*Robotics 2*

# Dynamic model of robots: Lagrangian approach


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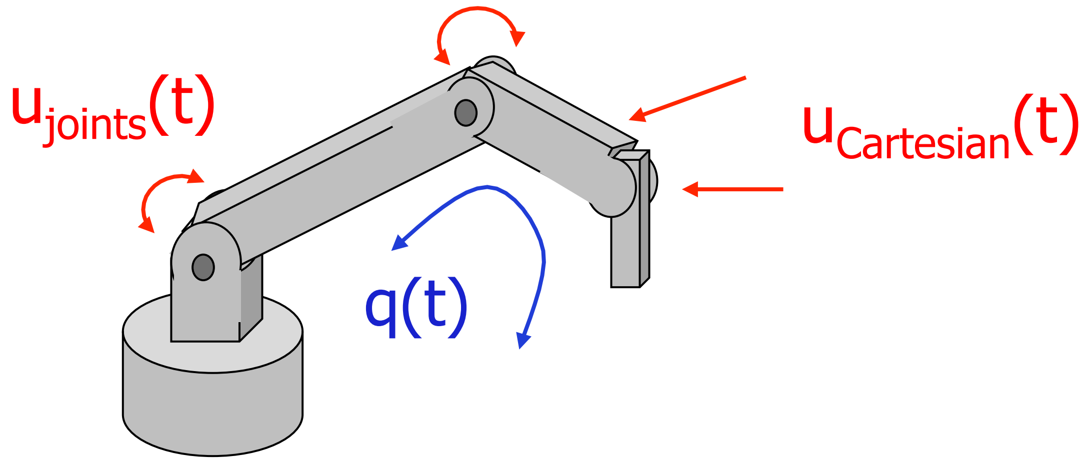
# Dynamic model

- provides the relation between

generalized forces $u(t)$ acting on the robot

robot motion, i.e.,
assumed configurations $q(t)$ over time



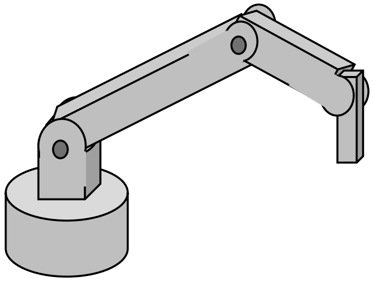

a system of 2nd order differential equations

$$\Phi(q,\dot{q},\ddot{q}) = u$$

# Direct dynamics

- direct relation

$$u(t) = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$$



$$q(t) = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}$$

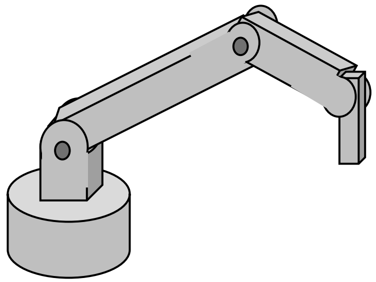
input for $t \in [0,T]$ + $q(0), \dot{q}(0)$

initial state at $t = 0$

- experimental solution
  - apply torques/forces with motors and measure joint variables with encoders (with sampling time $T_c$ )
- solution by simulation $\longleftrightarrow$ $\boxed{\Phi(q,\dot{q},\ddot{q}) = u}$
  - use dynamic model and integrate numerically the differential equations (with simulation step $T_s \leq T_c$ )

# Inverse dynamics

- inverse relation

$q_d(t), \dot{q}_d(t), \ddot{q}_d(t)$ $\Rightarrow$  $\Rightarrow$ $u_d(t)$

desired motion for $t \in [0,T]$

required input for $t \in [0,T]$

- experimental solution
  - repeated motion trials of direct dynamics using $u_k(t)$, with iterative learning of nominal torques updated on trial k+1 based on the error in [0,T] measured in trial k: $u_k(t) \Rightarrow u_d(t)$
- analytic solution $\longleftrightarrow$ $\boxed{\Phi(q,\dot{q},\ddot{q}) = u}$
  - use dynamic model and compute algebraically the values $u_d(t)$ at every time instant t

# Approaches to dynamic modeling

**Euler-Lagrange method (energy-based approach)** ⇌ **Newton-Euler method (balance of forces/torques)**

Euler-Lagrange method (energy-based approach):

- dynamic equations in symbolic/closed form
- best for study of dynamic properties and analysis of control schemes

Newton-Euler method (balance of forces/torques):

- dynamic equations in numeric/recursive form
- best for implementation of control schemes (inverse dynamics in real time)

- many formal methods based on basic principles in mechanics are available for the derivation of the robot dynamic model:
  - principle of d'Alembert, of Hamilton, of virtual works, …

# Euler-Lagrange method
## (energy-based approach)

basic assumption: the N links in motion are considered as **rigid bodies**
(+ possibly, **concentrated elasticity** at the joints)

$q \in \mathbb{R}^N$ generalized coordinates (e.g., joint variables, but not only!)

