M-II QB Key

Unit-IV

Neha.K

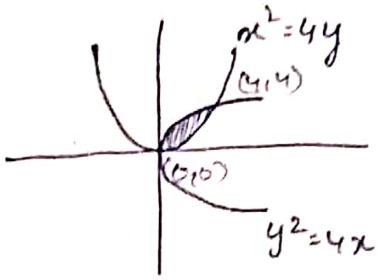
ECE-C

LAQ's:

1. $y^2 = 4x$ — ①

$x^2 = 4y$ — ②

Subs ② in ①



$$\left(\frac{x^2}{4}\right)^2 = 4x$$

$$\frac{x^4}{16} = 4x$$

$$x(x^3 - 64) = 0$$

$$x = 0, 4$$

At $x=0$, $y=0$ $(0,0)$

At $x=4$, $y=4$ $(4,4)$.

$\Rightarrow$ $x: 0 \to 4$

$y: \frac{x^2}{4} \to 2\sqrt{x}$

$$\iint_R y\,dx\,dy = \int_{x=0}^{4}\left(\int_{y=\frac{x^2}{4}}^{2\sqrt{x}} y\,dy\right)dx = \int_{x=0}^{4}\left[\frac{y^2}{2}\right]_{\frac{x^2}{4}}^{2\sqrt{x}} dx$$

$$= \int_{x=0}^{4}\left[\frac{\cancel{4}^{2}x}{\cancel{2}} - \frac{x^4}{32}\right]dx = \left[\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^5}{160}\right]_0^4$$

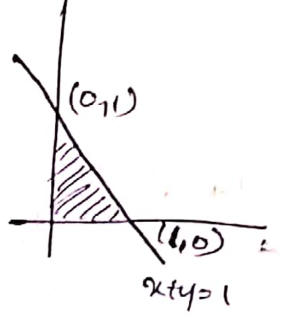
$$= 16 - \frac{1024}{160} = \frac{1536}{160}$$

$$= \frac{48}{5}$$

2. ~~At~~ $x+y=1$

At $x=0,\ y=1$ $(0,1)$

At $y=0,\ x=1$ $(1,0)$.



$x: 0 \to 1$

$y: 0 \to 1-x$.

$$\iint_R (x^2+y^2)\,dx\,dy = \int_{x=0}^{1}\left(\int_{y=0}^{1-x}(x^2+y^2)\,dy\right)dx$$

$$= \int_{x=0}^{1}\left[x^2y+\frac{y^3}{3}\right]_0^{1-x}dx$$

$$= \int_{x=0}^{1}\left(x^2(1-x)+\frac{(1-x)^3}{3}\right)dx$$

$$= \int_{x=0}^{1}\left(x^2-x^3+\frac{(1-x)^3}{3}\right)dx$$

$$= \left[\frac{x^3}{3}-\frac{x^4}{4}+\frac{(1-x)^4}{4(3)(-1)}\right]_0^1$$

$$= \left(\frac{1}{3}-\frac{1}{4}-0\right)-\left(\frac{1-0}{-12}\right)$$

$$= \frac{1}{3}-\frac{1}{4}+\frac{1}{12}$$

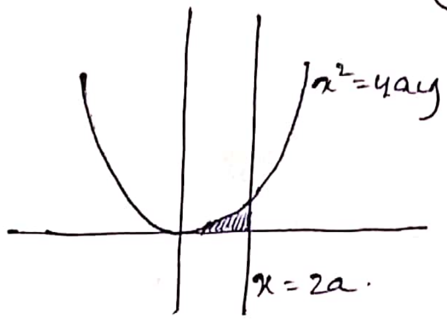
$$= \frac{4-3+1}{12}$$

$$= \frac{\cancel{2}\,1}{\cancel{12}\,6}$$

$$= \frac{1}{6}$$

3. $x^2 = 4ay$

$\Rightarrow y = \dfrac{x^2}{4a}$



$\therefore x : 0 \to 2a$

$y : 0 \to \dfrac{x^2}{4a}$

$$\iint_R xy\,dx\,dy = \int_{x=0}^{2a} \left(x \int_{y=0}^{\frac{x^2}{4a}} \cancel{x}y\,dy \right) dx$$

