{UNIT-4}

1. Verify Rolle's mean value theorem for the following

(i.) $f(x) = (x-a)^m (x-b)^n$ where m, n are positive integers in $[a, b]$.

Sol. Given $f(x) = (x-a)^m (x-b)^n$

1) It is continuous in $[a, b]$

$$f'(x) = m(x-a)^{m-1}(x-b)^n + (x-a)^m n(x-b)^{n-1}$$

2) $f(x)$ is derivable in (a, b)

3) $f(a) = 0$

$f(b) = 0$

$f(a) = f(b)$

All the conditions are satisfied, by rolles theorem, $f'(c) = 0$

$$m(c-a)^{m-1}(c-b)^n + (c-a)^m n(c-b)^{n-1} = 0$$

$$m\frac{(c-a)^m}{(c-a)}(c-b)^n + (c-a)^m n\frac{(c-b)^n}{(c-b)} = 0$$

$$(c-a)^m (c-b)^n \left[m(c-a)^{-1} + n(c-b)^{-1}\right] = 0$$

$$\frac{m}{c-a} + \frac{n}{c-b} = 0$$

$$m(c-b) + n(c-a) = 0$$

$$mc - mb + nc - na = 0$$

$$mc + nc = mb + na$$

$$c(m+n) = mb + na$$

$$c = \frac{mb + na}{m+n} \in (a, b)$$

(ii) $f(x) = x(x+3)e^{-x/2}$ in $[-3, 0]$

Sol. Given $f(x) = x(x+3)e^{-x/2}$

1) $f(x)$ is continuous in $[-3, 0]$

$$f'(x) = (2x+3)e^{-x/2} + (x^2+3x)e^{-x/2}\left(-\tfrac{1}{2}\right)$$

2) $f(x)$ is derivable in $(-3, 0)$

3) $f(-3) = 0$

$f(0) = 0$

$f(-3) = f(0)$

All the conditions of Rolles theorem are satisfied $f'(c) = 0$

$$(2c+3)e^{-c/2} + (c^2+3c)e^{-c/2}\left(-\tfrac{1}{2}\right) = 0$$

$$(2c+3) - \frac{1}{2}(c^2+3c) = 0$$

$$4c + 6 - c^2 - 3c = 0$$

$$-c^2 + c + 6 = 0$$

$c^2 - c - 6 = 0$

$(c-3)(c+2) = 0$

$c = 3 \quad c = -2$

$c = -2 \in [-3, 0]$

(iii) $f(x) = e^{-x} \sin x$ in $[0, \pi]$

Sol Given $f(x) = \dfrac{\sin x}{e^x}$

1) $f(x)$ is continuous in $[0, \pi]$

$$f'(x) = \frac{e^x (\cos x) - e^x (\sin x)}{e^{2x}}$$

$$= \frac{e^x (\cos x - \sin x)}{e^{2x}}$$

$$= e^{-2x} \cdot e^x (\cos x - \sin x)$$

$$= e^{-x} [\cos x - \sin x]$$

$$= \frac{(\cos x - \sin x)}{e^x}$$

2) $f(x)$ is derivable in $(0, \pi)$

3) $f(0) = \dfrac{\sin(0)}{e^0} = 0$

$f(\pi) = \dfrac{\sin(\pi)}{e^\pi} = 0$

$f(0) = f(\pi)$

All the conditions of Rolles theorem are satisfied

$$f'(c)=0$$

$$\frac{\cos c-\sin c}{e^c}=0$$

$$\cos c-\sin c=0$$

$$\cos c=\sin c$$

$$\tan c=1$$

$$c=\frac{\pi}{4}\in(0,\pi)$$

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2. (i) Prove that $\frac{\pi}{3}-\frac{1}{5\sqrt{3}}>\cos^{-1}\frac{3}{5}>\frac{\pi}{3}-\frac{1}{8}$ using Lagrange's mean value theorem.

