{\cancel{3}}(y+2z)\,\frac{dy\,dz}{\cancel{2}/\cancel{3}}$$

$2x+y+2z=6$

$x=0$      At $y=0$

$y+2z=6$      $2z=6$

$y=6-2z.$      $z=3.$

$y: 0\to 6-2z$      $z: 0\to 3.$

$$\int_S \bar{F}\cdot\hat{n}\,ds = \int_{z=0}^{3}\left(\int_{y=0}^{6-2z}(y^2+2yz)\,dy\right)dz$$

$$= \int_{z=0}^{3}\left[\frac{y^3}{3}+\frac{\cancel{2}y^2z}{\cancel{2}}\right]_0^{6-2z} dz$$

$$= \int_{z=0}^{3}\left(\frac{(6-2z)^3}{3}+(6-2z)^2 z\right)dz.$$

$$= \int_{z=0}^{3}\left[\frac{216-8z^3-216z+72z^2}{3}+z(36-24z+4z^2)\right]dz$$

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \int_{z=0}^{3}\left(\frac{216}{3} - \frac{8z^3}{3} - \frac{216z}{3} + \frac{72z^2}{3} + 36z + 4z^3 - 24z^2\right)dz$$

$$= \int_{z=0}^{3}\left(72 - \frac{8}{3}z^3 + 4z^3 - 72z + 36z + 24z^2 - 24z^2\right)dz$$

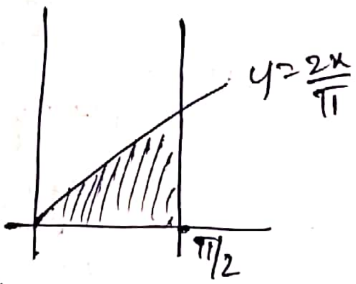
$$= \int_{z=0}^{3}\left(72 - \frac{20z^3}{3} - 36z\right)dz = \left[72z - \frac{\overset{5}{\cancel{20}}z^4}{\underset{3}{\cancel{12}}} - \frac{36z^2}{2}\right]_0^3$$

$$= 72(3) - \frac{5(\overset{27}{\cancel{81}})}{\cancel{3}} - \frac{\overset{18}{\cancel{36}}(9)}{\cancel{2}}$$

$$= 216 - 135 - 162 = \underline{81}.$$

8. $M = y - \sin x \qquad N = \cos x$

$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = -\sin x$




By Green's theorem, $x: 0 \to \pi/2$, $y: 0 \to \frac{2x}{\pi}$

$$\oint_C (y - \sin x)\,dx + \cos x\,dy = \iint_R (-\sin x - 1)\,dx\,dy$$

$$= -\int_{x=0}^{\pi/2}\int_{y=0}^{\frac{2x}{\pi}} (\sin x + 1)\,dy\,dx$$

$$= -\int_{x=0}^{\pi/2} (\sin x + 1)\,[y]_0^{\frac{2x}{\pi}}\,dx$$

$$= -\int_{x=0}^{\pi/2} (\sin x + 1)x\,dx$$

$$= -\frac{2}{\pi}\int_0^{\pi/2} (x\sin x + x)\,dx$$

$$= -\frac{2}{\pi}\left(\left[-x\cos x + \int \cos x\right]_0^{\pi/2} + \left[\frac{x^2}{2}\right]_0^{\pi/2}\right)$$

$$= -\frac{2}{\pi}\left[\left[-x\cos x + \sin x\right]_0^{\pi/2} + \frac{1}{2}\left(\frac{\pi^2}{4}\right)\right]$$

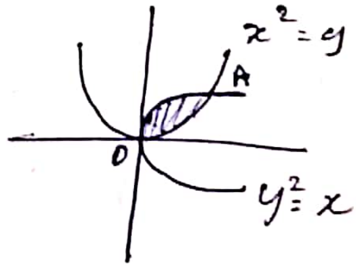
$$= -\frac{2}{\pi}\left[-\frac{\pi}{2}(0) + 1 + \frac{\pi^2}{8}\right] = -\frac{2}{\pi}\left[\frac{8+\pi^2}{8\cancel{4}}\right]$$

$$= -\left(\frac{2}{\pi} + \frac{\pi}{4}\right).$$

9. $$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

$M = 3x^2 - 8y^2$ $\quad$ $N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y$ $\quad$ $\frac{\partial N}{\partial x} = -6y.$