$$= \int_{x=0}^{2a} x \left[\frac{y^2}{2} \right]_0^{\frac{x^2}{4a}} dx = \int_{x=0}^{2a} x \left(\frac{x^2}{4a} \right)^2 \left(\frac{1}{2} \right) dx$$

$$= \int_{x=0}^{2a} x \left(\frac{x^4}{16a^2} \times \frac{1}{2} \right) dx$$

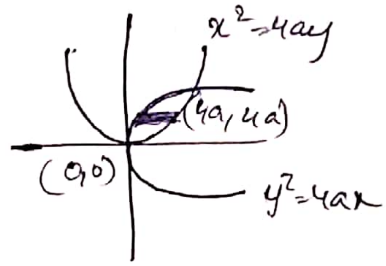
$$= \int_{x=0}^{2a} \frac{x^5}{32a^2}\,dx = \frac{1}{32a^2} \int_{x=0}^{2a} x^5\,dx$$

$$= \frac{1}{32a^2} \left[\frac{x^6}{6} \right]_0^{2a} = \frac{1}{32a^2} \left[\frac{(2a)^6}{6} \right]$$

$$= \frac{1}{\cancel{32}\cancel{a^2}} \; \frac{\cancel{64}^{\,2}\, a^{\cancel{6}\,4}}{\cancel{6}\,3}$$

$$= \frac{a^4}{3}$$

4. In the given integral, x limits are fixed and y varies.



$y = \frac{x^2}{4a}$ $\quad$ $y = 2\sqrt{ax}$

$\Rightarrow x^2 = 4ay$ (1) $\quad \Rightarrow y^2 = 4ax$ (2)

Substitute $y = \frac{x^2}{4a}$ in (2)

$$\frac{x^{\cancel{4}3}}{16a^2} = 4a\cancel{x}$$

$$x^3 = 64a^3$$

$$x^3 - (4a)^3 = 0 \Rightarrow x = 4a.$$

At $\quad y=0, x=0 \quad (0,0)$

At $\quad y=4a, x=4a \quad (4a,4a)$

$y: 0 \to 4a \qquad x: \frac{y^2}{4a} \to 2\sqrt{ay}$

After changing the order of integration, the given integral becomes:

$$\int_{y=0}^{4a}\left(\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\right)dy = \int_{y=0}^{4a}\left[x\right]_{y^2/4a}^{2\sqrt{ay}} dy = \int_{y=0}^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right)dy$$

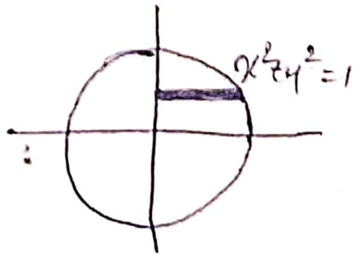
$$= \left[2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{1}{4a}\,\frac{y^3}{3}\right]_0^{4a}$$

$$= \frac{4\sqrt{a}\,(4a)^{3/2}}{3} - \frac{(4a)^3}{12a}$$

$$= \frac{4\sqrt{a}\;8a\sqrt{a}}{3} - \frac{\cancel{64}^{16}\,a^{\cancel{3}2}}{3\cancel{12}\cancel{a}}$$

$$= \frac{32a^2 - 16a^2}{3} = \frac{16a^2}{3}$$

5.