Sol. Let $f(x)=\cos^{-1}x$

1) $f(x)$ is continuous in $[0,\pi]$

$$f'(x)=\frac{-1}{\sqrt{1-x^2}}$$

2) $f(x)$ is derivable in $(0,\pi)$

3) $f'(c)=\frac{f(b)-f(a)}{b-a}$

$$\frac{-1}{\sqrt{1-c^2}}=\frac{\cos^{-1}b-\cos^{-1}a}{b-a}$$

$$a<b$$

$$a<c<b$$

$$a^2 < c^2 < b^2$$

$$\sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$$

$$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{-1}{\sqrt{1-a^2}} > \frac{-1}{\sqrt{1-c^2}} > \frac{-1}{\sqrt{1-b^2}}$$

$$\frac{-1}{\sqrt{1-a^2}} > \frac{\cos^{-1} b - \cos^{-1} a}{b-a} > \frac{-1}{\sqrt{1-b^2}}$$

$$\frac{-b+a}{\sqrt{1-a^2}} > \cos^{-1} b - \cos^{-1} a > \frac{-b+a}{\sqrt{1-b^2}}$$

$$b = \frac{3}{5} \qquad a = \frac{1}{2}$$

$$\frac{-\frac{3}{5}+\frac{1}{2}}{\sqrt{1-\left(\frac{1}{4}\right)}} > \cos^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{1}{2}\right) > \frac{-\frac{3}{5}+\frac{1}{2}}{\sqrt{1-\left(\frac{9}{25}\right)}}$$

$$\frac{\frac{-6+5}{10}}{\sqrt{3}/2} > \cos^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{1}{2}\right) > \frac{\frac{-6+5}{10}}{4/5}$$

$$\frac{-1/10}{\sqrt{3}/2} > \cos^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{1}{2}\right) > \frac{-1/10}{4/5}$$

$$\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) > \frac{\pi}{3} - \frac{1}{8}$$

(ii) For $0 < a < b$, prove that

$1 - \frac{a}{b} < \log\left(\frac{b}{a}\right) < \frac{b}{a} - 1$. Hence prove

that $\frac{1}{6} < \log\left(\frac{6}{5}\right) < \frac{1}{5}$.

Sol Given $f(x) = \log x$

1) $f(x)$ is continuous in $[a, b]$

$$f'(x) = \frac{1}{x}$$

2) $f(x)$ is derivable in (a, b)

3) $$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\frac{1}{c} = \frac{\log(b) - \log(a)}{b-a}$$

$$\frac{1}{c} = \frac{\log b/a}{b-a}$$

$$a < b$$

$$a < c < b$$

$$\frac{1}{a} > \frac{1}{c} > \frac{1}{b}$$

$$\frac{1}{a} > \frac{\log b/a}{b-a} > \frac{1}{b}$$

$$\frac{b-a}{a} > \log b/a > \frac{b-a}{b} \Rightarrow 1 - \frac{a}{b} < \log \frac{b}{a} < \frac{b}{a} - 1$$

$$b = 6 \qquad a = 5$$

$$1 - \frac{5}{6} < \log \frac{6}{5} < \frac{6}{5} - 1$$

$$\frac{1}{6} < \log \frac{6}{5} < \frac{1}{5}$$

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3. Prove that if

$0 < a < 1$, $0 < b < 1$ and $a < b$, then

$$\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}}$$ and hence

deduce that $\frac{\pi}{6} - \frac{1}{2\sqrt{3}} < \sin^{-1}\frac{1}{4} < \frac{\pi}{6} - \frac{1}{\sqrt{15}}$

Sol: Given $f(x) = \sin^{-1} x$

1) $f(x)$ is continuous in $[a, b]$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

2) $f'(x)$ is derivable in (a, b)

3) $$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1} b - \sin^{-1} a}{b-a}$$

$$a < b$$

$$a < c < b$$

$$a^2 < c^2 < b^2$$

$$\sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$$

$$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1}b-\sin^{-1}a}{b-a} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1}b-\sin^{-1}a < \frac{b-a}{\sqrt{1-b^2}}$$

$$b=\frac{1}{4} \quad a=\frac{1}{2}$$

$$\frac{1/4-1/2}{\sqrt{1-1/4}} < \sin^{-1}\frac{1}{4}-\frac{\pi}{6} < \frac{1/4-1/2}{\sqrt{1-1/16}}$$

$$\frac{\pi}{6}-\frac{1}{2\sqrt{3}} < \sin^{-1}\frac{1}{4} < \frac{\pi}{6}-\frac{1}{\sqrt{15}}$$