Lagrangian $\boxed{L(q,\dot{q}) = T(q,\dot{q}) - U(q)}$

kinetic energy – potential energy

- least action principle of Hamilton
- virtual works principle

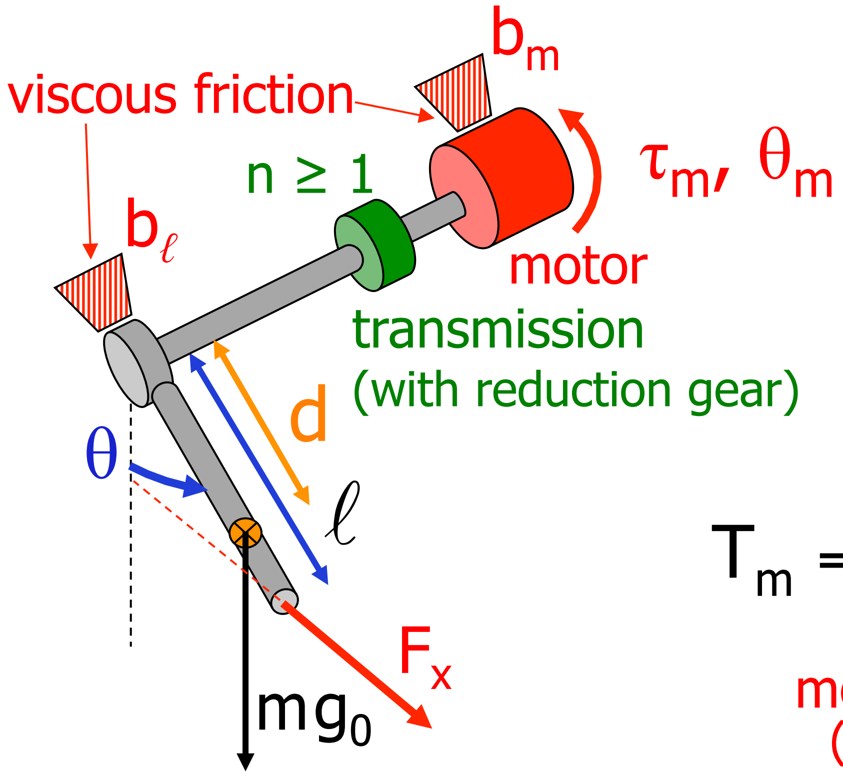
Euler-Lagrange equations

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = u_i} \qquad i = 1, \ldots, N$$

non-conservative (external or dissipative)
generalized forces performing work on $q_i$

# Dynamics of actuated pendulum

## a first example




$$\dot{\theta}_m = n\dot{\theta} \;\Rightarrow\; \theta_m = n\theta + \underbrace{\cancel{\theta_{m0}}}_{=\,0}$$

$$\tau = n\tau_m$$

$$\boxed{q = \theta} \quad (\text{or } q = \theta_m)$$

$$\boxed{T = T_m + T_\ell}$$

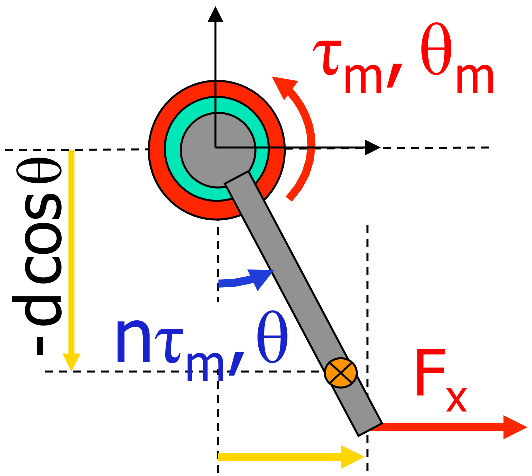
$$T_m = \frac{1}{2} I_m \dot{\theta}_m^2 \qquad T_\ell = \frac{1}{2}\left(I_\ell + md^2\right)\dot{\theta}^2$$

motor inertia (around its spinning axis)

link inertia (around the z-axis through its center of mass)

kinetic energy $\boxed{T = \frac{1}{2}\left(I_\ell + md^2 + n^2 I_m\right)\dot{\theta}^2 = \frac{1}{2} I\,\dot{\theta}^2}$

# Dynamics of actuated pendulum (cont)




$p_x = \ell \sin\theta$

$$\dot{p}_x = \ell \cos\theta \cdot \dot{\theta} = J_x \dot{\theta}$$

$$U = U_0 - m g_0 d \cos\theta$$ potential energy

$$L = T - U = \frac{1}{2} I \dot{\theta}^2 + m g_0 d \cos\theta - U_0$$

$$\frac{\partial L}{\partial \dot{\theta}} = I \dot{\theta} \qquad \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = I \ddot{\theta} \qquad \frac{\partial L}{\partial \theta} = -m g_0 d \sin\theta$$

