Numerical Analysis: Homework #5

Due on February 18, 2015

Professor Mohler MWF 2:15

Rick Sullivan

Problem 1

a) Let T be an $n \times n$ matrix with $\rho(T) < 1$. Show that $(I - T)^{-1} = I + T + T^2 + T^3 + T^4 + \dots$ (You may assume that both the inverse on the left and the infinite sum on the right are well defined, which happens with $\rho(T) < 1$).

b) Use part a) to verify that $\vec{x} = (1 + T + T^2 + T^3 + \dots)\vec{c}$ satisfies the fixed point equation

$$\vec{x} = T\vec{x} + \vec{c}$$

Part a

Part b

Problem 2

Let A be a diagonally dominant $n \times n$ matrix and T be the matrix constructed from A corresponding to Jacobi's method. Show that $||T||_\infty < 1$.

Solution

A diagonally dominant matrix has $|A_{ii}| \geq \sum_{j!=i} |A_{ij}|$ for all i. Therefore, the diagonal entries are gauranteed to have an absolute value at least equal to the cumulative magnitudes of the other values in each row.
When constructing T for Jacobi's method, diagonal entries are isolated on the left hand side. In this isolation, all other entries are divided by the diagonal coefficient. For example, a row representing $a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$ will become

$$\begin{aligned} a_ix_i &= -(a_1x_1 + \dots a_nx_n) + c \\ x_i &= (1/a_i)(-(a_1x_1 + \dots a_nx_n) + c) \end{aligned}$$

Where i refers to the diagonal term, and the right hand sides do not have an i term. a_i is guaranteed to be at least the absolute sums of the other coefficients, so the right hand side will always be less than or equal to 1.
The infinity norm of T is calculated by finding the maximum absolute sum of any row in T. As explained above, the sum for any row will be less than or equal to 1, so $||T||_\infty \leq 1$.
Note that this is not $||T||_\infty < 1$. That condition requires A to be *strictly* diagonally dominant, which ensures that $|A_{ii}| > \sum_{j!=i} |A_{ij}|$ for all i. The logic proving that is the same.

Problem 3

a) Find the first two iterations of the Jacobi method for the system $A\vec{x} = \vec{b}$ using $\vec{x}^{(0)} = \vec{0}$, where

$$A = \begin{bmatrix} 4 & 1 & -1 & 1 \\ 1 & 4 & -1 & -1 \\ -1 & -1 & 5 & 1 \\ 1 & -1 & 1 & 6 \end{bmatrix}$$

and

Problem 3 continued on next page. . .

$$\vec{b} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

b) Let T be the Jacobi matrix associated with part a). Calculate $||T||_\infty$ and use this to estimate the number of iterations required to achieve an accuracy of 10^{-6} in the $||\cdot||_\infty$ norm. Hint: use the error formula

$$||\vec{x}^{(k)} - \vec{x}||_\infty \leq \frac{||T||_\infty^k}{1 - ||T||_\infty} ||\vec{x}^{(1)} - \vec{x}^{(0)}||_\infty$$

Part a

Isolate diagonal terms on the left hand side.

$$4x_1 + x_2 - x_3 + x_4 = -2$$
$$4x_1 = -x_2 + x_3 - x_4 - 2$$
$$x_1 = (1/4)(-x_2 + x_3 - x_4 - 2)$$

$$x_1 + 4x_2 - x_3 - x_4 = -1$$
$$4x_2 = -x_1 + x_3 + x_4 - 1$$
$$x_2 = (1/4)(-x_1 + x_3 + x_4 - 1)$$

$$-x_1 - x_2 + 5x_3 + x_4 = 0$$
$$5x_3 = x_1 + x_2 - x_4$$
$$x_3 = (1/5)(x_1 + x_2 - x_4)$$

$$x_1 - x_2 + x_3 + 6x_4 = 1$$
$$6x_4 = -x_1 + x_2 - x_3 + 1$$
$$x_4 = (1/6)(-x_1 + x_2 - x_3 + 1)$$

Problem 3 continued on next page...

Applying two iterations

$$\begin{aligned}
x_1^{(1)} &= (1/4)(-0+0-0-2) \\
&= -1/2 \\
x_2^{(1)} &= (1/4)(-(-1/2)+0+0-1) \\
&= -1/8 \\
x_3^{(1)} &= (1/5)(-1/2-1/8-0) \\
&= -1/8 \\
x_4^{(1)} &= (1/6)(1/2-1/8+1/8+1) \\
&= 1/4
\end{aligned}$$

$$\begin{aligned}
x_1^{(2)} &= (1/4)(1/8-1/8-1/4-2) \\
&= (1/4)(-9/4) \\
&= -9/16 \\
x_2^{(2)} &= (1/4)(9/16-1/8-1/8-1) \\
&= (1/4)(9/16-5/4) \\
&= (1/4)(-11/16) \\
&= -11/64 \\
x_3^{(2)} &= (1/5)(-9/16-11/64-1/4) \\
&= (1/5)(-63/64) \\
&\approx -.196875 \\
x_4^{(2)} &= (1/6)(9/16-11/64-0.196875+1) \\
&\approx 0.198958
\end{aligned}$$

Part b

T from part a) is

$$T = \begin{bmatrix} 0 & -1/4 & 1/4 & -1/4 \\ -1/4 & 0 & 1/4 & 1/4 \\ 1/5 & 1/5 & 0 & -1/5 \\ -1/6 & 1/6 & -1/6 & 0 \end{bmatrix}$$

$$\begin{aligned}
||T||_\infty &= \max(3/4, 3/4, 3/5, 3/6) \\
&= 3/4
\end{aligned}$$

Applying the error formula with the given accuracy,

$$\begin{aligned}
||\vec{x}^{(k)} - \vec{x}||_\infty \le \frac{||T||_\infty^k}{1-||T||_\infty} ||\vec{x}^{(1)} - \vec{x}^{(0)}||_\infty &\le 10^{-6} \\
\frac{||T||_\infty^k}{1-3/4} ||\vec{x}^{(1)}||_\infty &\le 10^{-6} \\
\frac{(3/4)^k}{1/4} 1/2 &\le 10^{-6} \\
k &\approx 50.433
\end{aligned}$$

So we need a minimum of 51 iterations to guarantee our estimation is within 10^{-6} of the solution.