Performance Analysis of Bearing-Only Target Location Algorithms

MOTTI GAVISH, Member, IEEE
ELTA Electronics Industries Ltd. and Tel-Aviv University

ANTHONY J. WEISS, Senior Member, IEEE
Tel-Aviv University
Israel

The performance of two well known bearing only location techniques, the maximum likelihood (ML) and the Stansfield estimators, is examined. Analytical expressions are obtained for the bias and the covariance matrix of the estimation error, which permit performance comparison for any case of interest. It is shown that the Stansfield algorithm provides biased estimates even for large numbers of measurements, in contrast with the ML method. The rms error of the Stansfield technique is not necessarily larger than the rms of the ML technique. However, it is shown that the ML technique is superior to the Stansfield method when the number of measurements is large enough. Simulation results verify the predicted theoretical performance.

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Authors' addresses: M. Gavish, Communication Systems Division, ELTA Electronics Industries Ltd., P.O. Box 330, Ashdod 77102, Israel; A. J. Weiss, Department of Electronic Systems, Faculty of Engineering, Tel-Aviv University, Ramat-Aviv 69978, Israel.

I. INTRODUCTION

The estimation of the position of an emitting source from passive angle measurements is a widely investigated problem. Various aspects of this problem examined in the literature include estimation algorithms [1–4, 6–10], location accuracy [5, 6, 9, 13], and target observability [11, 13].

Bearing measurements taken from two or more points along the trajectory of a moving observer, or collected by fixed direction finding (DF) sensors, can be intersected to determine the emitting target location. Since the bearing measurements are noisy, statistical algorithms, sometimes called triangulation or fixing methods, are required in order to obtain optimal target position estimates.

The pioneering work in this field is that of Stansfield [1], who derived, under some simplifying assumptions, a location algorithm. The Stansfield approach, which can be viewed as a small error approximation of the maximum likelihood (ML) estimator, has been further generalized, e.g., [2–4, 6, 8] and has been implemented in many practical systems. The true ML estimator, which is identical to the nonlinear least-squares technique under the assumption of Gaussian noise, is also available in the literature, e.g., [7, 9]. These two methods seem to be the candidates for the implementation of a (batch) bearing only location system. In some applications, however, a recursive algorithm is desired [10].

The purpose of this work is to analyze the performance of the Stansfield and the ML two-dimensional location techniques. Analytical expressions are derived for the bias and the covariance matrix of the estimation error. It is shown that the Stansfield method leads to a biased estimator, while the ML is asymptotically unbiased. However, the variance of the Stansfield technique may be smaller or larger than that of the ML procedure, depending on the specific problem parameters. We illustrate the results by simple examples. Finally, the predicted theoretical performance is verified via Monte Carlo simulations.

The paper is organized as follows. Section II contains the problem formulation, a review of the corresponding Cramer–Rao lower bound (CRLB), and a brief description of the ML estimator and the Stansfield procedure. The performance analysis is presented in Section III, while most of the derivations are relegated to the Appendices. Section IV discusses special cases and simulation results. Section V summarizes the paper.

II. BEARING ONLY LOCATION ESTIMATION

In this section, the location problem from passive angle measurements is defined, and the CRLB on the

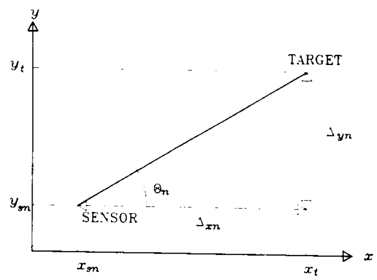

Fig. 1. Problem geometry.

achievable estimation accuracy is given. Then the ML and the Stansfield estimators are reviewed.

A. Problem Formulation

The two-dimensional bearing only location problem can be formulated as follows. Let $\mathbf{x} = (x_1, x_2)^{\mathrm{T}} = (x_t, y_t)^{\mathrm{T}}$ be the target coordinates vector to be estimated from bearing measurements $\Theta = (\theta_1, \theta_2, \ldots, \theta_N)^{\mathrm{T}}$, where $(\cdot)^{\mathrm{T}}$ denotes vector or matrix transposition. The target bearings are measured from an own-ship with known trajectory or from fixed DF sensors at known locations. Denote by (x_{sn}, y_{sn}) the sensor coordinates associated with the measurement θ_n. The problem geometry is depicted in Fig. 1. The angle measurements consist of the true bearings Θ_0 corrupted by additive noise $\delta\Theta = (\delta\theta_1, \ldots, \delta\theta_N)^{\mathrm{T}}$, which is assumed to be zero-mean Gaussian with $N \times N$ covariance matrix $S = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_N^2)$. Thus the problem is described by the nonlinear equation

$$\Theta = \mathbf{g}(\mathbf{x}) + \delta\Theta \tag{1}$$

where $\mathbf{g}(\mathbf{x}) = [g_1(\mathbf{x}), \ldots, g_N(\mathbf{x})]^{\mathrm{T}}$ and

$$\begin{aligned} g_n(\mathbf{x}) &= \arctan(\Delta_{yn}/\Delta_{xn}) \\ \Delta_{xn} &= x_t - x_{sn} \\ \Delta_{yn} &= y_t - y_{sn}, \qquad n = 1, 2, \ldots, N. \end{aligned} \tag{2}$$

It is useful to assume, without loss of generality, that $N = KM$, where K is the number of sensor positions, and M is the number of independent measurements (with equal noise variance) which are collected at each sensor location. Thus, the index n which takes values in the interval $[1, N]$ may be replaced by two indices m and k where $n = m + (k-1)M$. The index m indicates the measurements number at a specific sensor position and therefore it runs between 1 and M, while k indicates the sensor positions and it runs between 1 and K.

B. Cramer–Rao Lower Bound

The CRLB on the covariance of any unbiased estimator for the problem at hand is available in the literature, e.g., [5, 9, 13], and is given by the 2×2 matrix

$$C = (\mathbf{g}_x^{\mathrm{T}} S^{-1} \mathbf{g}_x)^{-1}. \tag{3}$$

The derivative $\mathbf{g}_x = \partial \mathbf{g}/\partial x$, evaluated at the true target position, is given by

$$\mathbf{g}_x = \begin{bmatrix} -\Delta_{y1}/r_1^2 & -\Delta_{y2}/r_2^2 & \cdots & -\Delta_{yK}/r_K^2 \\ \Delta_{x1}/r_1^2 & \Delta_{x2}/r_2^2 & \cdots & \Delta_{xK}/r_K^2 \end{bmatrix}^{\mathrm{T}} \otimes 1_M$$
$$\triangleq \tilde{\mathbf{g}}_x \otimes 1_M \tag{4}$$

where $r_k^2 \triangleq \Delta_{xk}^2 + \Delta_{yk}^2$; $k = 1, \ldots, K$. The notation $\otimes$ stands for the Kronecker product and 1_M is an $M \times 1$ column vector whose elements are all equal to one.

Defining $\tilde{S} \triangleq \mathrm{diag}(\sigma_1^2, \ldots, \sigma_K^2)$, and substituting (4) in (3), we obtain

$$C = \frac{1}{M}\tilde{C}$$

$$\tilde{C}^{-1} = \tilde{\mathbf{g}}_x^{\mathrm{T}} \tilde{S}^{-1} \tilde{\mathbf{g}}_x = \sum_{k=1}^{K} \frac{1}{\sigma_k^2 r_k^4} \begin{bmatrix} \Delta_{yk}^2 & -\Delta_{xk}\Delta_{yk} \\ -\Delta_{xk}\Delta_{yk} & \Delta_{xk}^2 \end{bmatrix}. \tag{5}$$

Thus, tilded symbols indicate that the corresponding quantities are evaluated for $M = 1$. As mentioned previously, the index k is associated with the different sensor positions.

The matrix C^{-1} is the Fisher information matrix. If the observability conditions are satisfied, i.e., the sensor positions do not lie on a straight line passing through the target [11, 13], the matrices C, $\tilde{C}$, C^{-1} and $\tilde{C}^{-1}$ are all positive definite.

The expressions on the main diagonal of C represent lower bounds on the variance of the estimation error of x_t and of y_t, respectively.

C. Maximum Likelihood Estimator

The ML estimator is attractive due to its properties, guaranteed by a well-known theorem of estimation theory. The theorem states that under mild regularity conditions [15, pp. 500–504], the ML estimator is unbiased and its covariance achieves the CRLB, provided that the number of measurements is large enough.