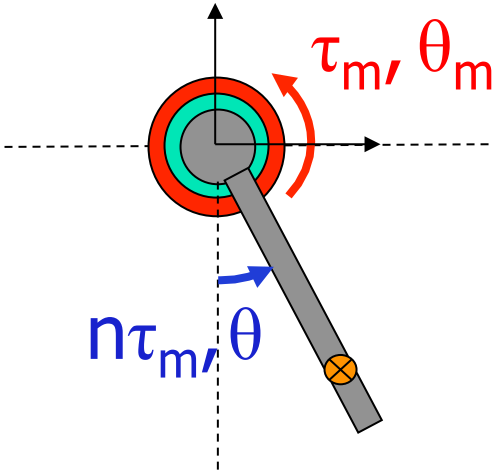
$$u = n\tau_m - b_\ell \dot{\theta} - n b_m \dot{\theta}_m + J_x^T F_x = n\tau_m - \left(b_\ell + n^2 b_m\right)\dot{\theta} + \ell \cos\theta \cdot F_x$$

applied or dissipated torques on motor side are multiplied by n when moved to the link side

equivalent joint torque due to force $F_x$ applied to the tip at point $p_x$

"sum" of non-conservative torques

# Dynamics of actuated pendulum (cont)




dynamic model in $q = \theta$

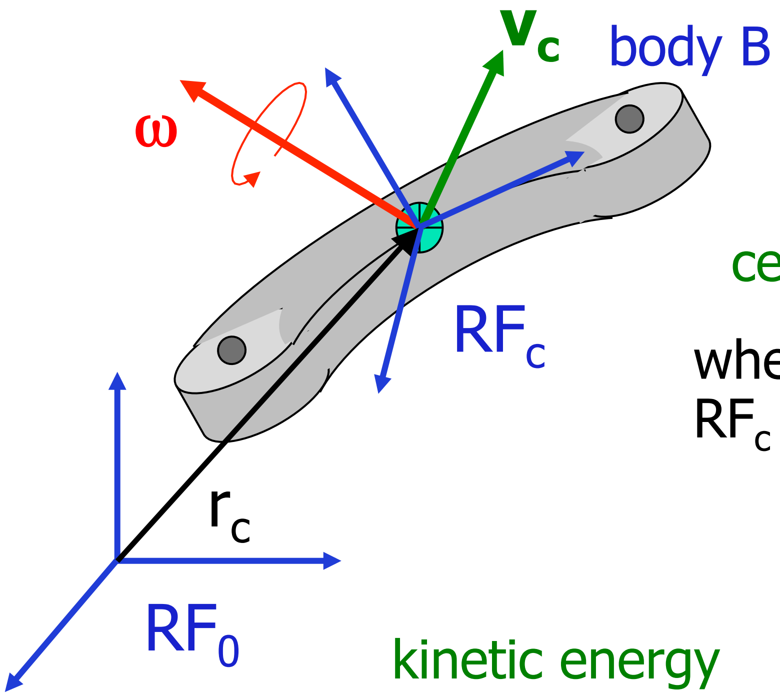
$$\boxed{I\ddot{\theta} + mg_0 d \sin\theta = n\tau_m - \left(b_\ell + n^2 b_m\right)\dot{\theta} + \ell \cos\theta \cdot F_x}$$

dividing by $n$ and substituting $\theta = \theta_m / n$

$$\boxed{\frac{I}{n^2}\ddot{\theta}_m + \frac{m}{n} g_0 d \sin\frac{\theta_m}{n} = \tau_m - \left(\frac{b_\ell}{n^2} + b_m\right)\dot{\theta}_m + \frac{\ell}{n}\cos\frac{\theta_m}{n} \cdot F_x}$$

dynamic model in $q = \theta_m$

# Kinetic energy of a rigid body




mass density

mass $\quad m = \int_B \rho(x,y,z)\,dx\,dy\,dz = \int_B dm$

position of center of mass (CoM) $\quad r_c = \frac{1}{m}\int_B r\,dm$

when all vectors are referred to a body frame $RF_c$ attached to the CoM, then

$$r_c = 0 \quad \Rightarrow \quad \int_B r\,dm = 0$$

kinetic energy $\quad T = \frac{1}{2}\int_B v^T(x,y,z)\,v(x,y,z)\,dm$

(fundamental) kinematic relation for a rigid body $\quad v = v_c + \omega \times r = v_c + S(\omega)r$

skew-symmetric matrix

# Kinetic energy of a rigid body (cont)

$$T = \frac{1}{2}\int_B [v_c + S(\omega)r]^T [v_c + S(\omega)r]\,dm$$

$$= \frac{1}{2}\int_B v_c^T v_c\,dm + \int_B v_c^T S(\omega) r\,dm + \frac{1}{2}\int_B r^T S^T(\omega) S(\omega) r\,dm$$

sum of elements on the diagonal of a matrix

$a^T b = \mathrm{trace}\{a b^T\}$

$$\boxed{= \frac{1}{2} m\, v_c^T v_c}$$

$$= v_c^T S(\omega) \int_B r\,dm = 0$$

$$= \frac{1}{2}\int_B \mathrm{trace}\{S(\omega) r \cdot r^T S^T(\omega)\}\,dm$$

$$= \frac{1}{2}\mathrm{trace}\left\{S(\omega)\left(\int_B r \cdot r^T dm\right) S^T(\omega)\right\}$$

$$= \frac{1}{2}\mathrm{trace}\{S(\omega)\, J_c\, S^T(\omega)\}$$

$$\boxed{= \frac{1}{2}\omega^T I_c\, \omega}$$

translational kinetic energy (point mass in CoM)

