# Numerical Analysis: Homework #5

Due on February 18, 2015

*Professor Mohler MWF 2:15*

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# Problem 1

a) Let $T$ be an $n \times n$ matrix with $\rho(T) < 1$. Show that $(I-T)^{-1} = I + T + T^2 + T^3 + T^4 + \dots$ (You may assume that both the inverse on the left and the infinite sum on the right are well defined, which happens with $\rho(T) < 1$).

b) Use part a) to verify that $\vec{x} = (1 + T + T^2 + T^3 + \dots)\vec{c}$ satisfies the fixed point equation

$$\vec{x} = T\vec{x} + \vec{c}$$

### Part a

If $\rho(T) < 1$, the largest absolute eigen value in $T$ is less than 1. $I - T$ then reverses the signs of each element in $T$ that does not lie on the diagonal. Every $a_{ii}$ for $i$ in the matrix then becomes $1 - a_{ii}$.

Multiplying both sides by $I - T$

$$\begin{aligned}(I-T)^{-1}(I-T) &= (I + T + T^2 + ..)(I-T)\\ I &= I(I-T) + T(I-T) + T^2(I-T)\\ &= I - T + T - T^2 + T^2 - T^3 + ...\\ I &= I\end{aligned}$$

Thus the statement holds.

### Part b

Substituting for $\vec{x}$

$$\begin{aligned}(1 + T + T^2 + T^3 + \dots)\vec{c} &= T(1 + T + T^2 + \dots)\vec{c} + \vec{c}\\ 1 + T + T^2 + T^3 + \dots &= T(1 + T + T^2 + \dots) + 1\\ T + T^2 + T^3 \dots &= T(1 + T + T^2 + \dots)\\ T + T^2 + T^3 \dots &= T + T^2 + T^3 + \dots\end{aligned}$$

The approximation will approach the unique solution because $\rho(T) < 1$.

# Problem 2

Let $A$ be a diagonally dominant $n \times n$ matrix and $T$ be the matrix constructed from $A$ corresponding to Jacobi's method. Show that $||T||_\infty < 1$.

### Solution

A diagonally dominant matrix has $|A_{ii}| \geq \sum_{j!=i} |A_{ij}|$ for all $i$. Therefore, the diagonal entries are gauranteed to have an absolute value at least equal to the cumulative magnitudes of the other values in each row.
When constructing $T$ for Jacobi's method, diagonal entries are isolated on the left hand side. In this isolation, all other entries are divided by the diagonal coefficient. For example, a row representing $a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$ will become

Problem 2 continued on next page. . . 

$$\begin{aligned} a_i x_i &= -(a_1 x_1 + \ldots a_n x_n) + c \\ x_i &= (1/a_i)(-(a_1 x_1 + \ldots a_n x_n) + c) \end{aligned}$$

Where $i$ refers to the diagonal term, and the right hand sides do not have an $i$ term. $a_i$ is guaranteed to be at least the absolute sums of the other coefficients, so the right hand side will always be less than or equal to 1.

The infinity norm of $T$ is calculated by finding the maximum absolute sum of any row in $T$. As explained above, the sum for any row will be less than or equal to 1, so $||T||_\infty \leq 1$.

Note that this is not $||T||_\infty < 1$. That condition requires $A$ to be *strictly* diagonally dominant, which ensures that $|A_{ii}| > \sum_{j!=i} |A_{ij}|$ for all $i$. The logic proving that is the same.

# Problem 3

a) Find the first two iterations of the Jacobi method for the system $A\vec{x} = \vec{b}$ using $\vec{x}^{(0)} = \vec{0}$, where

$$A = \begin{bmatrix} 4 & 1 & -1 & 1 \\ 1 & 4 & -1 & -1 \\ -1 & -1 & 5 & 1 \\ 1 & -1 & 1 & 6 \end{bmatrix}$$

and

$$\vec{b} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

b) Let $T$ be the Jacobi matrix associated with part a). Calculate $||T||_\infty$ and use this to estimate the number of iterations required to achieve an accuracy of $10^{-6}$ in the $||\cdot||_\infty$ norm. Hint: use the error formula

$$||\vec{x}^{(k)} - \vec{x}||_\infty \leq \frac{||T||_\infty^k}{1 - ||T||_\infty} ||\vec{x}^{(1)} - \vec{x}^{(0)}||_\infty$$

**Part a**

Problem 3 continued on next page. . . 