If the measurement noise is Gaussian with zero mean, as assumed previously, the ML estimator of the target position $\mathbf{x}$ is given by

$$\hat{\mathbf{x}}_{\mathrm{ML}} = \arg\min_{x} F_{\mathrm{ML}}(\mathbf{x}, \Theta) \tag{6}$$

where the cost function $F_{\mathrm{ML}}(\mathbf{x}, \Theta)$ has the form

$$\begin{aligned} F_{\mathrm{ML}}(\mathbf{x}, \Theta) &= \frac{1}{2}[\mathbf{g}(\mathbf{x}) - \Theta]^{\mathrm{T}} S^{-1} [\mathbf{g}(\mathbf{x}) - \Theta] \\ &= \frac{1}{2}\mathbf{f}^{\mathrm{T}} S^{-1} \mathbf{f} = \frac{1}{2}\sum_{n=1}^{N} \frac{f_n^2}{\sigma_n^2}. \end{aligned} \tag{7}$$

In the above equation, we used the notation

$$\mathbf{f} = (f_1,\dots,f_N)^{\mathrm{T}} \triangleq \mathbf{g}(\mathbf{x}) - \Theta. \tag{8}$$

Equation (6) involves a nonlinear least-squares minimization, which can be performed by the Newton-Gauss iterations:

$$\hat{\mathbf{x}}_{i+1} = \hat{\mathbf{x}}_i + (\mathbf{g}_x^{\mathrm{T}} S^{-1}\mathbf{g}_x)^{-1}\mathbf{g}_x^{\mathrm{T}} S^{-1}[\Theta - \mathbf{g}(\hat{\mathbf{x}}_i)], \quad i = 1,2,\dots. \tag{9}$$

The use of (9) requires an initial estimate, $\hat{\mathbf{x}}_0$, close enough to the true minimum of the cost function. Such an initial estimate may be available from prior information, or can be obtained using a simple (but suboptimal) procedure. Note that the partial derivatives involved in (9) are evaluated at the current estimated position, $\hat{\mathbf{x}}_i$. In practice, a number of 2–4 iterations are typically sufficient for convergence.

D. Stansfield Algorithm

Consider the ML cost function of (7). The quantities f_n represent the differences between the measured bearings and the bearings corresponding to a target at the estimated location. The Stansfield [1] approach is based on the assumption that the measurement errors are small enough to justify the replacement of f_n with $\sin f_n$ in (7). Thus, the following cost function is obtained

$$F_{\mathrm{ST}}(\mathbf{x},\Theta) = \frac{1}{2}\sum_{n=1}^{N}\frac{\sin^2 f_n}{\sigma_n^2}. \tag{10}$$

Using the relation

$$\sin f_n = \sin\left[\arctan\frac{\Delta_{yn}}{\Delta_{xn}} - \theta_n\right] = \frac{\Delta_{yn}\cos\theta_n - \Delta_{xn}\sin\theta_n}{r_n} = \frac{(y_t - y_{sn})\cos\theta_n - (x_t - x_{sn})\sin\theta_n}{r_n} \tag{11}$$

we can rewrite (10) as

$$F_{\mathrm{ST}}(\mathbf{x},\Theta) = \frac{1}{2}\sum_{n=1}^{N}\frac{[(y_t - y_{sn})\cos\theta_n - (x_t - x_{sn})\sin\theta_n]^2}{r_n^2\sigma_n^2} = \frac{1}{2}(A\mathbf{x} - \mathbf{b})^{\mathrm{T}} R^{-1} S^{-1}(A\mathbf{x} - \mathbf{b}) \tag{12}$$

where

$$A \triangleq \begin{bmatrix} \sin\theta_1 & -\cos\theta_1 \\ \vdots & \vdots \\ \sin\theta_N & -\cos\theta_N \end{bmatrix}; \quad \mathbf{b} \triangleq \begin{bmatrix} x_{s1}\sin\theta_1 - y_{s1}\cos\theta_1 \\ \vdots \\ x_{sN}\sin\theta_N - y_{sN}\cos\theta_N \end{bmatrix}; \quad R \triangleq \mathrm{diag}(r_1^2,\dots,r_N^2). \tag{13}$$

The minimization of (12) with respect to $\mathbf{x}$ is well known and it is given by

$$\hat{\mathbf{x}}_{\mathrm{ST}} = (A^{\mathrm{T}} R^{-1} S^{-1} A)^{-1} A^{\mathrm{T}} R^{-1} S^{-1}\mathbf{b} \tag{14}$$

provided that R is known. Stansfield assumed, implicitly, that even though R is not perfectly known, rough estimates of R can be used, without affecting significantly the solution, since the cost function is a weak function of R.

Note that the Stansfield solution is based on two approximations which lead to the closed form solution of (14). One may suggest to accept (12) but to reject the assumption of known R. This forces an iterative solution of (12) which has no advantage over the ML technique. We therefore regard (14) as the Stansfield location estimator.

III. PERFORMANCE ANALYSIS

We present in this section analytical expressions for the first two moments of the estimation error associated with the above mentioned bearing only location techniques. Our approach is based on evaluating the change in the location of the cost function minimum due to measurement noise. Results for an arbitrary cost function are summarized in Appendix A. Since the application of these results to the localization methods involve rather long manipulations, we present here only the final expressions, while the detailed derivations are relegated to Appendices B and C.

A. ML Estimator

As shown in Appendix B, the error bias for the ML location estimator is given by

$$E(\delta\mathbf{x}) \approx -\frac{1}{M}\tilde{C}\mathbf{h} \tag{15}$$

where $\mathbf{h}$ is a 2×1 column vector whose elements are

$$h_j = \mathrm{tr}\{\tfrac{1}{2}\tilde{S}^{-1}\tilde{\mathbf{g}}_x\tilde{C}\tilde{G}_{xx}^{(j)}\tilde{C}\tilde{\mathbf{g}}_x^{\mathrm{T}} + \tilde{\mathbf{g}}_x\tilde{C}\tilde{G}_{x\Theta}^{(j)}\}. \tag{16}$$

In the above equations, $\tilde{\mathbf{g}}_x$, $\tilde{S}$, and $\tilde{C}$ have been defined in (4)–(5), while $\tilde{G}_{xx}^{(j)}$ and $\tilde{G}_{x\Theta}^{(j)}$ are given by

$$\tilde{G}_{xx}^{(1)} = \sum_{k=1}^{K}\frac{1}{\sigma_k^2 r_k^6}\begin{bmatrix} -6\Delta_{xk}\Delta_{yk}^2 & -3\Delta_{yk}(\Delta_{yk}^2 - \Delta_{xk}^2) \\ \Delta_{yk}(5\Delta_{xk}^2 - \Delta_{yk}^2) & 2\Delta_{xk}(2\Delta_{yk}^2 - \Delta_{xk}^2) \end{bmatrix}$$

$$\tilde{G}_{xx}^{(2)} = \sum_{k=1}^{K}\frac{1}{\sigma_k^2 r_k^6}\begin{bmatrix} 2\Delta_{yk}(2\Delta_{xk}^2 - \Delta_{yk}^2) & \Delta_{xk}(5\Delta_{yk}^2 - \Delta_{xk}^2) \\ 3\Delta_{xk}(\Delta_{yk}^2 - \Delta_{xk}^2) & -6\Delta_{xk}^2\Delta_{yk} \end{bmatrix} \tag{17}$$

and

$$\tilde{G}_{x\theta}^{(1)} = -\sum_{k=1}^{K}\frac{1}{\sigma_k^2 r_k^4}\begin{bmatrix} 2\Delta_{xk}\Delta_{yk} \\ \Delta_{yk}^2 - \Delta_{xk}^2 \end{bmatrix}\tilde{\mathbf{e}}_k^{\mathrm{T}}$$

$$\tilde{G}_{x\theta}^{(2)} = -\sum_{k=1}^{K}\frac{1}{\sigma_k^2 r_k^4}\begin{bmatrix} \Delta_{yk}^2 - \Delta_{xk}^2 \\ -2\Delta_{xk}\Delta_{yk} \end{bmatrix}\tilde{\mathbf{e}}_k^{T}. \tag{18}$$

In (18), $\bar{\mathbf{e}}_k$ denotes the kth column of the $K \times K$ identity matrix I_K.

We observe that $\mathbf{h}$ defined in (16) does not depend on M. Thus the bias of (15) is of order $O(M^{-1})$, and the ML estimator is asymptotically (i.e., for large number of measurements) unbiased. Although this result has been expected, the expressions derived here permit the bias evaluation for finite M.

