THE FUTURE KID'S SCHOOL

{Affiliated to the Council for the Indian School Certificate Examinations (CISCE), New Delhi}

[School Affiliation code: AP-104 (ISC STD XI & XII), RAJAMAHENDRAVARAM]

This is to certify that the following candidate of class XII science has successfully completed her Mathematics project file. She has been taken proper care and at most sincerity in completion of her project. All the works related to her project was done by the candidate herself. The approach towards the subject has been sincere and scientific.

I certify that this project up to my expectation and as per guideline issued by the I.S.C.

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Class	: XII
Stream	: Science
Subject	: Mathematics
Topic	: Integration and the limit of sum, Vector Algebra
Submitted to	:
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External Examiner

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Principal

ACKNOWLEDGEMENT

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I would also like to my thank my parents and my friends who supported me and encouraged me in finishing this project within the stipulated timeframe.

I believe, this project not only helped me in my academic growth but also ignited the passion for doing a further research on this subject.

G. Vijaya Sri

Date:

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Maths Project 1

Integration as the limit of a summation

Introduction

In HELM 13, integration was introduced as the reverse of differentiation. A more rigorous treatment would show that integration is a process of adding or 'summation'. By viewing integration from this perspective it is possible to apply the techniques of integration to finding areas, volume, centres of gravity and many other important quantities.

The content of this section is important because it is here that integration is defined more carefully. A thorough understanding of the process involved is essential if you need to apply integration techniques to practical problems.

Prerequisites

Before starting this Section you should ...

- be able to calculate definite integrals

Learning Outcomes

On completion you should be able to ...

- explain integration as the limit of a sum
- evaluate the limit of a sum in simple cases

1. The limit of a sum

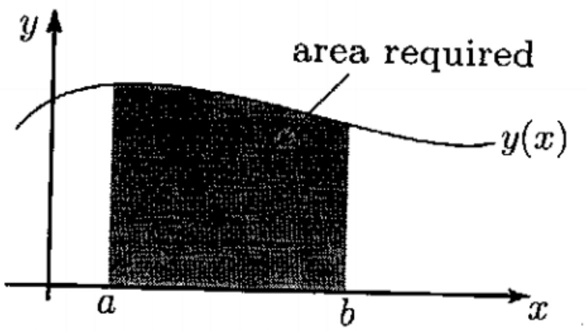
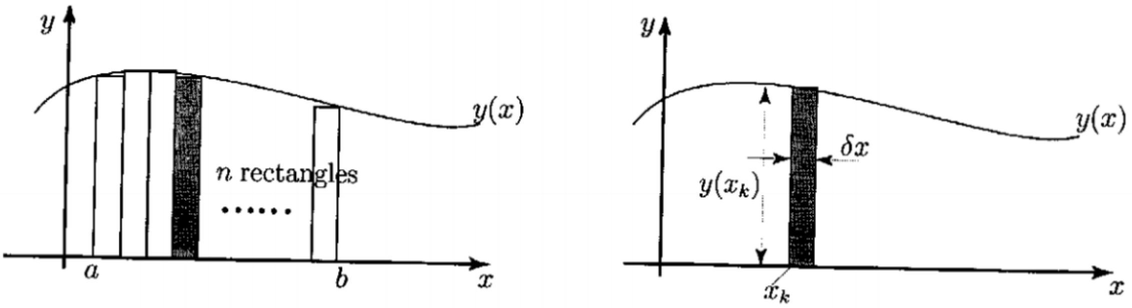


Figure 1: The area under a curve

Consider the graph of the positive function y(x) shown in Figure 1. Suppose we are interested in finding the area under the graph between x=a and x=b. One way in which this area can be approximated is to divide it into a number of rectangles of equal width, find the area of each rectangle, and then add up all these individual rectangular areas. This is illustrated in figure 2a ,which shows the area divided into n rectangles(with some discrepancies at the tops),and figure 2b which shows the dimensions of a typical rectangle which is located at x=xk.