Isolate diagonal terms on the left hand side.

$$\begin{aligned}
4x_1 + x_2 - x_3 + x_4 &= -2 \\
4x_1 &= -x_2 + x_3 - x_4 - 2 \\
x_1 &= (1/4)(-x_2 + x_3 - x_4 - 2)
\end{aligned}$$

$$\begin{aligned}
x_1 + 4x_2 - x_3 - x_4 &= -1 \\
4x_2 &= -x_1 + x_3 + x_4 - 1 \\
x_2 &= (1/4)(-x_1 + x_3 + x_4 - 1)
\end{aligned}$$

$$\begin{aligned}
-x_1 - x_2 + 5x_3 + x_4 &= 0 \\
5x_3 &= x_1 + x_2 - x_4 \\
x_3 &= (1/5)(x_1 + x_2 - x_4)
\end{aligned}$$

$$\begin{aligned}
x_1 - x_2 + x_3 + 6x_4 &= 1 \\
6x_4 &= -x_1 + x_2 - x_3 + 1 \\
x_4 &= (1/6)(-x_1 + x_2 - x_3 + 1)
\end{aligned}$$

Applying two iterations

$$\begin{aligned}
x_1^{(1)} &= (1/4)(-0 + 0 - 0 - 2) \\
&= -1/2 \\
x_2^{(1)} &= (1/4)(-(-1/2) + 0 + 0 - 1) \\
&= -1/8 \\
x_3^{(1)} &= (1/5)(-1/2 - 1/8 - 0) \\
&= -1/8 \\
x_4^{(1)} &= (1/6)(1/2 - 1/8 + 1/8 + 1) \\
&= 1/4
\end{aligned}$$

$$\begin{aligned}
x_1^{(2)} &= (1/4)(1/8 - 1/8 - 1/4 - 2) \\
&= (1/4)(-9/4) \\
&= -9/16 \\
x_2^{(2)} &= (1/4)(9/16 - 1/8 + 1/4 - 1) \\
&= (1/4)(9/16 - 7/8) \\
&= (1/4)(-5/16) \\
&= -5/64 \\
x_3^{(2)} &= (1/5)(-9/16 - 5/64 - 1/4) \\
&= (1/5)(-57/64) \\
&\approx -.178125 \\
x_4^{(2)} &= (1/6)(9/16 - 5/64 + 0.178215 + 1) \\
&\approx 0.27708
\end{aligned}$$

**Part b**

Problem 3 continued on next page...

$T$ from part a) is

$$
\begin{aligned}
T &= \begin{bmatrix} 0 & -1/4 & 1/4 & -1/4 \\ -1/4 & 0 & 1/4 & 1/4 \\ 1/5 & 1/5 & 0 & -1/5 \\ -1/6 & 1/6 & -1/6 & 0 \end{bmatrix} \\
||T||_\infty &= \max(3/4, 3/4, 3/5, 3/6) \\
&= 3/4
\end{aligned}
$$

Applying the error formula with the given accuracy,

$$
\begin{aligned}
||\vec{x}^{(k)} - \vec{x}||_\infty \leq \frac{||T||_\infty^k}{1 - ||T||_\infty} ||\vec{x}^{(1)} - \vec{x}^{(0)}||_\infty &\leq 10^{-6} \\
\frac{||T||_\infty^k}{1 - 3/4} ||\vec{x}^{(1)}||_\infty &\leq 10^{-6} \\
\frac{(3/4)^k}{1/4} 1/2 &\leq 10^{-6} \\
k &\approx 50.433
\end{aligned}
$$

So we need a minimum of 51 iterations to guarantee our estimation is within $10^{-6}$ of the solution.