Unit - V LAQ's.

1. (i) G.T

$$x+y+z = u \quad —① $$
$$y+z = uv \quad —②$$
$$z = uvw \quad —③$$

considering ①

$x+y+z = u$

$x+uv = u$ ($\because$ from ②)

$x = u-uv$

$x = u(1-v)$

considering ②

$y+z = uv$

$y+uvw = uv$ ($\because$ from ③)

$y = uv - uvw$

$y = uv(1-w)$

So,

$x = u(1-v)$ $\left(\because \frac{d}{dx}(x) = 1\right)$

$\frac{\partial x}{\partial u} = 1-v$

$\frac{\partial x}{\partial v} = u(-1) = -u$

$\frac{\partial x}{\partial w} = 0$

$y = uv(1-w)$

$\frac{\partial y}{\partial u} = v(1-w)$

$\frac{\partial y}{\partial v} = u(1-w)$

$\frac{\partial y}{\partial w} = uv(-1) = -uv$

$z = uvw$

$\frac{\partial z}{\partial u} = vw$

$\frac{\partial z}{\partial v} = uw$

$\frac{\partial z}{\partial w} = uv$

w.k.T
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} u-uv & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$= (1-v)\left[uv(u-uw)+(u^2v)(uw)\right] + u\left[(v-vw)uv + (uv)(vw)\right] + 0$

$= (1-v)\left[u^2v - \cancel{u^2vw} + \cancel{u^2vw}\right] + u\left[uv^2 - \cancel{uv^2w} + \cancel{uv^2w}\right]$

$= (1-v)(u^2v) + u(uv^2) = \cancel{u^2v} - \cancel{u^2v^2} + u^2v^2 = u^2v$

(iii) G.T $u = \frac{yz}{x}$ $v = \frac{zx}{y}$

$\frac{\partial u}{\partial x} = yz\left(\frac{-1}{x^2}\right)$ $\left(\because \frac{\partial}{\partial x}\left(\frac{1}{x}\right) = \frac{-1}{x^2}\right)$

$\frac{\partial u}{\partial y} = \frac{z}{x}(1)$ $\left(\because \frac{d}{dx}(x) = 1\right)$

$\frac{\partial u}{\partial z} = \frac{y}{x}(1)$

$\frac{\partial v}{\partial x} = \frac{z}{y}(1)$

$\frac{\partial v}{\partial y} = zx\left(\frac{-1}{y^2}\right)$

$\frac{\partial v}{\partial z} = \frac{x}{y}(1)$

$w = \frac{xy}{z}$

$\frac{\partial w}{\partial x} = \frac{y}{z}(1)$

$\frac{\partial w}{\partial y} = \frac{x}{z}(1)$

$\frac{\partial w}{\partial z} = xy\left(\frac{-1}{z^2}\right)$

W.K.T $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

$= \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$

$= \frac{1}{xyz}\begin{vmatrix} -\frac{yz}{x} & z & y \\ z & -\frac{zx}{y} & x \\ y & x & -\frac{xy}{z} \end{vmatrix}$

$= \frac{1}{xyz}\left[-\frac{yz}{x}\left(\frac{x^2yz}{yz} - x^2\right) - z\left(-\frac{xyz}{z} - xy\right) + y\left(xz + \frac{xyz}{y}\right)\right]$

$= \frac{1}{xyz}\left[-\frac{yz}{x}(\cancel{x^2} - \cancel{x^2}) - z(-xy - xy) + y(xz + xz)\right]$

$= \frac{1}{xyz}\left[2xyz + 2xyz\right] = \frac{4\cancel{xyz}}{\cancel{xyz}} = 4$

(iii) G.T $u = \frac{2yz}{x}$ ; $v = \frac{3zx}{y}$ ; $w = \frac{4xy}{z}$

$\frac{\partial u}{\partial x} = 2yz\left(\frac{-1}{x^2}\right)$ $\quad$ $\frac{\partial v}{\partial x} = \frac{3z}{y}(1)$ $\quad$ $\frac{\partial w}{\partial x} = \frac{4y}{z}(1)$

$\frac{\partial u}{\partial y} = \frac{2z}{x}(1)$ $\quad$ $\frac{\partial v}{\partial y} = 3zx\left(\frac{-1}{y^2}\right)$ $\quad$ $\frac{\partial w}{\partial y} = \frac{4x}{z}(1)$

$\frac{\partial u}{\partial z} = \frac{2y}{x}(1)$ $\quad$ $\frac{\partial v}{\partial z} = \frac{3x}{y}(1)$ $\quad$ $\frac{\partial w}{\partial z} = 4xy\left(\frac{-1}{z^2}\right)$

w.k.T $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

$$= \begin{vmatrix} -\frac{2yz}{x^2} & \frac{2z}{x} & \frac{2y}{x} \\ \frac{3z}{y} & -\frac{3zx}{y^2} & \frac{3x}{y} \\ \frac{4y}{z} & \frac{4x}{z} & -\frac{4xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{xyz}\begin{vmatrix} -\frac{2yz}{x} & 3z & 2y \\ 3z & -\frac{3zx}{y} & 3x \\ 4y & 4x & -\frac{4yx}{z} \end{vmatrix}$$

$$= \frac{1}{xyz}\left[-\frac{2yz}{x}\left(\frac{12x^2\cancel{yz}}{\cancel{yz}} - 12x^2\right) - 3z\left(\frac{-12xy\cancel{z}}{\cancel{z}} - 12xy\right) + 2y\left(12zx + \frac{12zx\cancel{y}}{\cancel{y}}\right)\right]$$

$$= \frac{1}{xyz}\left[0 - 3z(-24xy) + 2y(24xz)\right]$$

$$= \frac{1}{xyz}\left[+72xyz + 48xyz\right]$$

$$= \frac{120xyz}{xyz} = 120$$

w.k.T $\Rightarrow J \cdot J' = 1$

$J' = \frac{1}{J}$

$J' = \frac{1}{120}$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{120}.$$

