HiperLife Tutorial: NonLinear Thermal Conduction

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1 Problem Definition

Thermal conduction is the diffusion of thermal energy within one material or between materials in contact. Nowadays it is becoming more important to predict heat transport properties according to geometric structures or component materials which add complexity in form of nonlinearity to the PDEs. Here the simplest case of non-linear heat transfer equation has been chosen to show how to use nonlinear solver within the time discretization.

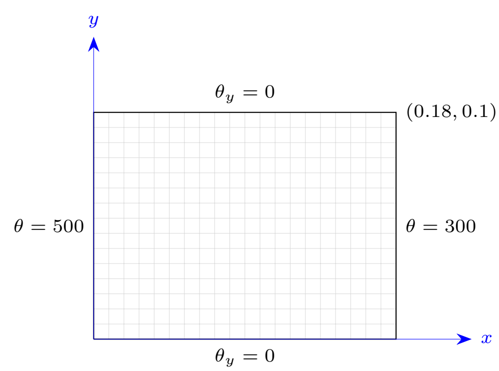

Figure 1: Geometry, BC and computational domain used for the analysis of nonlinear heat transfer.

2 Governing Equations

Consider the transient heat conduction equation [1]

$$\frac{\partial \theta}{\partial t} - \frac{\partial}{\partial x}(\kappa \frac{\partial \theta}{\partial x}) - \frac{\partial}{\partial y}(\kappa \frac{\partial \theta}{\partial y}) = 0 \quad \text{in } \Omega \,. \tag{1}$$

where θ is the temperature and κ is the conductivity. as it is demonstrated in Figure **??**, The domain Ω is a rectangle of dimensions $[1.18 \times 0.1]$ along the x and y coordinates, respectively; the conductivity κ is of the form

$$\kappa = \kappa_0 (1 + \beta \theta) \,. \tag{2}$$

where $\kappa_0 = 0.2$ W/(m °C) and $\beta = 2 \times 10^{-3}$ °C^{-1}.the boundary conditions to be

$$\theta(0, y, t) = 500 \text{ °C}, \quad \theta(0.18, y, t) = 300 \text{ °C}, \quad \theta_y(x, 0, t) = \theta_y(x, 0.1, t) = 0 \,. \tag{3}$$

The initial condition is assumed to be

$$\theta(x, y, 0) = 0 \,. \tag{4}$$

This is essentially a one-dimensional problem and $\Delta t = 0.005$ and we use Backward temporal approximation scheme.

3 Weak Form

The starting point for the development of the finite element models of Eq. (1) is their weak forms. By considering $\Delta t = t^{n+1} - t^n$, the α-family approximation for the temperature field takes the following form

$$\{\theta\}^{n+1} = \{\theta\}^n + \Delta t(1-\alpha)\{\theta_t\}^n + \Delta t\alpha\{\theta_t\}^{n+1} \,. \tag{5}$$

The time derivatives of θ on the right hand side of this equation can be calculated from Eq. (1) for each step, like this

$$\begin{aligned} \{\theta_t\}^{n+1} &= \nabla(\kappa\nabla\{\theta\}^{n+1})\,, \\ \{\theta_t\}^{n} &= \nabla(\kappa\nabla\{\theta\}^{n})\,. \end{aligned} \tag{6}$$

combining these two equations gives us

$$\{\theta\}^{n+1} - \Delta t\alpha[\nabla(\kappa\nabla\{\theta\}^{n+1})] - \{\theta\}^n - \Delta t(1-\alpha)[\nabla(\kappa\nabla\{\theta\}^n)] = 0\,. \tag{7}$$

The variation formulation of our model problem can be introduced as find $\theta \in V$ such that

$$\mathcal{F}(\theta; v) = 0 \quad \forall v \in \hat{V}\,. \tag{8}$$

where

$$\mathcal{F}(\theta; v) = \int_\Omega v\{\theta\}^{n+1} - \Delta t\alpha v\nabla\cdot(\kappa\nabla\{\theta\}^{n+1}) - v\{\theta\}^n - \Delta t(1-\alpha)v\nabla\cdot(\kappa\nabla\{\theta\}^n)\ \mathrm{d}\Omega\,. \tag{9}$$

and

$$\begin{aligned} \hat{V} &= \{v \in H^1(\Omega) : v = 0 \text{ on } x = 0 \text{ and } x = 0.18\}, \\ V &= \{v \in H^1(\Omega) : v = 500 \text{ on } x = 0 \text{ and } v = 300 \text{ on } x = 0.18\}\,. \end{aligned} \tag{10}$$

