Graph Theory: Homework #8

Due on February 25, 2015

Professor McGinley MWF 9:15

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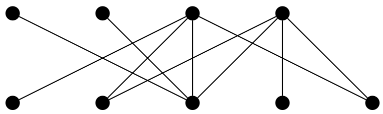
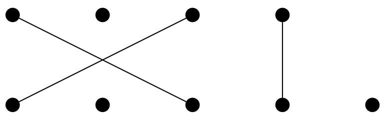
Problem 1

Find the maximum matching of G. Prove that it is maximum by giving the minimum vertex cover, of the same size as the matching.



Solution

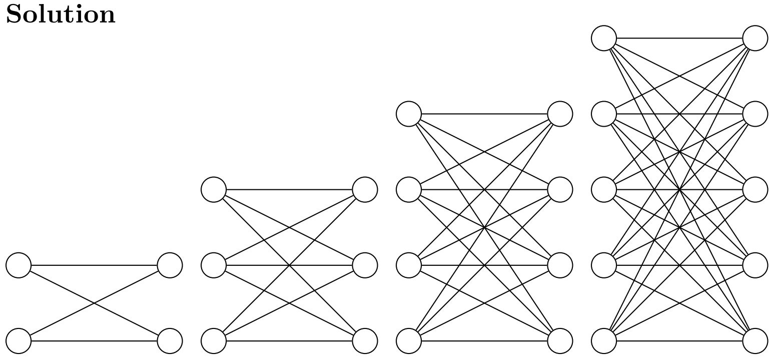


The minimum vertex cover consists of three vertices, either the top or bottom set of three non-isolated vertices. This is the same size as the matching given.

Problem 2

Determine the number of perfect matchings in $K_{p,p}$.

Solution



In all complete bipartite graphs, we can connect any vertex in one set to any vertex in the opposite set. When choosing a perfect matching, we have p choices for matching the first vertex, followed by $p-1$ for the next, and so on. Therefore, we have $p!$ perfect matchings.

Problem 3

Determine the number of perfect matchings in K_{2p}.

Solution

In constructing the perfect matching, we first have $2p-1$ choices to match any given vertex. The next vertex then has $2p-3$ choices to match with, and so on. In this manner, we can consider matching p of the vertices to another, with $(2p-1)*(2p-3)*...*(3)*(1)$ choices. There will be p elements with choices given in the manner

$$\prod_{i=1}^{p} 2p - 2i + 1$$

Which gives the total number of perfect matchings.

Problem 4

Show that a tree has at most one perfect matching.

Solution

Proof. If the tree has an odd number of vertices, it cannot have a perfect matching.

Consider a tree with an even number of vertices.

Base case: $n = 2$. The tree has one perfect matching on the single edge.

Inductive case: Assume a tree T has one or zero perfect matchings. If T has zero perfect matchings, adding one vertex to T can add at most one perfect matching, as it can only connect to a single vertex. If T has one matching, adding a vertex means that T has an odd number of vertices, so it has no perfect matchings.

Therefore, any tree will have at most one perfect matching. $\square$