(2) G.T $x = r\sin\theta\cos\phi$ $\quad y = r\sin\theta\sin\phi$ $\quad z = r\cos\theta$

$\frac{\partial x}{\partial r} = \sin\theta\cos\phi(1)$ $\quad \frac{\partial y}{\partial r} = \sin\theta\sin\phi(1)$ $\quad \frac{\partial z}{\partial r} = \cos\theta(1)$

$\frac{\partial x}{\partial \theta} = r\cos\phi(\cos\theta)$ $\quad \frac{\partial y}{\partial \theta} = r\sin\phi(\cos\theta)$ $\quad \frac{\partial z}{\partial \theta} = r(-\sin\theta)$

$\frac{\partial x}{\partial \phi} = r\sin\theta(-\sin\phi)$ $\quad \frac{\partial y}{\partial \phi} = r\sin\theta(\cos\phi)$ $\quad \frac{\partial z}{\partial \phi} = 0.$

$\because \frac{d}{dx}(x) = 1$

$\frac{d}{dx}(\sin\theta) = \cos\theta$

$\frac{d}{dx}(\cos\theta) = -\sin\theta.$

consider $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$

$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\sin\phi\cos\theta & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$

$= \cos\theta\left[(r\cos\theta\cos\phi)(r\sin\theta\cos\phi) + (r\sin\phi\cos\theta)(r\sin\theta\sin\phi)\right]$

$+ r\sin\theta\left[(\sin\theta\cos\phi)(r\sin\theta\cos\phi) + (r\sin\theta\sin\phi)(\sin\theta\sin\phi)\right]$

$= \cos\theta\left[r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi\right] + r\sin\theta\left[r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\right]$

$= \cos\theta \cdot r^2\sin\theta\cos\theta(\cos^2\phi + \sin^2\phi) + r\sin\theta \cdot r\sin^2\theta(\cos^2\phi + \sin^2\phi)$

$= r^2\sin\theta\cos^2\theta(1) + r^2\sin^3\theta(1)$ $\quad \because \sin^2\theta + \cos^2\theta = 1$

$= r^2\sin\theta\cos^2\theta + r^2\sin^3\theta$

$= r^2\sin\theta(\cos^2\theta + \sin^2\theta) = r^2\sin\theta.$ Hence proved.

W.K.T $J \times J' = 1 \Rightarrow J' = \frac{1}{J}$

$J' = \frac{1}{r^2\sin\theta}$

$\therefore \frac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = \frac{1}{r^2\sin\theta}$

(3.) G.T $u = x^2 - y^2$ $\quad v = 2xy$ & $x = r\cos\theta$ ; $y = r\sin\theta$

$$u = (r\cos\theta)^2 - (r\sin\theta)^2 \qquad v = 2(r\cos\theta)(r\sin\theta) \qquad \because \cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$= r^2\cos^2\theta - r^2\sin^2\theta \qquad v = r^2(2\sin\theta\cos\theta) \qquad 2\sin\theta\cos\theta = \sin 2\theta.$$

$$u = r^2(\cos^2\theta - \sin^2\theta) \qquad v = r^2\, 2\sin 2\theta$$

$$u = r^2\cos 2\theta$$

So, $u = r^2\cos 2\theta \qquad v = r^2\sin 2\theta \qquad \because \frac{d}{dx}(\cos 2x) = -2\sin 2x$

$$\frac{\partial u}{\partial r} = 2r(\cos 2\theta) \qquad \frac{\partial v}{\partial r} = 2r(\sin 2\theta) \qquad \frac{d}{dx}(x^n) = nx^{n-1}.$$

$$\frac{\partial u}{\partial \theta} = r^2(-2\sin 2\theta) \qquad \frac{\partial v}{\partial \theta} = r^2(2\cos 2\theta)$$

$$J\left(\frac{u,v}{r,\theta}\right) = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2r\cos 2\theta & 2r\sin 2\theta \\ -2r^2\sin 2\theta & 2r^2\cos 2\theta \end{vmatrix}$$

$$= (2r\cos 2\theta)(2r^2\cos 2\theta) - (2r\sin 2\theta)(-2r^2\sin 2\theta)$$

$$= 4r^3\cos^2 2\theta + 4r^3\sin^2 2\theta \qquad \because \sin^2\theta + \cos^2\theta = 1.$$

$$= 4r^3(\cos^2 2\theta + \sin^2 2\theta)$$

$$J\left(\frac{u,v}{r,\theta}\right) = 4r^3$$

Hence proved.

(4) G.T $x = r\cos\theta$ —(1) $y = r\sin\theta$ —(2)

squaring and adding (1) and (2)

$$x^2 = r^2\cos^2\theta$$
$$y^2 = r^2\sin^2\theta$$

$\because \sin^2\theta + \cos^2\theta = 1$

$$x^2+y^2 = r^2(\sin^2\theta+\cos^2\theta)$$

$$x^2+y^2 = r^2$$

The above equation is a circle equation with centre (0,0).

Consider. $x = r\cos\theta$ $y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta(1) \qquad \frac{\partial y}{\partial r} = \sin\theta(1)$$

$$\frac{\partial x}{\partial \theta} = r(-\sin\theta) \qquad \frac{\partial y}{\partial \theta} = r(\cos\theta)$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\sin^2\theta+\cos^2\theta) = 1$$

$\because \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

$\frac{d}{dx}(x^n) = nx^{n-1}$

consider $\frac{(1)}{(2)} \Rightarrow \frac{x}{y} = \frac{\cancel{r}\cos\theta}{\cancel{r}\sin\theta} \Rightarrow \frac{x}{y} = \cot\theta$.