**+**

rotational kinetic energy (of the whole body)

**König theorem**

Euler matrix

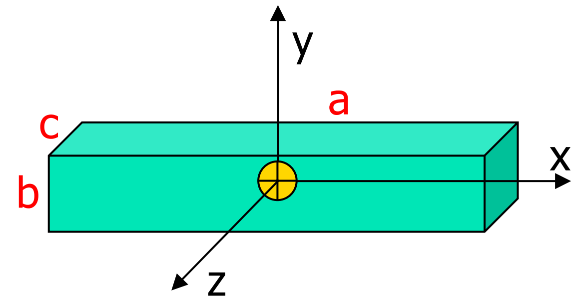
body inertia matrix (around the CoM)

**Ex #1:** provide the expressions of the elements of Euler matrix $J_c$

**Ex #2:** prove last equality and provide the expressions of the elements of inertia matrix $I_c$

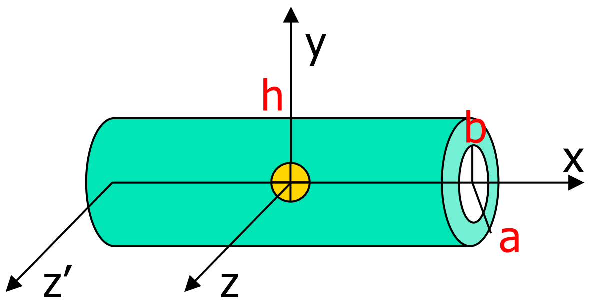
# Examples of body inertia matrices

## homogeneous bodies of mass m, with axes of symmetry




parallelepiped with sides
a (length/height), b, c (base)

$$I_c = \begin{pmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{pmatrix} = \begin{pmatrix} \frac{1}{12}m\left(b^2+c^2\right) & & \\ & \frac{1}{12}m\left(a^2+c^2\right) & \\ & & \frac{1}{12}m\left(a^2+b^2\right) \end{pmatrix}$$




empty cylinder with length h,
and external/internal radius a, b

$$I_c = \begin{pmatrix} \frac{1}{2}m\left(a^2+b^2\right) & & \\ & \frac{1}{12}m\left(3\left(a^2+b^2\right)+h^2\right) & \\ & & I_{zz} \end{pmatrix} \qquad I_{zz} = I_{yy}$$

$$I_{zz}' = I_{zz} + m(h/2)^2$$

(parallel) axis translation theorem

**Steiner theorem**

$$\boxed{I = I_c + m\left(r^T r \cdot E_{3\times 3} - r r^T\right) = I_c + m\,S^T(r)S(r)}$$

body inertia matrix relative to the CoM

identity matrix

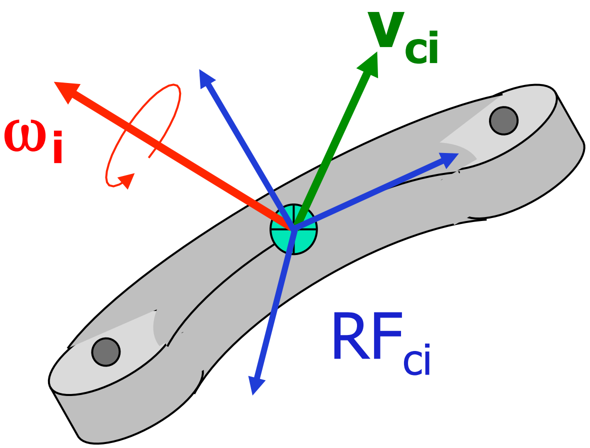
**Ex #3:** prove the last equality

its generalization:
changes on body inertia matrix due to a pure translation r of the reference frame

# Robot kinetic energy

$$T = \sum_{i=1}^{N} T_i$$ ⇐ N rigid bodies (+ fixed base)

$$T_i = T_i\left(q_j, \dot{q}_j, \underbrace{j \le i}\right)$$ ⇐ open kinematic chain




i-th link (body)
of the robot

König theorem

$$T_i = \frac{1}{2} m_i v_{ci}^T v_{ci} + \frac{1}{2} \omega_i^T I_{ci} \omega_i$$

absolute velocity of
the center of mass (CoM)