(a) The area approximated by n rectangles

(b) A typical rectangle

Figure 2

We wish to find an expression for the area under a curve based on the sum of many rectangles. Firstly,we note that the distance from x=a to x=b is b-a. In the figure 2a the area has been dividend into n rectangles. If n rectangles span the distance fron a to b the width of each rectangle is b-a/n:

It is conventional to label the width of each rectangle is x, i.e δx, i.e. $\delta x = \frac{b-a}{n}$. we label the x coordinates at the left hand side of the rectangles as x1,x2, up to xn (here x1=a and xn+1=b). A typical rectangle ,the kth rectangle, is shown in figure 2b. Note that its height is y(xk),so its area is y(xk)*

The sum of the areas of all n rectangles is then

$$y(x_1)\delta x + y(x_2)\delta x + y(x_3)\delta x + \cdots + y(x_n)\delta x$$

which we write concisely using sigma notation as

$$\sum_{k=1}^{n} y(x_k)\delta x$$

This quantity gives us estimate of the area under under the curve but is not exact .To improve the estimate we must take a large number of very thin rectangles .So, what we want to find is the value of this sum when n tends to infinity and

Example 1

The area under the graph of y= z* between x= 0 and x = 1 is to be found by approximating it by a large number of thin rectangles and finding the limit of the sum of their areas. From Equation (1) this is

$\lim_{\delta x \to 0} \sum_{x=0}^{x=1} y(x)\,\delta x$. Write down the integral which this sum defines and evaluate it to obtain the area under the curve.

Solution

The limit of the sum defines the integral $\int_0^1 y(x)dx$. Here $y = x^2$ and so $\int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

To show that the process of of taking the limit of a sum actually works we investigate the problem in detail. We use the idea of the limit of the sum to find the area under the graph of y=x between x=0 and x=1, as illustrated in figure 3.

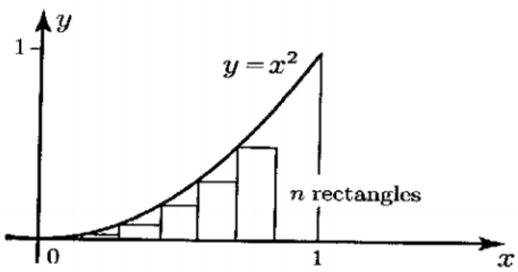
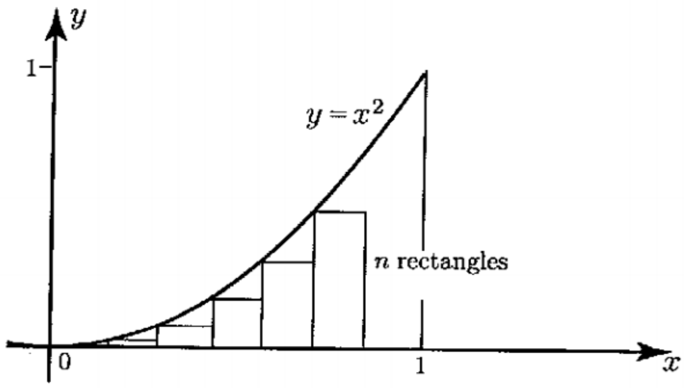


Figure 3: The area under $y = x^2$ is approximated by a number of thin rectangles

Refer to the diagram below to help you answer the questions below.



If the interval between x=0 and x=1 is divided into n rectangle what is the width of each rectangle ?

Your solution:

To find the total area A_n of the n rectangles we must add up all these individual rectangular areas:

$$A_n = \sum_{k=1}^{n} \frac{(k-1)^2}{n^3}$$

This sum can be simplified and then calculated as follows. You will need to make use of the formulas for the sum of the first n integers, and the sum of the squares of the first n integers:

$$\sum_{k=1}^{n} 1 = n, \qquad \sum_{k=1}^{n} k = \frac{1}{2}n(n+1), \qquad \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$$

