UNIT - 2

LONG ANSWER QUESTIONS :

1. Find the Eigen values and Eigen vectors of the following matrices :-

(i) $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Ans Consider the characteristic equation

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(5-\lambda)(1-\lambda) - (1)(1)] - 1[(1-\lambda)(1) - 3(1)] + 3[(1)(1) - 3(5-\lambda)] = 0$$

$$(1-\lambda)[5 - 5\lambda - \lambda + \lambda^2 - 1] - 1[1 - \lambda - 3] + 3[1 - 15 + 3\lambda] = 0$$

$$(1-\lambda)[\lambda^2 - 6\lambda + 4] - 1[-\lambda - 2] + 3[-14 + 3\lambda] = 0$$

$\cancel{\lambda^2 \lambda y}$

$\lambda^2 - 6\lambda + 4 - \lambda^3 + 6\lambda^2 - 4\lambda + \lambda + 2 - 42 + 9\lambda = 0$

$-\lambda^3 + 7\lambda^2 - 36 = 0$

$\Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0$

$\lambda = 6, -2, 3$

The eigen values are $\lambda = 6, -2, 3$

Case - 1

$\lambda = 6$

Consider the equation $(A - \lambda I)x = 0$

$\Rightarrow (A - 6I)x = 0$

$$\left[\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}\right] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 5R_2 + R_1 \quad ; \quad R_3 \to 5R_3 + 3R_1$

$$\begin{bmatrix} -5 & 1 & 3 \\ 0 & -4 & 8 \\ 0 & 8 & -16 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 / -4 \quad ; \quad R_3 \to R_3 / 8$

$$\begin{bmatrix} -5 & 1 & 3 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} -5 & 1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $\boxed{z = k_1}$

$-5x + y + 3z = 0$

$-5x + 2k_1 + 3k_1 = 0$

$-5x = -5k_1$

$\boxed{x = k_1}$

$y - 2z = 0$

$y - 2k_1 = 0$

$\boxed{y = 2k_1}$

The eigen vectors corresponding to $\lambda = 6$ is

$$X_1 = \begin{bmatrix} k_1 \\ 2k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Case - 2

$\lambda = -2$

Consider the equation $(A - \lambda I)x = 0$

$\Rightarrow (A + 2I)x = 0$

The

is

$$\left[\begin{bmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{bmatrix} + \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}\right]\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}3 & 1 & 3\\ 1 & 7 & 1\\ 3 & 1 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$R_2 \to 3R_2 - R_1$; $R_3 \to R_3 - R_1$

$$\begin{bmatrix}3 & 1 & 3\\ 0 & 20 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$R_2 \to R_2/20$

$$\begin{bmatrix}3 & 1 & 3\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Let $\boxed{z = k_2}$

$\Rightarrow \boxed{y = 0}$

$3x + y + 3z = 0$

$3x + 0 + 3k_2 = 0$

$3x = -3k_2$

$\boxed{x = -k_2}$

The eigen vectors corresponding to $\lambda = -2$ is

$$X_2 = \begin{bmatrix}-k_2\\ 0\\ k_2\end{bmatrix} = -k_2\begin{bmatrix}1\\ 0\\ -1\end{bmatrix}$$

Case - 3

$\lambda = 3$

Consider the equation $(A - \lambda I)X = 0$

$$\left[\begin{bmatrix}1&1&3\\1&5&1\\3&1&1\end{bmatrix} - \begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}\right]\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-2&1&3\\1&2&1\\3&1&-2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_2 \to 2R_2 + R_1$; $R_3 \to 2R_3 + 3R_1$

$$\begin{bmatrix}-2&1&3\\0&5&5\\0&5&5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix}-2&1&3\\0&5&5\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Let $\boxed{z = k_3}$

$5y + 5z = 0$

$5y + 5k_3 = 0$

$5y = -5k_3$

$\boxed{y = -k_3}$

$-2x + y + 3z = 0$

$-2x - k_3 + 3k_3 = 0$

$-2x + 2k_3 = 0$

$-2x = -2k_3$

$\Rightarrow \boxed{x = k_3}$

The eigen vectors corresponding to $\lambda = 3$ is

$$X_3 = \begin{bmatrix} k_3 \\ -k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

$$A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Consider the characteristic equation