absolute
angular velocity
of whole body

# Kinetic energy of a robot link

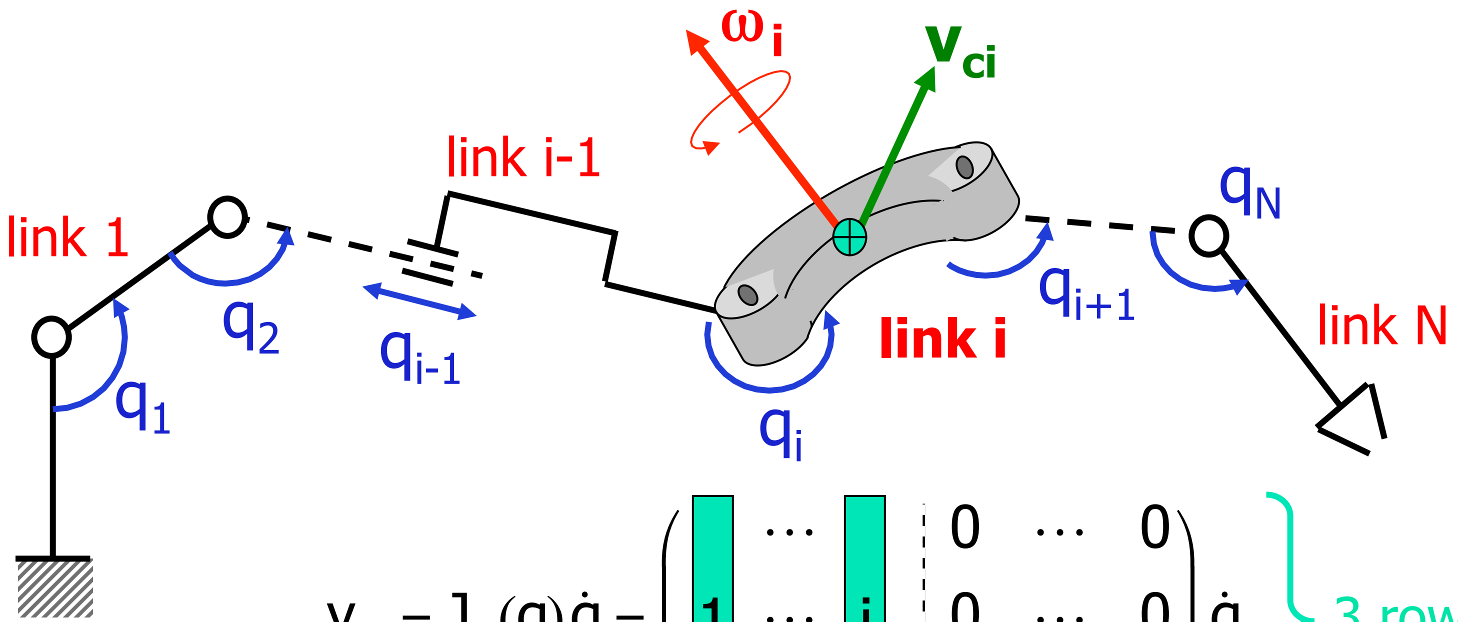
$$T_i = \frac{1}{2} m_i v_{ci}^T v_{ci} + \frac{1}{2} \omega_i^T I_{ci} \omega_i$$

$\boldsymbol{\omega_i}$, $\mathbf{I_{ci}}$ should be expressed in the **same reference frame**, but the product $\boldsymbol{\omega_i^T I_{ci} \omega_i}$ is **invariant** w.r.t. any chosen frame

in frame $RF_{ci}$ attached to (the center of mass of) link i

$$
\underset{\text{constant!}}{{}^{i}I_{ci}} = \begin{pmatrix} \int (y^2+z^2)dm & -\int xy\,dm & -\int xz\,dm \\ & \int (x^2+z^2)dm & -\int yz\,dm \\ \text{symm} & & \int (x^2+y^2)dm \end{pmatrix}
$$

# Dependence of T from q and $\dot{q}$




$$v_{ci} = J_{Li}(q)\dot{q} = \underbrace{\begin{pmatrix} & \cdots & & 0 & \cdots & 0 \\ 1 & \cdots & i & 0 & \cdots & 0 \\ & \cdots & & 0 & \cdots & 0 \end{pmatrix}}_{} \dot{q} \quad \Big\}\ \text{3 rows}$$

(partial) Jacobians
typically expressed in $RF_0$

$$\omega_i = J_{Ai}(q)\dot{q} = \begin{pmatrix} & \cdots & & 0 & \cdots & 0 \\ 1 & \cdots & i & 0 & \cdots & 0 \\ & \cdots & & 0 & \cdots & 0 \end{pmatrix} \dot{q} \quad \Big\}\ \text{3 rows}$$

# Final expression of T

$$\boxed{T = \frac{1}{2}\sum_{i=1}^{N}\left(m_i v_{ci}^T v_{ci} + \omega_i^T I_{ci}\,\omega_i\right)}$$

$$= \frac{1}{2}\dot{q}^T\left(\sum_{i=1}^{N} m_i J_{Li}^T(q) J_{Li}(q) + J_{Ai}^T(q) I_{ci} J_{Ai}(q)\right)\dot{q}$$

constant if $\boldsymbol{\omega}_i$ is expressed in $RF_{ci}$

else

$$\boxed{{}^0I_{ci}(q) = {}^0R_i(q)\,{}^iI_{ci}\,{}^0R_i^T(q)}$$

NOTE:
in practice, NEVER use this formula (or partial Jacobians) for computing T; a better method is available...