$\theta = \cot^{-1}\left(\frac{x}{y}\right)$ $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

consider $r = \sqrt{x^2+y^2}$ $\theta = \cot^{-1}\left(\frac{x}{y}\right)$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}}(2x) \qquad \frac{\partial \theta}{\partial x} = \frac{-1}{1+\left(\frac{x}{y}\right)^2}(1/y) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial r}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}}(2y) \qquad \frac{\partial \theta}{\partial y} = \frac{-1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(\frac{-x}{y^2}\right) = \frac{x}{x^2+y^2}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{+x}{(x^2+y^2)} \end{vmatrix}$$

$$= \frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}} + \frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}} \quad (\because r = \sqrt{x^2+y^2})$$

$$= \frac{(x^2+y^2)}{(x^2+y^2)^{3/2}} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}.$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = r \cdot \frac{1}{r} = 1.$$

Hence proved.

(5.) (i) G.T $u = \frac{x^2-y^2}{x^2+y^2}$ ; $v = \frac{2xy}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{2x(x^2+y^2) - 2x(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{\cancel{2x^3} + 2xy^2 - \cancel{2x^3} + 2xy^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = \frac{4xy^2}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{2y(x^2+y^2) - 2x(2xy)}{(x^2+y^2)^2}$$

$$= \frac{2x^2y + 2y^3 - 4x^2y}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{2y^3 - 2x^2y}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2y(x^2+y^2) - 2y(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{-2x^2y - \cancel{2y^3} - 2x^2y + \cancel{2y^3}}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-4x^2y}{}$$

$$\frac{\partial v}{\partial y} = \frac{2x(x^2+y^2) - 2y(2xy)}{(x^2+y^2)^2}$$

$$= \frac{2x^3 + 2xy^2 - 4xy^2}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{-2xy^2 + 2x^3}{}$$

consider $\dfrac{\partial(u,v,\cancel{w})}{\partial(x,y,\cancel{z})} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \cancel{\dfrac{\partial u}{\partial z}} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \cancel{\dfrac{\partial v}{\partial z}} \\ \cancel{\dfrac{\partial w}{\partial x}} & \cancel{\dfrac{\partial w}{\partial y}} & \cancel{\dfrac{\partial w}{\partial z}} \end{vmatrix}$

$$= \begin{vmatrix} \dfrac{4xy^2}{(x^2+y^2)^2} & \dfrac{-4x^2y}{(x^2+y^2)^2} \\ \dfrac{2y^3-2x^2y}{(x^2+y^2)^2} & \dfrac{2x^3-2xy^2}{(x^2+y^2)^2} \end{vmatrix}$$

$$\frac{4xy^2(2x^3-2xy^2)}{(x^2+y^2)^2} + \frac{4x^2y(2y^3-2x^2y)}{(x^2+y^2)^2}$$

$$\frac{8\cancel{x^4y^2}-8\cancel{x^2y^4}+8\cancel{x^2y^4}-8\cancel{x^4y^2}}{(x^2+y^2)^2} = 0$$

$\dfrac{\partial(u,v)}{\partial(x,y)} = 0$ . the functions are funtionally dependent.

(iii) G.T $u = x+y-z$ $\quad v = x-y+z \quad w = x^2+y^2+z^2-2yz$

$\dfrac{\partial u}{\partial x} = 1 \qquad \dfrac{\partial v}{\partial x} = 1 \qquad \dfrac{\partial w}{\partial x} = 2x$

$\dfrac{\partial u}{\partial y} = 1 \qquad \dfrac{\partial v}{\partial y} = -1 \qquad \dfrac{\partial w}{\partial y} = 2y-2z$

$\dfrac{\partial u}{\partial z} = -1 \qquad \dfrac{\partial v}{\partial z} = 1 \qquad \dfrac{\partial w}{\partial z} = 2z-2y$

consider $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2y-2z & 2z-2y \end{vmatrix}$

$$= (-\cancel{2z}+\cancel{2y}-\cancel{2y}+\cancel{2z}) - (2z-2y-2x) - (2y-2z+2x)$$

$$= 2\cancel{x}+2\cancel{y}-2\cancel{z}-2\cancel{x}-2\cancel{y}+2\cancel{z} = 0$$

$\therefore \dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 0$ the functions are functionally dependent.

(iii) G·T $u = xy + yz + zx$ $\quad v = x^2 + y^2 + z^2$ $\quad w = x + y + z$

$$\frac{\partial u}{\partial x} = y + z \qquad \frac{\partial v}{\partial x} = 2x \qquad \frac{\partial w}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = x + z \qquad \frac{\partial v}{\partial y} = 2y \qquad \frac{\partial w}{\partial y} = 1$$

$$\frac{\partial u}{\partial z} = y + x \qquad \frac{\partial v}{\partial z} = 2z \qquad \frac{\partial w}{\partial z} = 1$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} y+z & x+z & y+x \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$\cancel{= (y+z)(2y-2z)}$

$$= [2z(x+z) - 2y(x+y)] - [2z(y+z) - 2x(x+y)] + [2y(y+z) - 2x(x+z)]$$

$$= \cancel{2xz} + \cancel{2z^2} - \cancel{2xy} - \cancel{2y^2} - \cancel{2yz} - \cancel{2z^2} + \cancel{2x^2} + \cancel{2xy} + \cancel{2y^2} + \cancel{2yz} - \cancel{2x^2} - \cancel{2zx}$$

$= 0$

$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$ The functions are functionally dependent.

(8.) let $x, y, z$ be the dimensions of the box.

