# Heap structure and applications

*Data Structures and Algorithms*

**Dept. Computer Science**
*Faculty of Computer Science and Engineering*
*Ho Chi Minh University of Technology, VNU-HCM*

# Overview

1. **Heap Definition**
2. **Heap Structure**
3. **Basic Heap Algorithms**
   - ReheapUp
   - ReheapDown
4. **Heap Data Structure**
5. **Heap Algorithms**
   - ReheapUp
   - ReheapDown
   - Build a Heap
   - Insert a Node
   - Delete a Node
6. **Heap Applications**
   - Selection Algorithms
   - Priority Queues
7. **Heap Sort**

# Course learning outcomes

| | |
|---|---|
| L.O.1 | Determine the complexity of simple algorithms (polynomial time - nested loop - no recursive) |
| L.O.1.1 | Give definition of Big-O notation |
| L.O.1.2 | Determine complexity of simple polynomial algorithms |
| L.O.2 | Manipulate basic data structures such as list, tree and graph |
| L.O.2.1 | Describe and present basic data structures such as: array, linked list, stack, queue, tree, and graph |
| L.O.2.2 | Implement basic methods for each of basic data structures: array, linked list, stack, queue, tree, and graph |
| L.O.3 | Implement basic sorting and searching algorithms |
| L.O.3.1 | Illustrate how searching algorithms work on data structures: array, linked list, stack, queue, tree, and graph |
| L.O.3.2 | Illustrate how sorting algorithms work on an array |
| L.O.3.3 | Implement necessary methods and proposed algorithms on a given data structure for problem solving |

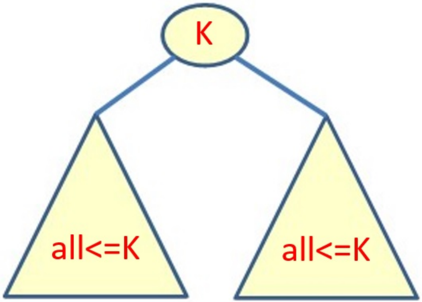
# Heap Definition

# Heap Definition

## Definition

A heap (max-heap) is a binary tree structure with the following properties:

1. The tree is complete or nearly complete.
2. The key value of each node is greater than or equal to the key value in each of its descendents.



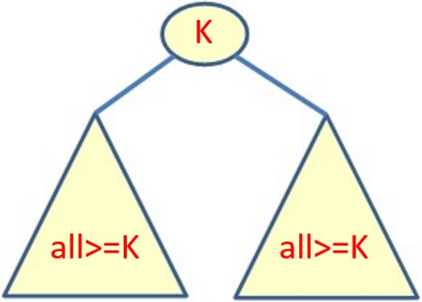

(Source: Data Structures - A Pseudocode Approach with C++)

# Heap Definition

## Definition

A min-heap is a binary tree structure with the following properties:

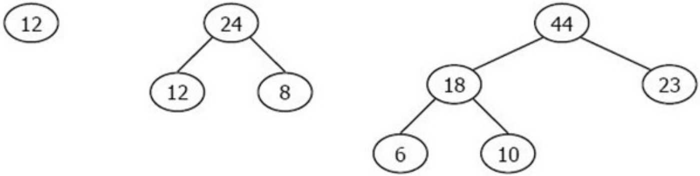
1. The tree is complete or nearly complete.
2. The key value of each node is less than or equal to the key value in each of its descendents.



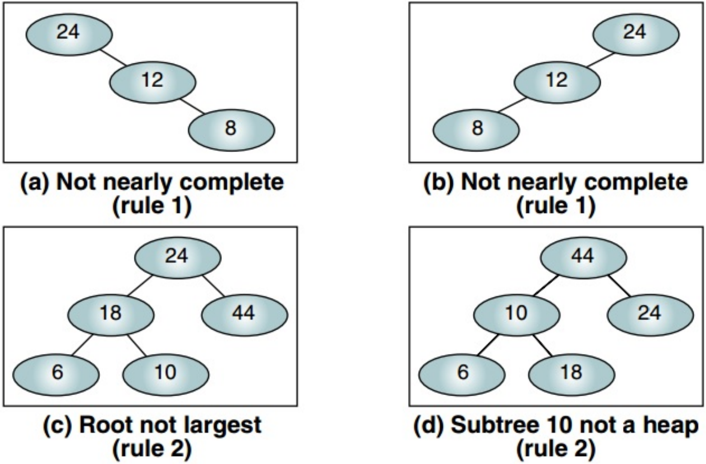

(Source: Data Structures - A Pseudocode Approach with C++)