According to Appendix B, the covariance of the ML estimator is

$$\operatorname{cov}(\delta\mathbf{x}) \approx \frac{1}{M}(\bar{C}^{-1} + \bar{D})^{-1}\bar{C}^{-1}(\bar{C}^{-1} + \bar{D})^{-1}\Big|_{(\mathbf{x}_b,\Theta_0)} \tag{19}$$

where $\bar{D}$ is given by

$$\bar{D} = \sum_{k=1}^{K} \frac{\arctan(\Delta_{yk}/\Delta_{xk}) - \theta_k}{\sigma_k^2 r_k^4} \times \begin{bmatrix} 2\Delta_{xk}\Delta_{yk} & \Delta_{yk}^2 - \Delta_{xk}^2 \\ \Delta_{yk}^2 - \Delta_{xk}^2 & -2\Delta_{xk}\Delta_{yk} \end{bmatrix}. \tag{20}$$

The notation in (19) means that the matrices $\bar{C}$ of (5) and $\bar{D}$ of (20) are evaluated at $(\mathbf{x}_b,\Theta_0)$, where $\mathbf{x}_b = \mathbf{x}_0 + E(\delta\mathbf{x})$, and a good approximation of $E(\delta\mathbf{x})$ is given in (15)–(16).

Since $\mathbf{x}_b$ approaches $\mathbf{x}_0$ and $\bar{D}$ approaches zero, the right-hand side of (19) approaches $(1/M)\bar{C} = C$ as M increases. This indicates that the ML covariance can be approximated by the CRLB for M which is not too small. While the asymptotic result is well known, (19) provides a formula for predicting the covariance when M is finite.

B. The Stansfield Estimator

As shown in Appendix C, the bias of the Stansfield estimator is

$$E(\delta\mathbf{x}) \approx -\frac{1}{M}\bar{C}\mathbf{h} \tag{21}$$

where the vector $\mathbf{h}$ is given by

$$\mathbf{h} = \sum_{k=1}^{K} \frac{1}{r_k^2} \left\{ \frac{1}{\sigma_k^2 r_k^4} \begin{bmatrix} 2\Delta_{xk}\Delta_{yk} & \Delta_{yk}^2 - \Delta_{xk}^2 \\ \Delta_{yk}^2 - \Delta_{xk}^2 & -2\Delta_{xk}\Delta_{yk} \end{bmatrix} \times \bar{C} \begin{bmatrix} -\Delta_{yk} \\ \Delta_{xk} \end{bmatrix} + M \begin{bmatrix} \Delta_{xk} \\ \Delta_{yk} \end{bmatrix} \right\}. \tag{22}$$

Since the first term in (22) does not depend on M, we have

$$\lim_{M\to\infty} E(\delta\mathbf{x}) \approx -\bar{C} \sum_{k=1}^{K} \begin{bmatrix} \Delta_{xk}/r_k^2 \\ \Delta_{yk}/r_k^2 \end{bmatrix}. \tag{23}$$

The right-hand side of (23) depends on K, but not on M. Thus the Stansfield estimator is biased, even for large number of measurements. This bias may be reduced by reducing the single measurement variance at all sensor positions (i.e., $\sigma_k \to 0, \forall k$). Note that one can identify singular configurations of sensor positions relative to the target position for which the bias vanishes. However, in practice, the target position is not known *a priori* and the estimates are biased.

According to Appendix C, the covariance of the Stansfield estimator is

$$\operatorname{cov}(\delta\mathbf{x}) \approx \frac{1}{M}\bar{C}\bar{H}\bar{S}^{-1}\bar{H}^{\mathrm{T}}\bar{C}\Big|_{(\mathbf{x}_b,\Theta_0)} \tag{24}$$

where the matrix $\bar{H}$ is given by

$$\bar{H} = \sum_{k=1}^{K} \frac{1}{r_k^2} \begin{bmatrix} \Delta_{xk}\sin 2\theta_k - \Delta_{yk}\cos 2\theta_k \\ -\Delta_{xk}\cos 2\theta_k - \Delta_{yk}\sin 2\theta_k \end{bmatrix} \bar{\mathbf{e}}_k^{\mathrm{T}}. \tag{25}$$

The matrices $\bar{C}$ and $\bar{H}$ in (24) are evaluated at $(\mathbf{x}_b,\Theta_0)$, where $\mathbf{x}_b = \mathbf{x}_0 + E(\delta\mathbf{x})$, and a good approximation of $E(\delta\mathbf{x})$ is given in (21)–(22). Recall that the ranges r_n are considered known, therefore the true values should be used, even for the "biased" point.

The existence of the bias eliminate the CRLB as a useful approximation for the covariance, and one must use (24). Note that we cannot say whether the covariance is smaller or bigger than the CRLB without knowing the specific details of the problem. The CRLB is a lower bound on the covariance of any unbiased estimator, but the Stansfield method is biased and therefore the bound does not apply in this case. Moreover, frequently the mean-square error (MSE) is selected as a performance criterion. It is not unlikely that a biased estimator will out perform an unbiased estimator using this criterion. Finally, note that for sufficiently large M, the dominant estimation error of Stansfield's technique is the bias given in (23). Thus, ignoring the bias can lead to wrong confidence intervals (i.e., uncertainty ellipse). In fact, in [1] the bias is ignored and therefore the 50% probability ellipses in [1] are misleading.

Concluding this section, it has been shown that, if an unbiased estimator is desired, the Stansfield algorithm is not a good candidate. Either the ML or the Stansfield algorithm can achieve smaller variance, depending on the scenario. This is true also for the total MSE, which can be defined as

$$\mathrm{MSE} = \operatorname{tr}\{\operatorname{cov}(\delta\mathbf{x})\} + E(\delta\mathbf{x}^{\mathrm{T}})E(\delta\mathbf{x}) \tag{26}$$

or for the total rms error i.e., $\mathrm{MSE}^{1/2}$. However, when the number of measurements M is large enough, the ML estimator provides a smaller MSE (which is approximately equal, in this case, to the trace of the CRLB matrix).

IV. EXAMPLES

In this section we concentrate on special cases that provide some useful insight and illustrate our claims. The results are verified via Monte Carlo simulations.

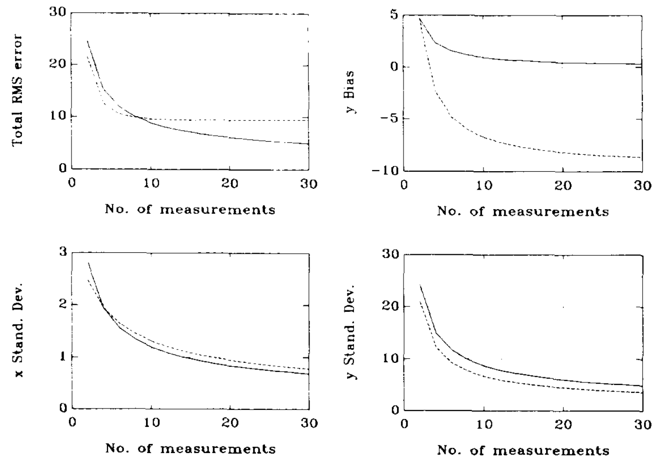

Fig. 2. Performance of ML (solid lines) and Stansfield (dashed lines) estimators for example 1. Total rms error, bias y component and standard deviation for x and y axes are plotted versus number of measurements $N = 2M$.

A. Example 1

Consider a target located at $(0, y_t)$ and two DF stations located at $(-a, 0)$ and $(a, 0)$. Each station collects M measurements, with associated variance of σ^2. Thus $K = 2$ and $N = 2M$.

The matrix C becomes

$$C = \frac{\sigma^2(y_t^2 + a^2)^2}{2M} \begin{bmatrix} y_t^{-2} & 0 \\ 0 & a^{-2} \end{bmatrix}. \tag{27}$$

For the ML bias, we evaluate the expressions (16)–(18), substitute the results in (15) and obtain

$$E(\delta\mathbf{x}) \approx \frac{\sigma^2(y_t^4 - a^4)}{2Ma^2 y_t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{28a}$$

$$\lim_{M\to\infty} E(\delta\mathbf{x}) = 0. \tag{28b}$$

For the Stansfield bias, we evaluate (22), substitute in (21) and get

$$E(\delta\mathbf{x}) \approx -\frac{\sigma^2 y_t(y_t^2 + a^2)}{a^2} \begin{bmatrix} 0 \\ 1 - \dfrac{3}{2M} + \dfrac{a^2}{2My_t^2} \end{bmatrix}, \tag{29a}$$

$$\lim_{M\to\infty} E(\delta\mathbf{x}) = -\frac{\sigma^2 y_t(y_t^2 + a^2)}{a^2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{29b}$$

Thus, for this special case, the x component of the bias is zero, while the y component is, in general, different from zero. Note that (28a) and (29a) give the same result for $M = 1$, expressing the fact that when only two DF measurements (one from each station) are available the ML and the Stansfield method coincide, yielding the same fix which is the intersection of the two lines of position. The sign of the ML bias depends on the ratio y_t/a, and this bias is zero for $y_t = a$. The Stansfield bias sign is always negative for $M > 1$, resulting in estimates which concentrate closer to the stations baseline than the true location. The absolute value of the bias is proportional to the noise variance, and for large M increases with y_t.