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4. If $a<b$ prove that $\frac{b-a}{1+b^2} < \tan^{-1}b-\tan^{-1}a < \frac{b-a}{1+a^2}$ using Lagrange's mean value theorem and hence deduce the following

(i) $\frac{\pi}{4}+\frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4}+\frac{1}{6}$

(ii) $\frac{5\pi+4}{20} < \tan^{-1}2 < \frac{\pi+2}{4}$

Sol: Given $f(x)=\tan^{-1}x$

1) $f(x)$ is continuous in $[a,b]$

$$f'(x) = \frac{1}{1+x^2}$$

2) $f(x)$ is derivable in (a, b)

3) $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$$\frac{1}{1+c^2} = \frac{\tan^{-1} b - \tan^{-1} a}{b-a}$$

$$a < b$$

$$a < c < b$$

$$1+a^2 < 1+c^2 < 1+b^2$$

$$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

$$\frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$$

$$\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2}$$

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$$

(i) $a = 1$ $\quad b = \frac{4}{3}$

$$\frac{4/3-1}{1+16/9} < \tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{4/3-1}{1+1}$$

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1} \frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$$

(ii) $b = 2 \quad a = 1$

$$\frac{2-1}{1+4} < \tan^{-1}(2) - \tan^{-1}(1) < \frac{2-1}{1+1}$$

$$\frac{1}{5} + \frac{\pi}{4} < \tan^{-1} 2 < \frac{\pi}{4} + \frac{1}{2}$$

$$\frac{5\pi + 4}{20} < \tan^{-1} 2 < \frac{\pi + 2}{4}$$

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5. Using Cauchy's mean value theorem.

(i) prove that $\dfrac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta$, $0 < \alpha < \beta < \frac{\pi}{2}$.

(ii) prove that the mean value 'c' of the functions $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$ is geometric mean of a and b, $a > 0$, $b > 0$.

(iii) Find 'c' for $\sin x$ and $\cos x$ in $[0, \frac{\pi}{2}]$

Sol (i) $f(x) = \sin x \qquad g(x) = \cos x$

1) $f(x)$ and $g(x)$ is continuous in $[0, \frac{\pi}{2}]$

$f'(x) = \cos x \qquad g'(x) = -\sin x$

2) $f(x)$ and $g(x)$ are derivable in $(0, \frac{\pi}{2})$

3) $g'(x) \neq 0$ $\quad\quad -\sin x \neq 0$

Then by Cauchy's mean value theorem

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{\cos c}{-\sin c} = \frac{\sin(\pi/2) - \sin(0)}{\cos(\pi/2) - \cos(0)}$$

$$-\cot c = \frac{1-0}{0-1}$$

$$\cot c = 1$$

$$c = \frac{\pi}{4} \in \left[0, \frac{\pi}{2}\right]$$

(ii) $f(x) = \sqrt{x}$ $\quad\quad g(x) = \frac{1}{\sqrt{x}}$

1) $f(x)$ and $g(x)$ are continuous in $[a,b]$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\quad\quad g'(x) = -\frac{1}{2} x^{-3/2}$

2) $f(x)$ and $g(x)$ are derivable in (a,b)

3) $g'(x) \neq 0$ $\quad\quad \frac{-1}{2\text{[redacted]}} x^{-3/2} \neq 0$

By Cauchy's mean value theorem

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{\frac{1}{2\sqrt{c}}}{-\frac{1}{2}c^{-3/2}} = \frac{\sqrt{b}-\sqrt{a}}{1/\sqrt{b} - 1/\sqrt{a}}$$

$$\frac{1}{2\sqrt{c}} \cdot c^{3/2} \cdot \frac{2}{(-1)} = \frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}}$$

$$-c^{3/2 - 1/2} = \frac{\cancel{(\sqrt{b}-\sqrt{a})}}{\frac{-\cancel{(\sqrt{b}-\sqrt{a})}}{\sqrt{ab}}}$$