Then, the total area of the rectangles is given by

$$\begin{aligned} A_n &= \sum_{k=1}^{n} \frac{(k-1)^2}{n^3} \\ &= \frac{1}{n^3}\sum_{k=1}^{n}(k-1)^2 \\ &= \frac{1}{n^3}\sum_{k=1}^{n}(k^2 - 2k + 1) \\ &= \frac{1}{n^3}\left(\sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n^3}\left(\frac{n}{6}(n+1)(2n+1) - 2\frac{n}{2}(n+1) + n\right) \\
&= \frac{1}{n^2}\left(\frac{(n+1)(2n+1)}{6} - (n+1) + 1\right) \\
&= \frac{1}{n^2}\left(\frac{(n+1)(2n+1)}{6} - n\right) \\
&= \frac{1}{6n^2}\left(2n^2 - 3n + 1\right) = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}
\end{aligned}
$$

Note that this is a formula for the **exact** total area of the n rectangles. It is an **estimate** of the area under the graph of $y = x^2$. However, as n gets larger, the terms $\frac{1}{2n}$ and $\frac{1}{6n^2}$ become small and will eventually tend to zero. If we let n tend to infinity we obtain the exact answer of $\frac{1}{3}$.

The required area is $\frac{1}{3}$. It has been found as **the limit of a sum** and of course agrees with that calculated by integration.

In the calculations which follow in subsequent sections the need to evaluate complicated limits like this is avoided by performing the integration using the technique of HELM 31. Nevertheless it will sometimes be necessary to go through the process of dividing a region into small sections, performing a calculation on each section and then adding the results, in order to formulate the integral required. When numerical methods of integration are studied (HELM 31) this summation method will prove fundamental.

Maths

Project

2

Vector Algebra

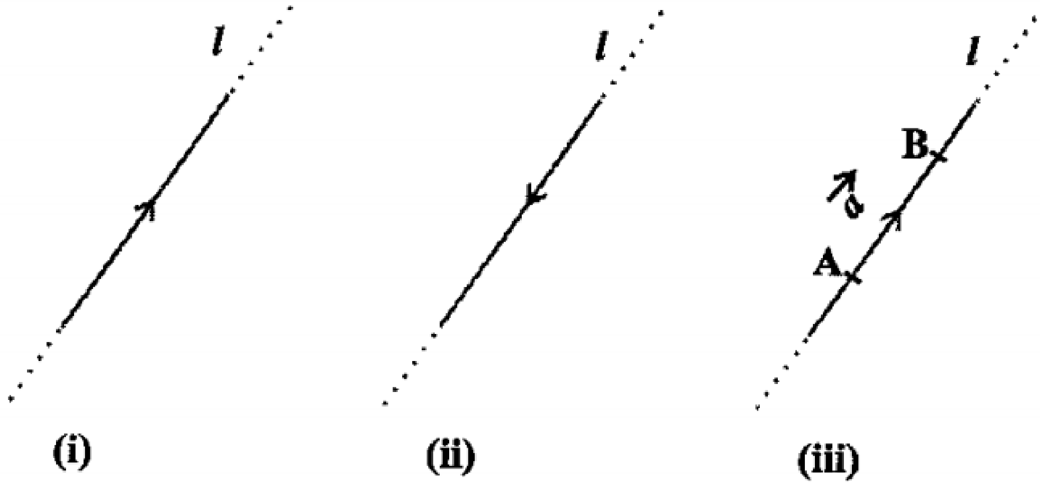


Fig 10.1

Now observe that if we restrict the line l to the line segment AB , then a magnitude is prescribed on the line l with one of the two directions ,so that we obtain a directed line segment (Fig 10.1(iii)) . thus ,a directed line segment has magnitude as well as direction.

Definition 1 A quantity that has magnitude as well as direction is called vector. Notice that a directed line segment is a vector (Fig 10.1(iii)), denoted as AB or simply as a, and read as 'vector AB' or 'vector a'.

The point A from where the vector AB starts is called its initial point, and point B where it ends is called its terminal point. The distance between initial and terminal points of a vector is called the magnitude (or length) of the vector ,denoted as |AB|, or |a|, or a. The arrow indicates the direction of the vector.

NOTE Since the length is never negative, the notation |a|<0 has no meaning.