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[(1-\lambda)(-1-\lambda) - 3] + 2[1(-1-\lambda) - 1] + 2[3 - 1(1-\lambda)] = 0$$

$$(2-\lambda)[-1-\cancel{\lambda}+\cancel{\lambda}+\lambda^2-3] + 2[-1-\lambda-1] + 2[3-1+\lambda] = 0$$

$$(2-\lambda)[-4+\lambda^2] + 2[-2-\lambda] + 2[2+\lambda] = 0$$

$$-8+2\lambda^2+4\lambda-\lambda^3-\cancel{4}-\cancel{2\lambda}+\cancel{4}+\cancel{2\lambda} = 0$$

$$-\lambda^3+2\lambda^2+4\lambda-8 = 0$$

$$\Rightarrow \lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$$

$\lambda = -2, 2, 2$

The eigen values are $\lambda = -2, 2, 2$

Case - 1

$\lambda = -2$

Consider the equation $(A - \lambda I)X = 0$

$\Rightarrow (A + 2I)X = 0$

$$\left[\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 4R_2 - R_1$; $R_3 \to 4R_3 - R_1$

$$\begin{bmatrix} 4 & -2 & 2 \\ 0 & 14 & 2 \\ 0 & 14 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to R_1/2$; $R_2 \to R_2/2$; $R_3 \to R_3/2$

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 7 & 1 \\ 0 & 7 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 7 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $\boxed{z = k_1}$

$7y + z = 0$

$7y = -k_1$

$\Rightarrow \boxed{y = \frac{-k_1}{7}}$

$2x - y + z = 0$

$2x + \frac{k_1}{7} + k_1 = 0$

$14x + k_1 + 7k_1 = 0$

$14x = -8k_1$

$\boxed{x = -\frac{4}{7}k_1}$

The eigen vector for $\lambda = -2$ is

$$X_1 = \begin{bmatrix} -4/7\, k_1 \\ -k_1/7 \\ k_1 \end{bmatrix} = -\frac{4}{7}k_1 \begin{bmatrix} 1 \\ 1/4 \\ -7/4 \end{bmatrix}$$

Case - 2

$\lambda = 2$

Consider the equation $(A - \lambda I)x = 0$

$(A - 2I)x = 0$

$$\left[\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 3 & -3 \\ 1 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$

$$\begin{bmatrix} 1 & 3 & -3 \\ 0 & -4 & 4 \\ 0 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 / -4 \; ; \; R_3 \rightarrow R_3 / -2$

$$\begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $\boxed{z = k_1}$

$y - z = 0$

$\Rightarrow \boxed{y = k_1}$

$x + 3y - 3z = 0$

$x + 3k_1 - 3k_1 = 0$

$\Rightarrow \boxed{x = 0}$

The eigen vector's for $\lambda = 2$ are

$$X_2 = \begin{bmatrix} 0 \\ k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(iii) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Consider the characteristic equation

$$|A - \lambda \bar{I}| = 0$$

$$\left| \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)[(1-\lambda)(-\lambda) - (-2)(-6)] - 2[2(-\lambda) - (-1)-6]$$
$$-3[(-2)(2) - (-1)(1-\lambda)] = 0$$

$$(-2-\lambda)[-\lambda^2 + \lambda^2 - 12] - 2[-2\lambda - 6] - 3[-3-\lambda] = 0$$