$y=0 \qquad y=\sqrt{1-x^2}$

$y^2 = 1-x^2$

$x^2+y^2=1 \qquad$ At $x=0,\ y=1$

$y=0,\ x=1$

$y: 0 \to 1 \qquad x: 0 \to \sqrt{1-y^2}$

After changing the order of integration, the given integral becomes

$$\int_{y=0}^{1}\left(\int_{x=0}^{\sqrt{1-y^2}} y^2\,dx\right)dy = \int_{y=0}^{1} y^2\,[x]_0^{\sqrt{1-y^2}}\,dy$$

$$= \int_{y=0}^{1} y^2\sqrt{1-y^2}\,dy$$

let $y=\sin\theta \Rightarrow dy=\cos\theta\,d\theta$

At $y=0\ \theta=0 \quad y=1\ \theta=\pi/2$

$$= \int_{y=0}^{\pi/2} \sin^2\theta\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta = \int_{y=0}^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$$

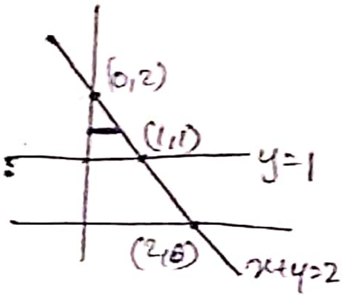
$$= \frac{1}{4}\int_0^{\pi/2}\sin^2 2\theta\,d\theta = \frac{1}{4}\int_0^{\pi/2}\frac{1-\cos 4\theta}{2}\,d\theta$$

$$= \frac{1}{4}\left[\frac{\theta}{2}-\frac{\sin 4\theta}{8}\right]_0^{\pi/2} = \frac{1}{4}\left[\frac{\pi}{4}-0\right]$$

$$= \frac{\pi}{16}$$

6. $y=1$ $\quad y=2-x$

$x+y=2$



At $y=1$, $x=2-1=1$ $(1,1)$.

$y: 1 \to 2 \quad x: 0 \to 2-y.$

After changing the order of integration, the given integral becomes:

$$\int_{y=1}^{2} y\left(\int_{x=0}^{2-y} x\,dx\right)dy = \int_{y=1}^{2} y\left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \int_{y=1}^{2} y\,\frac{(2-y)^2}{2}\,dy = \int_{y=1}^{2} \frac{y(4-4y+y^2)}{2}\,dy$$

$$= \frac{1}{2}\int_{y=1}^{2}(4y-4y^2+y^3)\,dy = \frac{1}{2}\left[\frac{\cancel{4}y^2}{\cancel{2}} - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{2}\left[2y^2 - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{2}\left[\left(2(2^2) - \frac{4(2^3)}{3} + \frac{2^4}{4}\right) - \left(2(1) - \frac{4(1)}{3} + \frac{1}{4}\right)\right]$$

$$= \frac{1}{2}\left[\left(8 - \frac{32}{3} + \frac{\cancel{16}^{4}}{\cancel{4}}\right) - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right]$$

$$= \frac{1}{2}\left[\frac{96-128+48-24+16-3}{12}\right] = \frac{1}{2}\left[\frac{5}{12}\right] = \frac{5}{24}$$

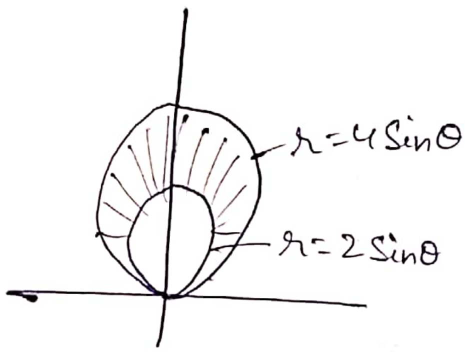
7. $$\int_{\theta=0}^{\pi}\left(\int_{r=0}^{a\sin\theta} r\,dr\right)d\theta = \int_{\theta=0}^{\pi}\left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta$$

$$= \int_0^{\pi} \frac{a^2\sin^2\theta}{2}\,d\theta = \frac{a^2}{2}\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \frac{a^2}{2}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi} = \frac{a^2}{2}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{a^2\pi}{4}$$

8.