Position Vector

From class X1,recall the three dimensional right handed rectangular coordinate system (fig 10.2(i)). Consider a point p in space ,having coordinates (x,y,z) with respect to the origin o (0,0,0).then the vector OP having o and p as its initial and terminal points respectively is called a position vector of the point P with respect to O. using distance formula the magnitude of OP (or r) is given by

$$|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$$

In practice the position vector of point A, B,C etc, with respect to the origin O are denoted by a, b, c, etc., respectively (fig 10.2(ii)).

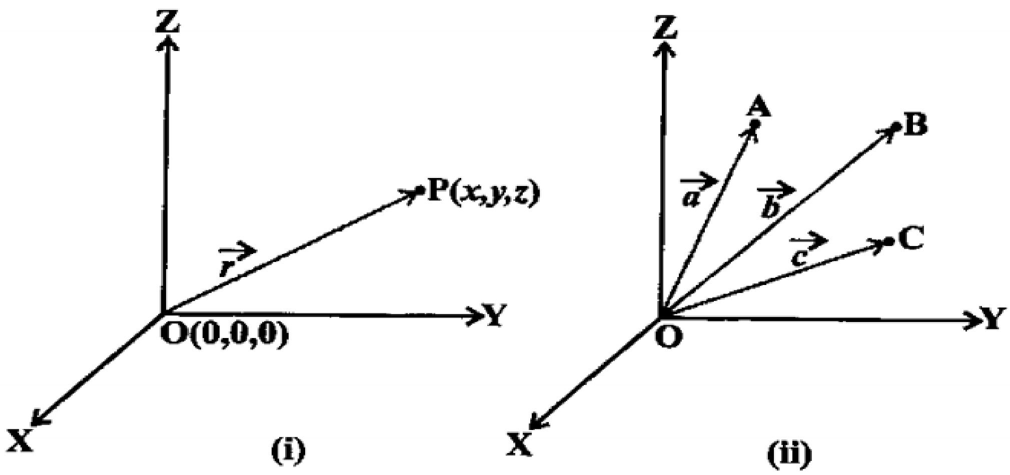


Fig 10.2

Direction cosines

Consider the position vector OP(or r) of a point P(x, y, z) as in fig 10.3. the angles alpha, beta, gamma made by the vector r with the positive direction of x, y, z and z axis respectively are called its direction angles. The cosine value of this angles I.e, cos alpha , cos beta ,and cos gamma are called direction cosine of vector r , and usually denoted by l, m, n respectively.

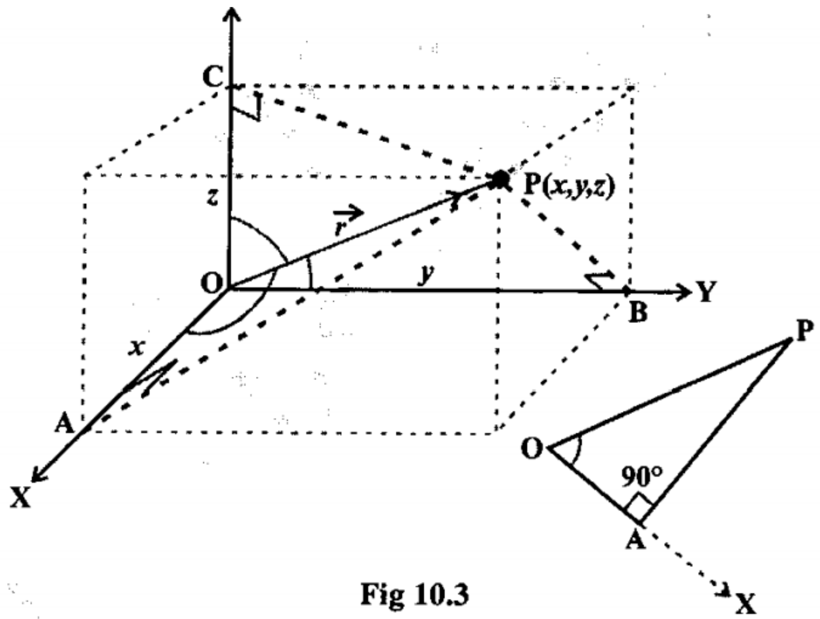


Fig 10.3

From Fig 10.3, one may note that the triangle OAP is right angled, and in it, we have $\cos\alpha = \frac{x}{r}$ (r stands for $|\vec{r}|$). Similarly, from the right angled triangles OBP and OCP, we may write $\cos\beta = \frac{y}{r}$ and $\cos\gamma = \frac{z}{r}$. Thus, the coordinates of the point P may

Also be expressed as (lr, mn, nr). The numbers lr ,mr, nr proportional to the direction cosine are direction ratios of vector r and denoted as a, b and c, respectively.