$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$

$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$\lambda = 5, -3, -3$

The Eigen values are 5, -3, -3

Case - 1

$\lambda = 5$

Consider the equation $(A - \lambda I)X = 0$

$\Rightarrow (A - 5I)X = 0$

$$\left[\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & 6 \\ 1 & -2 & 0 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2/2$; $R_3 \to R_3/-1$

$$\begin{bmatrix} -7 & 2 & -3 \\ 1 & -2 & -3 \\ 1 & 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & -2 & -3 \\ -7 & 2 & -3 \\ 1 & 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 7R_1$; $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & -2 & -3 \\ 0 & -12 & -24 \\ 0 & 4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2/-12$; $R_3 \to R_3/4$

$$\begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $\boxed{z = k_1}$

$x - 2y - 3z = 0$

$x + 4k_1 - 3k = 0$

$\Rightarrow \boxed{x = -k_1}$

$y + 2z = 0$

$\Rightarrow \boxed{y = -2k_1}$

The eigen vectors for $\lambda = 5$ is

$$X_1 = \begin{bmatrix} -k_1 \\ -2k_1 \\ k_1 \end{bmatrix} = -k_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Case - 2.

$\lambda = -3$

Consider the equation $(A - \lambda I)x = 0$

$\Rightarrow (A + 3I)x = 0$

$$\left(\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & 6 \\ -1 & -2 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1 \quad ; \quad R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $\boxed{y = k_2}$ & $\boxed{z = k_3}$

$x + 2y - 3z = 0$

$x + 2k_2 - 3k_3 = 0$

$\Rightarrow \boxed{x = -2k_2 + 3k_3}$

$\therefore$ The eigen values for $\lambda = -3$ is

$$X_2 = \begin{bmatrix} -2k_2 + 3k_3 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} -2k_2 + 3k_3 \\ k_2 + 0k_3 \\ 0k_2 + k_3 \end{bmatrix}$$

$$= \begin{bmatrix} -2k_2 \\ k_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3k_3 \\ 0 \\ k_3 \end{bmatrix}$$

$$\Rightarrow {}^{P}X_2 = -k_2 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

5. Verify the Cayley Hamilton theorem for $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$ and hence find it's inverse and A^4

Ans: The Characteristic equation is

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & -1 \\ 3 & 1-\lambda & 0 \\ -2 & 1 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(1-\lambda)(4-\lambda)\right] - 2\left[(3)(4-\lambda)\right] - 1\left[3+2(1-\lambda)\right] = 0$$

$$(1-\lambda)\left[\lambda^2 - 5\lambda + 4\right] - 2\left[12 - 3\lambda\right] - 1\left[5 - 2\lambda\right] = 0$$

$$\Rightarrow -\lambda^3+6\lambda^2-\lambda-25=0$$

$$\Rightarrow \lambda^3-6\lambda^2+\lambda+25=0$$

To verify theorem.

$$A^3-6A^2+A+25\underline{I}=0$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}^3 - 6\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}^2 + \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$$

$$+ \begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 25 \end{bmatrix} = 0$$

$$\begin{bmatrix} 28 & 16 & -29 \\ 33 & 16 & -18 \\ -40 & 5 & 79 \end{bmatrix} - 6\begin{bmatrix} 9 & 3 & -5 \\ 6 & 7 & -3 \\ -7 & 1 & 18 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$$

$$+ \begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

$$\begin{bmatrix} 28-54+1+25 & 16-18+2+0 & -29+30-1+0 \\ 33-36+3+0 & 16-42+1+25 & -18+18+0+0 \\ -40+42-2+0 & 5-6+1+0 & 79-108+4+20 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = RHS.$$

Cayley Hamilton theorem is verified

Finding A^{-1}

Consider,

$$A^3 - 6A^2 + A + 25\bar{I} = 0$$

Multiply both sides by A^{-1}

$$A^{-1}(A^3 - 6A^2 + A + 25\bar{I}) = 0$$

$$A^2 - 6A + \bar{I} + 25A^{-1} = 0$$

$$A^{-1} = \frac{1}{25}(6A - A^2 - \bar{I})$$

$$= \frac{1}{25}\left[6\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}^2 - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$= \frac{1}{25}\begin{bmatrix} 6-9-1 & 12-3-0 & -6+5-0 \\ 18-6-0 & 6-7-1 & 0+3-0 \\ -12-7-0 & 6-1-0 & 24-18-1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{25}\begin{bmatrix} -4 & 9 & -1 \\ 12 & -2 & 3 \\ -5 & 5 & 5 \end{bmatrix}$$