$$\boxed{T(q,\dot{q}) = \frac{1}{2}\dot{q}^T B(q)\dot{q}}$$

**robot (generalized) inertia matrix**

- symmetric
- positive definite, $\forall q \Rightarrow$ **always invertible**

# Robot potential energy

assumption: GRAVITY contribution only

$$U = \sum_{i=1}^{N} U_i$$ ⟸ N rigid bodies (+ fixed base)

$$U_i = U_i(q_j, \underbrace{j \le i})$$ ⟸ open kinematic chain

$$\boxed{U_i = -m_i g^T r_{0,ci}}$$

{ gravity acceleration vector ; position of the center of mass of link i } typically expressed in $RF_0$

dependence on q

$$\boxed{\begin{pmatrix} r_{0,ci} \\ 1 \end{pmatrix} = {}^0A_1(q_1)\,{}^1A_2(q_2)\cdots{}^{i-1}A_i(q_i)\begin{pmatrix} r_{i,ci} \\ 1 \end{pmatrix}}$$

constant in $RF_i$

NOTE: need to work with homogeneous coordinates

# Summarizing ...

kinetic energy

$$T = \frac{1}{2}\dot{q}^T B(q)\dot{q} = \frac{1}{2}\sum_{i,j} b_{ij}(q)\dot{q}_i\dot{q}_j \geq 0$$

positive definite quadratic form

$$\boxed{T = 0 \Leftrightarrow \dot{q} = 0}$$

potential energy

$$U = U(q)$$

Lagrangian

$$L(q,\dot{q}) = T(q,\dot{q}) - U(q)$$

Euler-Lagrange equations

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = u_k} \qquad k = 1,\ldots,N$$

non-conservative (active/dissipative) generalized forces **performing work** on $q_k$ coordinate

# Applying Euler-Lagrange equations

(the scalar derivation; see Appendix for vector format)

$$L(q,\dot{q}) = \frac{1}{2}\sum_{i,j} b_{ij}(q)\dot{q}_i\dot{q}_j - U(q)$$

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_j b_{kj}\dot{q}_j \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \sum_j b_{kj}\ddot{q}_j + \sum_{i,j}\frac{\partial b_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j$$

(dependences on q are not shown)

$$\frac{\partial L}{\partial q_k} = \frac{1}{2}\sum_{i,j}\frac{\partial b_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial U}{\partial q_k}$$

LINEAR terms in ACCELERATION $\ddot{q}$

QUADRATIC terms in VELOCITY $\dot{q}$

NONLINEAR terms in CONFIGURATION $q$

# k-th dynamic equation ...

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = u_k$$

$$\sum_j b_{kj}(q)\ddot{q}_j + \sum_{i,j}\left(\frac{\partial b_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial b_{ij}}{\partial q_k}\right)\dot{q}_i\dot{q}_j + \frac{\partial U}{\partial q_k} = u_k$$

exchanging indices i,j

$$\cdots + \sum_{i,j}\underbrace{\frac{1}{2}\left(\frac{\partial b_{kj}}{\partial q_i} + \frac{\partial b_{ki}}{\partial q_j} - \frac{\partial b_{ij}}{\partial q_k}\right)}_{c_{kij} = c_{kji}}\dot{q}_i\dot{q}_j + \cdots$$

Christoffel symbols
of the first kind

# … and interpretation of dynamic terms

$$\boxed{\sum_j b_{kj}(q)\ddot{q}_j} + \boxed{\sum_{i,j} c_{kij}(q)\dot{q}_i\dot{q}_j} + \boxed{\frac{\partial U}{\partial q_k}} = u_k \qquad k = 1,\dots,N$$

**INERTIAL** terms

**CENTRIFUGAL** (i=j) and **CORIOLIS** (i≠j) terms

**GRAVITY** terms $g_k(q)$

$b_{kk}(q)$ = inertia at joint k when joint k accelerates ($b_{kk} > 0$!!)