For the covariance, we have to evaluate (19) and (24). Since the expressions are rather complicated, we illustrate the results in Fig. 2 for the particular case $y_t = 70$, $a = 10$, and $\sigma = 3°$, as a function of the number of measurements $N = 2M$. The y component of the bias and the total rms error $\mathrm{MSE}^{1/2}$, with MSE defined in (26), are shown also in the figure. The solid lines correspond to ML, while the dashed lines correspond to the Stansfield algorithm. We observe that the Stansfield bias becomes negative for $M \geq 2$, and its magnitude increases with M, approaching the asymptotical value previously mentioned. A bias that increases with the number of measurements is undesired in most practical systems and therefore should be considered as a serious disadvantage of this algorithm. The y component of the Stansfield variance is smaller, while the x component is bigger than the corresponding values of the ML, if $M > 1$. If the total rms is taken as a criterion, the ML estimator performs better, unless $M < 4$. For large M the ML estimator rms is approximately equal to the square root of the trace of the CRLB matrix, while the dominant Stansfield error is the bias.

B. Example 2

This example presents simulation results and compares them to the analytic expressions derived in Section III.

Consider a target located at $x_t = 0$, $y_t = 50$. Own-ship moves with constant velocity along the x axis and its trajectory is defined by $x_s(t) = -50 + 0.15t$, $y_s(t) = 0$. Own-ship collects target bearing measurements every 25 time units. The distance,

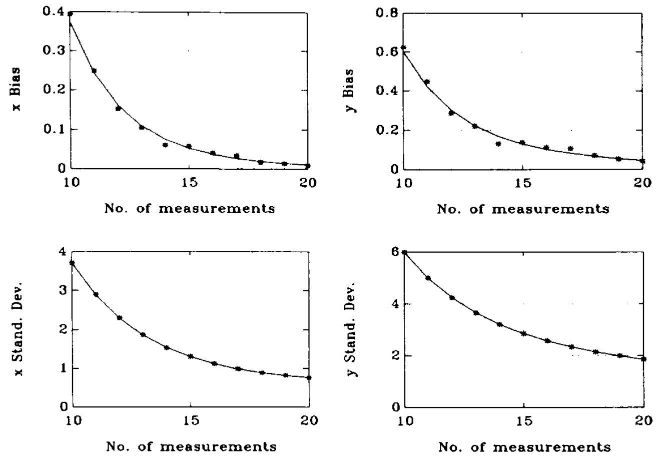

Fig. 3. Performance of ML estimator for example 2, theory (solid lines) and simulation (asterisks). Bias and standard deviation for x and y axes are plotted versus number of measurements $N = K$.

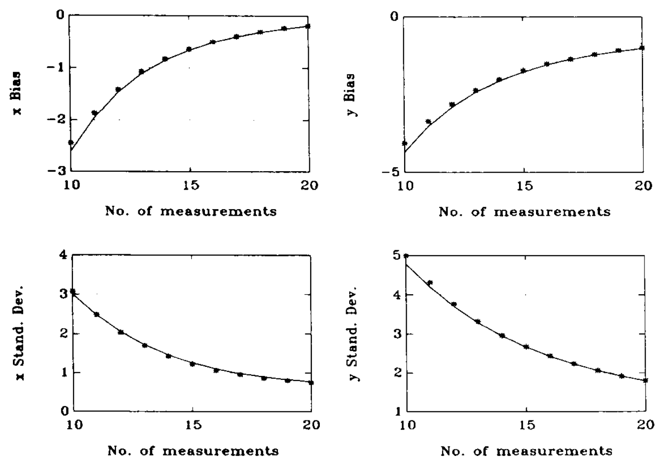

Fig. 4. Performance of Stansfield estimator for example 2, theory (solid lines) and simulation (asterisks). Bias and standard deviation for x and y axes are plotted versus number of measurements $N = K$.

velocity, and time units should be read in consistent units, e.g., kilometer, kilometer per second, and second, respectively. The measurement noise is Gaussian with constant standard deviation $\sigma_n = 3°, \forall n$. Let us look at observation times $0 \leq t \leq T$, where T is between 225 and 475 time units, i.e., $10 \leq N = K \leq 20$ and $M = 1$.

Figs. 3 and 4 show the error bias and standard deviation for the x and y components, for the ML, and for the Stansfield estimators, respectively. The solid lines correspond to the analytic predictions of Section III. The asterisks represent simulation results, where each point has been obtained from 50000 Monte Carlo experiments. (A large number of experiments is required for establishing high statistical confidence in the results, especially for the ML bias, which is smaller by an order of magnitude, or more, than the standard deviation.) The simulation results verify the theory. We note that, in this example, the Stansfield bias is larger than the ML bias, the Stansfield standard deviations are smaller, and the ML total rms error is smaller.

V. CONCLUSIONS

The performance of two well-known bearing-only location techniques, the ML, and the Stansfield estimators, has been analyzed. Analytic expressions have been derived for the bias and the covariance matrix of the estimation error. It has been shown that the Stansfield algorithm is biased and the bias does not vanish even when the number of measurements increases without limit. In some examples, the bias even increases with the number of measurements, an undesired feature in most practical systems. In contrast, the bias of the ML method decreases with the number of measurements, until it disappears. If the rms is used as a performance criterion, there are situations where the Stansfield method is better than the ML procedure. However, for large number of measurements the ML outperforms the Stansfield procedure. The theoretical results were verified via Monte Carlo simulations.

APPENDIX A. PERTURBATION OF THE MINIMUM OF A FUNCTION

Let $F(\mathbf{x},\Theta)$ be a real-valued function of a real vector $\mathbf{x} = (x_1,x_2,\ldots,x_L)^{\mathrm{T}}$, indexed by a real vector $\Theta = (\theta_1,\theta_2,\ldots,\theta_N)^{\mathrm{T}}$. Suppose that, for some fixed $\Theta = \Theta_0$, $F(\mathbf{x},\Theta)$ (regarded as a function of $\mathbf{x}$), attains a local minimum at $\mathbf{x} = \mathbf{x}_0$. We assume that the Hessian matrix $(\partial^2 F/\partial \mathbf{x}^2)$ is nonsingular in some neighborhood of $(\mathbf{x}_0,\Theta_0)$ and that all the partial derivatives used below exist.

Let Θ_0 be perturbed by an amount $\delta\Theta$. Then the point of local minimum of F is perturbed by a corresponding amount $\delta\mathbf{x}$. Assuming that $\delta\Theta$ is a random variable with zero mean and covariance matrix $S \triangleq E(\delta\Theta\,\delta\Theta^{\mathrm{T}})$, we derive in this appendix approximate expressions for the mean and the covarianceof $\delta\mathbf{x}$. Our approach follows Porat and Friedlander [12].

We start with a derivation of an expression for $E(\delta\mathbf{x})$. We need the second-order moments, which can be found as follows. Let us expand $\partial F/\partial x_i$ in a first-orderTaylor series around the point $(\mathbf{x}_0,\Theta_0)$:

$$\left.\frac{\partial F}{\partial x_i}\right|_{\Theta=\Theta_0+\delta\Theta,\mathbf{x}=\mathbf{x}_0+\delta\mathbf{x}} = \frac{\partial F}{\partial x_i} + \sum_{k=1}^{L} \frac{\partial^2 F}{\partial x_i \partial x_k}\delta x_k + \sum_{k=1}^{N} \frac{\partial^2 F}{\partial x_i \partial \theta_k}\delta\theta_k + \epsilon_1 \tag{A1}$$

where the partial derivatives in the right-hand side are understood to be evaluated at $(\mathbf{x}_0,\Theta_0)$. Since $(\mathbf{x}_0,\Theta_0)$ and $(\mathbf{x}_0+\delta\mathbf{x},\Theta_0+\delta\Theta)$ are both points of local minimum, $\partial F/\partial x_i$ is zero at both points.