$$c = \sqrt{ab} \in [a, b]$$

(iii) $f(x) = \sin x$ $\quad g(x) = \cos x$

1) $f(x)$ and $g(x)$ are continuous in $[0, \frac{\pi}{2}]$

$f'(x) = \cos x$ $\quad g'(x) = -\sin x$

2) $f(x)$ and $g(x)$ are derivable in $(0, \frac{\pi}{2})$

3) $g'(x) \neq 0$ $\quad -\sin x \neq 0$

By Cauchy's mean value theorem

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{\cos c}{-\sin c} = \frac{\sin \pi/2 - \sin(0)}{\cos \pi/2 - \cos 0}$$

$$-\cot c = \frac{1-0}{0-1}$$

$$\cot c = 1$$

$$c = \frac{\pi}{4}$$

6. Prove that $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, where $m>0,\ n>0$

Sol. $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx \longrightarrow (1)$

Let $x = y^2$

$dx = 2y\,dy$

$\Gamma(n) = \int_0^\infty e^{-y^2} (y^2)^{n-1}\, 2y\,dy$

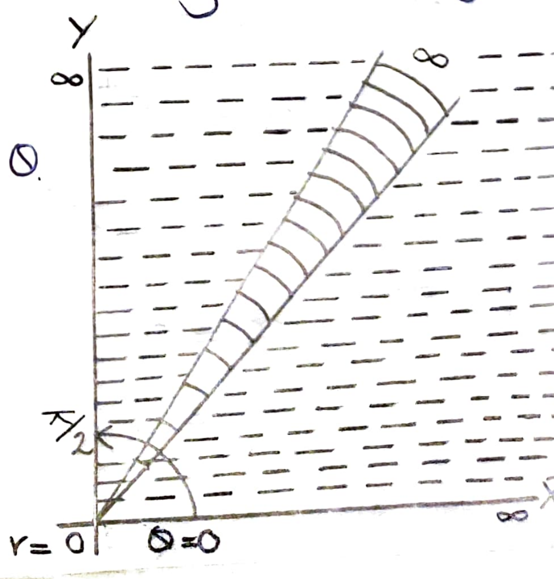
$\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-2+1}\,dy$

$\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1}\,dy \longrightarrow (2)$

$\Gamma(m) = 2\int_0^\infty e^{-x^2} x^{2m-1}\,dx \longrightarrow (3)$

(2) × (3)

$\Gamma(m)\Gamma(n) = 4\int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-x^2} e^{-y^2} x^{2m-1} y^{2n-1}\,dx\,dy$



put $x = r\cos\theta \quad y = r\sin\theta$

$x^2+y^2 = r^2$

$dx\,dy = r\,dr\,d\theta$

Limits $r = 0$ to ∞

$\theta = 0$ to $\pi/2$

$$= 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} \, r\,dr\,d\theta$$

$$= 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} \cdot r^{2m-1+2n-1+1} \cdot \cos^{2m-1}\theta \cdot \sin^{2n-1}\theta \, dr\,d\theta$$

$$= 2 \int_{\theta=0}^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \, d\theta \cdot 2\int_{r=0}^{\infty} e^{-r^2} r^{2(m+n)-1} \, dr$$

$$\Gamma(m)\Gamma(n) = B(m,n)\,\Gamma(m+n)$$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

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7. Show that $\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$ where n is positive integer and $m > -1$, evaluate $\int_0^1 x(\log x)^3 dx$.

Sol. $\int_0^1 x^m (\log x)^n dx$

Let $\log x = -t$

$x = e^{-t}$

$dx = -e^{-t} dt$

Limits $x = 0 \quad t = \infty$

$x = 1 \quad t = 0$

$$= \int_{\infty}^{0} (e^{-t})^m (-t)^n (-e^{-t})dt$$

$$= \int_{0}^{\infty} e^{-t(m+1)} (-1)^n (t^n)\,dt$$

$$t(m+1) = x$$

$$t = \frac{x}{m+1}$$

$$dt = \frac{dx}{m+1}$$

$$= \int_{0}^{\infty} e^{-x} (-1)^n \left(\frac{x}{m+1}\right)^n \frac{dx}{m+1}$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \int_{0}^{\infty} e^{-x} \cdot x^{n+1-1}\, dx$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \Gamma(n+1)$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \cdot n!$$