NOTE One may note that l2 + m2 + n2 = 1 but a2 +b2+ c2is not equal to 1, in general.

10.3 Types of vectors

Zero vector A vector whose initial and terminal points coincide is called a zero vector (or null vector) and denoted as 0. Zero vector cannot be assigned a definite direction as it has zero magnitude. Or alternatively otherwise it may be regarded as having any direction. The vectors AA, BB represent the zero vector .

Coinitial vector Two or more vectors are said to be coinitial vectors.

Collinear vector Two or more vectors are said to be collinear if they are parallel to the same line, irrespective of their magnitudes and directions.

Equal vectors Two vectors a and b are said to be equal , if they have the same magnitude and direction regardless of the position of their initial points ,and written as a=b.

Negative of a vector A vector whose magnitude is the same as that of a given vector (say AB) but direction is opposite to that of it is called negative after given vector. For example ,vector BA is negative of the vector AB and written as BA=-AB.

Remark The vector defined above are such that any one them may be subject to its parallel displacement without changing its magnitude and direction .such vectors are called free vectors . Throughout this chapter we will be dealing with free vectors only.

Example 1 Represent graphically a displacement of 40 km, 30° west of south.

Solution The vector $\overrightarrow{OP}$ represents the required displacement (Fig 10.4).

Example 2 Classify the following measures as scalars and vectors.

(i) 5 seconds

(ii) $1000\ cm^3$

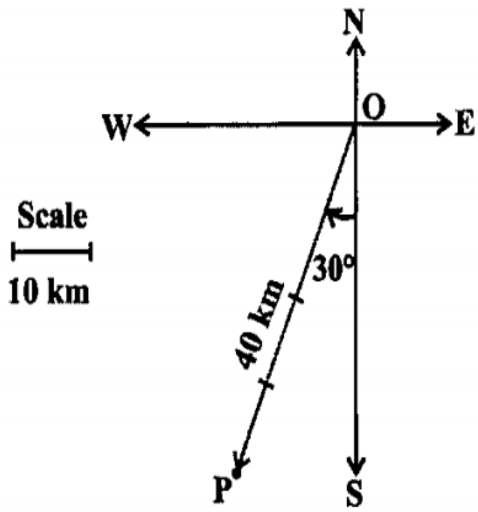


Fig 10.4

10.4 Addition of Vectors

A vector $\overrightarrow{AB}$ simply means the displacement from a point A to the point B. Now consider a situation that a girl moves from A to B and then from B to C (Fig 10.7). The net displacement made by the girl from point A to the point C, is given by the vector $\overrightarrow{AC}$ and expressed as

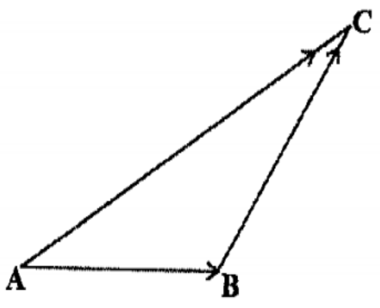


Fig 10.7

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

This is known as the *triangle law of vector addition*.

In general, if we have two vectors a and b (Fig 10.8(i)), then to add them they are positioned so that the initial point of one coincides with the terminal point of the other (Fig 10.8(ii)).