$\theta : 0 \to \pi \qquad r : 2\sin\theta \to 4\sin\theta.$

$$\iint_R r^3\,dr\,d\theta = \int_{\theta=0}^{\pi}\left[\int_{r=2\sin\theta}^{4\sin\theta} r^3\,dr\right]d\theta = \int_0^{\pi}\left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta.$$

$$= \frac{1}{4}\int_0^{\pi}(256\sin^4\theta - 16\sin^4\theta)\,d\theta.$$

$$= \frac{16}{4}\int_0^{\pi} 15\sin^4\theta\,d\theta.$$

$$= 60\int_0^{\pi}(\sin^2\theta)^2\,d\theta.$$

$$= 60 \int_0^{\pi} \left(\frac{1-\cos 2\theta}{2} \right)^2 d\theta = \frac{15}{\cancel{60}}{\cancel{4}} \int_0^{\pi} (1 - 2\cos\theta + \cos^2 2\theta)\, d\theta.$$

$$= 15 \left[\theta - \frac{2\sin\theta}{2} \right]_0^{\pi} + 15 \int_0^{\pi} \left(\frac{1+\cos 4\theta}{2} \right) d\theta$$

$$= 15(\pi) + 15 \left[\frac{\theta}{2} + \frac{1}{2} \frac{\sin 4\theta}{4} \right]_0^{\pi} = 15\pi + \frac{15\pi}{2}$$

$$= \frac{45\pi}{2}$$

9. $x = r\cos\theta \qquad y = r\sin\theta$

$y = \sqrt{a^2 - x^2}$

$y^2 = a^2 - x^2 \rightarrow x^2 + y^2 = a^2$

$r^2 = a^2$

$r = \pm a \qquad r = a.$

$r : 0 \rightarrow a \qquad \theta : 0 \rightarrow \pi/2.$

$x = 0$

$r\cos\theta = 0$

$\theta = \pi/2$

$x = a$

$r\cos\theta = a$

$\cos\theta = 1 \qquad \theta = 0.$

$$\int_{r=0}^{a} \int_{\theta=0}^{\pi/2} r\sin\theta \cdot \sqrt{r^2}\; r\,dr\,d\theta = \int_{r=0}^{a} r^3 \left[\int_{\theta=0}^{\pi/2} \sin\theta\, d\theta \right] dr$$

$$= \int_{r=0}^{a} r^3 \left[-\cos\theta \right]_0^{\pi/2} dr = \int_{r=0}^{a} r^3\, dr = \left[\frac{r^4}{4} \right]_0^{a}$$

$$= \frac{a^4}{4}$$

10. $\int_{x=0}^{1}\int_{z=y^2}^{1}\int_{y=0}^{1-x} x\,dz\,dx\,dy = \int_{z=0}^{1}\int_{x=y^2}^{1} x\left(\int_{y=0}^{1-x} dy\right)dx\,dz$

$$= \int_{z=0}^{1}\int_{x=y^2}^{1} x\,[y]_0^{1-x}\,dx\,dz = \int_{z=0}^{1}\left(\int_{x=y^2}^{1} x(1-x)\,dx\right)dz$$

$$= \int_{z=0}^{1}\left(\int_{x=y^2}^{1} (x-x^2)\,dx\right)dz = \int_{z=0}^{1}\left[\frac{x^2}{2}-\frac{x^3}{3}\right]$$

$$\int_{y=0}^{1}\int_{x=y^2}^{1} x\int_{z=0}^{1-x} dz\,dx\,dy = \int_{y=0}^{1}\left(\int_{x=y^2}^{1} x(1-x)\,dx\right)dy$$

$$= \int_{y=0}^{1}\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{y^2}^{1} dy = \int_{y=0}^{1}\left[\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{y^4}{2}-\frac{y^6}{3}\right)\right]dy$$

$$= \left[\frac{y}{2}-\frac{y}{3}-\frac{y^5}{10}+\frac{y^7}{21}\right]_0^1$$

$$= \frac{1}{2}-\frac{1}{3}-\frac{1}{10}+\frac{1}{21}$$

$$= \frac{105-70-21+10}{210}$$

$$= \frac{\cancel{24}\;24}{\cancel{210}\;\cancel{105}\;35} = \frac{4}{35}$$

11. $\displaystyle\int_{z=-1}^{1}\int_{x=0}^{z}\int_{y=(x-z)}^{x+z}(x+y+z)\,dx\,dy\,dz$