Neglecting ϵ_1, multiplying (A1) by $\delta\theta_j$, and taking expected values, we get

$$\sum_{k=1}^{L} \frac{\partial^2 F}{\partial x_i \partial x_k} E(\delta x_k \delta\theta_j) \approx -\sum_{k=1}^{N} \frac{\partial^2 F}{\partial x_i \partial\theta_k} E(\delta\theta_k \delta\theta_j)$$

$$i = 1,\ldots,L, \quad j = 1,\ldots,N \tag{A2}$$

or

$$E(\delta\mathbf{x}\,\delta\Theta^{\mathrm{T}}) \approx -\left(\frac{\partial^2 F}{\partial \mathbf{x}^2}\right)^{-1} \left(\frac{\partial^2 F}{\partial \mathbf{x}\,\partial\Theta}\right) S. \tag{A3}$$

Next, multiplying (A1) by δx_j and taking expected values yields

$$\sum_{k=1}^{L} \frac{\partial^2 F}{\partial x_i \partial x_k} E(\delta x_k \delta x_j) \approx -\sum_{k=1}^{N} \frac{\partial^2 F}{\partial x_i \partial\theta_k} E(\delta\theta_k \delta x_j),$$

$$i = 1,\ldots,L, \quad j = 1,\ldots,L \tag{A4}$$

or

$$E(\delta\mathbf{x}\,\delta\mathbf{x}^{\mathrm{T}}) \approx -\left(\frac{\partial^2 F}{\partial \mathbf{x}^2}\right)^{-1} \left(\frac{\partial^2 F}{\partial \mathbf{x}\,\partial\Theta}\right) E(\delta\Theta\,\delta\mathbf{x}^{\mathrm{T}}). \tag{A5}$$

Substituting (A3) into (A5), we obtain

$$E(\delta\mathbf{x}\,\delta\mathbf{x}^{\mathrm{T}}) \approx \left(\frac{\partial^2 F}{\partial \mathbf{x}^2}\right)^{-1} \left(\frac{\partial^2 F}{\partial \mathbf{x}\,\partial\Theta}\right) S \left(\frac{\partial^2 F}{\partial \mathbf{x}\,\partial\Theta}\right)^{\mathrm{T}} \left(\frac{\partial^2 F}{\partial \mathbf{x}^2}\right)^{-1}. \tag{A6}$$

For the evaluation of the first moment of $\delta\mathbf{x}$, since (A1) predicts the trivial result $E(\delta\mathbf{x}) \approx 0$, we use a second-order Taylor series expansion:

$$\begin{aligned}\left.\frac{\partial F}{\partial x_i}\right|_{\Theta=\Theta_0+\delta\Theta,\mathbf{x}=\mathbf{x}_0+\delta\mathbf{x}} &= \frac{\partial F}{\partial x_i} + \sum_{k=1}^{L} \frac{\partial^2 F}{\partial x_i \partial x_k}\delta x_k \\ &+ \sum_{k=1}^{N} \frac{\partial^2 F}{\partial x_i \partial\theta_k}\delta\theta_k \\ &+ \frac{1}{2}\sum_{k=1}^{L}\sum_{l=1}^{L} \frac{\partial^3 F}{\partial x_i \partial x_k \partial x_l}\delta x_k \delta x_l \\ &+ \frac{1}{2}\sum_{k=1}^{L}\sum_{l=1}^{N} \frac{\partial^3 F}{\partial x_i \partial x_k \partial\theta_l}\delta x_k \delta\theta_l \\ &+ \frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{L} \frac{\partial^3 F}{\partial x_i \partial\theta_k \partial x_l}\delta\theta_k \delta x_l \\ &+ \frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N} \frac{\partial^3 F}{\partial x_i \partial\theta_k \partial\theta_l}\delta\theta_k \delta\theta_l + \epsilon_2.\end{aligned} \tag{A7}$$

Neglecting ϵ_2 and taking the expected value of (A7), we have

$$\begin{aligned}-\sum_{k=1}^{L} \frac{\partial^2 F}{\partial x_i \partial x_k} E(\delta x_k) &\approx \frac{1}{2}\sum_{k=1}^{L}\sum_{l=1}^{L} \frac{\partial^3 F}{\partial x_i \partial x_k \partial x_l} E(\delta x_k \delta x_l) \\ &+ \frac{1}{2}\sum_{k=1}^{L}\sum_{l=1}^{N} \frac{\partial^3 F}{\partial x_i \partial x_k \partial\theta_l} E(\delta x_k \delta\theta_l) \\ &+ \frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{L} \frac{\partial^3 F}{\partial x_i \partial\theta_k \partial x_l} E(\delta\theta_k \delta x_l) \\ &+ \frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N} \frac{\partial^3 F}{\partial x_i \partial\theta_k \partial\theta_l} E(\delta\theta_k \delta\theta_l),\end{aligned}$$

$$i = 1,\ldots,L. \tag{A8}$$

Substituting (A3) and (A6) in (A8), we obtain the following expression for $E(\delta\mathbf{x})$:

$$E(\delta\mathbf{x}) \approx -\left(\frac{\partial^2 F}{\partial\mathbf{x}^2}\right)^{-1}\mathbf{h} \tag{A9}$$

where $\mathbf{h}$ is the $L\times 1$ vector whose ith component is

$$h_i = \mathrm{tr}\left\{\left[\frac{1}{2}\left(\frac{\partial^2 F}{\partial\mathbf{x}\partial\Theta}\right)^{\mathrm{T}}\left(\frac{\partial^2 F}{\partial\mathbf{x}^2}\right)^{-1}G_{xx}^{(i)}\left(\frac{\partial^2 F}{\partial\mathbf{x}^2}\right)^{-1}\left(\frac{\partial^2 F}{\partial\mathbf{x}\partial\Theta}\right)\right.\right.$$
$$\left.\left. -\left(\frac{\partial^2 F}{\partial\mathbf{x}\partial\Theta}\right)^{\mathrm{T}}\left(\frac{\partial^2 F}{\partial\mathbf{x}^2}\right)^{-1}G_{x\Theta}^{(i)}+\frac{1}{2}G_{\Theta\Theta}^{(i)}\right]S\right\} \tag{A10}$$

and

$$G_{xx}^{(i)} \triangleq \frac{\partial}{\partial x_i}\left(\frac{\partial^2 F}{\partial\mathbf{x}^2}\right);$$
$$G_{x\Theta}^{(i)} \triangleq \frac{\partial}{\partial x_i}\left(\frac{\partial^2 F}{\partial\mathbf{x}\partial\Theta}\right); \tag{A11}$$
$$G_{\Theta\Theta}^{(i)} \triangleq \frac{\partial}{\partial x_i}\left(\frac{\partial^2 F}{\partial\Theta^2}\right).$$

For the evaluation of the covariance of $\delta\mathbf{x}$, define $\mathbf{x}_b = \mathbf{x}_0 + E(\delta\mathbf{x})$ and expand again $\partial F/\partial x_i$ in a first-order Taylor series, this time around $(\mathbf{x}_b,\Theta_0)$. The corresponding equation is similar to (A1), however the partial derivatives in the right-hand side are evaluated now at $(\mathbf{x}_b,\Theta_0)$. We repeat the steps performed to derive (A6), and note that the term $\partial F/\partial x_i$ (which is not zero any more) vanishes, after multiplying by $\delta\theta_j$ or by δx_j and taking the expectation. For this new Taylor expansion, $E(\delta\mathbf{x}\,\delta\mathbf{x}^{\mathrm{T}})$ can be replaced by $\mathrm{cov}(\delta\mathbf{x})$, and instead of (A6) we obtain

$$\mathrm{cov}(\delta\mathbf{x}) \approx \left(\frac{\partial^2 F}{\partial\mathbf{x}^2}\right)^{-1}\left(\frac{\partial^2 F}{\partial\mathbf{x}\partial\Theta}\right)S\left(\frac{\partial^2 F}{\partial\mathbf{x}\partial\Theta}\right)^{\mathrm{T}}\left(\frac{\partial^2 F}{\partial\mathbf{x}^2}\right)^{-1} \tag{A12}$$

where the partial derivatives are evaluated at $(\mathbf{x}_b,\Theta_0)$ and $\mathbf{x}_b$ is calculated using (A9).

APPENDIX B. PERFORMANCE ANALYSIS OF ML ESTIMATOR

We now apply the formulas derived in Appendix A to evaluate the bias and the covariance of the ML location estimator.

For the ML estimator, the function $F(\mathbf{x},\Theta)$ to be minimized is given by (7). Let us evaluate first the derivatives of F as functions of $\mathbf{f}$ and its derivatives. The first-order derivatives are

$$\frac{\partial F}{\partial\mathbf{x}} = \mathbf{f}^{\mathrm{T}}S^{-1}\frac{\partial\mathbf{f}}{\partial\mathbf{x}} = \sum_{n=1}^{N}\frac{1}{\sigma_n^2}f_n\left(\frac{\partial f_n}{\partial\mathbf{x}}\right) \tag{B1}$$

and

$$\frac{\partial F}{\partial\Theta} = \mathbf{f}^{\mathrm{T}}S^{-1}\frac{\partial\mathbf{f}}{\partial\Theta} = -\mathbf{f}^{\mathrm{T}}S^{-1}I_N = -\sum_{n=1}^{N}\frac{1}{\sigma_n^2}f_n\mathbf{e}_n^{\mathrm{T}} \tag{B2}$$

where $\mathbf{e}_n$ denotes the nth column of the $N\times N$ identity matrix I_N.