Now, $\int_{0}^{1} x (\log x)^3\, dx$

Here $m=1$ $n=3$

$$= \frac{(-1)^3\; 3!}{(1+1)^{3+1}} = \frac{-6}{16} = -\frac{3}{8}$$

8. Prove that

(i) $$\int_0^{\pi/2} \sin^2\theta \cos^4\theta \, d\theta = \frac{\pi}{32}$$

Sol. * $$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$2m-1 = 2$ $\qquad$ $2n-1 = 4$

$2m = 3$ $\qquad$ $2n = 5$

$m = 3/2$ $\qquad$ $n = 5/2$

$$= \int_0^{\pi/2} \sin^{2(3/2)-1}\theta \cos^{2(5/2)-1}\theta \, d\theta$$

$$= \frac{1}{2} \beta(3/2, 5/2)$$

$$= \frac{1}{2}\left[\frac{\Gamma(3/2)\Gamma(5/2)}{\Gamma(3/2+5/2)}\right]$$

$$= \frac{1}{2}\left[\frac{\Gamma(3/2)\Gamma(5/2)}{\Gamma(4)}\right]$$

$$= \frac{\pi}{32}$$

(ii) $$\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta} \, d\theta = \pi$$

Sol. $$\int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta \, d\theta \times \int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta \, d\theta$$

$\because \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta.$

(i) $2m-1 = -\frac{1}{2}$ $\qquad$ $2n-1 = 0$

$2m = \frac{1}{2}$ $\qquad$ $2n = 1$

$m = \frac{1}{4}$ $\qquad$ $n = \frac{1}{2}$

(ii) $2m-1 = \frac{1}{2}$ $\qquad$ $2n-1 = 0$

$2m = \frac{3}{2}$ $\qquad$ $2n = 1$

$m = \frac{3}{4}$ $\qquad$ $n = \frac{1}{2}$

$$= \int_0^{\pi/2} \sin^{2(1/4)-1}\theta \cos^{2(1/2)-1}\theta \, d\theta \times \int_0^{\pi/2} \sin^{2(3/4)-1}\theta \cos^{2(1/2)-1}\theta \, d\theta$$

$$= \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right) \times \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4}\left\{\frac{\Gamma(\frac{1}{4})\Gamma(1/2)}{\Gamma(1/4 + 1/2)} \times \frac{\Gamma(3/4)\Gamma(1/2)}{\Gamma(3/4 + 1/2)}\right\}$$

$$= \frac{1}{4}\left\{\frac{\Gamma(1/4)\sqrt{\pi}}{\cancel{\Gamma(3/4)}} \cdot \frac{\cancel{\Gamma(3/4)}\sqrt{\pi}}{\Gamma(5/4)}\right\}$$

$$= \frac{\pi}{4}\left[\frac{\Gamma(1/4)}{\Gamma(\frac{1}{4}+1)}\right]$$

$$= \frac{\pi}{4}\left[\frac{\cancel{\Gamma(1/4)}}{1/4\,\cancel{\Gamma(1/4)}}\right] \qquad \{\because \Gamma(n+1) = n\Gamma(n)\}$$

$$= \pi$$

9. Prove that

(i.) $\int_0^\infty x^{-3/2}(1-e^{-x})\,dx = 2\sqrt{\pi}$

Sol. $\left[\left\{[1-e^{-x}]\dfrac{x^{-3/2+1}}{-3/2+1}\right\}_0^\infty - \int_0^\infty e^{-x}\dfrac{x^{-3/2+1}}{-3/2+1}\right] dx$

$= [\{0-0\} + 2\int_0^\infty e^{-x}\, x^{-1/2+1-1}\, dx$

$= 2\int_0^\infty e^{-x}\, x^{1/2-1}\, dx$

$= 2\Gamma(1/2)$

$= 2\sqrt{\pi}$

(ii) $\int_0^1 \dfrac{x^2}{\sqrt{1-x^4}}\,dx \times \int_0^1 \dfrac{1}{\sqrt{1+x^4}}\,dx = \dfrac{\pi}{4\sqrt{2}}$