The discrete problem arises as usual by restricting V and $\hat{V}$ to a pair of discrete spaces.with $\theta = \sum_{j=1}^N \theta_j\phi_j$. Since $\mathcal{F}$ is a nonlinear function of θ, the variational statement gives rise to a system of nonlinear algebraic equations. applying integration by part, Using Gauss's theorem and also defining heat flux density as $q_{\mathfrak{n}} = [\nabla(\kappa\theta)]\cdot\mathfrak{n}$, the weak form takes the following form

$$\begin{aligned} \mathcal{F}(\theta; v) = &\int_\Omega v\{\theta\}^{n+1} + \Delta t\alpha\kappa\nabla v\nabla(\{\theta\}^{n+1})\ \mathrm{d}\Omega - \int_\Omega v\{\theta\}^n + \Delta t(1-\alpha)\kappa\nabla v\nabla(\{\theta\}^n)\ \mathrm{d}\Omega \\ &- v\Delta t\int_\Gamma \alpha v q_{\mathfrak{n}}^{n+1} + (1-\alpha)v q_{\mathfrak{n}}^n\ \mathrm{d}\Gamma\,. \end{aligned} \tag{11}$$

Applying boundary conditions ($\Gamma = \Gamma_N \cup \Gamma_D$):

$$\begin{aligned} v &= 0 \quad \text{on } \Gamma_D\,, \\ \kappa\nabla[\theta]\cdot n &= 0 \quad \text{on } \Gamma_N\,. \end{aligned} \tag{12}$$

we get the final form for $\mathcal{F}$

$$\mathcal{F}(\theta; v) = \int_\Omega v\{\theta\}^{n+1} + \Delta t\alpha\kappa\nabla v\nabla(\{\theta\}^{n+1})\ d\Omega - \int_\Omega v\{\theta\}^n + \Delta t(1-\alpha)\kappa\nabla v\nabla(\{\theta\}^n)\ \mathrm{d}\Omega\,. \tag{13}$$

In order to linearize our discretized nonlinear PDE problem, we may use Newton's method. which for the system $\mathcal{F}_i(\Theta_1, \ldots, \Theta_j) = 0$ it can be formulated by the first terms of a Taylor series approximation for the value of the variational as

$$\begin{aligned} \sum_{j=1}^N \frac{\partial}{\partial\Theta_j}\mathcal{F}_i(\Theta_1^k, \ldots, \Theta_N^k)\delta\Theta_j &= -\mathcal{F}_i(\Theta_1^k, \ldots, \Theta_N^k), \quad i = 1, \ldots, N, \\ \Theta_j^{k+1} &= \Theta_j^k + \delta\Theta_j, \quad j = 1, \ldots, N\,. \end{aligned} \tag{14}$$

where k is an iteration index and n is time step. An initial guess θ^0 must be provided to start the algorithm. We need to compute the $\partial\mathcal{F}_i/\partial\Theta_j$ and the right-hand side vector $-\mathcal{F}_i$. Our present problem has $\mathcal{F}_i$ given by above. the Hessian is given by

$$\mathcal{J}(\theta; v) = \int_\Omega v\frac{\partial\{\theta\}^{n+1}}{\partial\theta_j} + \Delta t\alpha\nabla v\left[\frac{\partial\kappa}{\partial\theta_j}\nabla\{\theta\}^{n+1} + \kappa\nabla\frac{\partial\{\theta\}^{n+1}}{\partial\theta_j}\right]\ \mathrm{d}\Omega \tag{15}$$

4 Finite Element Model

Since we are developing the Ritz-Galerkin finite element model, the choice of the weight functions is restricted to the spaces of approximation functions used for the solution field. Suppose that the dependent variable θ approximated by expansions of the form

$$\theta(\mathbf{x}, t) = \sum_{m=1}^{M} \phi_m(\mathrm{x})\theta^m(t) = \mathbf{\Phi}^T \boldsymbol{\theta}\,, \tag{16}$$

Lets rewrite this equation for known and unknown variables.