The second-order derivatives are

$$\frac{\partial^2 F}{\partial\mathbf{x}^2} = \sum_{n=1}^{N}\frac{1}{\sigma_n^2}\left[\left(\frac{\partial f_n}{\partial\mathbf{x}}\right)^{\mathrm{T}}\left(\frac{\partial f_n}{\partial\mathbf{x}}\right)+f_n\left(\frac{\partial^2 f_n}{\partial\mathbf{x}^2}\right)\right]$$
$$= C^{-1}+\sum_{n=1}^{N}\frac{f_n}{\sigma_n^2}\left(\frac{\partial^2 f_n}{\partial\mathbf{x}^2}\right) \triangleq C^{-1}+D \triangleq C^{-1}+M\bar{D} \tag{B3}$$

where C is the CRLB defined in (3) and (5), and we used the fact that $\partial\mathbf{f}/\partial\mathbf{x} = \mathbf{g}_x$. Since $f_n(\mathbf{x}_0,\Theta_0) = 0$, (B3) gives

$$\left.\frac{\partial^2 F}{\partial\mathbf{x}^2}\right|_{(\mathbf{x}_0,\Theta_0)} = C^{-1}. \tag{B4}$$

Next,

$$\frac{\partial^2 F}{\partial\mathbf{x}\partial\Theta} = \sum_{n=1}^{N}\frac{1}{\sigma_n^2}\left[\left(\frac{\partial f_n}{\partial\mathbf{x}}\right)^{\mathrm{T}}\left(\frac{\partial f_n}{\partial\Theta}\right)+f_n\left(\frac{\partial^2 f_n}{\partial\mathbf{x}\partial\Theta}\right)\right]$$
$$= -\sum_{n=1}^{N}\frac{1}{\sigma_n^2}\left(\frac{\partial f_n}{\partial\mathbf{x}}\right)^{\mathrm{T}}\mathbf{e}_n^{\mathrm{T}} = -\left(\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right)^{\mathrm{T}}S^{-1} \tag{B5}$$

and

$$\frac{\partial^2 F}{\partial\Theta^2} = -\mathrm{diag}\left\{\frac{1}{\sigma_n^2}\frac{\partial f_n}{\partial\theta_n}\right\} = S^{-1}. \tag{B6}$$

We evaluate now the third-order derivatives required, and omit terms including f_n, since these derivatives are calculated at $(\mathbf{x}_0,\Theta_0)$. The derivative of (B3) with respect to x_j gives

$$G_{xx}^{(j)} = \sum_{n=1}^{N}\frac{1}{\sigma_n^2}\frac{\partial}{\partial x_j}\left[\left(\frac{\partial f_n}{\partial\mathbf{x}}\right)^{\mathrm{T}}\left(\frac{\partial f_n}{\partial\mathbf{x}}\right)+f_n\left(\frac{\partial^2 f_n}{\partial\mathbf{x}^2}\right)\right]$$
$$= \sum_{n=1}^{N}\frac{1}{\sigma_n^2}\left[2\left(\frac{\partial f_n}{\partial\mathbf{x}}\right)^{\mathrm{T}}\frac{\partial}{\partial x_j}\left(\frac{\partial f_n}{\partial\mathbf{x}}\right)+\left(\frac{\partial f_n}{\partial x_j}\right)\left(\frac{\partial^2 f_n}{\partial\mathbf{x}^2}\right)\right]. \tag{B7}$$

The derivative of (B5) with respect to x_j gives

$$G_{x\Theta}^{(j)} = -\sum_{n=1}^{N}\frac{1}{\sigma_n^2}\frac{\partial}{\partial x_j}\left(\frac{\partial f_n}{\partial\mathbf{x}}\right)^{\mathrm{T}}\mathbf{e}_n^{\mathrm{T}} = -\frac{\partial}{\partial x_j}\left(\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right)^{\mathrm{T}}S^{-1}. \tag{B8}$$

The derivative of (B6) with respect to x_j gives

$$G_{\Theta\Theta}^{(j)} = 0. \tag{B9}$$

We need now the derivatives of $\mathbf{f}$. From (2) and (9) we obtain

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial \mathbf{x}} \\ \vdots \\ \frac{\partial f_N}{\partial \mathbf{x}} \end{bmatrix}, \tag{B10}$$

$$\frac{\partial f_n}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} \end{bmatrix} = \frac{1}{r_n^2}[-\Delta_{yn} \quad \Delta_{xn}], \tag{B11}$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial f_n}{\partial \mathbf{x}}\right) = \frac{1}{r_n^4}[2\Delta_{xn}\Delta_{yn} \quad \Delta_{yn}^2 - \Delta_{xn}^2], \tag{B12}$$

$$\frac{\partial}{\partial x_2}\left(\frac{\partial f_n}{\partial \mathbf{x}}\right) = \frac{1}{r_n^4}[\Delta_{yn}^2 - \Delta_{xn}^2 \quad -2\Delta_{xn}\Delta_{yn}], \tag{B13}$$

$$\frac{\partial^2 f_n}{\partial \mathbf{x}^2} = \frac{1}{r_n^4}\begin{bmatrix} 2\Delta_{xn}\Delta_{yn} & \Delta_{yn}^2 - \Delta_{xn}^2 \\ \Delta_{yn}^2 - \Delta_{xn}^2 & -2\Delta_{xn}\Delta_{yn} \end{bmatrix}. \tag{B14}$$

Substituting (B11)–(B14) in (B7), we obtain

$$G_{xx}^{(1)} = \sum_{n=1}^{N} \frac{1}{\sigma_n^2 r_n^6}\left\{\begin{bmatrix} -2\Delta_{yn} \\ 2\Delta_{xn} \end{bmatrix}[2\Delta_{xn}\Delta_{yn} \quad \Delta_{yn}^2 - \Delta_{xn}^2] - \Delta_{yn}\begin{bmatrix} 2\Delta_{xn}\Delta_{yn} & \Delta_{yn}^2 - \Delta_{xn}^2 \\ \Delta_{yn}^2 - \Delta_{xn}^2 & -2\Delta_{xn}\Delta_{yn} \end{bmatrix}\right\}$$

$$= M\sum_{k=1}^{K} \frac{1}{\sigma_k^2 r_k^6}\begin{bmatrix} -6\Delta_{xk}\Delta_{yk}^2 & -3\Delta_{yk}(\Delta_{yk}^2 - \Delta_{xk}^2) \\ \Delta_{yk}(5\Delta_{xk}^2 - \Delta_{yk}^2) & 2\Delta_{xk}(2\Delta_{yk}^2 - \Delta_{xk}^2) \end{bmatrix}$$

$$\triangleq M\tilde{G}_{xx}^{(1)} \tag{B15a}$$

and

$$G_{xx}^{(2)} = \sum_{n=1}^{N} \frac{1}{\sigma_n^2 r_n^6}\left\{\begin{bmatrix} -2\Delta_{yn} \\ 2\Delta_{xn} \end{bmatrix}[\Delta_{yn}^2 - \Delta_{xn}^2 \quad -2\Delta_{xn}\Delta_{yn}] + \Delta_{xn}\begin{bmatrix} 2\Delta_{xn}\Delta_{yn} & \Delta_{yn}^2 - \Delta_{xn}^2 \\ \Delta_{y_n^2} - \Delta_{x_n^2} & -2\Delta_{xn}\Delta_{yn} \end{bmatrix}\right\}$$

$$= M\sum_{k=1}^{K} \frac{1}{\sigma_k^2 r_k^6}\begin{bmatrix} 2\Delta_{yk}(2\Delta_{xk}^2 - \Delta_{yk}^2) & \Delta_{xk}(5\Delta_{yk}^2 - \Delta_{xk}^2) \\ 3\Delta_{xk}(\Delta_{yk}^2 - \Delta_{xk}^2) & -6\Delta_{xk}^2\Delta_{yk} \end{bmatrix}$$

$$\triangleq M\tilde{G}_{xx}^{(2)}. \tag{B15b}$$

Substituting (B10)–(B13) in (B8), we obtain

$$G_{x\theta}^{(1)} = -\sum_{n=1}^{N} \frac{1}{\sigma_n^2 r_n^4}\begin{bmatrix} 2\Delta_{xn}\Delta_{yn} \\ \Delta_{yn}^2 - \Delta_{xn}^2 \end{bmatrix}\mathbf{e}_n^{\mathrm{T}}$$

$$= -\sum_{k=1}^{K} \frac{1}{\sigma_k^2 r_k^4}\begin{bmatrix} 2\Delta_{xk}\Delta_{yk} \\ \Delta_{yk}^2 - \Delta_{xk}^2 \end{bmatrix}\tilde{\mathbf{e}}_k^{\mathrm{T}} \otimes 1_M^{\mathrm{T}} \triangleq \tilde{G}_{x\theta}^{(1)} \otimes 1_M^{\mathrm{T}} \tag{B16a}$$

$$G_{x\theta}^{(2)} = -\sum_{n=1}^{N} \frac{1}{\sigma_n^2 r_n^4}\begin{bmatrix} \Delta_{yn}^2 - \Delta_{xn}^2 \\ -2\Delta_{xn}\Delta_{yn} \end{bmatrix}\mathbf{e}_n^{\mathrm{T}}$$

$$= -\sum_{k=1}^{K} \frac{1}{\sigma_k^2 r_k^4}\begin{bmatrix} \Delta_{yk}^2 - \Delta_{xk}^2 \\ -2\Delta_{xk}\Delta_{yk} \end{bmatrix}\tilde{\mathbf{e}}_k^{\mathrm{T}} \otimes 1_M^{\mathrm{T}} \triangleq \tilde{G}_{x\theta}^{(2)} \otimes 1_M^{\mathrm{T}} \tag{B16b}$$

where $\tilde{\mathbf{e}}_k$ is the kth column of the identity $K \times K$ matrix I_K.

