Graph Theory: Homework #3

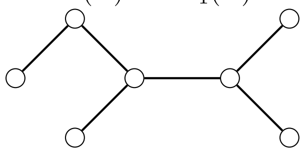
Due on January 21, 2015

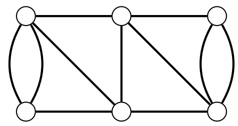
Professor McGinley MWF 9:15

Rick Sullivan

Problem 1

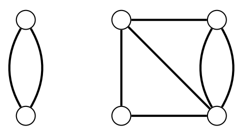
Give $k(G)$ and $k_1(G)$ for the following graphs:



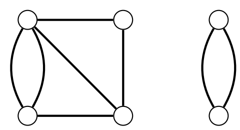


Solution
The first graph has $k(G) = 1$ and $k_1(G) = 1$. These can be found by removing any vertex with degree ¿ 1, or removing any edge.

The second graph has $k(G) = 2$ and $k_1(G) = 3$. $k(G)$ requires removing the two central vertices, while $k_1(G)$ requires removing either set of three edges like so:



or

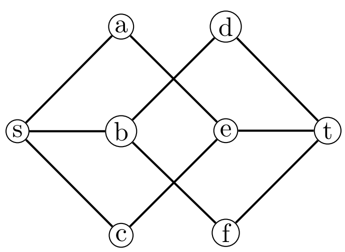


Problem 2

For the graph below:

a) Give three edges whose removal separates s from t. Is this the minimum number of such edges?

b) What is the maximum number of edge-disjoint paths from s to t and why?



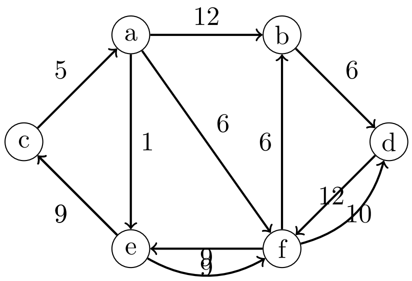
Part a
The edge set $s-a, s-b, s-c$ separates s from t when removed. This is not the minimum number of such edges; the set $s-b, e-t$ does the same.

Part b
The maximum number of edge-disjoint paths from s to t is 2: every path must take either edge $s-b$ or $e-t$. This is confirmed by the second answer of part a. Because the graph can be separated by removing those two edges, one of them must be traversed to travel from one subgraph to the other.

Problem 3

Find the shortest path from a to the other vertices.

Problem 3 continued on next page...



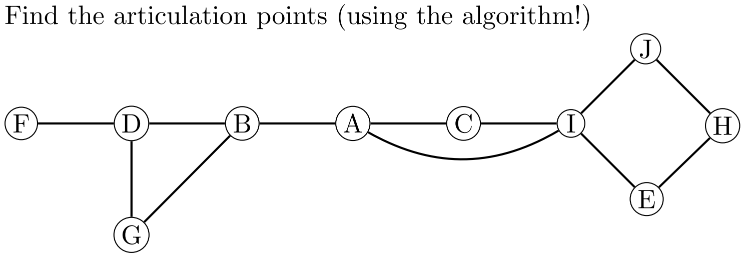
Solution

Using Djikstra's algorithm:

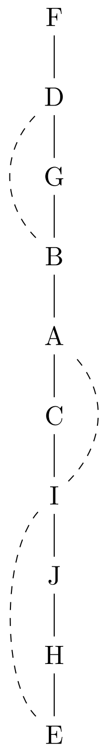
	a	b	c	d	e	f
Initializing:	0	∞	∞	∞	∞	∞
At a:	0	12	∞	∞	1	6
At e:	0	12	10	18	1	6
At f (no change):	0	12	10	18	1	6
At e (no change):	0	12	10	18	1	6
At b (no change):	0	12	10	18	1	6
At d (no change):	0	12	10	18	1	6

Problem 4

Find the articulation points (using the algorithm!)



Solution
My constructed DFS tree:



Initializing our n and c values to the order a node was encountered on our search.

	n	c
A	5	5
B	4	4
C	6	6
D	2	2
E	10	10
F	1	1
G	3	3
H	9	9
I	7	7
J	8	8

Begin traversing the tree backwards, comparing c values according to the algorithm.
At E, c(I) is less than n(E). Therefore, update c(E), c(H), and c(J) (J and H can take forward edges to the back edge).

	n	c
A	5	5
B	4	4
C	6	6
D	2	2
E	10	**7**
F	1	1
G	3	3
H	9	**7**
I	7	7
J	8	**7**

At I, c(A) is less than n(I).

	n	c
A	5	5
B	4	4
C	6	**5**
D	2	2
E	10	7
F	1	1
G	3	3
H	9	7
I	7	**5**
J	8	7

At B, c(D) is less than n(B).

	n	c
A	5	5
B	4	**2**
C	6	5
D	2	2
E	10	7
F	1	1
G	3	**2**
H	9	7
I	7	5
J	8	7

In our final table, nodes A, B, D, and I are not root nodes and have n values less than or equal to the c value of one of their children. F, our root node, has only one child, so it is not an articulation point. The articulation points are A, B, D, and I.

Problem 5

Use induction on p to show that if G is a connected graph of order p, then the size of G is at least $p-1$.

Solution

Problem 5 continued on next page. . .

Proof. Base case: With $p = 1$, G is connected trivially with size 0. In this case, the size of G is at least $p-1$.

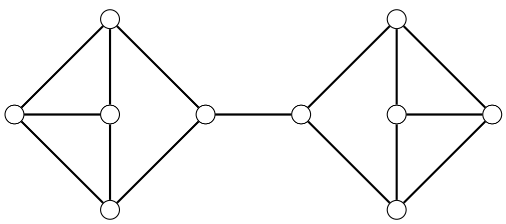
Inductive case: Assume that we have a connected graph G of order p with size $s_0 \geq p-1$. If we add another vertex to this graph and want to connect it to the graph, we must also add at least one edge. Therefore, a new graph with an added connected vertex will have size $s_1 \geq s_0 + 1 \geq p - 1 + 1$. This new graph of order $p+1$ therefore also satisfies the size condition. □

Problem 6

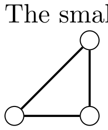
Find the smallest 3-regular graph simple graph having $k(G) = 1$.

Solution

Through experimentation, I have found that the smallest 3-ragular simple graph with a vertex cut set size of 1 is a bridged graph with 10 vertices like the following.



I am not sure how to rigorously prove this concept, but here is my approach:
The smallest 3-regular graph is a simple triangle:



Problem 7

Prove or disprove: If G is 2-connected, then for an *arbitrary* $u - v$ path P, there is *some other* $u - v$ path edge-disjoint from P.

Solution

Proof. If a graph is 2-connected, there does not exist any cut vertex that would disconnect the graph upon removal. In other words, G has no articulation points. If G has no articulation points, there is no point that exists in every path between two or more subgraphs of G. Therefore, every path in G has an alternative vertex-disjoint path. Path edges are defined by the vertices at each edge's endpoints. Therefore, because every path in G has an alternative vertex-disjoint path, it must also have an alternative edge-disjoint path. □