$$\mathcal{J}(\theta;\phi) = \int_\Omega \phi\phi^T + \Delta t\alpha\kappa_0 \nabla\phi\Big[\beta\phi^T\nabla\{\theta\}^{n+1} + (1+\beta\{\theta\}^{n+1})\nabla\phi^T\Big]\mathrm{d}\Omega\,. \tag{17}$$

The above equations can be written symbolically in matrix form as we are going to solve it,

$$\sum_j \mathcal{J}(\theta;\phi)\{\delta\theta_j\}^{n+1} = -\mathcal{F}(\theta;\phi)\,. \tag{18}$$

where $\mathcal{F}(\theta;\phi)$ is

$$\mathcal{F}(\theta;\phi) = \int_\Omega \phi\{\theta\}^{n+1} + \kappa_0\Delta t\alpha(1+\beta\{\theta\}^{n+1})\nabla\phi\nabla\{\theta\}^{n+1} - \phi\theta^n + \kappa_0\Delta t\alpha(1+\beta\{\theta\}^n)\nabla\phi\nabla\{\theta\}^n\mathrm{d}\Omega\,. \tag{19}$$

The elemental representation of the vector and matrix required for filling function in hiperlife would be like

$$\begin{aligned}
Ak(i,j) =& jac \times \Big[\phi_i\phi_j + \Delta t\alpha\kappa_0\nabla\phi_i\Big(\beta\phi_j\nabla\{\theta\}^{n+1} + (1+\beta\{\theta\}^{n+1})\nabla\phi_j\Big)\Big]\,,\\
Bk(i) =& jac \times \Big[\phi_i\{\theta\}^{n+1} + \kappa_0\Delta t\alpha(1+\beta\{\theta\}^{n+1})\nabla\phi_i\nabla\{\theta\}^{n+1} - \phi_i\theta^n + \kappa_0\Delta t\alpha(1+\beta\{\theta\}^n)\nabla\phi_i\nabla\{\theta\}^n\Big]\,.
\end{aligned} \tag{20}$$

Note that $\mathrm{d}A = \mathrm{d}x_1 \times \mathrm{d}x_2 = jac\ \mathrm{d}\xi\mathrm{d}\eta$, which $jac = \det(Jacobian)$.

5 Choice of Elements

Thus, for this simple problem every Lagrange and serendipity family of interpolation functions are admissible for the interpolation of the temperature field, our choice is would be quadratic quadrilateral elements consist of 9 nodes. The shape function with respect to the reference element are given for the vertex nodes ($I = 1, 2, 3, 4$):

$$\mathbf{\Phi}_I(\xi,\eta) = \frac{1}{4}(\xi^2 + \xi_I\xi)(\eta^2 + \eta_I\eta) \tag{21}$$

-the middle edge nodes ($I = 5, 6, 7, 8$):

$$\mathbf{\Phi}_I(\xi,\eta) = \frac{1}{2}\xi_I^2(\xi^2 + \xi_I\xi)(1-\eta^2) + \frac{1}{2}\eta_I^2(\eta^2 + \eta_I\eta)(1-\xi^2) \tag{22}$$

-and the middle node ($I = 9$):

$$\mathbf{\Phi}_I(\xi,\eta) = (1-\xi^2)(1-\eta^2) \tag{23}$$

where ξ_I and η_I are the corner coordinates at element T in domain of $\Omega_T \in (-1,1)^2$. As it shown in Figure 2 we are using 2×2 Gauss–Legendre quadrature integration. We also chose a uniform mesh of size 10×10 to model the domain of our problem.

6 Implementation

In this section, we present the implementation of our solution in the Hiperlife. The program is divided into three separate files, main part which we create our problem by the Hiperlife headers, auxiliary header where we introduce parameters and declare defined functions, and at last auxiliary file, where we define some functions which provide required matrices like the Jacobian and the Hessian.

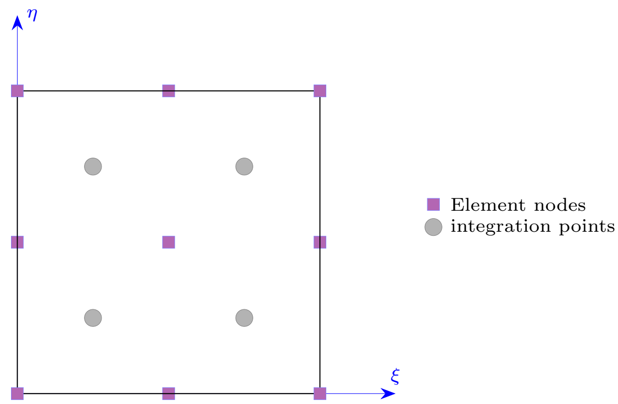

Figure 2: Quadratic quadrilateral element used for finite element model.