To find A^4

Consider

$$A^3 - 6A^2 + A + 25I = 0$$

$$A^3 = 6A^2 - A - 25I$$

Multiply both sides by A

$$A^4 = 6A^3 - A^2 - 25A$$

$$A^4 = 6(6A^2 - A - 25I) - A^2 - 25A$$

$$A^4 = 35A^2 - 31A - 150I$$

$$A^4 = \begin{bmatrix} 134 & 43 & -144 \\ 117 & 64 & -105 \\ -183 & 4 & 356 \end{bmatrix}$$

6 Find A^{-1} and A^{4} for A by Cayley-Hamilton theorem. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

→ The Characteristic equation is

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(4-\lambda)(6-\lambda)-25]-2[2(6-\lambda)-15]$$
$$+3[10-3(4-\lambda)]=0$$

$$(1-\lambda)[24-4\lambda-6\lambda+\lambda^2-25]-2[12-2\lambda-15]$$
$$+3[10-12+3\lambda]=0$$

$$(1-\lambda)[\lambda^2-10\lambda-1]-2[-2\lambda-3]+3[-2+3\lambda]=0$$

$$\lambda^2-\cancel{10\lambda}-1-\lambda^3+10\lambda^2+\cancel{\lambda}+4\lambda+6-6+9\lambda=0$$

$$-\lambda^3+11\lambda^2+4\lambda-11=0$$

$$\Rightarrow \lambda^3-11\lambda^2-4\lambda+11=0$$

To verify

$$A^3-11A^2-4A+1I=0$$

LHS

$$\begin{bmatrix}1&2&3\\2&4&5\\3&5&6\end{bmatrix}^3-11\begin{bmatrix}1&2&3\\2&4&5\\3&5&6\end{bmatrix}^2-4\begin{bmatrix}1&2&3\\2&4&5\\3&5&6\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}157&283&353\\283&510&636\\353&636&793\end{bmatrix}-\begin{bmatrix}154&275&341\\275&495&616\\341&616&770\end{bmatrix}$$

$$-\begin{bmatrix}4&8&12\\8&16&20\\12&20&24\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = RHS$$

Thus cayley hamilton theorem is verified

. To find A^{-1}

Consider

$$A^3 - 11A^2 - 4A + \underline{I} = 0$$

Multiply on both sides with A^{-1}

$$A^{-1}(A^3 - 11A^2 - 4A + I) = 0$$

$$A^2 - 11A - 4\underline{I} + A^{-1} = 0$$

$$A^{-1} = [11A - A^2 + 4\underline{I}]$$

$$= 11\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}^2 + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 22 & 33 \\ 22 & 44 & 55 \\ 33 & 55 & 66 \end{bmatrix} - \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

To ~~A~~ Find A^4

Consider

$$A^3 - 11A^2 - 4A + I = 0$$

$$A^3 = 11A^2 + 4A - I \quad \text{— (1)}$$

Multiply on both sides by A

$$A^4 = 11A^3 + 4A^2 - A$$

Substitute (1)

$$A^4 = 11[11A^2 + 4A - I] + 4A^2 - A$$

$$= 121A^2 + 44A - 11I + 4A^2 - A$$

$$= 125A^2 + 43A - 11I$$

$$\Rightarrow A^4 = 125A^2 + 43A - 11I$$

$$A^4 = \begin{bmatrix} 1782 & 3211 & 4004 \\ 3211 & 5786 & 7215 \\ 4004 & 7215 & 8997 \end{bmatrix}$$

8) Diagonalize the matrix & hence find A^4

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Ans) The Given matrix is symmetric

Therefore we can apply orthogonal transformation.

