Graph Theory: Homework #1

Due on January 9, 2015

Professor McGinley MWF 9:15

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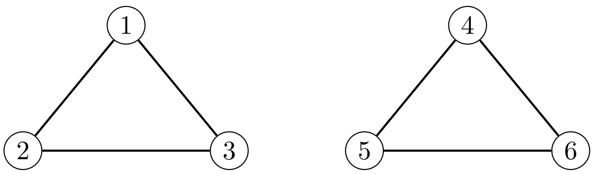
Problem 1

Prove or disprove: if every vertex of a simple graph G has degree two, the G is a cycle.

Solution

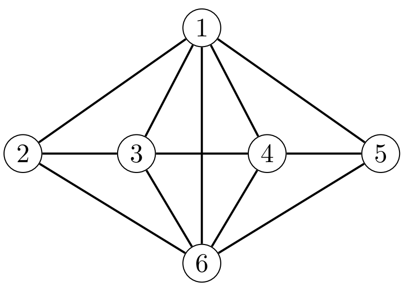
Consider the following graph:



Each node has degree two and the graph has cycles contained within it, but the graph as a whole is not a cycle.

Problem 2

Determine the maximum size of a clique and the maximum size of an indepenedent set of the graph below.

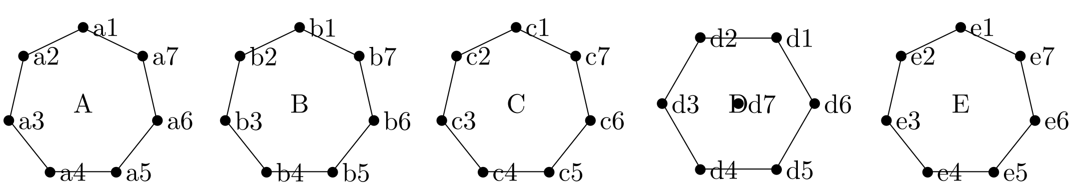


Solution

The maximum clique size is 4, consisting of nodes 1, 6 and any other two adjacent nodes from the middle row.
The maximum size of an independent set is only 2, either the set {2, 4} or the set {3, 5}.

Problem 3

Determine which pairs of graphs below are isomorphic.



Solution

Graphs A, B, and E are all isomorphic, as are graphs C and D.

The isomorphisms can be constructed using the following mappings:

A	B
a1	b1
a2	b6
a3	b7
a4	b5
a5	b4
a6	b3
a7	b2

A	E
a1	e1
a2	e4
a3	e7
a4	e3
a5	e6
a6	e2
a7	e5

C	D
c1	d3
c2	d5
c3	d6
c4	d2
c5	d1
c6	d7
c7	d4

Problem 4

Prove that $G_1 \times G_2$ is isomorphic to $G_2 \times G_1$.

Solution

Proof. Consider two graphs, G_1 and G_2, with vertices $v_{a1}, v_{a2}, ..., v_{an}$ and $v_{b1}, v_{b2}, ..., v_{bn}$, and edges $e_{a1}, e_{a2}, ..., e_{an}$ and $e_{b1}, e_{b2}, ..., e_{bn}$, respectively.

The product $G_1 \times G_2$ will have vertices $(v_{a1}, v_{b1}), (v_{a1}, v_{b2}), ..., (v_{an}, v_{bn}), (v_{b1}, v_{a1}), ...(v_{bn}, v_{an})$. Edges will exist between any two vertices (v_1, w_1) and (v_2, w_2) if and only if either $v_1 = v_2$ and w_1 and w_2 shared an edge in G_2, or vice versa.

The product $G_2 \times G_1$ follows a symmetrical pattern. In this case, the vertices of the product will be $(v_{b1}, v_{a1}), (v_{b1}, v_{a2}), ..., (v_{bn}, v_{an}), (v_{a1}, v_{b1}), ...(v_{an}, v_{bn})$. Edges will similarly be symmetrical.

We can produce an isomorphism by mapping vertex pairs in one cartesian product to the same vertex in the other. For example, the vertex (v_{a1}, v_{b1}) in $G_1 \times G_2$ will actually map to (v_{a1}, v_{b1}) in $G_2 \times G_1$. □

Problem 5

Prove that a p, q graph is complete if and only if $q = \binom{p}{2}$.

Proving completeness leads to $\binom{p}{2}$ edges

Proof. Assume that a graph G with p vertices is complete. By definition of graph completeness, all possible edges will exist. Edges, in a simple undirected graph, can be represented as a set of two vertices that make up the edge's endpoints. Therefore, the total number of possible edges is the total possible number of two-vertex sets from a vertex set of size p, or $\binom{p}{2}$. Therefore, complete graph G must have $\binom{p}{2}$ edges. □

Proving $\binom{p}{2}$ edges leads to completeness

Proof. Assume that a graph G with p vertices has $q = \binom{p}{2}$ edges. The maximum possible number of edges in a graph of order p, as explained above, is $q = \binom{p}{2}$. Therefore, every possible edge in G exists. By definition, G is complete. □