$b_{kj}(q)$ = inertia “seen” at joint k when joint j accelerates

$c_{kii}(q)$ = coefficient of the centrifugal force at joint k when joint i is moving ($c_{iii} = 0,\ \forall i$)

$c_{kij}(q)$ = coefficient of the Coriolis force at joint k when both joint i and joint j are moving

# Robot dynamic model
## in vector formats

1. $$B(q)\ddot{q} + c(q,\dot{q}) + g(q) = u$$

$$c_k(q,\dot{q}) = \dot{q}^T C_k(q)\dot{q}$$

k-th component of vector c

k-th column of matrix B(q)

$$C_k(q) = \frac{1}{2}\left(\frac{\partial b_k}{\partial q} + \left(\frac{\partial b_k}{\partial q}\right)^T - \frac{\partial B}{\partial q_k}\right)$$

symmetric matrix

2. $$B(q)\ddot{q} + S(q,\dot{q})\dot{q} + g(q) = u$$

NOTE: these models are in the form $\Phi(q,\dot{q},\ddot{q}) = u$ as expected

NOT a symmetric matrix

$$s_{kj}(q,\dot{q}) = \sum_i c_{kij}(q)\dot{q}_i$$

factorization of c by S is **not unique!**

# A structural property

matrix $\dot{B} - 2S$ is skew-symmetric
(when using Christoffel symbols to define matrix S)

**Proof**

$$\dot{b}_{kj} = \sum_i \frac{\partial b_{kj}}{\partial q_i}\dot{q}_i \qquad 2s_{kj} = 2\sum_i c_{kji}\dot{q}_i = 2\sum_i \frac{1}{2}\left(\frac{\partial b_{kj}}{\partial q_i} + \frac{\partial b_{ki}}{\partial q_j} - \frac{\partial b_{ij}}{\partial q_k}\right)\dot{q}_i$$

$$\Rightarrow \quad \dot{b}_{kj} - 2s_{kj} = \sum_i \left(\frac{\partial b_{ij}}{\partial q_k} - \frac{\partial b_{ki}}{\partial q_j}\right)\dot{q}_i = n_{kj}$$

$$n_{jk} = \dot{b}_{jk} - 2s_{jk} = \sum_i \left(\frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{ji}}{\partial q_k}\right)\dot{q}_i = -n_{kj}$$

because of the symmetry of B

$$\Rightarrow \quad \boxed{x^T\left(\dot{B} - 2S\right)x = 0, \quad \forall x}$$

# Energy conservation

- total robot energy

$$E = T + U = \frac{1}{2}\dot{q}^T B(q)\dot{q} + U(q)$$

- its evolution over time (using the dynamic model)

$$\dot{E} = \dot{q}^T B(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{B}(q)\dot{q} + \frac{\partial U}{\partial q}\dot{q}$$

$$= \dot{q}^T\left(u - S(q,\dot{q})\dot{q} - g(q)\right) + \frac{1}{2}\dot{q}^T \dot{B}(q)\dot{q} + \dot{q}^T g(q)$$

$$= \dot{q}^T u + \frac{1}{2}\dot{q}^T\left(\dot{B}(q) - 2S(q,\dot{q})\right)\dot{q}$$

here, any factorization of vector c by a matrix S can be used

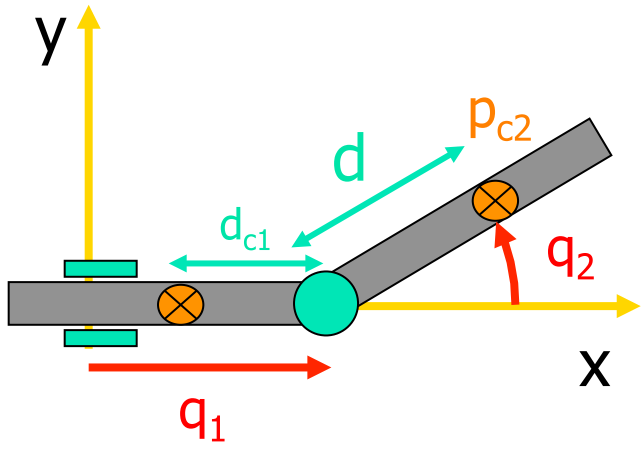
- if $u \equiv 0$, total energy is constant (no dissipation or increase)

$$\dot{E} = 0 \quad \Rightarrow \quad \boxed{\dot{q}^T\left(\dot{B} - 2S\right)\dot{q} = 0, \ \forall q, \dot{q}} \quad \Rightarrow \quad \dot{E} = \dot{q}^T u$$

weaker than skew-symmetry, as the external velocity is the same that appears in the internal matrices

in general, the variation of the total energy is equal to the work of non-conservative forces

# Dynamic model of a PR robot




$$T = T_1 + T_2 \qquad U = \text{constant}$$

(on horizontal plane)

$$p_{c1} = \begin{pmatrix} q_1 - d_{c1} \\ 0 \\ 0 \end{pmatrix} \Rightarrow \|v_{c1}\|^2 = \dot{p}_{c1}^T \dot{p}_{c1} = \dot{q}_1^2$$

$$\boxed{T_1 = \frac{1}{2} m_1 \dot{q}_1^2}$$

$$T_2 = \frac{1}{2} m_2 v_{c2}^T v_{c2} + \frac{1}{2} \omega_2^T I_{c2} \omega_2$$

$$p_{c2} = \begin{pmatrix} q_1 + d\cos q_2 \\ d\sin q_2 \\ 0 \end{pmatrix} \Rightarrow v_{c2} = \begin{pmatrix} \dot{q}_1 - d\sin q_2\, \dot{q}_2 \\ d\cos q_2\, \dot{q}_2 \\ 0 \end{pmatrix} \qquad \omega_2 = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_2 \end{pmatrix}$$

$$\boxed{T_2 = \frac{1}{2} m_2 \left( \dot{q}_1^2 + d^2 \dot{q}_2^2 - 2d\sin q_2\, \dot{q}_1 \dot{q}_2 \right) + \frac{1}{2} I_{c2,zz} \dot{q}_2^2}$$