The Characteristic Equation is

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(5-\lambda)(1-\lambda)-1] - 1[1(1-\lambda)-3]$$
$$+3[1-3(5-\lambda)] = 0$$

$$(1-\lambda)[\lambda^2 - 6\lambda + 4] - 1[-\lambda-2] + 3[3\lambda - 14] = 0$$

$$-\lambda^3 + 7\lambda^2 - 36 = 0$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0$$

$\lambda = -2, 6, 3$

Case - 1

$\lambda = -2$

$(A + 2I)X = 0$

$$\left[\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 7 & 1 \\ 3 & 1 & 3 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$; $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 7 & 1 \\ 0 & -20 & 0 \\ 0 & -20 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 7 & 1 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 / -20$

$$\begin{bmatrix} 1 & 7 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $z = k_1$

$x + 7y + z = 0$

$\boxed{x = -k_1}$

$\Rightarrow \boxed{y = 0}$

The eigen vector for $\lambda = -2$ is

$$X_1' = \begin{bmatrix} -k_1 \\ 0 \\ k_1 \end{bmatrix} = -k_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\text{Length} = \sqrt{1^2 + 0^2 (-1)^2}$$
$$= \sqrt{2}$$

The normalized eigen vector for $\lambda = -2$

is $X_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$

· Case - 2

$\lambda = 6$

$(A - 6I)X = 0$

$$\left[\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & -1 & 1 \\ -5 & 1 & 3 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 5R_1$; $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -4 & 8 \\ 0 & 4 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 / -4$; $R_3 \to R_3 / 4$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $\boxed{z = k_2}$

$x - y + z = 0$

$x - 2k_2 + k_2 = 0$

$y - 2z = 0$

$\Rightarrow \boxed{y = 2k_2}$

$\Rightarrow \boxed{x = k_2}$

The eigen vector for $\lambda = 6$ is

$$X_2' = \begin{bmatrix} k_2 \\ 2k_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{Length} = \sqrt{1^2 + 2^2 + 1^2}$$
$$= \sqrt{6}.$$

The normalized eigen vector for $\lambda = 6$ is

$$X_2 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

Case - 3

$\lambda = 3$

$(A - 3\bar{I})X = 0$

$$\left[\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 3 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 2R_1$; $R_3 \rightarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $\boxed{z = k_3}$

$5y + 5z = 0$

$5y = -5k_3$

$\Rightarrow \boxed{y = -k_3}$

$x + 2y + z = 0$

$x - 2k_3 + k_3 = 0$

$\Rightarrow \boxed{x = k_3}$

The eigen vector for $\lambda = 3$ is

$$X_3' = \begin{bmatrix} k_3 \\ -k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Length $= \sqrt{1^2 + (-1)^2 + 1^2}$

$= \sqrt{3}$

The normalized eigen vector for $\lambda = 3$

is $X_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

Modal matrix,

$$B = [X_1 \quad X_2 \quad X_3]$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

B is orthogonal

① $X_1^T X_2 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

$= \frac{1}{\sqrt{12}} - \frac{1}{\sqrt{12}} = 0$

② $X_2^T X_3 = \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$= \frac{1}{\sqrt{18}} - \frac{2}{\sqrt{18}} + \frac{1}{\sqrt{18}} = 0$

③ $X_1^T X_3 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$= \frac{1}{\sqrt{6}} \quad 0 \quad - \frac{1}{\sqrt{6}} = 0$

$$\Delta = B^T A B$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}^T \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A^n = B \Delta^n B^{-1}$$

$$A^4 = B \Delta^4 B^{-1}$$

$$A^4 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}^4 \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 16 & 0 & 0 \\ 0 & 1296 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}^{-1}$$

$$A^4 = \begin{bmatrix} 251 & 405 & 235 \\ 408 & 891 & 405 \\ 235 & 408 & 251 \end{bmatrix}$$

Q Reduce the Quadratic form $3x^2+2y^2+3z^2-2xy-2yz$ into canonical form by orthogonal transformation

Ans Consider

$$3x^2+2y^2+3z^2-2xy-2yz$$

Compare with

$$a_{11}x_1^2+a_{22}x_2^2+a_{33}x_3^2+2a_{12}x_1x_2+2a_{23}x_2x_3+2a_{13}x_1x_3$$