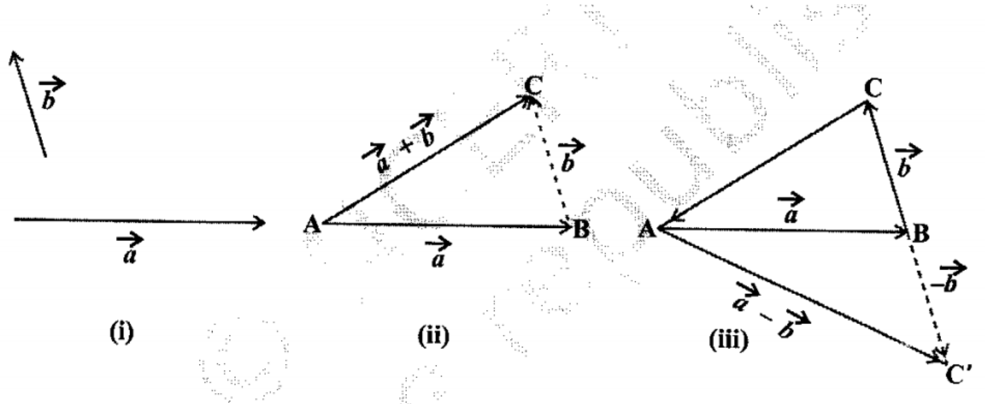


Fig 10.8

For example, in fig 10.8(ii) we have shifted vector b without changing its magnitude and direction so that its initial point coincide with the terminal point of a .Then the vector a+b, represented by the third side AC of the triangle ABC, gives us the sum of the vectors a and b in the triangle ABC (fig 10.8(ii)) we have

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

Now again, since $\overrightarrow{AC} = -\overrightarrow{CA}$, from the above equation, we have

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AA} = \vec{0}$$

This means that when the sides of a triangle are taken in order ,it leads to zero resultant as the initial and terminal points get coincide(Fig 10.8(iii)).

Now, construct a vector BC so that its magnitude is same as the vector BC but the direction opposite to that direction opposite to that of it (Fig 10.8(iii))

$$\overrightarrow{BC'} = -\overrightarrow{BC}$$

Then, on applying triangle law from the Fig 10.8 (iii), we have

$$\overrightarrow{AC'} = \overrightarrow{AB} + \overrightarrow{BC'} = \overrightarrow{AB} + (-\overrightarrow{BC}) = \vec{a} - \vec{b}$$

Now consider a boat in a river going from one bank of the river toother in a direction perpendicular to the flow of the river . Then it is acted upon by to velocity vectors one is the velocity imparted to the boat by its engine and other one is the velocity of the flow of river water . under the simultaneous influence of these two velocities the boat in actual starts travelling with a different velocity . to have a preci idea about the effective speed and direction (i.e., the resultant velocity) of the boat, we have the following law of vector addition.

If we have two vectors $\vec{a}$ and $\vec{b}$ represented by the two adjacent sides of a parallelogram in magnitude and direction (Fig 10.9), then their sum $\vec{a}+\vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram through their common point. This is known as the *parallelogram law of vector addition.*

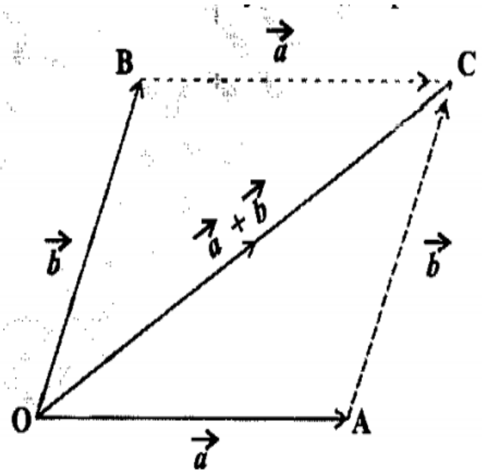


Fig 10.9

Note

From fig 10.9 using the triangle law one may note that

$$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$$

$$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$$

Which is parallelogram law .Thus we may say that the two laws of vector addition are equivalent to each other .

