Numerical Analysis: Homework #7

Due on May 4, 2015

Professor Mohler MWF 2:15

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Problem 1

Consider the nonlinear system

$$\begin{aligned} 5x_1^2 - x_2^2 &= 0 \\ x_2 - (\sin(x_1) + \cos(x_2))/4 &= 0 \end{aligned}$$

a) Find a function $G(\vec{x})$ and a set D in $\mathbb{R}^2$ such that G has a unique fixed point in D.

b) Estimate the number of iterations required to approximate the exact solution within 10^{-5} in the $||\cdot||_\infty$ norm, given any initial guess in D.

Part a

Solving the system of equations for the appropriate x's gives us G.

$$G(\vec{x}) = \begin{bmatrix} x_2/\sqrt{5} \\ (\sin(x_1) + \cos(x_2))/4 \end{bmatrix}$$

Guess:
$x_1 \in [0, 1]$ and $x_2 \in [0, 1]$

The first equation will be contained in D, as it maps to $[0, 1/\sqrt{5}]$. The second equation will always be less than $1/2$, so we can conservatively consider it mapping from D to $[0, 1/2]$, which is contained in D. Therefore, this G will have a unique fixed point in this chosen D.

Part b

Speed of convergence of our fixed point iteration is determined by the norm of G'.

$$G'(\vec{x}) = \begin{bmatrix} 0 & 1/\sqrt{5} \\ \cos(x_1)/4 & -\sin(x_2)/4 \end{bmatrix}$$

The largest $||G'||_\infty$ can be in D is when $x_1 = 0$ and $x_2 = 1$, which gives us a norm of approximately 0.460368.

The error is determined by

$$\begin{aligned} &\frac{k^m}{1-k}||\vec{x}^{(1)} - \vec{x}^{(0)}||_\infty \\ &\frac{0.460368^m}{0.539632}||\vec{x}^{(1)} - \vec{x}^{(0)}||_\infty \end{aligned}$$

Because all iterations of $\vec{x}$ will map to D, $||\vec{x}^{(1)} - \vec{x}^{(0)}||_\infty$ will always be less than 1. Using this, we can solve for an upper bound on the required number of iterations with the accuracy 10^{-5}.

$$\begin{aligned} \frac{0.460368^m}{0.539632} &= 10^{-5} \\ m &= 15.6366 \end{aligned}$$

Therefore, with 16 iterations, we will reach our required accuracy (although it could potentially converge faster).

Problem 2

Use two iterations of Newton's method with initial guess $vec0$ to approximate the solution to

$$3x - \cos(yz) - 1/2 = 0$$
$$4x^2 - 625y^2 + 2y - 1 = 0$$
$$e^{-xy} + 20z + \frac{10\pi - 3}{3} = 0$$

Solution

$$J = \begin{bmatrix} 3 & z\sin(yz) & y\sin(yz) \\ 8x & -1250y + 2 & 0 \\ -ye^{-xy} & -xe^{-xy} & 20 \end{bmatrix}$$

First iteration

$$J(\vec{0}) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 20 \end{bmatrix}$$
$$J(\vec{0})\vec{y} = -\vec{F}(\vec{0})$$
$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 20 \end{bmatrix}\vec{y} = -\vec{F}(\vec{0})$$
$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 20 \end{bmatrix}\vec{y} = \begin{bmatrix} 3/2 \\ 1 \\ -\frac{10\pi}{3} \end{bmatrix}$$
$$\vec{y} = \begin{bmatrix} 1/2 \\ 1/2 \\ -\frac{\pi}{6} \end{bmatrix}$$
$$\vec{x}^{(1)} = \vec{x}^{(0)} + \vec{y}$$
$$= \begin{bmatrix} 1/2 \\ 1/2 \\ -\frac{\pi}{6} \end{bmatrix}$$

Problem 2 continued on next page. . .

Second iteration

$$J(\vec{x}^{(1)}) = \begin{bmatrix} 3 & 0.135517 & -0.12941 \\ 4 & -623 & 0 \\ -0.3894 & -0.3894 & 20 \end{bmatrix}$$

$$J(\vec{x}^{(1)})\vec{y} = -\vec{F}(\vec{x}^{(1)})$$

$$\begin{bmatrix} 3 & 0.135517 & -0.12941 \\ 4 & -623 & 0 \\ -0.3894 & -0.3894 & 20 \end{bmatrix} \vec{y} = -\vec{F}(\vec{x}^{(1)})$$

$$\begin{bmatrix} 3 & 0.135517 & -0.12941 \\ 4 & -623 & 0 \\ -0.3894 & -0.3894 & 20 \end{bmatrix} \vec{y} = \begin{bmatrix} -0.034074 \\ 155.25 \\ 0.221199 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} 0.0001667 \\ -0.249196 \\ 0.0062113 \end{bmatrix}$$

$$\vec{x}^{(2)} = \vec{x}^{(1)} + \vec{y}$$

$$= \begin{bmatrix} 0.5001667 \\ 0.2508036 \\ -0.523432 \end{bmatrix}$$