w.k.T surface area of rectangular box,

$$S = 2xy + 2yz + 2zx.$$

But, the box is open at top,

$$S = xy + 2yz + 2zx$$

$$\therefore f(x,y,z) = xy + 2yz + 2zx.$$

G.T volume of rectangular box $= 32$ cub. ft

$$V = xyz \Rightarrow 32 = xyz$$

$$\therefore \phi(x,y,z) = xyz - 32.$$

considering new function,

$$F(x,y,z) = f(x,y,z) + \lambda \cdot \phi(x,y,z)$$

$$F(x,y,z) = (xy + 2yz + 2zx) + \lambda(xyz - 32)$$

consider $\frac{\partial F}{\partial x} = 0$

$$(y + 2z) + \lambda(yz) = 0$$

$$-\lambda = \frac{+(y+2z)}{yz} \quad \text{---} (1)$$

consider $\frac{\partial F}{\partial y} = 0.$

$$(x + 2z) + \lambda(xz) = 0$$

$$-\lambda = \frac{(x+2z)}{zx} \quad \text{---} (2)$$

consider $\frac{\partial F}{\partial z} = 0$

$$(2y + 2x) + \lambda(xy) = 0$$

$$-\lambda = \frac{2(x+y)}{xy} \quad \text{---} (3)$$

equating ①, ②, ③

$$\frac{y+2z}{yz} = \frac{x+2z}{xz} = \frac{2(x+y)}{xy}$$

Since, $\frac{y+2z}{yz} = \frac{x+2z}{xz}$

$(y+2z)(\cancel{z}x) = (x+2z)(y\cancel{z})$

$xy\cancel{z} + 2xz = xy\cancel{z} + 2yz$

~~$2z^2(x-y) = 0$~~

$\cancel{2}\cancel{z}x = \cancel{2}y\cancel{z}$

$\therefore x = y$

$\frac{x+2z}{xz} = \frac{2(x+y)}{xy}$

$\frac{x+2z}{\cancel{x}z} = \frac{2(x+y)}{\cancel{x}y}$

$(x+2z)y = 2z(x+y)$

$xy + \cancel{2yz} = 2zx + \cancel{2yz}$

$\cancel{x}y = 2z\cancel{x}$

$\therefore y = 2z$.

So, $x = y = 2z$.

Substituting the above in $xyz = 32$

$(2z)(2z)z = 32$

$4z^3 = 32$

$z^3 = 8$

$\boxed{z = 2.}$

$\therefore$

$x = 2z$

$\boxed{x = 4}$

$y = 2z$

$\boxed{y = 4}$

$\therefore$ The dimension of the box are $(x, y, z) = (4, 4, 2)$.

(10.) Let $x, y, z$ be the length, breadth and height of rectangular parallelopiped as well as ellipsoid.

G.T rectangular parallelopiped is inscribed in ellipsoid.

then length, breadth, height will be doubled.

W.K.T volume $= l \cdot b \cdot h$

$$V = (2x)(2y)(2z)$$

$$f(x,y,z) = 8xyz.$$

G.T $\phi(x,y,z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1$

Define. $F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$

$$F(x,y,z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

consider $\dfrac{\partial F}{\partial x} = 0$

$$8yz + \lambda\left(\frac{2x}{a^2}\right) = 0$$

$$-\frac{\lambda}{4} = \frac{\cancel{4}a^2yz}{x} \quad \text{— (1)}$$

consider $\dfrac{\partial F}{\partial y} = 0$

$$8xz + \lambda\left(\frac{2y}{b^2}\right) = 0$$

$$-\frac{\lambda}{4} = \frac{\cancel{4}b^2zx}{y} \quad \text{— (2)}$$

consider $\dfrac{\partial F}{\partial z} = 0$

$$8xy + \lambda\left(\frac{2z}{c^2}\right) = 0$$

$$-\frac{\lambda}{4} = \frac{\cancel{4}c^2xy}{z} \quad \text{— (3)}$$

equating ①, ②, ③

$$\frac{a^2yz}{x} = \frac{b^2zx}{y} = \frac{c^2xy}{z}$$

since, $\frac{a^2yz}{x} = \frac{b^2zx}{y}$

$$a^2y^2\cancel{z} = b^2x^2\cancel{z}$$

$$\frac{y^2}{b^2} = \frac{x^2}{a^2}$$

$$\frac{b^2zx}{y} = \frac{c^2xy}{z}$$

$$b^2z^2\cancel{x} = c^2\cancel{x}y^2$$

$$\frac{z^2}{c^2} = \frac{y^2}{b^2}.$$

$$\therefore \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}.$$

Substituting above in $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1$$

$$\frac{3x^2}{a^2} = 1$$

$$x^2 = \frac{a^2}{3}$$

$$x = \pm\frac{a}{\sqrt{3}}.$$

Similarly $y = \frac{\pm b}{\sqrt{3}}$ $z = \frac{\pm c}{\sqrt{3}}$.

Since, length, breadth, height cannot be measured in negative.

$$\therefore x = \frac{a}{\sqrt{3}} \; ; \; y = \frac{b}{\sqrt{3}} \; ; \; z = \frac{c}{\sqrt{3}}.$$

Hence, maximum volume $= 8xyz$

$$= 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right)$$

$$= \frac{8abc}{3\sqrt{3}} \text{ cub. ft.}$$

LAQ's

6 Find the maxima and minima values of the following functions.

i) $f(x,y) = 2(x^2-y^2) - x^4 + y^4$

Sol: Given function $f(x,y) = 2x^2 - 2y^2 - x^4 + y^4$

$$\frac{\partial f}{\partial x} = 4x - 0 - 4x^3 + 0 = 4x(1-x^2)$$

$$\frac{\partial f}{\partial y} = 0 - 4y + 4y^3 = 4y(y^2-1)$$

to get max & min value equate $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} = 0$

$4x(1-x^2) = 0 \Rightarrow 4x = 0 \mid 1-x^2 = 0$

$4y(y^2-1) = 0 \Rightarrow 4y = 0 / y^2 - 1 = 0$

$x = 0, x^2 = 1 \Rightarrow x = \pm 1;$

$y = 0, y^2 = 1 \Rightarrow y = \pm 1$

Let the points be $A(0,0)$ $B(0,1)$ $C(0,-1)$

$D(1,0)$ $E(1,1)$ $F(1,-1)$

$G(-1,0)$ $H(-1,1)$ $I(-1,-1)$.