$$= \int_{z=-1}^{1}\int_{x=0}^{z}\left[xy+\frac{y^2}{2}+zy\right]_{x-z}^{x+z} dx\,dz$$

$$= \int_{z=-1}^{1}\int_{x=0}^{z}\left[\left(x(x+z)+\frac{(x+z)^2}{2}+z(x+z)\right)-\left(x(x-z)+\frac{(x-z)^2}{2}+z(x-z)\right)\right]dx\,dz$$

$$= \int_{z=-1}^{1}\int_{x=0}^{z}\left[\left(x^2+xz+\frac{x^2+2xz+z^2}{2}+zx+z^2\right)-\left(x^2-xz+\frac{x^2-2xz+z^2}{2}+zx-z^2\right)\right]dx\,dz$$

$$= \int_{z=-1}^{1}\int_{x=0}^{z}\left(\cancel{x^2}+xz+\cancel{\frac{x^2}{2}}+xz+\cancel{\frac{z^2}{2}}+\cancel{zx}+z^2-\cancel{x^2}+xz-\cancel{\frac{x^2}{2}}+xz-\cancel{\frac{z^2}{2}}-\cancel{zx}+z^2\right)dx\,dz$$

$$= \int_{z=-1}^{1}\int_{x=0}^{z}(2z^2+4xz)\,dx\,dz = \int_{z=-1}^{1}\left[2xz^2+\frac{4x^2z}{2}\right]_0^z dz$$

$$= \int_{z=-1}^{1}\left(2z^3+\frac{\cancel{4}^{2}z^3}{\cancel{2}}\right)dz = \int_{z=-1}^{1}4z^3\,dz$$

$$= 4\left[\frac{z^4}{4}\right]_{-1}^{1} = 4\left[\frac{1}{4}-\frac{1}{4}\right]$$

$$= 0$$

12. $\phi = x^2yz + 4xz^2$

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2yz + 4xz^2)$$

$$= 2xyz\,i + 4z^2 i + x^2z\,j + x^2y\,k + 8xz\,k.$$

$$\nabla\phi_{(1,-2,-1)} = 2(1)(-2)(-1)i + 4(1)i + (1)(-1)j + (1)(-2)k + 8(1)(-1)k$$

$$= 4i + 4i - j - 2k - 8k$$

$$= 8i - j - 10k.$$

$$\bar{e} = \frac{2i - j - k}{\sqrt{4+1+1}} = \frac{2i - j - k}{\sqrt{6}}$$

$$d.d = \bar{e}\cdot\nabla\phi = \frac{2i - j - k}{\sqrt{6}}(8i - j - 10k)$$

$$= \frac{16+1+10}{\sqrt{6}} = \frac{27}{\sqrt{6}}$$

13. $\phi = 4xy^2 + 2x^2yz$

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(4xy^2 + 2x^2yz)$$

$$= i(4y^2 + 4xyz) + j(8xy + 2x^2z) + k(2x^2y)$$

$$\nabla\phi_{(1,2,3)} = i(4(2^2) + 4(1)(2)(3)) + j(8(1)(2) + 2(1)(3)) + k(2(1)(2))$$

$$= 40i + 22j + 4k.$$

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$$\bar{e} = \frac{(5i+4k)-(i+2j+3k)}{\sqrt{(5-1)^2+(2)^2+(4-3)^2}} = \frac{4i-2j+k}{\sqrt{21}}$$

$$dd = \nabla\phi\cdot\bar{e} = \frac{(40i+22j+4k)(4i-2j+k)}{\sqrt{21}}$$

$$= \frac{160-44+4}{\sqrt{21}} = \frac{120}{\sqrt{21}}$$

14. $\phi = x^2yz + 4xz^2$

$\nabla\phi = 8i - j - 10k$ (Refer LAQ 12)

