

Numerical Analysis: Homework #4

Due on February 9, 2015

Professor Mohler MWF 2:15

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Problem 1

- a) Use the Gaussian elimination algorithm discussed in class to put the following augmented matrix into triangular (reduced) form.

$$A = \begin{bmatrix} 1 & -1 & 3 & 2 \\ 3 & -3 & 1 & -1 \\ 1 & 1 & 0 & 3 \end{bmatrix}$$

- b) Use backward substitution to solve the system.

Part a

$$\begin{aligned} A &= \begin{bmatrix} 1 & -1 & 3 & 2 \\ 3 & -3 & 1 & -1 \\ 1 & 1 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 3 & 2 \\ 1 & -1 & 1/3 & -1/3 \\ 1 & 1 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 0 & 8/3 & 7/3 \\ 1 & 1 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 0 & 1 & 7/8 \\ 1 & 1 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 3 & 2 \\ 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 7/8 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & 1 & 7/8 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & -3/2 & 1/2 \\ 0 & 0 & 1 & 7/8 \end{bmatrix} \end{aligned}$$

Part b

$$\begin{aligned} A &= \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & -3/2 & 1/2 \\ 0 & 0 & 1 & 7/8 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & 0 & 29/16 \\ 0 & 0 & 1 & 7/8 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 19/16 \\ 0 & 1 & 0 & 29/16 \\ 0 & 0 & 1 & 7/8 \end{bmatrix} \end{aligned}$$

Problem 2

a) Factor the matrix

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 8 & 2 \\ 4 & 4 & 16 \end{bmatrix}$$

into the product of a lower and upper triangular matrix.

b) Solve the linear system $Ax = b$, where b is the vector of all 1's, using $A = LU$ from part a) and forward/backward substitution.

Part a

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 8 & 2 \\ 4 & 4 & 16 \end{bmatrix} = \begin{bmatrix} l_{1,1} & 0 & 0 \\ l_{2,1} & l_{2,2} & 0 \\ 0 & l_{3,2} & l_{3,3} \end{bmatrix} \begin{bmatrix} 1 & u_{1,2} & 0 \\ 0 & 1 & u_{2,3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$l_{1,1} = 4$$

$$2 = l_{2,1} * 1$$

$$l_{2,1} = 2$$

$$0 = 4 * u_{1,2}$$

$$u_{1,2} = 0$$

So we now have

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 2 & l_{2,2} & 0 \\ 0 & l_{3,2} & l_{3,3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & u_{2,3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$8 = 2 * 0 + l_{2,2} * 1$$

$$l_{2,2} = 8$$

$$2 = 8 * u_{2,3}$$

$$u_{2,3} = 1/4$$

$$4 = 1 * l_{3,2}$$

$$l_{3,2} = 4$$

$$16 = 4 * (1/4) + 1 * l_{3,3}$$

$$l_{3,3} = 15$$

Finally,

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 8 & 0 \\ 0 & 4 & 15 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix}$$

Part b

$$Ax = LUx = b$$

First solving for y in $Ly = b$.

$$\begin{bmatrix} 4 & 0 & 0 \\ 2 & 8 & 0 \\ 0 & 4 & 15 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$1 = 4y_1$$

$$y_1 = 1/4$$

$$1 = 2(1/4) + 8y_2$$

$$y_2 = 1/16$$

$$1 = 4(1/16) + 15y_3$$

$$y_3 = 1/20$$

Now solving for x in $Ux = y$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/16 \\ 1/20 \end{bmatrix}$$

$$x_1 = 1/4$$

$$x_3 = 1/20$$

$$x_2 = 1/16 - 1/4(1/20)$$

$$= 1/20$$

$$x = \begin{bmatrix} 1/4 \\ 1/16 \\ 1/20 \end{bmatrix}$$

Problem 3

Let

$$A = \begin{bmatrix} \alpha & 1 & 0 \\ \beta & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Find all values of α and β such that

- a) A is singular (non-invertible)
- b) A is symmetric
- c) A is strictly diagonally dominant
- d) A is positive definite

Solution

1. A singular matrix has a determinant of 0.

$$|A| = 0$$

$$(\alpha * 2 * 2) - (1 * 1 * \alpha + 2 * 1 * \beta) = 0$$

$$4\alpha - \alpha - 2\beta = 0$$

$$\alpha = (2/3)\beta$$

A is singular for all α and β where $\alpha = (2/3)\beta$.

2. A is symmetric for $\alpha = 2$ and $\beta = 1$.

3. A is strictly diagonally dominant if its determinant is nonzero. From a), A is strictly diagonally dominant when $\alpha \neq (2/3)\beta$.

4. Whaaa?

Problem 4

a) Find $\|x\|_\infty$ and $\|x\|_2$ for $x = [\cos(k), \sin(k), e^k]^T$ and a k a positive integer.

b) Find $\|A\|_\infty$ for

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Part a

Part b