Graph Theory: Homework #7

Due on February 18, 2015

Professor McGinley MWF 9:15

Rick Sullivan

Problem 1

Determine all r, s such that $K_{r,s}$ is planar.

Solution

$K_{r,s}$ is planar whenever either r or s is less than or equal to 2. When either is 1, the graph is a tree, so it is planar. With either r or s equal to 2, the graph can be drawn with the larger set in the middle, with either of the pair on each side.

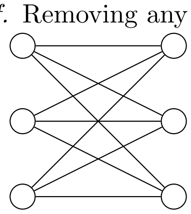
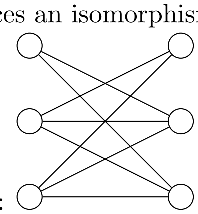
When both are greater than 2, the graph contains a subdivision of $K_{3,3}$, so it is not planar.

Problem 2

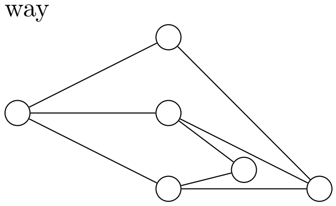
Show that the graph obtained by deleting one edge of $K_{3,3}$ is planar.

Solution

Proof. Removing any edge from $K_{3,3}$ produces an isomorphism of the same graph.

$K_{3,3}$:  Remove any one edge: 

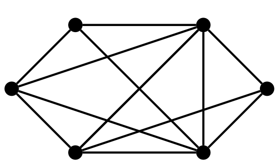
Moving the top and bottom vertices in the left set, we can embed the graph in the plane in the following way



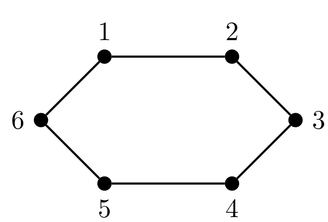
□

Problem 3

Determine (using the algorithm given in class) if the graph below is planar. If so, give its planar embedding.



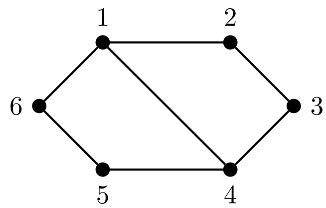
Solution

Choosing the exterior cycle H_1 : 

Problem 3 continued on next page...

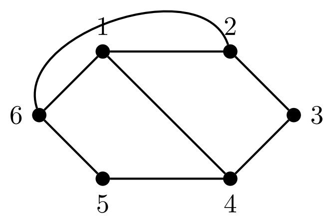
Any segment can be placed in either r_1 or r_2, so choose s_1 in r_1.

H_2 :



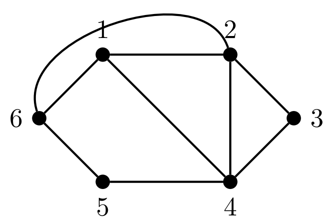
$R(s_1, H_2) = r_3$. Choose s_1 now because it can only be placed in one region.

H_3 :



Once again, s_1 can only be placed in one region, r_4, so preemptively choose it for simplicity.

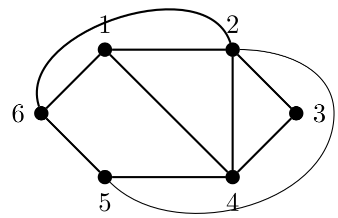
H_4 :



$R(s_1, H_4) = r_5, r_1$
$R(s_2, H_4) = r_5$
$R(s_3, H_4) = r_5$

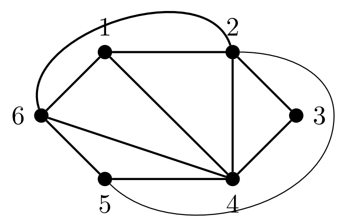
Choose s_2

Problem 3 continued on next page. . .



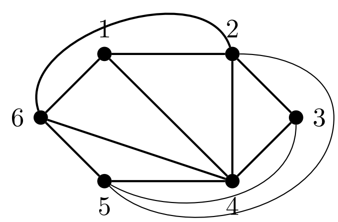
H_5 :

s_1 can be placed in r_1.



H_6 :

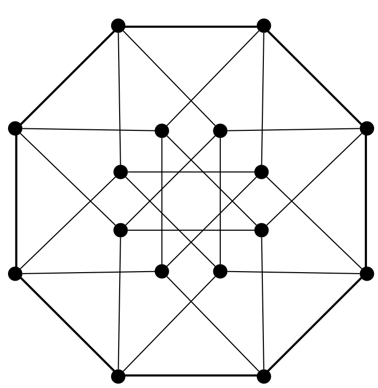
The final edge can be placed in r_5. Our final planar graph is



G :

Problem 4

Is the graph below planar? Use one of the theorems we talked about in class, not the algorithm.



Solution

Proof. According to a corollary of Euler's formula, for a graph to be planar, $e(G) \leq 3n(G) - 6$.

$$\begin{aligned} n(G) &= 16 \\ e(G) &= 32 \\ e(G) &\leq 3n(G) - 6 \\ 32 &\leq 48 - 6 \\ 32 &\leq 42 \end{aligned}$$

For graphs with no triangular faces, it also must be true that $e(G) \leq 2n(G) - 4$. This graph has no triangular faces, so

$$\begin{aligned} e(G) &\leq 2n(G) - 4 \\ 32 &\leq 32 - 4 \\ 32 &\leq 28 \end{aligned}$$

This is not true, so this graph is not planar. □