By Green's theorem,

$x : 0 \to 1$

$y : x^2 \to \sqrt{x}$.

$$\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy = \iint_R (-6y + 16y)\,dy\,dx.$$

$$= \int_{x=0}^{1}\left(\int_{y=x^2}^{\sqrt{x}} 10y\,dy\right)dx$$

$$= \int_{x=0}^{1}\left[\frac{\overset{5}{\cancel{10}}y^2}{\cancel{2}}\right]_{x^2}^{\sqrt{x}} dx = \int_{x=0}^{1}(5x - 5x^4)\,dx$$

$$= \left[\frac{5x^2}{2} - \frac{\cancel{5}x^5}{\cancel{5}}\right]_0^1 = \frac{5}{2} - \frac{1}{1} = \frac{3}{2} \quad (1)$$

$$\oint M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AO} M\,dx + N\,dy.$$

$$= I_1 + I_2.$$

OA: $x^2 = y$, $dy = 2x\,dx$; $y : 0 \to x^2$; $x : 0 \to 1$

AO: $y^2 = x$, $y = \sqrt{x}$, ~~dx~~ $2y\,dy$, $dy = \frac{1}{2\sqrt{x}}\,dx$; $y : 0 \to \sqrt{x}$; $x : 1 \to 0$.

$$I_1 = \int_{x=0}^{1} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int_{x=0}^{1} (3x^2 - 8(x^2)^2)\,dx + (4x^2 - 6x(x^2))\,2x\,dx.$$

$$= \int_{x=0}^{1} (3x^2 - 8x^4 + 8x^3 - 12x^4)\,dx$$

$$= \left[\frac{3x^3}{3} - \frac{8x^5}{5} + \frac{8x^4}{4} - \frac{12x^5}{5}\right]_0^1$$

$$= 1 - \frac{8}{5} + 2 - \frac{12}{5} = -1.$$

$$I_2 = \int_{x=1}^{0} (3x^2 - 8(\sqrt{x})^2)\,dx + (4\sqrt{x} - 6x(\sqrt{x}))\,\frac{dx}{2\sqrt{x}}$$

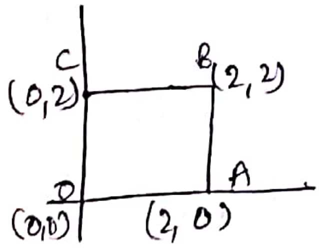
$$= \int_{x=1}^{0} (3x^2 - 8x + 2 - 3x)\,dx = \left[\frac{3x^3}{3} - \frac{8x^2}{2} + 2x - \frac{3x^2}{2}\right]_1^0$$

$$= -\left[1 - 4 + 2 - \frac{3}{2}\right] = +\frac{5}{2}$$

$$\oint (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = I_1 + I_2 = -1 + \frac{5}{2} = \frac{3}{2} \quad \text{— (2)}$$

From (1) & (2), Green's theorem is verified.

10. $\oint (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$




$M = x^2 - xy^3$ $\quad\quad N = y^2 - 2xy$

$\frac{\partial M}{\partial y} = -3xy^2$ $\quad\quad \frac{\partial N}{\partial x} = -2y$

$x : 0 \to 2$

$y : 0 \to 2$

By Green's theorem,

$$\oint (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy = \iint_R (-2y + 3xy^2)\,dx\,dy.$$

$$= \int_0^2 \int_0^2 (-2y + 3xy^2)\,dx\,dy$$

$$= \int_0^2 \left[-2xy + \frac{3x^2}{2}y^2\right]_0^2 dy$$

$$= \int_0^2 (-4y + 6y^2)\,dy = \left[-2y^2 + 2y^3\right]_0^2 = -8 + 16$$

$$\iint_R (-2y + 3xy^2)\,dx\,dy = 8 \quad \text{---} (1)$$

To evaluate $\oint_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$, we shall take C in four different segments