Sol. Let $x^4 = t \rightarrow x = t^{1/4}$

$x^2 = t^{1/2}$

$4x^3 dx = dt$

$dx = \dfrac{dt}{4x^3} = \dfrac{dt}{4 \cdot t^{3/4}}$

$= \int_0^1 \dfrac{1}{\sqrt{1-t}} \cdot \sqrt{t} \cdot \dfrac{dt}{4 \cdot t^{3/4}}$

$= \dfrac{1}{4}\int_0^1 t^{1/2-3/4}\,(1-t)^{-1/2}\,dt$

$= \dfrac{1}{4}\int_0^1 t^{-1/4+1-1}\,(1-t)^{-1/2+1-1}\,dt$

$$= \frac{1}{4}\int_0^1 t^{3/4-1}(1-t)^{1/2-1}\,dt$$

$$= \frac{1}{4}\beta\left(3/4 \cdot 1/2\right)$$

$$= \frac{1}{4}\,\frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma(3/4+1/2)}$$

$$= \frac{1}{4}\,\frac{\Gamma(3/4)\sqrt{\pi}}{\Gamma(5/4)} \longrightarrow (1)$$

Now, $\displaystyle\int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx$

Let $x^2 = \tan\theta$

$$2x\,dx = \sec^2\theta\,d\theta$$

$$dx = \frac{\sec^2\theta}{2x}\,d\theta$$

$$dx = \frac{\sec^2\theta}{2\sqrt{\tan\theta}}\,d\theta$$

Limits $\quad x=0 \quad \theta=0$

$\quad x=1 \quad \theta=\pi/4$

$$= \int_0^{\pi/4} \frac{1}{\sqrt{1+\tan^2\theta}}\cdot\frac{\sec^2\theta}{2\sqrt{\tan\theta}}\,d\theta$$

$$= \int_0^{\pi/4} \frac{1}{\sec\theta}\cdot\frac{\sec^2\theta}{2\sqrt{\tan\theta}}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \sec\theta\cdot\frac{1}{\sqrt{\tan\theta}}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{\sqrt{\sin\theta}\,\sqrt{\cos\theta}}\,d\theta$$

$$= \frac{\sqrt{2}}{2}\int_0^{\pi/4} \frac{1}{\sqrt{2\sin\theta\cos\theta}}\,d\theta$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{1}{\sqrt{\sin 2\theta}}\,d\theta$$

Let $2\theta = x$ Limits $\theta = 0$ $x = 0$

$\theta = \frac{x}{2}$ $\theta = \pi/4$ $x = \pi/2$

$d\theta = \frac{dx}{2}$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/2} \frac{1}{\sqrt{\sin x}}\cdot\frac{dx}{2}$$

$$= \frac{1}{2\sqrt{2}}\int_0^{\pi/2} (\sin x)^{-1/2}\cos^0 x\,dx$$

$-\frac{1}{2} = 2m-1$ $0 = 2n-1$

$-\frac{1}{2}+1 = 2m$ $1 = 2n$

$m = 1/4$ $n = 1/2$

$$= \frac{1}{2\sqrt{2}}\cdot\frac{1}{2}\int_0^{\pi/2} 2\sin^{2(1/4)-1} x\,\cos^{2(\frac{1}{2})-1} x\,dx$$

$$= \frac{1}{4\sqrt{2}}\cdot\beta\left(\frac{1}{4},\frac{1}{2}\right)$$

$$= \frac{1}{4\sqrt{2}} \cdot \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)}$$

$$= \frac{1}{4\sqrt{2}} \cdot \frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\Gamma\left(3/4\right)} \longrightarrow (2)$$

From (1) & (2)

$$= \frac{1}{4} \frac{\cancel{\Gamma\left(3/4\right)}\sqrt{\pi}}{\Gamma\left(5/4\right)} \times \frac{1}{4\sqrt{2}} \frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\cancel{\Gamma\left(3/4\right)}}$$

$$= \frac{\pi}{16\sqrt{2}} \cdot \frac{\Gamma\left(1/4\right)}{\Gamma\left(1/4+1\right)}$$

$$= \frac{\pi}{16\sqrt{2}} \cdot \frac{\Gamma\left(1/4\right)}{\frac{1}{4}\Gamma\left(1/4\right)}$$

$$= \frac{\pi}{16\sqrt{2}} \times 4 = \frac{\pi}{4\sqrt{2}}$$

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10. Show that

(i) $\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy = \frac{\Gamma(p)}{q^p} \quad (p, q > 0)$