6.1 HeatTransferNonL.cpp

```
/*
* Heat Transfer conduction nonlinear
*/
// cpp headers
#include <iostream>
#include <fstream>
#include <cmath>

// hiperlife headers
#include "hl_Core.h"
#include "hl_ParamStructure.h"
#include "hl_Parser.h"
#include "hl_TypeDefs.h"
#include "hl_GlobalBasisFunctions.h"
#include "hl_StructMeshGenerator.h"
#include "hl_DistributedMesh.h"
#include "hl_FillStructure.h"
#include "hl_DOFsHandler.h"
#include "hl_SurfLagrParam.h"
#include "hl_HiPerProblem.h"
#include "hl_LinearSolver_Iterative_AztecOO.h"
#include "hl_NonlinearSolver_NewtonRaphson.h"
#include <hl_ConsistencyCheck.h>

// Header to auxiliary functions
#include "AuxHeatTransferNonL.h"

// ------------------------------------------------------------------------//
/// -------------------     MAIN FUNCTION     ------------------------------///
// ------------------------------------------------------------------------//

int main(int argc, char** argv)
{
        using namespace std;
        using namespace hiperlife;
        using namespace hiperlife::Tensor;

        // --------------------------------------------------------------------//
        /// *****                   INITIALIZATION                     *****///
        // --------------------------------------------------------------------//

        // Initialize MPI
        hiperlife::Init(argc, argv);

        // --------------------------------------------------------------------//
```

```
    /// *****                    DATA INPUT                    *****///
    // ----------------------------------------------------------------//
    // Time-related parameters
    int maxSteps = 1000;
    double maxTime  = 0.1;
    double maxDeltat = 1.0;
    double adaptiveFactor = 1.;

    // number of output files
    int nSave = 10;

    // Put parameters in the user structure
    SmartPtr<ParamStructure> paramStr = CreateParamStructure<HeatParams>();

    // Data
    paramStr->setRealParameter(HeatParams::delta_t, 0.005);
    paramStr->setRealParameter(HeatParams::alpha, 0.5);
    double delta_t = paramStr->getRealParameter(HeatParams::delta_t);

    // ----------------------------------------------------------------//
    /// *****                    MESH CREATION                 *****///
    // ----------------------------------------------------------------//

    // Create a rectangular structured mesh
    SmartPtr<StructMeshGenerator> structMesh = Create<StructMeshGenerator>();
    structMesh->setNDim(3);
    structMesh->setBasisFuncType(BasisFuncType::Lagrangian);
    structMesh->setBasisFuncOrder(2);
    structMesh->setElemType(ElemType::Square);
    structMesh->genRectangle(4, 2, 0.18, 0.1);

    // Distributed mesh
    SmartPtr<DistributedMesh> disMesh = Create<DistributedMesh>();
    disMesh->setMesh(structMesh);
    disMesh->setBalanceMesh(true);
    disMesh->setElementLocatorEngine(ElementLocatorEngine::BoundingVolumeHierarchy);
    disMesh->Update();

    // checking mesh
    disMesh->printFileLegacyVtk("mesh");

    // ----------------------------------------------------------------//
    /// *****              DOFsHANDLER CREATION                *****///
    // ----------------------------------------------------------------//

    // DOFHandler
    SmartPtr<DOFsHandler> dofHand = Create<DOFsHandler>(disMesh);
    dofHand->setNameTag("dofHand");
    dofHand->setNumDOFs(1);
    dofHand->setDOFs({"theta"});
    dofHand->Update();
    // ---------------- Initial conditions-------------------- //
    //---------------------------------------------------------------//
    double f;
    for (int i = 0; i < disMesh->loc_nPts(); i++)
    {
        // Coordinate
        std::vector<double> x = disMesh->nodeCoords(i, IndexType::Local);
        f = 0.0 * x[i];
        // Initial condition
        dofHand->nodeDOFs->setValue("theta", i, IndexType::Local, f);
        // -------------------- Boundary condition ------------- //
        //---------------------------------------------------------------//
        if (x[0] < 1e-5)
        {
            dofHand->nodeDOFs->setValue("theta", i, IndexType::Local,500.0);
            dofHand->setConstraint("theta",i, IndexType::Local,0.0);
        }
```

```
            if (x[0] > (0.18-1e-5))
            {
                dofHand->nodeDOFs->setValue("theta", i, IndexType::Local,300.0);
                dofHand->setConstraint("theta",i, IndexType::Local,0.0);
            }
        }
        // Update
        dofHand->nodeDOFs0->setValue(dofHand->nodeDOFs);
        dofHand->nodeDOFs0->UpdateGhosts();
        dofHand->UpdateGhosts();
        // checking initial and boundary condition
        dofHand->printFileLegacyVtk("HeatTransferNonL0");
        // --------------------------------------------------------------------//
        /// *****              HIPERPROBLEM CREATION              *****///
        // --------------------------------------------------------------------//

        SmartPtr<HiPerProblem> hiperProbl = Create<HiPerProblem>();
        hiperProbl->setParameterStructure(paramStr);
        hiperProbl->setDOFsHandlers({dofHand});
        hiperProbl->setIntegration("Integ", {"dofHand"});
        hiperProbl->setCubatureGauss("Integ", 4);
        hiperProbl->setElementFillings("Integ", LS);
        if (true)
        {
            hiperProbl->setConsistencyDOFs("dofHand", {"theta"});
            hiperProbl->setElementFillings("Integ", ConsistencyCheck<LS>);
            hiperProbl->setConsistencyCheckType(ConsistencyCheckType::Hessian);
        }
        hiperProbl->Update();

        // --------------------------------------------------------------------//
        /// *****                SOLVER CREATION                  *****///
        // --------------------------------------------------------------------//

        SmartPtr<AztecOOIterativeLinearSolver> linsolver=
        Create<AztecOOIterativeLinearSolver>();
        linsolver->setHiPerProblem(hiperProbl);
        linsolver->setTolerance(1.E-8);
        linsolver->setMaxNumIterations(500);
        linsolver->setSolver(AztecOOIterativeLinearSolver::Solver::Gmres);
        linsolver->setPreconditioner(AztecOOIterativeLinearSolver::Preconditioner::None);
        linsolver->setDefaultParameters();
        linsolver->setVerbosity(AztecOOIterativeLinearSolver::Verbosity::None);
        linsolver->Update();

        // Create nonlinear solver
        SmartPtr<NewtonRaphsonNonlinearSolver> nonLinSolver =
        Create<NewtonRaphsonNonlinearSolver>();
        nonLinSolver->setLinearSolver(linsolver);
        nonLinSolver->setConvRelTolerance(true);
        nonLinSolver->setMaxNumIterations(15);
        nonLinSolver->setResTolerance(1e-6);
        nonLinSolver->setSolTolerance(1e-6);
        nonLinSolver->setResMaximum(1e5);
        nonLinSolver->setSolMaximum(1e5);
        nonLinSolver->setExitRelMaximum(true);
        nonLinSolver->setLineSearch(false);
        nonLinSolver->setPrintSummary(false);
        nonLinSolver->setPrintIntermInfo(true);
        nonLinSolver->Update();
        // --------------------------------------------------------------------//
        /// *****              SOLVE HIPERPROBLEM                 *****///
        // --------------------------------------------------------------------//
        // Load loop
        int timeStep{1};
        double time{delta_t};
        double delta_t0 = delta_t;
        while (timeStep <= maxSteps and time <= maxTime)
```

```
    {
        // Print info
        if (hiperProbl->myRank() == 0)
        cout<<endl<<endl<<"Time step:"<<timeStep<<"/"<<maxSteps<<" deltat"<<delta_t
        <<" time"<<time<<"/"<<maxTime<<endl;

        paramStr->setRealParameter(HeatParams::delta_t, delta_t);
        paramStr->setRealParameter(HeatParams::time, time);
        paramStr->setIntParameter(HeatParams::timeStep, timeStep);
        // Initial guess
        dofHand->nodeDOFs->setValue(dofHand->nodeDOFs0);
        hiperProbl->UpdateGhosts();

        bool converged = nonLinSolver->solve();

        // Check convergence
        if (converged)
        {
            // Save solution
            dofHand->nodeDOFs0->setValue(dofHand->nodeDOFs);
            dofHand->nodeDOFs0->UpdateGhosts();

            // Save results each nSave time steps
            if (timeStep%2 == 0)
            {
                // Output results
                string solName = "CondNL." + to_string(timeStep);
                dofHand->printFileLegacyVtk(solName, true);
            }

            // Update load variables
            timeStep++;
            int iter = nonLinSolver->numberOfIterations();
            if (iter < 4)
            delta_t /= adaptiveFactor;
            else if (iter == 5)
            delta_t *= adaptiveFactor;
            if (delta_t > maxDeltat)
            delta_t = maxDeltat;
            if (time < maxTime && time + delta_t >= maxTime)
            delta_t = maxTime - time;
            time += delta_t;
        }
        else
        {
            // End if no adaptivity
            if (adaptiveFactor == 1.0)
            {
                cout << endl << endl<< "Not converged" << endl;
                break;
            }

            // Steady state or non-convergence
            if (delta_t < 1e-8)
            {
                cout <<endl<<endl<<"Steady state or non-convergence"<<endl;
                break;
            }
            time -=delta_t;
            delta_t *= adaptiveFactor;
            time += delta_t;
        }
    }
    hiperlife::Finalize();
    return 0;
}
```

