# Numerical Analysis: Homework #2

Due on January 21, 2015

*Professor Mohler MWF 2:15*

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# Problem 1

Let $f(x) = -x^3 - \cos x$ and $p_0 = 1$. Use two iterations of Newtown's method to approximate the root of $f(x)$.

**Solution**
Iteration one:

$$f'(x) = -3x^2 + \sin x$$

Use the slope at $x = p_0 = 1$ to approximate the root for our next iteration.

$$\begin{aligned}
f(1) &\approx -1.5403 \\
f'(1) &\approx -2.15853 \\
p_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\
&= 1 - \frac{-1.5403}{-2.15853} \\
&= 0.2864
\end{aligned}$$

Iteration two with $p_1 = 0.2864$:

$$\begin{aligned}
f(0.2864) &\approx -0.982759 \\
f'(0.2864) &\approx 0.0364182 \\
p_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\
&= 0.2864 - \frac{-0.982759}{0.0364182} \\
&= 27.98538
\end{aligned}$$

Our final approximation is 27.9854. This is very innacurate because the slope at our initial choice is nearly 0, pushing our estimation far from the actual root.

# Problem 2

The derivative of a function can be approximated by

$$f'(x) \approx \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h}$$

Show that the error of this approximation is $O(h^2)$.

**Solution**
The Taylor series for $f(x+h)$ and $f(x+2h)$ are as follows.

$$\begin{aligned}
f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + Ch^4 \\
f(x+2h) &= f(x) + 2hf'(x) + 2h^2 f''(x) + (4/3)h^3 f'''(x) + Ch^4
\end{aligned}$$

Using those expansions to calculate the error of the approximation

Problem 2 continued on next page. . .

$$
\begin{aligned}
err_h &= f'(x) - \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h} \\
err_h &= f'(x) - \frac{-3f(x) + 4\left[f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + Ch^4\right]}{2h} \\
&\quad - \frac{\left[f(x) + 2hf'(x) + 2h^2f''(x) + (4/3)h^3f'''(x) + Ch^4\right]}{2h} \\
&= f'(x) - \frac{4hf'(x) + 2h^2f''(x) + \frac{2h^3}{3}f'''(x) - 2hf'(x) - 2h^2f''(x) - \frac{4h^3}{3}f'''(x) + Ch^4}{2h} \\
&= f'(x) - \frac{2hf'(x) - \frac{2h^3}{3}f'''(x) + Ch^4}{2h} \\
&= f'(x) - f'(x) - \frac{h^2}{3}f'''(x) + Ch^3 \\
&= \frac{h^2}{3}f'''(x) + Ch^3
\end{aligned}
$$

If h is small, the first term will be much more significant than any other term in the expansion. Therefore, the error of this approximation is $O(h^2)$.

## Problem 3

Use the approximation for $f''(x)$ given in class to approximate the second derivative of

$$f(x) = \cos\left(e^{x^2} + x^3\right)$$

at $x = 1$ using $h = .1, .05, .025$. How does the decrease in error compare to the order of accuracy of the method (is the decrease what we should expect)?

**Solution**

The equation for $f''(x)$ given in class:

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Using the specified inputs:

$$
\begin{aligned}
f''(1)_{0.1} &= \frac{f(1+0.1) - 2f(1) + f(1-0.1)}{0.1^2} \\
&\approx 66.2173 \\
f''(1)_{0.05} &= \frac{f(1+0.05) - 2f(1) + f(1-0.05)}{0.05^2} \\
&\approx 70.4458 \\
f''(1)_{0.025} &= \frac{f(1+0.025) - 2f(1) + f(1-0.025)}{0.025^2} \\
&\approx 71.3510
\end{aligned}
$$

In class, we determined that the calculated errror of this approach is $O(h^2)$. Therefore, estimation with choice $h_1$ should have 4 times the error of an estimation with choice $h_2 = (1/2)h_1$.

$$
\begin{aligned}
\frac{f''(1)_{0.1} - f''(1)_{0.05}}{f''(1)_{0.05} - f''(1)_{0.025}} &= \frac{66.2173 - 70.4458}{70.4458 - 71.3510} \\
&= 4.6713
\end{aligned}
$$

Problem 3 continued on next page. . . 

As expected, our approximation seems to have an accuracy of order $O(h^2)$.

# Problem 4

Use the composite trapezoidal rule and Simpson's rule with $n = 4$ to approximate

$$\int_0^1 x^2 e^{-x}$$

What is the error of each method?

**Solution**
With $n = 4$, we will divide the interval $[0, 1]$ into 4 segments with $h = 0.25$. Using the composite trapezoidal rule:

$$\begin{aligned}
\int_0^1 x^2 e^{-x} &= \sum_{i=0}^{3} \frac{1}{4} \frac{f(x_i) + f(x_{i+1})}{2} \\
&= \frac{1}{8} \left[ f(0) + 2f(1/4) + 2f(1/2) + 2f(3/4) + f(1) \right] \\
&= \frac{1}{8} \left[ 0 + \frac{1}{8e^{1/4}} + \frac{1}{2e^{1/2}} + \frac{9}{8e^{3/4}} + \frac{1}{e} \right] \\
&\approx 0.1598
\end{aligned}$$

Using the composite Simpson's rule:

$$\begin{aligned}
\int_0^1 x^2 e^{-x} &= \sum_{i=0}^{1} \frac{0.25}{3} \left[ f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2}) \right] \\
&= \frac{1}{12} \left[ (0 + \frac{1}{4e^{1/4}} + \frac{1}{4e^{1/2}}) + (\frac{1}{4e^{1/2}} + \frac{9}{4e^{3/4}} + \frac{1}{e}) \right] \\
&\approx 0.1608
\end{aligned}$$

The actual solution is close to 0.1606. The error of the composite trapezoidal estimation is then about 0.0008. And the error using the composite Simpson's rule is about 0.0002.

# Problem 5

Determine $c_0$, $c_1$, and $c_2$ such that

$$\int_0^2 p(x)dx = c_0 p(0) + c_1 p(1) + c_2 p(2)$$

for all polynomials $p(x)$ of degree 2 or less.

**Solution**
Simpson's rule provides exact estimation for polynomials up to degree 3.

$$\begin{aligned}
\int_0^2 p(x)dx &\approx \frac{1}{3} \left[ p(0) + 4p(1) + p(2) \right] \\
&= \frac{1}{3} p(0) + \frac{4}{3} p(1) + \frac{1}{3} p(2)
\end{aligned}$$

Therefore, $c_0 = 1/3$, $c_1 = 4/3$, and $c_2 = 1/3$.