$x_1=x$

$x_2=y$

$x_3=z$

$a_{11}=3$ $a_{22}=2$ $a_{33}=3$ $2a_{12}=-2$

$a_{12}=-1$

$2a_{23}=-2$ $2a_{13}=0$ $a_{13}=0$

$a_{23}=-1$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

The Characteristic equation is

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(2-\lambda)(3-\lambda) - 1\right] + 1\left[-1(3-\lambda)\right] = 0$$

$$(3-\lambda)\left[6 - 5\lambda + \lambda^2 - 1 - 1\right] = 0$$

$$(3-\lambda)\left[\lambda^2 - 5\lambda + 4\right] = 0$$

$\lambda = 3$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$\lambda^2 - 4\lambda - \lambda + 4 = 0$$

$$(\lambda - 4)(\lambda - 1)$$

$$\lambda = 4, 1$$

The eigen values are 1, 4, 3

Case - 1

$\lambda = 1$

$(A - I)x = 0$

$$\left[\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \longleftrightarrow R_2$

$$\begin{bmatrix} -1 & 1 & -1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 2R_1$

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $z = k_1$

$y - 2z = 0$
$\Rightarrow y = 2k_1$

$-x + y - z = 0$
$\Rightarrow x = k_1$

The eigen vector for $\lambda = 1$ is

$$x_1' = \begin{bmatrix} k_1 \\ 2k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{Length} = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$$

The normalized eigen vector for $\lambda = 1$ is

$$x_1 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

- Case - 2

$\lambda = 4$

$$(A - 4I)x = 0$$

$$\left[\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $z = k_2$ $\quad -y - z = 0 \Rightarrow y = -k_2$ $\quad -x - y = 0 \Rightarrow x = k_2$

The eigen vector for $\lambda = 4$ is

$$X_2' = \begin{bmatrix} k_2 \\ -k_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{Length} = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$

The normalized eigen vector for $\lambda = 4$ is

$$X_2 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

- Case - 3

$\lambda = 3$

$$(A - 3I)X = 0$$

$$\left[\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to R_1/-1 \quad R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $z = k_3$

$y = 0$

$x + y + z = 0$

$\Rightarrow \boxed{x = -k_3}$

- The Eigen vector for $\lambda = 3$ is

$$x_3' = \begin{bmatrix} -k_3 \\ 0 \\ k_3 \end{bmatrix} = -k_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\text{Length} = \sqrt{1^2 + 0^2 + (-1)^2}$$

$$= \sqrt{2}$$

- The normalized eigen vector for $\lambda = 3$ is

$$\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

· Modal matrix

$$B = [x_1 \quad x_2 \quad x_3]$$

$$= \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}$$

· To ensure B is orthogonal

(a) $x_1^T x_2 = \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = 1/\sqrt{18} - 2/\sqrt{18} + 1/\sqrt{18}$
$= 0$

(b) $x_2^T x_3 = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = 1/\sqrt{6} - 1/\sqrt{6}$
$= 0$

(c) $x_3^T x_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = 1/\sqrt{12} - \frac{1}{\sqrt{12}}$
$= 0$

B is orthogonal.

· $D = B^T A B$

$$= \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Canonical form $Y^T D Y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$= y_1^2 + 4y_2^2 + 3y_3^2$$

10. Reduce the Quadratic form $3x^2+5y^2+3z^2$
$-2xy-2yz+2zx$

to the canonical form by orthogonal transformation.

Ans

Consider

$$3x^2+5y^2+3z^2-2xy-2yz+2zx$$

compare with

$$a_{11}x_1^2+a_{22}x_2^2+a_{33}x_3^2+2a_{12}x_1x_2+2a_{23}x_2x_3+2a_{13}x_1x_3$$

$$x_1 = x \;;\; x_2 = y \;;\; x_3 = z$$

$$a_{11}x^2+a_{22}y^2+a_{33}z+2a_{12}xy+2a_{23}yz+2a_{13}xz$$

$a_{11} = 3$ $\quad$ $2a_{12} = -2$ $\quad$ $2a_{23} = -2$ $\quad$ $2a_{13} = 2$

$a_{22} = 5$ $\quad$ $a_{12} = -1$ $\quad$ $a_{23} = -1$ $\quad$ $a_{13} = 1$

$a_{33} = 3$ $\quad$ $a_{21} = -1$ $\quad$ $a_{32} = -1$ $\quad$ $a_{31} = 1$

By symmetry $a_{12}=a_{21}$; $a_{13}=a_{31}$; $a_{32}=a_{23}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation is