Consider:

$$l = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial}{\partial x}(4x(1-x^2)) = 4 - 12x^2$$

$$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0$$

$$n = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}(-4y + 4y^3)$$

$$= -4 + 12y^2$$

Consider

$$ln - m^2 = (4 - 12x^2)(-4 + 12y^2)$$

$$= -16 - 144x^2y^2 + 48y^2 + 48x^2$$

$$= -16 - 144x^2y^2 + 48x^2 + 48y^2$$

At point $A(0,0)$ $D(1,0)$ $E(1,1)$ $F(1,-1)$ $H(-1,1)$ $I(-1,-1)$ $ln - m^2 < 0$ here they are saddle points

Consider

At $B(0,1) = ln - m^2 = -16 - 0 + 48 + 0$

$$= 32 > 0$$

$$l = 4 - 12 \times 0 = 4 > 0$$

It has min value and the min points is $(0,1)$

min value $= f(0,1)$

$$= 2(0 - (1)^2) - (0)^4 + (1)^4$$

$$= 2(-1) + 1 = -2 + 1 = 1$$

At $C(0,-1) = ln - m^2 = -16 - 0 + 48(-1)^2 + 48(0)$

$= 16 + 48 = 32 > 0$

$l = 4 - 12 \times 0 = 4 > 0$

It has a min point at $C(0,-1)$.

Min value is $= f(0,-1)$

$= 2(0 - (-1)^2) - 0^4 + (-1)^4$

$= 2(-1) - 0 + 1 = -2 + 1 = -1$

At $D(1,0) = ln - m^2 = -16 - 0 + 0 + 48$

$= 32 > 0$

$l = (4 - 12(1)) = 4 - 12 = -8 < 0$

It has a max point at $D(1,0)$

max value $= f(1,0) = 2(1-0) - 1 + 0 = 2 - 1 = 1$

At $G(-1,0) = ln - m^2 = -16 - 0 + 48(0)^2 + 48(-1)^2$

$= 16 + 48 = 32 > 0$

$l = (4 - 12(-1)^2) = 4 - 12 = -8 < 0$

It has max point at $G(-1,0)$

Max value $= G(-1,0) = \cancel{2(1,0)}$

$2(1-0) - 1 + 0$

$= 2 - 1 = 1$

$\therefore$ Max value $= 1$ and min value $= -1$

ii) $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 \cancel{-72y} - 72x$

Sol :- Given $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 - 72x$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x - 0 + 72$$

$$= 3x^2 + 3y^2 - 30x + 72$$

$$\frac{\partial f}{\partial y} = 0 + 6xy - 0 - 30y + 0$$

$$= 6xy - 30y$$

to find min and max value equate

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} = 0$$

$$3x^2 + 3y^2 - 30x + 72 = 0 \quad - ①$$

$$6xy - 30y = 0 \quad \rightarrow ②$$

$6y = 0$ | $x - 5 = 0$

$\boxed{y = 0}$ | $\boxed{x = 5}$

Sub $y = 0$ in ①

$$3x^2 - 30x + 72 = 0$$

$$x^2 - 10x + 24 = 0$$

$$\boxed{x = 6, 4}$$

Point are $A(6,0)$ $B(4,0)$

Sub $x = 5$ in ①

$$3(25) + 3y^2 - 30(5) + 72 = 0$$

$75 + 3y^2 - 150 + 72 = 0$

$147 + 3y^2 - 150 = 0$

$3y^2 = 3 \quad \boxed{y = \pm 1}$

Points are $C(5,1)$ $D(5,-1)$

Consider

$$l = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2 + 3y^2 - 30x + 72)$$

$$= 6x + 0 - 30 = 6x - 30$$

$$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(6xy - 30y)$$

$$= 6x - 30$$

Let

$$ln - m^2 = (6x-30)(6x-30) - 6y^2$$

$$= (6x-30)^2 - 36y^2$$

at $A(6,0) \Rightarrow ln - m^2$

$$(6(6) - 30)^2 - 0$$

$$= (36-30)^2 = 6^2 = 36 > 0$$

$$l = (6(6) - 30) = 36 - 30 = 6 > 0$$

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$\because ln - m^2 > 0$ ; $l > 0$ hence $(6,0)$ is a min point.

min value $= f(6,0) = 6^3 + 3(6)(0) - 15(6^2)$
$- 15(0)^2 + 72(6)$

$= 216 + 0 - 540 + 432$

$= 108$

At $B(4,0)$ $= ln - m^2 =$
$(6(4) - 30)^2 - 0$

$= (24-30)^2 = 6^2 = 36 > 0$

$l = (6 \times 4) - 30) = 6^2 = 36 > 0$

$l = (6 \times (4) - 30) = 24 - 30 = \cancel{6 > 0}$
$-6 < 0$

~~$\because l = (6 \times 4 - 30) = 24 - 30 = -6 >$~~

$\because ln - m^2 > 0$ ; $l < 0$ here $(4,0)$ is a max point.

max value $= f(4,0) = 4^3 + 3(4)(0) - 15(4)^2$
$- 15\, 0^2 + 72(4)$

$= 64 + 0 - 15(16) - 0 + 288$

$= 64 - 240 + 288$

$= 122$

At

At C $(5,1)$ $= ln - m^2$

$(6(5)-30)^2 - 36(1)^2$

$= (30-30)^2 - 36$

$= -36 < 0$

$\because ln - m^2 < 0$, hence $(+5,1)$ is a Saddle point

At D $(5,-1)$ $= ln - m^2$

$= (6(5)-30)^2 - 36(-1)^2$

$= -36 < 0$

$\therefore ln - m^2 < 0$, hence $(5,-1)$ is a Saddle point.

iii) $f(x,y) = x^3y^2(1-x-y)$

Sol:- Given $f(x,y) = x^3y^2 - x^4y^2 - x^3y^3$

$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$

$\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$

to get the min and max value $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} = 0$

