Graph Theory: Homework #2

Due on January 14, 2015

Professor McGinley MWF 9:15

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Problem 1

Let G be a loopless graph. For $v \in V(G)$ and $e \in E(G)$, describe the adjacency matrix of $G - e$ and $G - v$ in terms of $A(G)$.

Solution The adjacency matrix is an nxn matrix, where n is $|V(G)|$. Each index i on each axis in $A(G)$ corresponds to v_i in $V(G)$ (i.e. the i-th vertex in G).

Each pair of indices x, y contains the number of edges from vertex v_x to vertex v_y. Assuming the graph is undirected, the element at index x, y will be the same as the one at y, x. Similarly, if duplicate edges are not allowed, each element will be either 0 or 1.

Because the graph is loopless, all elements with the same x and y indices will be 0, as there are no edges from a vertex to itself.

Problem 2

Decide if the following sequences are graphical. If the sequence is graphical, produce a graph with that degree sequence (using the algorithm!).

(a) 5, 5, 4, 3, 2, 2, 2, 1

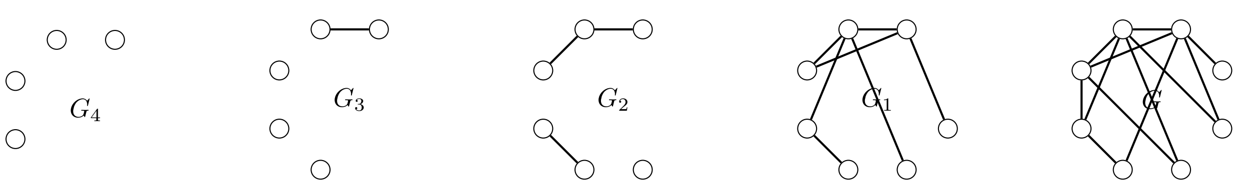
(b) 5, 5, 5, 4, 2, 1, 1, 1

Part a

First, check that the sequence is graphical.

S	5	5	4	3	2	2	2	1
S_1		4	3	2	1	1	2	1
		4	3	2	2	1	1	1
S_2			2	1	1	0	1	1
			2	1	1	1	1	0
S_3				0	0	1	1	0
				1	1	0	0	0
S_4					0	0	0	0

A row of 0s indicates that this sequence is graphical. Now, moving backward to construct the graph.



Part b

First, check that the sequence is graphical.

Problem 2 continued on next page. . .

S	5	5	5	4	2	1	1	1
S_1		4	4	3	1	0	1	1
		4	4	3	1	1	1	0
S_2			3	2	0	0	1	0
			3	2	1	0	0	0
S_3				1	0	-1	0	0

S_3 contains a negative number, so it is not graphical. Therefore, S is not graphical.

Problem 3

Prove that every simple graph with at least two vertices has two vertices with the same degree.

Solution

Proof. Consider a graph G with n vertices, where n is at least 2. Because the graph is simple, the maximum degree of any vertex in G is $n-1$ (one edge with every other node). There are two cases to consider:

1. One node has degree $n-1$. Because this node is adjacent to every other node, no node has degree 0.
2. There is no node with degree $n-1$.

In each case, there are $n-1$ possible buckets of degree for a vertex in G. By the pigeon-hole principle, one of these buckets will have at least 2 vertices. Therefore, there are two vertices with the same degree in G. □

Problem 4

Prove that a graph with exactly two vertices of odd degree has a path from one to the other. (Hint: what would happen if the graph did not have a path from one to the other?)

Solution

Proof. Assume that a graph G has exactly two vertices v_a, v_b of odd degree. Also assume that these vertices do not have a path from one to the other. Therefore, G must have at least two completely disjoint subgraphs, with v_a and v_b existing in separate subgraphs. Because there are only two vertices of odd degree in G, the subgraphs containing v_a and v_b must contain exactly one vertex of odd degree. By theorem 1.1.1 in the text, however, any graph must contain an even number of vertices of odd degree. By contradiction, therefore, G must contain a path from v_a to v_b. □