According to (A9), the bias of the ML estimator is given by

$$E(\delta\mathbf{x}) \approx -\left(\frac{\partial^2 F}{\partial \mathbf{x}^2}\right)^{-1}\mathbf{h} = -C\mathbf{h} \tag{B17}$$

where the elements of the 2×1 vector h are found substituting (B4)–(B6) in (A10). We have

$$h_j = \mathrm{tr}\left\{\frac{1}{2}S^{-1}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)CG_{xx}^{(j)}C\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^{\mathrm{T}} + S^{-1}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)CG_{x\Theta}^{(j)}S\right\}. \tag{B17}$$

Note that the factors S^{-1} and S cancel each other, when taking the trace of the second term in (B17). Furthermore, $\partial\mathbf{f}/\partial\mathbf{x}$ can be replaced by $\mathbf{g}_x$. Substituting the third-order derivatives found in (B15)–(B16), and $C = (1/M)\tilde{C}$, we obtain the final expression (16).

For the covariance, substituting (B3) and (B5) in (A12), we have

$$\mathrm{cov}(\delta\mathbf{x}) \approx (C^{-1} + D)^{-1}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^{\mathrm{T}}S^{-1}SS^{-1}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)(C^{-1} + D)^{-1}$$

$$= (C^{-1} + D)^{-1}C^{-1}(C^{-1} + D)^{-1}$$

$$= \frac{1}{M}(\tilde{C}^{-1} + \tilde{D})^{-1}\tilde{C}^{-1}(\tilde{C}^{-1} + \tilde{D})^{-1}\Big|_{(\mathbf{x}_b,\Theta_0)} \tag{B18}$$

meaning that the matrices $\tilde{C}$ of (5) and $\tilde{D}$ of (B3) are evaluated at $(\mathbf{x}_b,\Theta_0)$, and $\mathbf{x}_b = \mathbf{x}_0 + E(\delta\mathbf{x})$.

APPENDIX C. PERFORMANCE ANALYSIS OF THE STANSFIELD ALGORITHM

We now use the results of Appendix A to evaluate the bias and the covariance of the Stansfield algorithm estimation error.

For the Stansfield estimator, the function $F(\mathbf{x},\Theta)$ to be minimized is given by (12)–(13), where the matrix

R is assumed known. Let us evaluate the first-order derivatives of F:

$$\frac{\partial F}{\partial \mathbf{x}} = (A\mathbf{x}-\mathbf{b})^{\mathrm{T}} R^{-1} S^{-1} A$$

$$= \sum_{n=1}^{N} \frac{1}{\sigma_n^2 r_n^2} (\Delta_{xn}\sin\theta_n - \Delta_{yn}\cos\theta_n)[\sin\theta_n, -\cos\theta_n]$$

$$= \sum_{n=1}^{N} \frac{1}{\sigma_n^2 r_n^2} \begin{bmatrix} \Delta_{xn}\sin^2\theta_n - \frac{1}{2}\Delta_{yn}\sin 2\theta_n \\ -\frac{1}{2}\Delta_{xn}\sin 2\theta_n + \Delta_{yn}\cos^2\theta_n \end{bmatrix}^{\mathrm{T}} \quad \text{(C1)}$$

and

$$\frac{\partial F}{\partial \Theta} = \frac{1}{2}\sum_{n=1}^{N} \frac{1}{\sigma_n^2 r_n^2} \frac{\partial}{\partial \theta_n} (\Delta_{xn}\sin\theta_n - \Delta_{yn}\cos\theta_n)^2 \mathbf{e}_n^{\mathrm{T}}$$

$$= \sum_{n=1}^{N} \frac{1}{\sigma_n^2 r_n^2} (\Delta_{xn}\sin\theta_n - \Delta_{yn}\cos\theta_n) \times (\Delta_{xn}\cos\theta_n + \Delta_{yn}\sin\theta_n)\mathbf{e}_n^{\mathrm{T}}$$

$$= \sum_{n=1}^{N} \frac{1}{\sigma_n^2 r_n^2} \left[\frac{1}{2}(\Delta_{xn}^2 - \Delta_{yn}^2)\sin 2\theta_n - \Delta_{xn}\Delta_{yn}\cos 2\theta_n\right] \mathbf{e}_n^{\mathrm{T}}. \quad \text{(C2)}$$

The second-order derivatives of F are

$$\frac{\partial^2 F}{\partial \mathbf{x}^2} = A^{\mathrm{T}} R^{-1} S^{-1} A = \sum_{n=1}^{N} \frac{1}{\sigma_n^2 r_n^2} \times \begin{bmatrix} \sin^2\theta_n & -\sin\theta_n\cos\theta_n \\ -\sin\theta_n\cos\theta_n & \cos^2\theta_n \end{bmatrix}, \quad \text{(C3)}$$

$$\left.\frac{\partial^2 F}{\partial \mathbf{x}^2}\right|_{(\mathbf{x}_0,\Theta_0)} = \sum_{n=1}^{N} \frac{1}{\sigma_n^2 r_n^4} \begin{bmatrix} \Delta_{yn}^2 & -\Delta_{xn}\Delta_{yn} \\ -\Delta_{xn}\Delta_{yn} & \Delta_{xn}^2 \end{bmatrix} = C^{-1}, \quad \text{(C4)}$$

$$\frac{\partial^2 F}{\partial \mathbf{x}\partial \Theta} = \sum_{n=1}^{N} \frac{1}{\sigma_n^2 r_n^2} \begin{bmatrix} \Delta_{xn}\sin 2\theta_n - \Delta_{yn}\cos 2\theta_n \\ -\Delta_{xn}\cos 2\theta_n - \Delta_{yn}\sin 2\theta_n \end{bmatrix} \mathbf{e}_n^{\mathrm{T}} \triangleq HS^{-1}$$

$$= \sum_{k=1}^{K} \frac{1}{\sigma_k^2 r_k^2} \begin{bmatrix} \Delta_{xk}\sin 2\theta_k - \Delta_{yk}\cos 2\theta_k \\ -\Delta_{xk}\cos 2\theta_k - \Delta_{yk}\sin 2\theta_k \end{bmatrix} \bar{\mathbf{e}}_k^{\mathrm{T}} \otimes 1_M^{\mathrm{T}}$$

$$\triangleq (\bar{H}\bar{S}^{-1}) \otimes 1_M^{\mathrm{T}}, \quad \text{(C5)}$$

$$\left.\frac{\partial^2 F}{\partial \mathbf{x}\partial \Theta}\right|_{(\mathbf{x}_0,\Theta_0)} = \sum_{n=1}^{N} \frac{1}{\sigma_n^2 r_n^4} \begin{bmatrix} 2\Delta_{xn}^2\Delta_{yn} - \Delta_{xn}^2\Delta_{yn} + \Delta_{yn}^3 \\ -\Delta_{xn}^3 + \Delta_{xn}\Delta_{yn}^2 - 2\Delta_{xn}\Delta_{yn}^2 \end{bmatrix} \mathbf{e}_n^{\mathrm{T}}$$

$$= \sum_{n=1}^{N} \frac{1}{\sigma_n^2 r_n^2} \begin{bmatrix} \Delta_{yn} \\ -\Delta_{xn} \end{bmatrix} \mathbf{e}_n^{\mathrm{T}} = -\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^{\mathrm{T}} S^{-1}, \quad \text{(C6)}$$

$$\frac{\partial^2 F}{\partial \Theta^2} = -\mathrm{diag}\left\{\frac{1}{\sigma_n^2 r_n^2}[(\Delta_{xn}^2 - \Delta_{yn}^2)\cos 2\theta_n + 2\Delta_{xn}\Delta_{yn}\sin 2\theta_n]\right\}. \quad \text{(C7)}$$