$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$ — ①

$2x^3y - 2x^4y - 3x^3y^2 = 0 \rightarrow$ ②

From ①

$\Rightarrow x^2(3-4x-3y)=0$

$x^2y^2 \;\Big|\; 3-4x-3y=0$

$\qquad\quad 3-4x-3y=0$

From ② $\Rightarrow x^3y(2-2x-3y)=0$ — ③

$2-2x-3y=0$ — ④

Solving ③ & ④

$3-4x-3y=0$

$2-2x-3y=0$

$(-)\ (+)\ (+)$

$1-2x+0=0$

$1=2x \Rightarrow x=1/2$

Sub $x=1/2$ In ③

$3-\cancel{4}^{2}\left(\frac{1}{\cancel{2}}\right)-3y=0$

$3-2-3y=0$

$1-3y=0$

$-3y=-1 \quad y=1/3$

$x,y=\left(\frac{1}{2},1/3\right)$

possible extreme pts

A (0,0) , B ( $x = 0, 2 - 2x - 3y = 0$ ), ~~E~~ ,

C ( $y = 0, 2 - 2x - 3y = 0$ ) D ( $3 - 4x - 3y = 0, x = 0$ )

$\Rightarrow$ A (0,0) B (0, 2/3) C (0,1) D (1,0) E(3/4, 0)

P (1/2, 1/3)

Consider

$$l = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(3x^2y^2 - 4x^3y^2 - 3x^2y^3\right)$$

$$= 6xy^2 - 12x^2y^2 - 6xy^3$$

$$m = \frac{\partial^2 y}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(2x^3y - 2x^4y - 3x^3y^2\right)$$

$$= 6x^2y - 8x^3y - 9x^2y^2$$

$$n = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(2x^3y - 2x^4y - 3x^3y^2\right)$$

$$= 2x^3 - 2x^4 - 6x^3y$$

Consider $ln - m^2$

$= (6xy^2 - 12x^2y^2 - 6xy^3)(2x^3 - 2x^4 - 6x^3)$
$- (6x^2y - 8x^3y - 9x^2y^2)^2$

$\therefore$ at $A(0,0)$ $B(0, 2/3)$ $C(0,1)$ $D(1,0)$ $E(3/4, 0)$

$ln - m^2 = 0$

$\cancel{6xy^2 - 12x^2y^2 - 6xy}$

hence it can't be solved.

considered at $F(1/2, 1/3) \Rightarrow ln - m^2 =$

$\cancel{6}^{3}\left(\frac{1}{\cancel{2}}\right)\left(\frac{1}{3}\right)^2 - 12\left(\frac{1}{4}\right)\left(\frac{1}{9}\right) - \cancel{6}^{3}\left(\frac{1}{2}\right)\left(\frac{1}{27}\right)$

$\left(2\left(\frac{1}{8}\right) - \cancel{2}^{1}\left(\frac{1}{\cancel{16}_8}\right) - \cancel{6}^{2}\left(\frac{1}{8}\right)\left(\frac{1}{3}\right)\right) -$

$\left(6\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) - 8\left(\frac{1}{8}\right)\left(1/3\right) - 9\left(\frac{1}{4}\right)\left(\frac{1}{9}\right)\right)$

$\left(\frac{3}{9} - \frac{3}{9} - \frac{3}{27}\right)\left(\frac{1}{4} - \frac{1}{8} - \frac{2}{8}\right) - \left(\frac{2}{4} - \frac{1}{3} - \frac{1}{4}\right)^2$

$= \left(\frac{3}{9} - \frac{3}{9} - \frac{1}{9}\right)\left(\frac{1}{4} - \frac{1}{8} - \frac{\cancel{2}\,1}{\cancel{8}\,4}\right) - \left(\frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right)$

$= \left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right) - \left(\frac{6-4-3}{12}\right)^2$

$= \frac{1}{72} - \left(\frac{-1}{12}\right)^2$

$= \frac{1}{72} - \frac{1}{144} = \frac{2-1}{144} = \frac{1}{144} > 0$

$$\left(\frac{3}{9}-\frac{3}{9}-\frac{3}{27}\right)\left(\frac{1}{4}-\frac{1}{8}-\frac{2}{8}\right)-\left(\frac{2}{4}-\frac{1}{3}-\frac{1}{4}\right)^2$$

$$=\left(\frac{3}{9}-\frac{3}{9}-\frac{1}{9}\right)\left(\frac{1}{4}-\frac{1}{8}-\frac{\cancel{2}^{1}}{\cancel{8}_4}\right)-\left(\frac{1}{2}-\frac{1}{3}-\frac{1}{4}\right)^2$$

$$=\left(\frac{1}{9}\right)\left(-\frac{1}{8}\right)-\left(\frac{6-4-3}{12}\right)^2$$

$$=\frac{1}{72}-\left(-\frac{1}{2}\right)^2$$

$$=\frac{1}{72}-\frac{1}{144}=\frac{2-1}{144}=\frac{1}{144}>0$$

$$l=\cancel{6}^{3}\left(\frac{1}{\cancel{2}}\right)\left(\frac{1}{3}\right)^2-12\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2-6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^3$$

$$=\frac{3}{9}-\frac{1}{3}-\frac{1}{9}$$

$$=\frac{3-3-1}{9}=-1/9<0$$

$\therefore\ ln-m^2>0$; $l<0$ hence $(1/2, 1/3)$ is max point.

$\therefore$ max value $=f(1/2, \frac{1}{3})$

$$=\left(\frac{1}{2}\right)^3\left(\frac{1}{9}\right)\left(1-\frac{1}{2}-1/3\right)$$

$$=\frac{1}{8}\times\frac{1}{9}\left(\frac{6-3-2}{6}\right)$$

$$=\frac{1}{9}\times\frac{1}{8}\left(\frac{1}{6}\right)=\frac{1}{8}\times\frac{1}{9}\times\frac{1}{6}=\frac{1}{432}$$

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(IV) $f(x,y) = x^3 + y^3 - 3axy$

sol:- given $f(x,y) = x^3+y^3-3axy$

$$\frac{\partial f}{\partial x} = 3x^2+0-3ay = 3x^2-3ay$$

$$\frac{\partial f}{\partial y} = 3y^2-3ax$$

to get min & Max value equate $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} = 0$

$$3x^2-3ay = 0 \rightarrow (1)$$

$$3y^2-3ax = 0 \rightarrow (2)$$

$3(x^2-ay) = 0 \Rightarrow x^2 = ay \quad y = x^2/a$

$$\frac{\partial f}{\partial y} = \frac{x^4}{a^2} - ax = 0$$

$$\frac{x^4}{a^2} - ax = 0$$

$$x^4 - a^3x = 0$$

$\boxed{x=0}$ $\quad x^3 = a^3 \Rightarrow \boxed{x=a}$

sub $x = 0$ in (1)