$$|A-\lambda I| = 0$$

$$\left| \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)[(5-\lambda)(3-\lambda) - (-1)(-1)] + 1[(-1)(3-\lambda) - (-1)] + 1[1-(5-\lambda)] = 0$$

$$(3-\lambda)[15 - 5\lambda - 3\lambda + \lambda^2 - 1] + 1[-3+\lambda+1] + 1[1-5+\lambda] = 0$$

$$(3-\lambda)[\lambda^2 - 8\lambda + 14] + 1[\lambda - 2] + 1[\lambda - 4] = 0$$

$$3\lambda^2 - 24\lambda + 42 - \lambda^3 + 8\lambda^2 - 14\lambda + \lambda - 2 + \lambda - 4 = 0$$

$$-\lambda^3 + 11\lambda^2 - 36\lambda + 36 = 0$$

$$\Rightarrow \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\lambda = 6, 3, 2$$

The Eigen values are 6, 3, 2.

Case - 1

$\lambda = 6$

$(A - 6I)x = 0$

$$\left[\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}\right] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} -1 & -1 & -1 \\ -3 & -1 & 1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to R_1 / -1$; $\cancel{R_2 \to R_2 \times -1}$

$$\begin{bmatrix} 1 & 1 & 1 \\ -3 & -1 & +1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 3R_1$; $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $\boxed{z = k_1}$

$2y + 4z = 0$

$2y = -4k_1$

$\boxed{y = -2k_1}$

$x + y + z = 0$

$x - 2k_1 + k_1 = 0$

$\boxed{x = k_1}$

The eigen vector for $\lambda = 6$ is

$$X_1' = \begin{bmatrix} k_1 \\ -2k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{Length} = \sqrt{1^2 + (-2)^2 + 1^2}$$
$$= \sqrt{6}$$

The normalized eigen vector for $\lambda = 6$ is

$$X_1 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

Case - 2

$\lambda = 3$

$(A - \lambda I)x = 0$

$\implies (A - 3I)x = 0$

$$\left[\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

~~$R_3 \to R_3$~~

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $\boxed{z = k_2}$

$-y + z = 0$

$\Rightarrow \boxed{y = k_2}$

$-x + 2y - z = 0$

$-x + 2k_2 - k_2 = 0$

$-x = -k_2$

$\Rightarrow \boxed{x = k_2}$

The eigen vector for $\lambda = 3$ is

$$X_2' = \begin{bmatrix} k_2 \\ k_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Length} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

The normalized eigen vector is

$$X_2 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

- Case - 3

$\lambda = 2$

$(A - 2I)X = 0$

$$\left[\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$; $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $\boxed{z = k_3}$ $\qquad$ $x - y + z = 0$

$\Rightarrow \boxed{y = 0}$ $\qquad$ $\boxed{x = -k_3}$

The eigen vector for $\lambda = 2$ is

$$X_3' = \begin{bmatrix} -k_3 \\ 0 \\ k_3 \end{bmatrix} = -k_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\text{Length} = \sqrt{1^2 + 0^2 (-1)^2}$$
$$= \sqrt{2}$$

The normalized eigen vector is

$$X_3 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

· Modal matrix

$$B = [X_1 \quad X_2 \quad X_3]$$

$$= \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}$$

· To ensure B is orthogonal

ⓐ $x_1^T x_2 = \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$$= \frac{1}{\sqrt{18}} - \frac{2}{\sqrt{18}} + \frac{1}{\sqrt{18}} = 0$$

ⓑ $x_2^T x_3 = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$

$$= \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} = 0$$

ⓒ $x_3^T x_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

$$= \frac{1}{\sqrt{12}} - \frac{1}{\sqrt{12}} = 0 .$$

B is orthogonal

$$D = B^T A B$$

$$= \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}^T \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Canonical form, $Y^T D Y$

$$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 6y_1^2 + 3y_2^2 + 2y_3^2$$