Numerical Analysis: Homework #4

Due on February 11, 2015

Professor Mohler MWF 2:15

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Problem 1

a) Use the Gaussian elimination algorithm discussed in class to put the following augmented matrix into triangular (reduced) form.

$$A = \begin{bmatrix} 1 & -1 & 3 & 2 \\ 3 & -3 & 1 & -1 \\ 1 & 1 & 0 & 3 \end{bmatrix}$$

b) Use backward substitution to solve the system.

Part a

$$\begin{aligned} A &= \begin{bmatrix} 1 & -1 & 3 & 2 \\ 3 & -3 & 1 & -1 \\ 1 & 1 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 3 & 2 \\ 1 & -1 & 1/3 & -1/3 \\ 1 & 1 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 0 & 8/3 & 7/3 \\ 1 & 1 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 0 & 1 & 7/8 \\ 1 & 1 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 3 & 2 \\ 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 7/8 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & 1 & 7/8 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & -3/2 & 1/2 \\ 0 & 0 & 1 & 7/8 \end{bmatrix} \end{aligned}$$

Part b

$$\begin{aligned} A &= \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & -3/2 & 1/2 \\ 0 & 0 & 1 & 7/8 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & 0 & 29/16 \\ 0 & 0 & 1 & 7/8 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 19/16 \\ 0 & 1 & 0 & 29/16 \\ 0 & 0 & 1 & 7/8 \end{bmatrix} \end{aligned}$$

Problem 2

a) Factor the matrix

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 8 & 2 \\ 4 & 4 & 16 \end{bmatrix}$$

into the product of a lower and upper triangular matrix.

b) Solve the linear system $Ax = b$, where b is the vector of all 1's, using $A = LU$ from part a) and forward/backward substitution.

Part a

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 8 & 2 \\ 4 & 4 & 16 \end{bmatrix} = \begin{bmatrix} l_{1,1} & 0 & 0 \\ l_{2,1} & l_{2,2} & 0 \\ 0 & l_{3,2} & l_{3,3} \end{bmatrix} \begin{bmatrix} 1 & u_{1,2} & 0 \\ 0 & 1 & u_{2,3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} l_{1,1} &= 4 \\ 2 &= l_{2,1} * 1 \\ l_{2,1} &= 2 \\ 0 &= 4 * u_{1,2} \\ u_{1,2} &= 0 \end{aligned}$$

So we now have

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 2 & l_{2,2} & 0 \\ 0 & l_{3,2} & l_{3,3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & u_{2,3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} 8 &= 2 * 0 + l_{2,2} * 1 \\ l_{2,2} &= 8 \\ 2 &= 8 * u_{2,3} \\ u_{2,3} &= 1/4 \\ 4 &= 1 * l_{3,2} \\ l_{3,2} &= 4 \\ 16 &= 4 * (1/4) + 1 * l_{3,3} \\ l_{3,3} &= 15 \end{aligned}$$

Finally,

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 8 & 0 \\ 0 & 4 & 15 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix}$$

Part b

$$Ax = LUx = b$$

Problem 2 continued on next page. . .

First solving for y in $Ly = b$.

$$\begin{bmatrix} 4 & 0 & 0 \\ 2 & 8 & 0 \\ 0 & 4 & 15 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} 1 &= 4y_1 \\ y_1 &= 1/4 \\ 1 &= 2(1/4) + 8y_2 \\ y_2 &= 1/16 \\ 1 &= 4(1/16) + 15y_3 \\ y_3 &= 1/20 \end{aligned}$$

Now solving for x in $Ux = y$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/16 \\ 1/20 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 1/4 \\ x_3 &= 1/20 \\ x_2 &= 1/16 - 1/4(1/20) \\ &= 1/20 \\ x &= \begin{bmatrix} 1/4 \\ 1/16 \\ 1/20 \end{bmatrix} \end{aligned}$$

Problem 3

Let

$$A = \begin{bmatrix} \alpha & 1 & 0 \\ \beta & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Find all values of α and β such that

a) A is singular (non-invertible)

b) A is symmetric

c) A is strictly diagonally dominant

d) A is positive definite

Solution

a) A singular matrix has a determinant of 0.

$$\begin{aligned} |A| &= 0 \\ (\alpha * 2 * 2) - (1 * 1 * \alpha + 2 * 1 * \beta) &= 0 \\ 4\alpha - \alpha - 2\beta &= 0 \\ \alpha &= (2/3)\beta \end{aligned}$$

A is singular for all α and β where $\alpha = (2/3)\beta$.

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b) A is symmetric for $\alpha = 2$ and $\beta = 1$.

c) A is strictly diagonally dominant if its determinant is nonzero. From a), A is strictly diagonally dominant when $\alpha \neq (2/3)\beta$.

d) A is positive definite if every principal submatrix of A has a positive determinant. From a), the largest submatrix has a positive determinant when $\alpha > (2/3)\beta$. The second submatrix is

$$A_b = \begin{bmatrix} \alpha & 1 \\ \beta & 2 \end{bmatrix}$$

And the determinant is $2\alpha - \beta$, so this is positive when $\alpha > (1/2)\beta$. The final submatrix is simply

$$A_c = \begin{bmatrix} \alpha \end{bmatrix}$$

Which has positive determinant when $\alpha > 0$.

Therefore, A is positive definite when $\alpha > 0$ and $\alpha > (2/3)\beta$.

Problem 4

a) Find $||x||_\infty$ and $||x||_2$ for $x = [\cos(k), \sin(k), e^k]^T$ and a k a positive integer.

b) Find $||A||_\infty$ for

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Part a

$||\vec{x}||_\infty$ is determined by

$$||\vec{x}||_\infty = \max_{1 \leq i \leq n} |x_i|$$

$\cos(k)$ and $\sin(k)$ will oscillate between -1 and 1, but e^k will increase as k increases. Therefore, $||\vec{x}||_\infty = e^k$.

$||\vec{x}||_2$, the Euclidean norm, is

$$||\vec{x}||_2 = \sqrt{\cos^2(k) + \sin^2(k) + e^{2k}}$$

Part b

$$\begin{aligned} ||A||_\infty &= \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| \\ &= \max(3, 4, 3) \\ &= 4 \end{aligned}$$