$0 - 3ay = 0$

$ay = 0$

$\boxed{y=0}$

$\therefore A(0,0)$

sub $x = a$ in ①

$3a^2 - 3ay = 0$

$3a(a-y) = 0$

$a - y = 0$ $\boxed{y = a}$

$\therefore B = (a, a)$

Consider $l = \frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dx}(3x^2 - 3ay$

$= 6x - 0 = 6x$

$m = \frac{d^2f}{dxdy} = \frac{d}{dx}\left(\frac{df}{dy}\right) = \frac{d}{dx}(3y^2 - 3ax)$

$= 0 - 3a = -3a$

$n = \frac{d^2f}{dy^2} = \frac{d}{dy}\left(\frac{df}{dy}\right) = \frac{d}{dy}(3y^2 - 3ax)$

$= 6y - 0 = 6y$

Consider

$ln - m^2 = 36xy - 9a^2$

At $A(0,0)$ $ln - m^2 = 36(0)(0) - 9a^2$

$-9a^2 < 0$

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$\therefore$ A(0,0) is a Saddle point

At B(a,a) $ln - m^2 = 36(a)(a) - 9a^2$

$36a^2 - 9a^2 = 27a^2 > 0$

$l = 6a$

If $a > 0$ then $l > 0$ hence a min point obtained

If $a < 0$ then $l < 0$ hence a max point obtained

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7(i) Find max value of $x+y+z$ subject to $\frac{1}{x} + \frac{1}{y} + 1/z = 1$

Given $f(x,y,z) = x+y+z$ ;

$\phi(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1$

define $F(x,y,z) = f(x,y,z) + \lambda\, \phi(x,y,z)$

$= x+y+z + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1\right)$ - ①

$\frac{df}{dx} = 1 + \lambda\left(-\frac{1}{x^2}\right) = 0 \Rightarrow x^2 - \lambda = 0$

$\Rightarrow \boxed{x^2 = \lambda}$ - ②

$$\frac{\partial f}{\partial y} = 1+\lambda\left(-\frac{1}{y^2}\right) = 0 \Rightarrow y^2-\lambda = 0 \Rightarrow \boxed{y^2=\lambda} \rightarrow 3$$

$$\frac{\partial f}{\partial y} = 1+\lambda\left(-\frac{1}{z^2}\right) = 0 \Rightarrow z^2-\lambda = 0 \Rightarrow \boxed{z^2=\lambda} \Rightarrow 4$$

from 2 3 4

$\therefore x^2 = y^2 = z^2$

$x = y = z = \lambda$

Sub $\lambda$ In $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 1$

$$\frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}} = 1$$

$$\frac{3}{\sqrt{\lambda}} = 1$$

$\sqrt{\lambda} = 3 \Rightarrow \lambda = 9$

$\therefore x = y = z = 3$

$\therefore$ max value $= f(x,y,z)$

$3+3+3 = 9.$

(ii) Find the minimum value of $x^2+y^2+z^2$ given $xyz = a^3$

Sol:-

given $f(x,y,z) = x^2+y^2+z^2$, $\phi(x,y,z) = xyz - a^3$

define

$$F(x,y,z) = f(x,y,z) + \lambda \phi(x,y,z)$$

$$\frac{\partial F}{\partial x} = 2x + \lambda yz = 0 \Rightarrow 2x = -\lambda yz$$

$$\frac{x}{yz} = -\lambda/2 \rightarrow (2)$$

$$\frac{\partial F}{\partial y} = 2y + \lambda xz = 0 \Rightarrow 2y = -\lambda xz$$

$$\frac{y}{xz} = -\lambda/2 \rightarrow (3)$$

$$\frac{\partial F}{\partial z} = 2z + \lambda xy = 0 \Rightarrow 2z = -\lambda xy$$

$$\frac{z}{xy} = -\lambda/2 \rightarrow (4)$$

from (2), (3), (4)

(2) = (3) = (4)

$$\frac{x}{yz} = \frac{y}{xz} = \frac{z}{xy}$$

(2) = (3)

$$\frac{x}{y\cancel{z}} = \frac{y}{x\cancel{z}}$$

$$x^2 = y^2$$

(3) = (2)

$$\frac{y}{x\cancel{z}} = \frac{z}{xy}$$

$$y^2 = z^2$$

$$\therefore x^2 = y^2 = z^2 \Rightarrow x = y = z$$

Sub in $xyz = a^3$

$$x^3 = a^3 \Rightarrow x = a$$

Similarly $y = a, z = a$

$\therefore f(x,y,z) = a, a, a$

minimum value $= f(a,a,a)$

$= a^2 + a^2 + a^2 = 3a^2$

(iii) Find the minimum value of $u = x^2 y^3 z^4$

If $2x + 3y + 4z = a$

Sol: Given $f(x,y,z) = x^2 y^3 z^4$ ; ~~$\phi(x)$~~ $=$

$\phi(x,y,z) = 2x + 3y + 4z - a$

$F(x,y,z) = x^2 y^3 z^4 + \lambda(2x + 3y + 4z - a)$ - ①

$\frac{\partial F}{\partial x} = 2x + \lambda(2) = 0 \Rightarrow 2x = -2\lambda$

$\Rightarrow \boxed{x = -\lambda} \rightarrow$ ②

$\frac{\partial F}{\partial y} = 3y + \lambda(3) = 0 \Rightarrow 3y = -3\lambda$

$\boxed{y = -\lambda} \rightarrow$ ③

$\frac{\partial F}{\partial y} = 4z + \lambda(4) = 0 \Rightarrow 4z = -4\lambda$

$\Rightarrow \boxed{z = -\lambda} \rightarrow$ ④

from ②, ③ & ④

$x = y = z$

Sub $x = y = z$ in $2x + 3y + 4z = a$

$2x + 3x + 4x = a$

$$9x = a$$
$$x = \frac{a}{9}$$

Similarly $y = a/9$ $\quad z = a/9$

$\therefore$ minimum value $f(x,y,z)$

$$= \left(\frac{a}{9}\right)^2 \left(\frac{a}{9}\right)^3 \left(\frac{a}{9}\right)^4$$
$$= \frac{a^9}{9^9} = \left(\frac{a}{9}\right)^9$$

9 i) Divide 24 into three points such that continued product of the first, square of second and cube of third is maximum.