# Heap Structure

# Heap trees



# Invalid Heaps

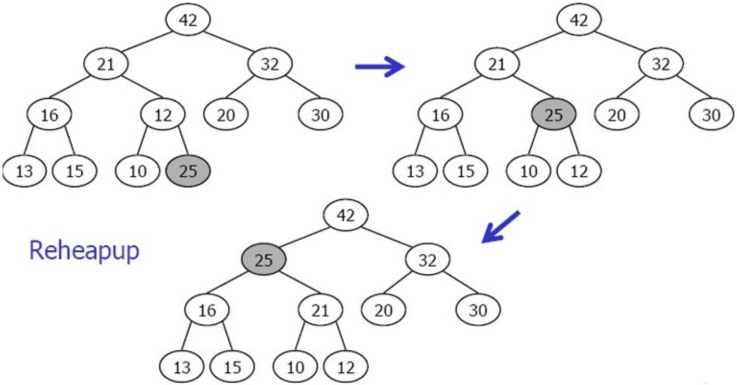



(Source: Data Structures - A Pseudocode Approach with C++)

# Basic Heap Algorithms

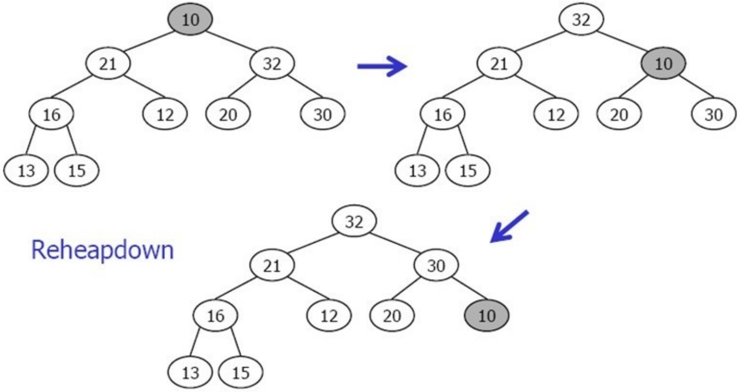
# ReheapUp

The reheapUp operation repairs a "broken" heap by floating the last element up the tree until it is in its correct location in the heap.



# ReheapDown

The reheapDown operation repairs a "broken" heap by pushing the root down the tree until it is in its correct location in the heap.



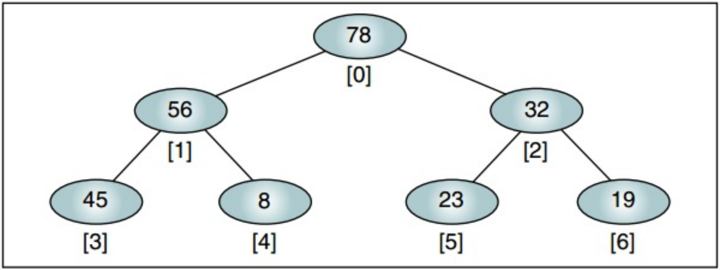
# Heap Data Structure

## Properties of Heaps

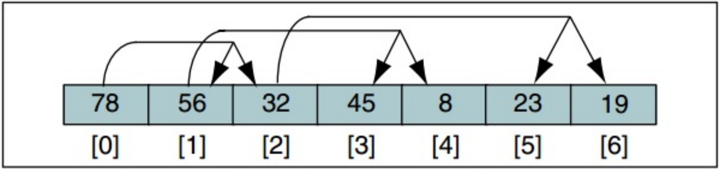
- A complete or nearly complete binary tree.
- If the height is $h$, the number of nodes $N$ is between $2^{h-1}$ and $2^h - 1$.
- Complete tree: $N = 2^h - 1$ when last level is full.
- Nearly complete: All nodes in the last level are on the left.

$\rightarrow$ Heap can be represented in an array.

# Heap in arrays




**(a) Heap in its logical form**




**(b) Heap in an array**

(Source: Data Structures - A Pseudocode Approach with C++)

# Heap Data Structure

The relationship between a node and its children is fixed and can be calculated:

1. For a node located at index $i$, its children are found at
    - Left child: $2i + 1$
    - Right child: $2i + 2$
2. The parent of a node located at index $i$ is located at $\lfloor (i - 1)/2 \rfloor$.
3. Given the index for a left child, $j$, its right sibling, if any, is found at $j + 1$. Conversely, given the index for a right child, $k$, its left sibling, which must exist, is found at $k - 1$.
4. Given the size, $N$, of a complete heap, the location of the first leaf is $\lfloor N/2 \rfloor$.
5. Given the location of the first leaf element, the location of the last nonleaf element is 1 less.