Sol. Let $\log\frac{1}{y} = t$

$\frac{1}{y} = e^t$

$y = e^{-t}$

$dy = -e^{-t}\,dt$

Limits

$x=0$	$t=\infty$
$x=1$	$t=0$

$$= \int_0^1 y^{q-1} \left[\log \frac{1}{y}\right]^{p-1} dy$$

$$= \int_\infty^0 (e^{-t})^{q-1} (t)^{p-1} - e^{-t} dt$$

$$= \int_0^\infty (e^{-t})^{q-1+1} . (t)^{p-1} dt$$

$$= \int_0^\infty e^{-qt} \, t^{p-1} dt$$

$$qt = x$$

$$q\,dt = dx$$

$$= \int_0^\infty e^{-x} \left(\frac{x}{q}\right)^{p-1} \frac{dx}{q}$$

$$= \frac{1}{q^{p-1+1}} \int_0^\infty e^{-x} x^{p-1} dx$$

$$= \frac{\Gamma(p)}{q^p}$$

(ii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Sol. Proof :- By the Definition of Gamma function we know that $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \rightarrow (1)$

Let $x = y^2$

$dx = 2y\,dy$

Limits $x = 0 \qquad y = 0$

$x = \infty \qquad y = \infty$

$$\Gamma(n) = \int_0^\infty e^{-y^2} (y^2)^{n-1} \, 2y \, dy$$

$$\Gamma(n) = 2\int_0^\infty e^{-y^2} \, y^{2n-2+1} \, dy$$

$$\Gamma(n) = 2\int_0^\infty e^{-y^2} \cdot y^{2n-1} \, dy \longrightarrow 2$$

$n = 1/2$

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-y^2} \cdot y^{2\left(\frac{1}{2}\right)-1} \, dy$$

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-y^2} \, dy \longrightarrow 3$$

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-x^2} \, dx \rightarrow 4$$

Multiplying Eq 3 and Eq 4

$$\left[\Gamma(1/2)\right]^2 = 4\int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} \, dx \, dy$$

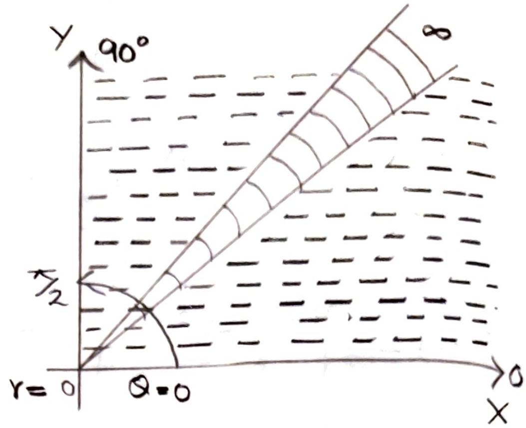
$$= 4\int_{x=0}^\infty \int_{y=0}^\infty e^{-(x^2+y^2)} \, dx \, dy$$

put $x = r\cos\theta \qquad y = r\sin\theta$

$$x^2 + y^2 = r^2$$

$$\theta = \tan^{-1}(y/x)$$

r, θ



limits $r = 0$ to ∞

$\theta = 0$ to $\pi/2$

$$dx\,dy = |J|\,dr\,d\theta$$

$$= r\,dr\,d\theta$$

$$= 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

$$= 4 \int_{\theta=0}^{\pi/2} d\theta \int_{r=0}^{\infty} e^{-r^2}\, r\,dr$$

Let $r^2 = t$

$2r\,dr = dt$

$r\,dr = \frac{dt}{2}$

$$= 4 \int_{\theta=0}^{\pi/2} d\theta \int_{r=0}^{\infty} e^{-t}\, \frac{dt}{2}$$

$$= 4\,[\theta]_0^{\pi/2} \cdot \frac{1}{2}\left[\frac{e^{-t}}{-1}\right]_0^{\infty}$$

$$= 4\left[\frac{\pi}{2} - 0\right] \cdot \frac{1}{2}\left[0 - \frac{1}{-1}\right] = \pi$$

$$= \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \pi \qquad \therefore \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$