6.2 AuxHeatTransferNonL.h

```
#ifndef AUXHeat_H
#define AUXHeat_H

// C headers
#include <iostream>

// hiperlife headers
#include "hl_Core.h"
#include "hl_ParamStructure.h"
#include "hl_Parser.h"
#include "hl_TypeDefs.h"
#include "hl_GlobalBasisFunctions.h"
#include "hl_StructMeshGenerator.h"
#include "hl_DistributedMesh.h"
#include "hl_FillStructure.h"
#include "hl_DOFsHandler.h"
#include "hl_HiPerProblem.h"
#include "hl_SurfLagrParam.h"
#include "hl_LinearSolver_Iterative_AztecOO.h"
#include "hl_NonlinearSolver_NewtonRaphson.h"

// alpha = 0.5: crank nicolson; delta_t = 0.05
// alpha = 1: backward euler; delta_t = 0.05
struct HeatParams
{
        enum RealParameters
        {
                delta_t,
                time,
                alpha,
        };
        enum IntParameters
        {
                timeStep,
        };
        HL_PARAMETER_LIST DefaultValues
        {
                {"delta_t,", 0.005},
                {"time,", 0.005},
                {"alpha,", 0.5},
                {"timeStep", 0},
        };
};

void LS(hiperlife::FillStructure& fillStr);

#endif
```