Sol: Given $xy^2z^3 = max$

$$f(x,y,z) = xy^2z^3$$
$$\phi(x,y,z) = x+y+z = 24 \Rightarrow x+y+z-24$$

define

$$F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$$
$$F(x,y,z) = xy^2z^3 + \lambda(x+y+z-24) \rightarrow ①$$

$\frac{dF}{dx} = y^2 z^3 + \lambda = 0 \Rightarrow y^2 z^3 = -\lambda \rightarrow (2)$

$\frac{dF}{dy} = 2xyz^3 + \lambda = 0 \Rightarrow 2xyz^3 = -\lambda \rightarrow (3)$

$\frac{dF}{dz} = 3xy^2z^2 + \lambda = 0 \Rightarrow 3xy^2z^2 = -\lambda \rightarrow (4)$

from (2), (3), & (4)

(2) = (3)

$y^2 z^3 = 2xyz^3$

$2x = y$

$x = y/2$

$y = 2x$

(3) = (4)

$2xyz^3 = 3xy^2z^2$

$2z = y$

$z = 3y/2$

$y = 2z/3$

Sub In $x + y + z = 24$

$\frac{y}{2} + y + \frac{3y}{2} = 24$

$\frac{y + 2y + 3y}{2} = 24$

$\frac{\cancel{6}^{3}y}{\cancel{2}} = 24$

$y = \frac{24}{3} = 8$

$x = \frac{y}{2} = \frac{8}{2} = 4$

$z = \frac{3y}{2} = \frac{3 \times \cancel{8}^{4}}{\cancel{2}} = 12$

$\therefore (x,y,z) = (4,8,12)$

Max value $f(4,8,12)$

$$= 4 \times 8^2 \times 12^3 = 442368$$

(ii) Find the three positive numbers numbers whose sum is 100 and whose product is maximum

So:- Given $f(x,y,z) = x \cdot y z$ ($\because$ product is max)

$$\phi(x,y,z) = x+y+z-100$$

($\because$ sum of $x+y+z=100$)

$$F(x,y,z) = f(x,y,z) + \lambda \phi(x,y,z)$$

$$F(x,y,z) = xyz + \lambda(x+y+z-100) \quad -(1)$$

$$\frac{\partial F}{\partial x} = yz + \lambda = 0 \Rightarrow yz = -\lambda \rightarrow (2)$$

$$\frac{\partial F}{\partial y} = xz + \lambda = 0 \Rightarrow xz = -\lambda \rightarrow (3)$$

$$\frac{\partial F}{\partial z} = xy + \lambda = 0 \Rightarrow xy = -\lambda \rightarrow (4)$$

from ②, ③, ④

② = ③ | ③ = ④

$yz = xz$ | $xz = xy$

$\boxed{x = y}$ | $\boxed{y = z}$

$\therefore \; x = y = z$ sub in $x + y + z = 100$

$$3x = 100$$

$$x = \frac{100}{3}$$

Similarly $y = \frac{100}{3}$, $z = \frac{100}{3}$

$\therefore \; (x, y, z) \left( \frac{100}{3}, \frac{100}{3}, \frac{100}{3} \right)$

max value $f\left( \frac{100}{3}, \frac{100}{3}, \frac{100}{3} \right)$

$$= \frac{1000000}{27}$$

(iii) Show that rectangular parallelopipe of max volume that can be inscribed in the given ~~stop~~ sphere is a cube.

Sol:- let $(x, y, z)$ be the dimension of parallelopiped

W.K.T

The equation of sphere with centre $(0,0,0)$ is

$$x^2+y^2+z^2 = a^2$$

$$\phi(x, y, z) = x^2+y^2+z^2-a^2$$

the volume of a parallelopiped is $= x \times y \times z$

$$\therefore f(x, y, z) = xyz$$

define

$$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$$

$$F(x, y, z) = xyz + \lambda(x^2+y^2+z^2-a^2) \rightarrow ①$$

$$\frac{\partial F}{\partial x} = yz + \lambda(2x) = 0 \Rightarrow \frac{yz}{x} = -2\lambda \rightarrow ②$$

$$\frac{\partial F}{\partial y} = xz + \lambda(2y) = 0 \Rightarrow \frac{xz}{y} = -2\lambda \rightarrow ③$$

$$\frac{\partial F}{\partial z} = xy + \lambda(2z) = 0 \Rightarrow$$

$$\frac{xy}{z} = -2\lambda \rightarrow ④$$

from ② ③ & ④

$$\frac{yz}{x} = \frac{xz}{y} = \frac{xy}{z}$$

from ① & ②

$$\frac{yz}{x} = \frac{xz}{y}$$

from ③ & ④

$$\frac{xz}{y} = \frac{xy}{z}$$

$$y^2 = z^2$$

$\therefore x^2 = y^2 = z^2 \Rightarrow x = y = z$

Sub in $x^2+y^2+z^2 = a^2$

$3x^2 = a^2 \Rightarrow x = \frac{a}{\sqrt{3}}$

Similarly $y = \frac{a}{\sqrt{3}}$ ; $z = \frac{a}{\sqrt{3}}$

$\therefore (x, y, z) = \left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$

Volume of parallelopiped

$= f\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$

$= \frac{a^3}{3\sqrt{3}}$ //

7. ~~Verify~~ the

hence the rectangular parallelopiped of max volume that can be inscribed in the given sphere is a cube.