Graph Theory: Homework #11

Due on March 11, 2015

Professor McGinley MWF 9:15

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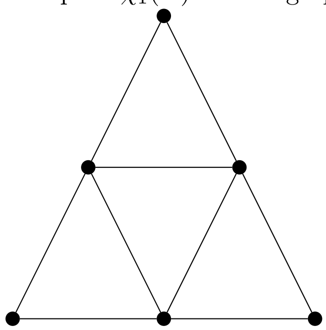
Problem 1

Compute $\chi_1(G)$ for the graph below.

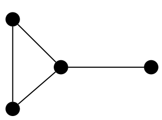


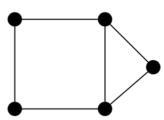
Solution

Proof. $\Delta(G) = 4$, so $\chi(G)$ will be either 4 or 5. If it is 4, G has a 1-factorization. Assume $\chi(G) = 4$ and consider one of the edges on the inner triagular cycle in G. This must be placed into a perfect matching. Both end vertices on this edge are the only two neighbors of one of the corner vertices in G. Therefore, this vertex cannot be placed into a perfect matching if this edge is part of the matching. This means that $\chi(G)! = 4$, so $\chi(G) = 5$. □

Problem 2

Compute the chromatic polynomials for the graphs below.





Graph a

We have k choices for the top vertex, with $k-1$ and then $k-2$ for the two other vertices in the three-cycle. The final vertex can be any color but the adjacent, so we have $k-1$ choices. Therefore, the chromatic polynomial is $(k)(k-1)^2(k-2)$.

Graph b

Starting with coloring the rightmost vertex, we have k options, followed by $k-1$ and $k-2$ colorings for the adjacent vertices, like above. We have two cases for each of the two left-side vertices. If one of them is the color of the opposing vertex in the square-cycle, the remaining vertex has $k-1$ color options. Otherwise, the remaining vertex will have $k-2$ possible colorings.
Therefore, our chromatic polynomial is

$$
\begin{aligned}
&(k)(k-1)(k-2)\left((k-1)+(k-1)(k-2)\right) \\
&= (k)(k-1)^2(k-2)\left(1+(k-2)\right) \\
&= (k)(k-1)^3(k-2)
\end{aligned}
$$

Problem 3

Prove that $k^4 - 4k^3 + 3k^2$ is not a chromatic polynomial.

Solution

By the theorem we covered in class, the highest power in $\chi(G, k)$ gives the order of G, so G has 4 vertices. The theorem also states that the exponent of the last nonzero term is the number of components of G. Additionally, the coefficient of the second term must be the negative number of edges in G. Therefore, G has four vertices and four edges in two components. Consider the two cases of components:

1. Both components have 2 vertices. In this case, G could only have 2 possible edges.
2. One component has 1 vertex, the other has 3 vertices. The component with 1 vertex has no edges, and a 3 vertex graph cannot have 4 edges.

Neither of these cases are possible, so $k^4 - 4k^3 + 3k^2$ must not be a chromatic polynomial.

Problem 4

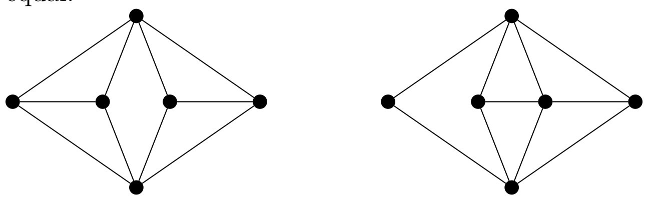
Prove that the sum of the coefficients of $\chi(G, k)$ is 0 unless G has no edges. (Hint: when a function is a polynomial, how can one obtain the sum of the coefficients?)

Solution

Proof. Calculating $\chi(G, 1)$ gives us the sum of the coefficients of the chromatic polynomial. Assume $\chi(G, 1)$ is nonzero. Therefore, there exists a vertex coloring of G with exactly one color. A viable coloring requires that no two same-colored vertices are adjacent, so no vertices are adjacent. Therefore, if $\chi(G, 1)$, the sum of the coefficients of $\chi(G, k)$, is nonzero, G has no edges. $\square$

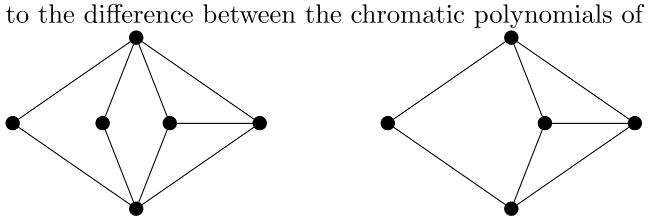
Problem 5

Without computing them, give a short proof that the chromatic polynomials of the two graphs below are equal.



Solution

Proof. By applying a single iteration of the algorithm discussed in class, both of these graphs can be reduced to the difference between the chromatic polynomials of the following two graphs



so both graphs have the same chromatic polynomial. □