# Heap Algorithms

# ReheapUp Algorithm

**Algorithm** reheapUp(ref heap <array>, val position <integer>)
Reestablishes heap by moving data in `position` up to its correct location.

**Pre:** All data in the heap above this `position` satisfy key value order of a heap, except the data in `position`

**Post:** Data in `position` has been moved up to its correct location.

# ReheapUp Algorithm

```
if position > 0 then
    parent = (position-1)/2
    if heap[position].key > heap[parent].key then
        swap(position, parent)
        reheapUp(heap, parent)
    end
end
return
End reheapUp
```

# ReheapDown Algorithm

**Algorithm** reheapDown(ref heap <array>, val position <integer>, val lastPosition <integer>)

Reestablishes heap by moving data in `position` down to its correct location.

**Pre:** All data in the subtree of `position` satisfy key value order of a heap, except the data in `position`

`lastPosition` is an index to the last element in heap

**Post:** Data in `position` has been moved down to its correct location.

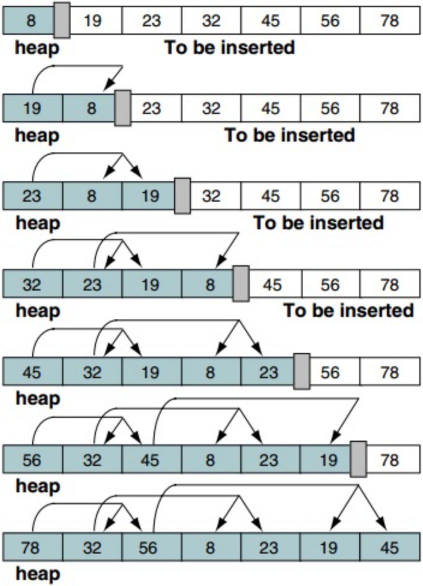
# ReheapDown Algorithm

```
leftChild = position * 2 + 1
rightChild = position * 2 + 2
if leftChild <= lastPosition then
    if (rightChild <= lastPosition) AND
      (heap[rightChild].key > heap[leftChild].key
      then
        largeChild = rightChild
    else
        largeChild = leftChild
    end
    if heap[largeChild].key > heap[position].key
      then
        swap(largeChild, position)
        reheapDown(heap, largeChild,
          lastPosition)
    end
end
return
```

# Build a Heap

- Given a filled array of elements in random order, to build the heap we need to rearrange the data so that each node in the heap is greater than its children.
- We begin by dividing the array into two parts, the left being a heap and the right being data to be inserted into the heap. Note the "wall" between the first and second parts.
- At the beginning the root (the first node) is the only node in the heap and the rest of the array are data to be inserted.
- Each iteration of the insertion algorithm uses reheap up to insert the next element into the heap and moves the wall separating the elements one position to the right.

# Build a Heap




(Source: Data Structures - A Pseudocode Approach with C++)

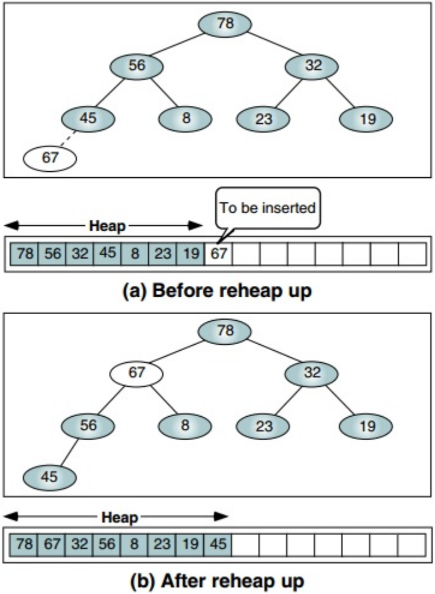
# Build a Heap

```
Algorithm buildHeap(ref heap <array>, val size <integer>)
Given an array, rearrange data so that they form a heap.

Pre: heap is array containing data in nonheap order
size is number of elements in array

Post: array is now a heap.

walker = 1
while walker < size do
    reheapUp(heap, walker)
    walker = walker + 1
end
End buildHeap
```

# Insert a Node into a Heap

- To insert a node, we need to locate the first empty leaf in the array.
- We find it immediately after the last node in the tree, which is given as a parameter.
- To insert a node, we move the new data to the first empty leaf and reheap up.