6.3 AuxHeatTransferNonL

```
// Header to auxiliary functions
#include "AuxHeatTransferNonL.h"

// Hiperlife headers
#include "hl_Core.h"
#include "hl_ParamStructure.h"
#include "hl_Parser.h"
#include "hl_TypeDefs.h"
#include "hl_GlobalBasisFunctions.h"
#include "hl_StructMeshGenerator.h"
#include "hl_DistributedMesh.h"
#include "hl_FillStructure.h"
#include "hl_DOFsHandler.h"
#include "hl_HiPerProblem.h"
```

```cpp
#include "hl_SurfLagrParam.h"
#include "hl_LinearSolver_Iterative_AztecOO.h"
#include "hl_NonlinearSolver_NewtonRaphson.h"

using namespace std;
using namespace hiperlife;
using namespace hiperlife::Tensor;


// Conduction

void LS(hiperlife::FillStructure& fillStr)
{
        double alpha   = fillStr.getRealParameter(HeatParams::alpha);
        double delta_t = fillStr.getRealParameter(HeatParams::delta_t);
        double k0{0.2};
        double beta{0.002};

        //------------------------------------------------------------------------------//
        // ------------------------------ INPUT DATA ------------------------------//
        //------------------------------------------------------------------------------//

        // Dimensions
        SubFillStructure& subFill = fillStr["dofHand"];
        int nDOFs = subFill.numDOFs;
        int eNN   = subFill.eNN;
        int nDim  = subFill.nDim;
        int pDim  = subFill.pDim;

        // Nodal values at Gauss points
        wrapper<double,1> nborDOFs(subFill.nborDOFs.data(),eNN);
        wrapper<double,1> nborDOFs0(subFill.nborDOFs0.data(),eNN);

        // Shape functions and derivatives at Gauss points

        double jac; // dx * dy = jac * dxi deta
        wrapper<double,1>  bf(subFill.nborBFs(), eNN); //basis function
        tensor<double,2> Dbf(eNN,pDim);                 //gradient of basis function
        GlobalBasisFunctions::gradients(Dbf, jac, subFill);

        //------------------------------------------------------------------------------//
        // ------------------- Initializing Hessian & Jacobian -------------------//
        //------------------------------------------------------------------------------//
        wrapper<double,2> Ak(fillStr.Ak(0, 0).data(), eNN, eNN);
        wrapper<double,1> Bk(fillStr.Bk(0).data(),eNN);

        //------------------------------------------------------------------------------//
        // ------------------------------ VARIABLES ------------------------------//
        //------------------------------------------------------------------------------//

        // temp
        double theta = bf*nborDOFs;
        double theta0 = bf*nborDOFs0;
        // grad temp
        tensor<double,1> grad_t(pDim);
        tensor<double,1> grad_t0(pDim);
        for (int i = 0; i < eNN; i++)
        {
                for (int d = 0; d < pDim; d++)
                {
                        grad_t(d) += Dbf(i,d)*nborDOFs(i);
                        grad_t0(d) += Dbf(i,d)*nborDOFs0(i);
                }
        }
        // (gradient of the bf) *  (gradient of the bf)
        tensor<double,2> DbfDbf = product(Dbf,Dbf,{{1,1}});
        // (gradient of the bf) *  (gradient of theta)
        tensor<double,1> DtDbf = product(grad_t,Dbf,{{0,1}});
```

```
        tensor<double,1> Dt0Dbf = product(grad_t0,Dbf,{{0,1}});
        //------------------------------------------------------------------------//
        //--------------------- Filling Hessain and Jacobian ---------------------//
        //------------------------------------------------------------------------//
        for (int i = 0; i < eNN; i++)            //i for basis functions
        {
                // Fill Jacobian
                Bk(i) += jac * (bf(i)*theta + k0*delta_t*alpha*DtDbf(i)*(1.0+2.0*beta*theta)
                - bf(i)*theta0 - k0*delta_t*(1.0-alpha)*Dt0Dbf(i)*(1.0+2.0*beta*theta0));
                for (int j = 0; j < eNN; j++)       //j for variable.
                {
                        // Fill Hessian
                        Ak(i,j) += jac * (bf(i)*bf(j) + delta_t*alpha*k0
                        * (DbfDbf(i,j) * (1+2.0*beta*theta) + 2.0*beta*bf(j)*DtDbf(i)));
                }
        }
}
```