(i) OA $\quad x : 0 \to 2 \quad y = 0 \quad dy = 0$

$$\int_0^2 (x^2 - xy^3)\,dx = \left[\frac{x^3}{3} - \frac{x^2 y^3}{2}\right]_0^2 = \frac{8}{3}$$

(ii) AB $\quad y : 0 \to 2 \quad x = 2 \quad dx = 0.$

$$\int_0^2 (y^2 - 2xy)\,dy = \left[\frac{y^3}{3} - \frac{\cancel{2}xy^2}{\cancel{2}}\right]_0^2 = \frac{8}{3} - 8 = -\frac{16}{3}$$

(iii) BC $\quad x: 2 \to 0 \quad y = 2,\ dy = 0$

$$\int_{x=2}^{0} (x^2 - xy^3)\,dx = \left[\frac{x^3}{3} - \frac{x^2y^3}{2}\right]_2^0 = -\left[\frac{8}{3} - 16\right]$$

$$= 16 - \frac{8}{3} = \frac{40}{3}.$$

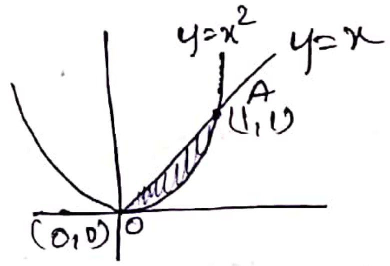
(iv) CO $\quad y: 2 \to 0 \quad x = 0 \quad dy = 0$

$$\int_{y=2}^{0} (y^2 - 2xy)\,dy = \left[\frac{y^3}{3} - \frac{2xy^2}{2}\right]_2^0 = -\left[\frac{8}{3} - 0\right] = \frac{-8}{3}$$

$$\oint (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy = \cancel{\frac{8}{3}} - \frac{16}{3} + \frac{40}{3} - \cancel{\frac{8}{3}} = \frac{24}{3} = 8 \quad \text{—(2)}$$

Comparing eq (1) & (2), Green's theorem is verified.

11. $\oint (xy + x^2)\,dx + x^2\,dy$.




$M = xy + x^2 \qquad N = x^2$

$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = 2x.$

$x: 0 \to 1$

$y: x^2 \to x$

By Green's theorem,

$$\oint (xy + x^2)\,dx + x^2\,dy = \iint_R (2x - x)\,dx\,dy.$$

$$= \int_{x=0}^{1} x\left(\int_{y=x^2}^{x} dy\right) dx = \int_{x=0}^{1} x\,[y]_{x^2}^{x}\,dx$$

$$= \int_{x=0}^{1} (x^2 - x^3)\,dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$\iint_R x\,dx\,dy = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad \text{—(1)}$$

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$$\oint (xy+x^2)dx + x^2 dy = \int_{OA} (xy+x^2)dx + x^2 dy + \int_{AO} (xy+x^2)dx + x^2 dy$$

$$= I_1 + I_2 .$$

$I_1$ : $x : 0 \to 1$ $\quad y = x^2$ $\quad dy = 2x\,dx$

$$= \int_{x=0}^{1} (x(x^2) + x^2)\,dx + x^2(2x)\,dx$$

$$= \int_{x=0}^{1} (x^3 + x^2 + 2x^3)\,dx = \left[\frac{3x^4}{4} + \frac{x^3}{3}\right]_0^1$$

$$= \frac{3}{4} + \frac{1}{3} = \frac{9+4}{12} = \frac{13}{12} .$$

$I_2$ : $x : 1 \to 0$ $\quad y = x$ $\quad dy = dx .$

$$= \int_{x=1}^{0} (x(x) + x^2)\,dx + x^2\,dx$$

$$= \int_{x=1}^{0} 2x^2 + x^2\,dx = \int_{x=1}^{0} 3x^2\,dx$$

$$= \left[\frac{\cancel{3}x^3}{\cancel{3}}\right]_1^0 = -1$$

$$I_1 + I_2 = \frac{13}{12} - 1 = \frac{1}{12} \quad — (2)$$

from eq^n (1) & (2), Green's theorem is verified.