# Dynamic model of a PR robot (cont)

$$B(q) = \begin{pmatrix} \boxed{\begin{matrix} m_1 + m_2 \\ -m_2 d \sin q_2 \end{matrix}} & \boxed{\begin{matrix} -m_2 d \sin q_2 \\ I_{c2,zz} + m_2 d^2 \end{matrix}} \end{pmatrix}$$

$$\qquad\quad b_1 \qquad\qquad\qquad b_2$$

$$c(q,\dot{q}) = \begin{pmatrix} c_1(q,\dot{q}) \\ c_2(q,\dot{q}) \end{pmatrix}$$

$$c_k(q,\dot{q}) = \dot{q}^T C_k(q) \dot{q}$$

where $C_k(q) = \frac{1}{2}\left( \frac{\partial b_k}{\partial q} + \left(\frac{\partial b_k}{\partial q}\right)^T - \frac{\partial B}{\partial q_k} \right)$

$$C_1(q) = \frac{1}{2}\left( \begin{pmatrix} 0 & 0 \\ 0 & -m_2 d \cos q_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -m_2 d \cos q_2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

$$c_1(q,\dot{q}) = -m_2 d \cos q_2 \, \dot{q}_2^2$$

$$C_2(q) = \frac{1}{2}\left( \begin{pmatrix} 0 & -m_2 d \cos q_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -m_2 d \cos q_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -m_2 d \cos q_2 \\ -m_2 d \cos q_2 & 0 \end{pmatrix} \right)$$

$$= 0$$

$$c_2(q,\dot{q}) = 0$$

# Dynamic model of a PR robot (cont)

$$\boxed{B(q)\ddot{q} + c(q,\dot{q}) = u}$$

$$\begin{pmatrix} m_1 + m_2 & -m_2 d \sin q_2 \\ -m_2 d \sin q_2 & I_{c2,zz} + m_2 d^2 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -m_2 d \cos q_2 \, \dot{q}_2^2 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

NOTE: the $b_{NN}$ element (here, for N=2) is always a constant!

Q1: why Coriolis terms are not present?

Q2: when applying a force $u_1$, does the second joint accelerate? … always?

Q3: what is the expression of a factorization matrix S? … is it unique?

Q4: which is the configuration with "maximum inertia"?

# Appendix: Vector format derivation of dynamic model

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)^T - \left(\frac{\partial L}{\partial q}\right)^T = u} \qquad L = \frac{1}{2}\dot{q}^T B(q)\dot{q} - U(q)$$

$$B(q) = \begin{bmatrix} b_1(q) & \dots & b_i(q) & \dots & b_N(q) \end{bmatrix} = \sum_{i=1}^{N} b_i(q)\, e_i^T \qquad e_i^T = \begin{bmatrix} 0 & \dots & 1 & \dots & 0 \end{bmatrix}$$

i-th position

dyadic expansion

$$\left(\frac{\partial L}{\partial \dot{q}}\right)^T = \left(\dot{q}^T B(q)\right)^T = B(q)\dot{q} \;\Rightarrow\; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)^T = B(q)\ddot{q} + \dot{B}(q)\dot{q} = B(q)\ddot{q} + \sum_{i=1}^{N}\left(\frac{\partial b_i}{\partial q}\right)\dot{q}_i\,\dot{q}$$

$$\left(\frac{\partial L}{\partial q}\right)^T = \left(\frac{1}{2}\dot{q}^T\left(\sum_{i=1}^{N}\frac{\partial b_i}{\partial q}e_i^T\right)\dot{q} - \frac{\partial U}{\partial q}\right)^T = \left(\frac{1}{2}\dot{q}^T\left(\sum_{i=1}^{N}\frac{\partial b_i}{\partial q}\dot{q}_i\right) - \frac{\partial U}{\partial q}\right)^T = \frac{1}{2}\sum_{i=1}^{N}\left(\frac{\partial b_i}{\partial q}\right)^T\dot{q}_i\,\dot{q} - \left(\frac{\partial U}{\partial q}\right)^T$$

$$\Rightarrow\; B(q)\ddot{q} + \underbrace{\left[\sum_{i=1}^{N}\left(\frac{\partial b_i}{\partial q} - \frac{1}{2}\left(\frac{\partial b_i}{\partial q}\right)^T\right)\dot{q}_i\right]}_{S(q,\dot{q})}\dot{q} + \underbrace{\left(\frac{\partial U}{\partial q}\right)^T}_{g(q)} = u$$