We proceed with the third-order derivatives of F. The derivative of (C3) with respect to x_j gives

$$G_{\mathbf{xx}}^{(j)} = 0. \quad \text{(C8)}$$

The derivative of (C5) with respect to x_1 and x_2 gives

$$G_{\mathbf{x}\Theta}^{(1)} = \sum_{n=1}^{N} \frac{1}{\sigma_n^2 r_n^2} \left.\begin{bmatrix} \sin 2\theta_n \\ -\cos 2\theta_n \end{bmatrix}\right|_{(\mathbf{x}_0,\Theta_0)} \mathbf{e}_n^{\mathrm{T}}$$

$$= \sum_{n=1}^{N} \frac{1}{\sigma_n^2 r_n^4} \begin{bmatrix} 2\Delta_{xn}\Delta_{yn} \\ \Delta_{yn}^2 - \Delta_{xn}^2 \end{bmatrix} \mathbf{e}_n^{\mathrm{T}}, \quad \text{(C9a)}$$

$$G_{\mathbf{x}\Theta}^{(2)} = \sum_{n=1}^{N} \frac{1}{\sigma_n^2 r_n^2} \left.\begin{bmatrix} -\cos 2\theta_n \\ -\sin 2\theta_n \end{bmatrix}\right|_{(\mathbf{x}_0,\Theta_0)} \mathbf{e}_n^{\mathrm{T}}$$

$$= \sum_{n=1}^{N} \frac{1}{\sigma_n^2 r_n^4} \begin{bmatrix} \Delta_{yn}^2 - \Delta_{xn}^2 \\ -2\Delta_{xn}\Delta_{yn} \end{bmatrix} \mathbf{e}_n^{\mathrm{T}}. \quad \text{(C9b)}$$

The derivative of (C7) with respect to x_1 and x_2 gives

$$G_{\Theta\Theta}^{(1)} = \mathrm{diag}\left\{\left.\frac{2}{\sigma_n^2 r_n^2}(\Delta_{xn}\cos 2\theta_n + \Delta_{yn}\sin 2\theta_n)\right|_{(\mathbf{x}_0,\Theta_0)}\right\}$$

$$= \mathrm{diag}\left\{\frac{2\Delta_{xn}}{\sigma_n^2 r_n^2}\right\}, \quad \text{(C10a)}$$

$$G_{\Theta\Theta}^{(2)} = \mathrm{diag}\left\{\left.\frac{2}{\sigma_n^2 r_n^2}(-\Delta_{yn}\cos 2\theta_n + \Delta_{xn}\sin 2\theta_n)\right|_{(\mathbf{x}_0,\Theta_0)}\right\}$$

$$= \mathrm{diag}\left\{\frac{2\Delta_{yn}}{\sigma_n^2 r_n^2}\right\}. \quad \text{(C10b)}$$

Substituting (C4), (C6), and (C8) in (A10), we obtain

$$h_j = \mathrm{tr}\left\{\left[S^{-1}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right) C G_{\mathbf{x}\Theta}^{(j)} + \frac{1}{2}G_{\Theta\Theta}^{(j)}\right] S\right\}$$

$$= \mathrm{tr}\left\{\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right) C G_{\mathbf{x}\Theta}^{(j)}\right\} + \frac{1}{2}\mathrm{tr}\left\{G_{\Theta\Theta}^{(j)} S\right\}. \quad \text{(C11)}$$

Using (B10)–(B11) and (C9)–(C10) in (C11), we get

$$h_1 = \sum_{n=1}^{N} \left\{\frac{1}{\sigma_n^2 r_n^6}[-\Delta_{yn} \quad \Delta_{xn}] C \begin{bmatrix} 2\Delta_{xn}\Delta_{yn} \\ \Delta_{yn}^2 - \Delta_{xn}^2 \end{bmatrix} + \frac{\Delta_{xn}}{r_n^2}\right\} \quad \text{(C12a)}$$

$$h_2 = \sum_{n=1}^{N} \left\{\frac{1}{\sigma_n^2 r_n^6}[-\Delta_{yn} \quad \Delta_{xn}] C \begin{bmatrix} \Delta_{yn}^2 - \Delta_{xn}^2 \\ -2\Delta_{xn}\Delta_{yn} \end{bmatrix} + \frac{\Delta_{xn}}{r_n^2}\right\}. \quad \text{(C12b)}$$

Equation (C12) can be written as

$$\mathbf{h} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \sum_{n=1}^{N} \frac{1}{r_n^2} \left\{ \frac{1}{\sigma_n^2 r_n^4} \begin{bmatrix} 2\Delta_{xn}\Delta_{yn} & \Delta_{yn}^2 - \Delta_{xn}^2 \\ \Delta_{yn}^2 - \Delta_{xn}^2 & -2\Delta_{xn}\Delta_{yn} \end{bmatrix} \times C \begin{bmatrix} -\Delta_{yn} \\ \Delta_{xn} \end{bmatrix} + \begin{bmatrix} \Delta_{xn} \\ \Delta_{yn} \end{bmatrix} \right\}$$

$$= \sum_{k=1}^{K} \frac{1}{r_k^2} \left\{ \frac{1}{\sigma_k^2 r_k^4} \begin{bmatrix} 2\Delta_{xk}\Delta_{yk} & \Delta_{yk}^2 - \Delta_{xk}^2 \\ \Delta_{yk}^2 - \Delta_{xk}^2 & -2\Delta_{xk}\Delta_{yk} \end{bmatrix} \times \bar{C} \begin{bmatrix} -\Delta_{yk} \\ \Delta_{xk} \end{bmatrix} + M \begin{bmatrix} \Delta_{xk} \\ \Delta_{yk} \end{bmatrix} \right\}. \tag{C13}$$

The final result for the bias is found by substituting (C4) and (C13) in (A9), and it is given in (21) and (22).

For the covariance, we define $\mathbf{x}_b = \mathbf{x}_0 + E(\delta\mathbf{x})$, substitute (C4) and (C6), evaluated at $(\mathbf{x}_b, \Theta_0)$, in (A12), and obtain

$$\text{cov}(\delta\mathbf{x}) \approx CHS^{-1}SS^{-1}H^{\mathrm{T}}C = \frac{1}{M}\bar{C}\bar{H}\bar{S}^{-1}\bar{H}^{\mathrm{T}}\bar{C}\Big|_{(\mathbf{x}_b,\Theta_0)} \tag{C14}$$

where the matrices H and $\bar{H}$ are defined in (C5). However, note that for Stansfield, even when evaluating the expressions at the "biased" point, r_n are the true ranges, which are assumed known.

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Motti Gavish (M'86) was born in Bucharest, Romania. He received the B.Sc. degree from the Technion–Israel Institute of Technology, Haifa, in 1978, and the M.Sc. degree from Tel-Aviv University, Israel, in 1984, both in electrical engineering.

During 1978–1981 he served in the Israel Defense Forces as an electronic engineer. Between 1981 and 1982 he was a communication equipment development engineer with Tadiran Ltd. Communication Division, Holon, Israel. From 1982 to 1986 he was employed by ELTA Electronics Industries, Ltd., Ashdod, Israel (a subsidiary of Israel Aircraft Industries Ltd.) as a signal processing engineer in the Tracking Systems Department. During 1985–1986 he was also a Lecturer of Electrical Engineering at the Center for Technological Education, Holon, Israel. Between 1986 and 1990 he was an expert engineer at Tadiran Ltd., Systems Division, Holon. Since 1990, he has been with ELTA Electronics Industries, Ltd., Ashdod, as a senior scientist in the Communication Systems Division. He is also working towards the Ph.D. degree at Tel-Aviv University. His research interests are in the areas of estimation theory, signal and array processing, with application to digital communication, electronic warfare, radar and passive localization and tracking systems.

Mr. Gavish has authored and coauthored over fifteen papers in his fields of interest.

Anthony J. Weiss (S'84—M'86—SM'86) received the B.Sc. from the Technion–Israel Institute of Technology, Haifa, Israel, in 1973 and the M.Sc. and Ph.D. degrees from Tel-Aviv University, Tel-Aviv, Israel, in 1982 and 1985, all in electrical engineering.

From 1973 to 1983 he was involved in research and development of numerous projects in the fields of communications, tracking systems, command and control, and direction finding. In 1985 he joined the Department of Electrical Engineering Systems, Tel-Aviv University. During the academic year 1986/1987 he was a visiting scientist with the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA. He currently spends his summer vacations with Signal Processing Technology, Ltd., of Palo-Alto, CA. His research activities are detection and estimation theory, signal processing, and array data processing, with applications to radar and sonar.

Dr. Weiss held a Rothschild Foundation Fellowship from 1986 to 1987 and a Yigaal Alon Fellowship from 1985 to 1988. He was a corecipient of the IEEE Acoustics, Speech, and Signal Processing Society's 1983 Senior Award for a paper excerpted from his Master's thesis.