7 Results

In this section, we present the results of our solution. The contour demonstration of temperature in the whole domain is shown in Figure 3.

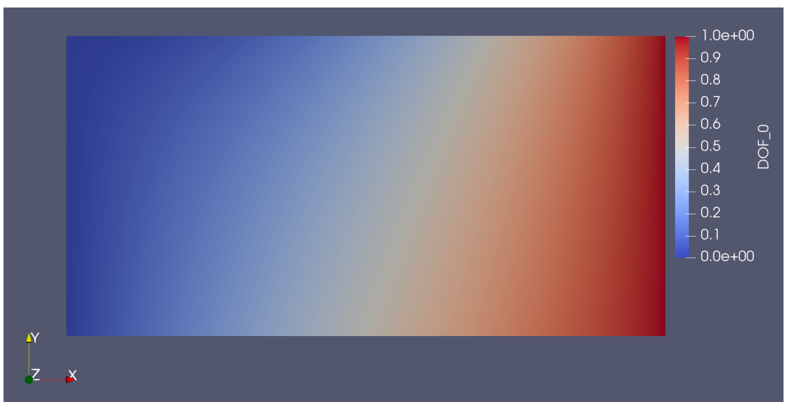

Figure 3: Distrobution of θ in the problem domain.

References

[1] Junuthula Narasimha Reddy. *An Introduction to Nonlinear Finite Element Analysis Second Edition: with applications to heat transfer, fluid mechanics, and solid mechanics*. OUP Oxford, 2014.