Properties of vector addition

Property 1 for any two vectors a and b

a+b=b+a

Proof Consider the parallelogram ABCD (Fig 10.10). Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$, then using the triangle law, from triangle ABC, we have

$$\overrightarrow{AC} = \vec{a} + \vec{b}$$

Now, since the opposite sides of a parallelogram are equal and parallel, from Fig 10.10, we have, $\overrightarrow{AD} = \overrightarrow{BC} = \vec{b}$ and $\overrightarrow{DC} = \overrightarrow{AB} = \vec{a}$. Again using triangle law, from triangle ADC, we have

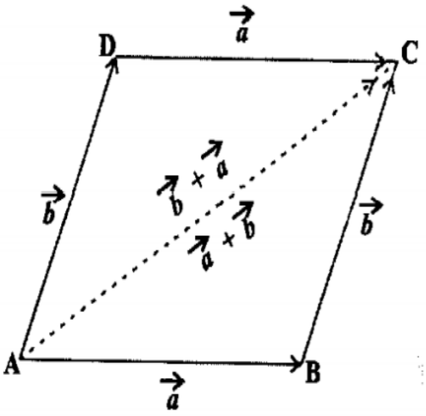


Fig 10.10

$$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \vec{b} + \vec{a}$$

Hence $\qquad \vec{a} + \vec{b} = \vec{b} + \vec{a}$

Property 2 For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \qquad \text{(Associative property)}$$

Proof Let the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ be represented by $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{RS}$, respectively, as shown in Fig 10.11(i) and (ii).

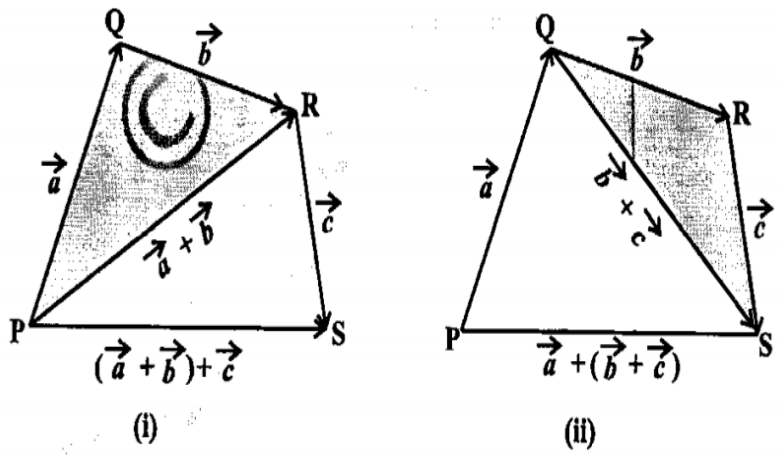


Fig 10.11

Then $\qquad \vec{a} + \vec{b} = \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

and $\qquad \vec{b} + \vec{c} = \overrightarrow{QR} + \overrightarrow{RS} = \overrightarrow{QS}$

So $\qquad (\vec{a} + \vec{b}) + \vec{c} = \overrightarrow{PR} + \overrightarrow{RS} = \overrightarrow{PS}$

- The vector sum of the three sides of a triangle taken in order is zero.
- The vector sum of three coinitial vectors is given by the diagonal of the parallelogram whose adjacent sides are the given vectors.
- The multiplication of a given vector by a scalar changes the magnitude of the vector by the multiple and keeps the direction same (or makes it opposite) according as the value of lamda is positive (or negative).
- For a given vector the vector in the direction of a.
- The position vector of a point R dividing a line segment joining the points P and Q whose position vectors are a and b respectively, in the ratio m, n

  (i) internally, is given by

$$\frac{n\vec{a}+m\vec{b}}{m+n}.$$

  (ii)Externally, is given by

$$\frac{m\vec{b}-n\vec{a}}{m-n}.$$

- The scalar product of two given vectors a and b having angle theta between them is defined as

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta.$$

  Also when a.b is given the angle theta between the vectors a and b may be

$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

- If theta is the angle between two vectors a and b then their cross product is given as

$$\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$$

- Where n is a unit vector perpendicular to the plane containing a and b. such that a, b, n form right handed system of coordinate axes.
- If we have two vectors a and b, given in component form as

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \text{ and } \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \lambda \text{ any scalar,}$$