Numerical Analysis: Homework #3

Due on February 2, 2015

Professor Mohler MWF 2:15

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Problem 1

Use Modified Euler's method to approximate the solution to

$$y' = \cos(yt), y(0) = 3$$

on the interval $[0, 2]$ with $h = .5$.

Solution

$$\begin{aligned}
\frac{w_{i+1} - w_i}{h} &= \frac{f(t_{i+1}, \widetilde{w}_{i+1}) + f(t_i, w_i)}{2} \\
f(t_{i+1}, \widetilde{w}_{i+1}) &= \frac{2(w_{i+1} - w_i)}{h} - f(t_i, w_i) \\
\widetilde{w}_{i+1} &= w_i + hf(t_i, w_i) \\
w_1 &= 3 + .5(\cos(0 * 3)) \\
&= 3 + .5 \\
&= 3.5 \\
y(.5) &= \frac{2(3.5 - 3)}{.5} - y(0) \\
&= 2 - 3 \\
&= -1 \\
w_2 &= 3.5 + .5(-1) \\
&= 3 \\
y(1) &= \frac{2(3 - 3.5)}{.5} - (-1) \\
&= -1 + 1 \\
&= 0 \\
w_3 &= 3 + .5(0) \\
&= 3 \\
y(1.5) &= \frac{2(3 - 3)}{.5} - 0 \\
&= 0 \\
w_4 &= 3 - .5(0) \\
&= 3 \\
y(2) &= \frac{2(3 - 3)}{.5} - 0 \\
&= 0
\end{aligned}$$

The approximation of the solution converges to 0 on the given interval.

Problem 2

Consider the initial value problem

$$y' = -200y + \cos(t), y(0) = 1$$

a) What happens to Forward Euler's method if $h = .1$ is used? Compute a few iterations and discuss the results.

b) What condition is needed on h for Forward Euler's method to give reasonable results?

c) What is a method that doesn't have this restriction on h? Use this method to approximate $y(.2)$.

Part a

$$\begin{aligned}
y_{i+1} &= y_i + hy'(t_i, y_i) \\
y(.1) &= y(0) + .1y'(0, 1) \\
&= 1 + .1(-199) \\
&= -18.9 \\
y(.2) &= y(.1) + .1f(.1, -18.9) \\
&= -18.9 + .1(3781) \\
&= 359.2 \\
y(.3) &= y(.2) + .1f(.2, 359.2) \\
&= 359.2 + .1(-71839) \\
&= -6824.7
\end{aligned}$$

The estimations are diverging and oscillating.

Part b

Forward Euler can be unstable for certain problems if we do not choose a small enough values for h. In general, for equations of the form $y' = ky$, we want the product hk to have an absolute value of between 0 and 1. The first term of the equation in this problem has $k = -200$, so h would have to be less than or equal to .005.

Part c

The modified (midpoint) Euler method we have covered eliminates this restriction on h. Using that method, we have

$$
\begin{aligned}
w_0 &= 0 \\
w_1 &= 0 + .1(1) \\
&= .1 \\
y(.1) &= \frac{2(.1-0)}{.1} - 1 \\
&= 1 \\
w_2 &= .1 + .1(1) \\
&= .2 \\
y(.2) &= \frac{2(.2-1)}{.1} - 1 \\
&= 1
\end{aligned}
$$

Problem 3

Lotka-Volterra equations are used to model predator-prey systems. For example

$$c' = .01cr - .5c$$

and

$$r' = 2r - .02cr$$

might model a population of coyotes and rabbits. Use Forward Euler and $h = .5$ to estimate the populations at time $t = 1$ if $c(0) = r(0) = 100$.

Solution

$$\begin{aligned}
c(.5) &= 100 + .5(.01(100)(100) - .5(100)) \\
&= 100 + .5(50) \\
&= 125 \\
r(.5) &= 100 + .5(2(100) - .02(100)(100)) \\
&= 100 + .5(200 - 200) \\
&= 100 \\
c(1) &= c_{.5} + .5(.01c_{.5}r_{.5} - .5c_{.5}) \\
&= 125 + .5(.01(125)(100) - .5(125)) \\
&= 125 + .5(125 - 62.5) \\
&= 125 + 31.25 \\
&= 156.25 \\
r(1) &= r_{.5} + .5(2r_{.5} - .02r_{.5}c_{.5}) \\
&= 100 + .5(2(100) - .02(100)(125)) \\
&= 100 + .5(200 - 250) \\
&= 75
\end{aligned}$$

Our final estimations using Forward Euler have $c(1) = 156.25$ and $r(1) = 75$.