12. $\operatorname{div} f = \nabla f = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)(4xi - 2y^2 j + z^2 k)$

$= 4 - 4y + 2z$

By divergence theorem

$x^2 + y^2 = 4$

At $y=0$, $x = \pm 2$

$x : -2 \to 2$

$y^2 = 4 - x^2$

$y : -\sqrt{4-x^2} \to \sqrt{4-x^2}$

$\iint_S \bar{F} \cdot ds = \int_V \operatorname{div} \bar{F} \, dv$

$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{3} (4 - 4y + 2z) \, dx \, dy \, dz$

$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz + \frac{2z^2}{2}\right]_0^3 dy \, dx$

$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9) \, dy \, dx$

$= \int_{-2}^{2} \left(\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y) \, dy\right) dx$

$= \int_{-2}^{2} \left[21y - \frac{12y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$

$= \int_{-2}^{2} \left(-21\sqrt{4-x^2} - 6(4-x^2) - \left(21\sqrt{4-x^2} - 6(4-x^2)\right)\right) dx$

$= -\int_{-2}^{2} 42\sqrt{4-x^2} \, dx = -42 \int_{-2}^{2} \sqrt{4-x^2} \, dx$

$= -42 \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2}\right]_{-2}^{2}$

$= -42 \left[\sqrt{4-4} + 2\sin^{-1} 1 - 0 - 2\sin^{-1}(-1)\right]$

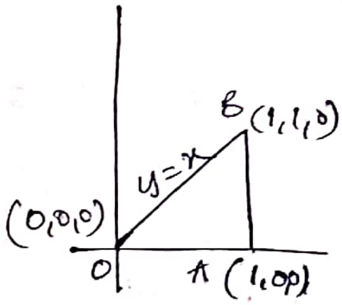
$$= -42(-2\pi) = \underline{\underline{84\pi}}.$$

13. LAQ 12

14. $\oint_C f_1\,dx + f_2\,dy + f_3\,dz = \int_S \text{curl}\,\bar{F}\cdot\hat{n}\,ds$

$f_1 = x+y \quad f_2 = 2x-z \quad f_3 = y+z.$

$$\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$




$$= i(1-(-1)) - j(0-0) + k(2-1)$$

$$= 2i + k.$$

$x: 0 \to 1$

$y: 0 \to x$

$$\int_{x=0}^{1}\int_{y=0}^{x} (2i+k)\,\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \int_{x=0}^{1}\int_{y=0}^{x} (2i+k)\,\hat{k}\,\frac{dx\,dy}{|\hat{k}\cdot\hat{k}|}$$

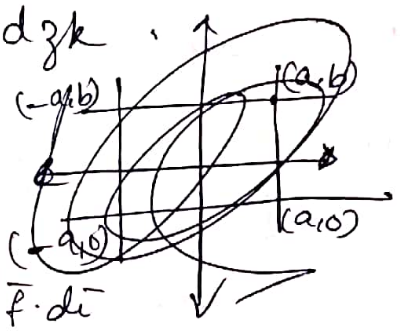
$$= \int_{x=0}^{1}\int_{y=0}^{x} dy\,dx = \int_{x=0}^{1} [y]_0^x\,dx$$

$$= \int_{x=0}^{1} x\,dx = \left[\frac{x^2}{2}\right]_0^1$$

$$= \underline{\underline{\frac{1}{2}}}.$$

15. $\bar{f} = (x^2+y^2)i - 2xyj$

$\bar{r} = xi + yj + zk \quad d\bar{r} = dx\,i + dy\,j + dz\,k$.