$f(x,y,z) = x\log z - y^2$

$\nabla f = i(\log z) - j(2y) + k\left(\frac{x}{z}\right)$

$\nabla f_{(-1,2,1)} = (\log 1)\,i - 4j + k(-1)$

$= -4j - k$

$|\nabla f| = \sqrt{16+1} = \sqrt{17}$

$$\text{normal} = \bar{e} = \frac{-4j-k}{\sqrt{17}}$$

$$dd = \nabla\phi\cdot\bar{e} = \frac{(8i-j-10k)(-4j-k)}{\sqrt{17}}$$

$$= \frac{4+10}{\sqrt{17}} = \frac{14}{\sqrt{17}}$$

15. $f = xy^2 + yz^3$

$$\nabla f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) (xy^2 + yz^3)$$

$$\nabla f = i(y^2) + j(2xy + z^3) + k(3yz^2)$$

$$\nabla f_{(2,-1,1)} = i(1) + j(2(2)(-1) + 1) + k(3(-1)(1))$$

$$= i + j(-\cancel{8}4 + 1) + k(-3)$$

$$= i - 3j - 3k.$$

$$\bar{e} = \frac{i + 2j + 2k}{\sqrt{1+4+4}} = \frac{i + 2j + 2k}{\sqrt{9}}$$

$$d.d = \nabla f \cdot \bar{e} = \frac{(i - 3j - 3k)(i + 2j + 2k)}{3}$$

$$= \frac{1 - 6 - 6}{3} = \frac{-11}{3}$$

16. $\phi = xy - z^2$

$$\nabla \phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) (xy - z^2)$$

$$= i(y) + j(x) + k(-2z)$$

$$= yi + xj - 2zk.$$

$$\nabla \phi_1 = i + 4j - 4k$$

$$\nabla \phi_2 = 3i + 3j + 6k.$$

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|} = \frac{(i+4j-4k)\cdot(3i+3j+6k)}{\sqrt{1+16+16}\,\sqrt{9+9+36}}$$

$$= \frac{3+12-24}{\sqrt{33}\,\sqrt{54}} = \frac{-9}{\sqrt{33}\,\sqrt{54}}$$

17. $\vec{r} = xi + yj + zk$.

$r^2 = x^2 + y^2 + z^2$

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$

$2r\frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}$.

$$\operatorname{div}\vec{f} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\left(r^n(xi+yj+zk)\right)$$

$$= \frac{\partial}{\partial x}(\underset{u}{r^n}\underset{v}{x}) + \frac{\partial}{\partial y}(\underset{u}{r^n}\underset{v}{y}) + \frac{\partial}{\partial z}(\underset{u}{r^n}\underset{v}{z})$$

$$= x n r^{n-1}\frac{\partial r}{\partial x} + r^n + y n r^{n-1}\frac{\partial r}{\partial y} + r^n + z n r^{n-1}\frac{\partial r}{\partial z} + r^n$$

$$= n r^{n-1}\left(x\left(\frac{x}{r}\right) + y\left(\frac{y}{r}\right) + z\left(\frac{z}{r}\right)\right) + 3r^n$$

$$= n r^{n} \left(\cancel{\frac{x^2+y^2+z^2}{r^2}}\right) + 3r^n = n r^{n} + 3r^n$$

Solenoidal $\Rightarrow \operatorname{div}\vec{f} = 0 \Rightarrow nr^n = -3r^n$

$$\boxed{n = -3}$$

18. irrotational $\Rightarrow$ curl $\bar{f} = 0$.

$$\text{curl } \bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= i(-x+x) + j(-y+y) + k(-z+z) = \underline{\underline{0}}$$

Hence proved.