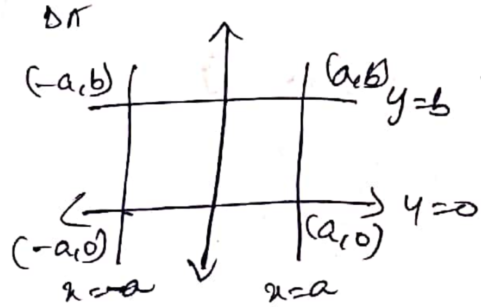



$\bar{f}\cdot d\bar{r} = dx(x^2+y^2) - dy(2xy)$

LHS: $\int_C \bar{f}\cdot d\bar{r} = \int_{AB}\bar{f}\cdot d\bar{r} + \int_{BC}\bar{f}\cdot d\bar{r} + \int_{CD}\bar{f}\cdot d\bar{r} + \int_{DA}\bar{f}\cdot d\bar{r}$

(i) AB: $x = a \Rightarrow dx = 0$

$y : 0 \to b$




$\int_{AB}\bar{f}\cdot d\bar{r} = \int_{y=0}^{b} (x^2+y^2)dx + 2xy\,dy$

$= \int_0^b \left((a^2+y^2)0 - 2ay\right)dy = \int_{y=0}^{b} -2ay\,dy = \left[-\frac{\cancel{2}ay^2}{\cancel{2}}\right]_0^b$

$= -ab^2$

(ii) BC: $x : a \to -a$

$y = b \Rightarrow dy = 0$

$\int_{BC}\bar{f}\cdot d\bar{r} = \int_{x=a}^{-a} (x^2+b^2)dx = \left[\frac{x^3}{3} + b^2x\right]_a^{-a} =$

$= -\left[\frac{1}{3}(a^3+a^3) + b^2(a+a)\right] = -\left(\frac{2a^3}{3} + 2ab^2\right)$

(iii) CD: $y : b \to 0$

$x = -a \Rightarrow dx = 0$

$\int_{CD}\bar{f}\cdot d\bar{r} = \int_{y=b}^{0} -2xy\,dy = \int_b^0 -2ay\,dy = -\left[\frac{2ay^2}{\cancel{2}}\right]_0^b$

$= -ab^2.$

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(iv) DA $\quad x : -a \to a$

$y = 0 \Rightarrow dy = 0.$

$$\int_{DA} \bar{f} \cdot d\bar{r} = \int_{x=-a}^{a} (x^2+y^2)\, dx = \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{a^3}{3} + \frac{a^3}{3}$$

$$= \frac{2a^3}{3}.$$

$$\int \bar{f} \cdot d\bar{r} = -ab^2 - \cancel{\frac{2a^3}{3}} - 2ab^2 - ab^2 + \cancel{\frac{2a^3}{3}} = -4ab^2 \quad (1).$$

RHS By stoke's theorem,

$$\int_C \bar{f} \cdot d\bar{r} = \int_S \text{curl } \bar{f}\, \hat{n}\, ds$$

$$\text{curl } \bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= i(0) - j(0) + k(-2y-2y) = -4yk.$$

$$\int_S \text{curl } \bar{f}\, \hat{n}\, ds = \int_S \text{curl } \bar{f}\, \hat{n} \frac{dx\,dy}{|\hat{n}\,\hat{k}|}$$

$$= \int_{x=a}^{a} \int_{y=0}^{b} (-4yk)\hat{k}\, dy\, dx.$$

$$= \int_{x=-a}^{a} \left( \int_{y=0}^{b} -4y\, dy \right) dx$$

$$= \int_{x=-a}^{a} \left[ \frac{-4y^2}{2} \right]_0^b dx.$$

$$= \int_{x=-a}^{a} -2b^2\,dx = \left[-2b^2 x\right]_{-a}^{a}$$

$$= -2b^2(a+a) = -4ab^2 \quad -(2)$$

LHS = RHS

from (1) & (2), Stoke's theorem is verified.