$$\bar{f} = \nabla \phi$$

$$(x^2 - yz)\,i + (y^2 - zx)\,j + (z^2 - xy)\,k = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}.$$

$$\frac{\partial \phi}{\partial x} = x^2 - yz \qquad \frac{\partial \phi}{\partial y} = y^2 - zx \qquad \frac{\partial \phi}{\partial z} = z^2 - xy$$

$$\phi_1 = \int (x^2 - yz)\,dx \qquad \phi_2 = \int (y^2 - zx)\,dy \qquad \phi_3 = \int (z^2 - xy)\,dz$$

$$\phi_1 = \frac{x^3}{3} - xyz + c_1 \qquad \phi_2 = \frac{y^3}{3} - xyz + c_2 \qquad \phi_3 = \frac{z^3}{3} - xyz + c_3$$

$$\phi = \underline{\underline{\frac{x^3 + y^3 + z^3}{3} - 3xyz + c}}.$$

19. Irrotational $\Rightarrow$ curl $\bar{f} = 0$

$$\text{curl } \bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2 y & 0 \end{vmatrix}$$

$$= i(0) - j(0) + k(2xy - 2xy) = \underline{0}$$

Hence proved

$$\bar{f} = \nabla\phi$$

$$(x^2+xy^2)i + (y^2+x^2y)j = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = x^2+y^2 \qquad \frac{\partial\phi}{\partial y} = y^2+x^2y \qquad \frac{\partial\phi}{\partial z} = 0.$$

$$\phi_1 = \int(x^2+y^2)\,dx \qquad \phi_2 = \int(y^2+x^2y)\,dy$$

$$\phi_1 = \frac{x^3}{3} + \frac{x^2y^2}{2} + c_1 \qquad \phi_2 = \frac{y^3}{3} + \frac{x^2y^2}{2} + c_2$$

$$\phi = \frac{x^3+y^3}{3} + x^2y^2 + C.$$

20. $\bar{r} = xi + yj + zk \qquad r^m = (xi+yj+zk)^m$.

LHS

$$\text{grad } r^m = \nabla(r^m) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cancel{(xi+yj+zk)^m}\, r^m$$

$$= mr^{m-1}\frac{\partial r}{\partial x} + mr^{m-1}\frac{\partial r}{\partial y} + mr^{m-1}\frac{\partial r}{\partial z}.$$

$$r^2 = x^2+y^2+z^2$$

$$\cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\cancel{2}r\frac{\partial r}{\partial y} = \cancel{2}y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\cancel{2}r\frac{\partial r}{\partial z} = \cancel{2}z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}.$$

$$\Rightarrow \text{grad } r^m = mr^{m-1}\left(\frac{x}{r}\right) + mr^{m-1}\left(\frac{y}{r}\right) + mr^{m-1}\left(\frac{z}{r}\right)$$

$$= mr^{m-2}(x+y+z) = mr^{m-2}(\bar{r}).$$

$$\operatorname{div}(\operatorname{grad} r^m) = \frac{\partial}{\partial x}(m x r^{m-2}) + \frac{\partial}{\partial y}(m y r^{m-2}) + \frac{\partial}{\partial z}(m z r^{m-2})$$

$$= m\left[\frac{\partial}{\partial x}(x r^{m-2}) + \frac{\partial}{\partial y}(y r^{m-2}) + \frac{\partial}{\partial z}(z r^{m-2})\right]$$

$$= m\left[\left(x(m-2)r^{m-3}\frac{\partial r}{\partial x} + r^{m-2}\right) + \left(y(m-2)r^{m-3}\frac{\partial r}{\partial y} + r^{m-2}\right) + \left(z(m-2)r^{m-3}\frac{\partial r}{\partial z} + r^{m-2}\right)\right]$$

$$= m\left[r^{m-2}\left[x(m-2)r^{-1}\left(\frac{x}{r}\right) + 1 + y(m-2)r^{-1}\left(\frac{y}{r}\right) + 1 + z(m-2)r^{-1}\left(\frac{z}{r}\right) + 1\right]\right]$$

$$= m r^{m-2}\left[(m-2)\left(\frac{x^2}{r^2}\right) + (m-2)\frac{y^2}{r^2} + (m-2)\frac{z^2}{r^2} + 3\right]$$

$$= m r^{m-2}\left[(m-2)\left(\cancel{\frac{x^2+y^2+z^2}{r^2}}\right) + 3\right]$$

$$= m r^{m-2}\left[m-2+3\right]$$

$$= m r^{m-2}\left[m+1\right]$$

$$= m(m+1)r^{m-2} = RHS$$

Hence Proved!