# Insert a Node into a Heap




(Source: Data Structures - A Pseudocode Approach with C++)

# Insert a Node into a Heap

**Algorithm** insertHeap(ref heap <array>, ref last <integer>, val data <dataType>)
Inserts `data` into heap.

**Pre:** `heap` is a valid heap structure
`last` is reference parameter to last node in heap
`data` contains data to be inserted

**Post:** `data` have been inserted into heap.

**Return** `true` if successful; `false` if array full

# Insert a Node into a Heap

```
if heap full then
    return false
end
last = last + 1
heap[last] = data
reheapUp(heap, last)
return true
End insertHeap
```

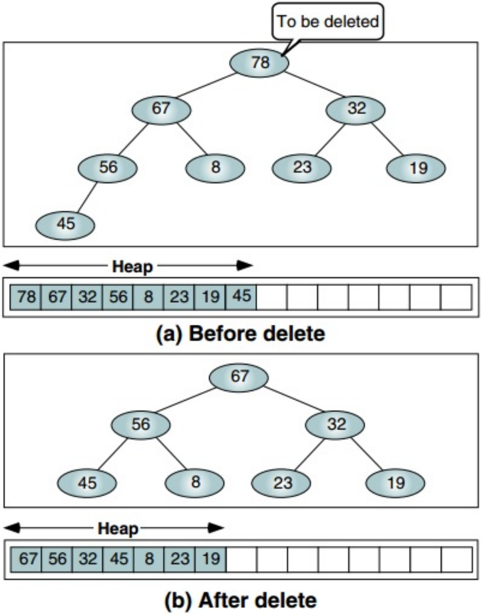
# Delete a Node from a Heap

- When deleting a node from a heap, the most common and meaningful logic is to delete the root.
- After it has been deleted, the heap is thus left without a root.
- To reestablish the heap, we move the data in the last heap node to the root and reheap down.



Dept. Computer Science

# Delete a Node from a Heap




(Source: Data Structures - A Pseudocode Approach with C++)

# Delete a Node from a Heap

```
Algorithm deleteHeap(ref heap <array>, ref last
  <integer>, ref dataOut <dataType>)
Deletes root of heap and passes data back to caller.
Pre: heap is a valid heap structure
last is reference parameter to last node
dataOut is reference parameter for output data
Post: root deleted and heap rebuilt
root data placed in dataOut
Return true if successful; false if array empty
```

# Delete a Node from a Heap

```
if heap empty then
    return false
end
dataOut = heap[0]
heap[0] = heap[last]
last = last - 1
reheapDown(heap, 0, last)
return true
End deleteHeap
```

# Complexity of Binary Heap Operations

- ReheapUp: $O(\log_2 n)$
- ReheapDown: $O(\log_2 n)$
- Build a Heap: $O(n \log_2 n)$
- Insert a Node into a Heap: $O(\log_2 n)$
- Delete a Node from a Heap: $O(\log_2 n)$

# Heap Applications

# Heap Applications

Three common applications of heaps are:

1. selection algorithms,
2. priority queues,
3. and sorting.

We discuss heap sorting in Chapter 10 and selection algorithms and priority queues here.

# Selection Algorithms

## Problem

**Determining the $k^{th}$ element in an unsorted list.**

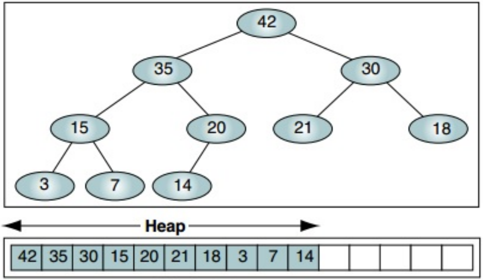
Two solutions:

1. Sort the list and select the element at location $k$. The complexity of a simple sorting algorithm is $O(n^2)$.
2. Create a heap and delete $k-1$ elements from the heap, leaving the desired element at the top. The complexity is $O(n \log_2 n)$.

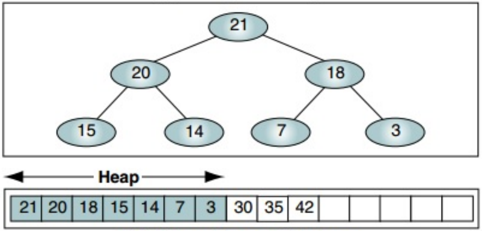
Rather than simply discarding the elements at the top of the heap, a better solution would be to place the deleted element at the end of the heap and reduce the heap size by 1.

After the $k^{th}$ element has been processed, the temporarily removed elements can then be inserted into the heap.

# Selection Algorithms




(a) Original heap




(b) After three deletions

(Source: Data Structures - A Pseudocode Approach with C++)

# Selection Algorithms

**Algorithm** selectK(ref heap <array>, ref k <integer>, ref last <integer>)
Select the k-th largest element from a list.

**Pre:** `heap` is an array implementation of a heap
`k` is the ordinal of the element desired
`last` is reference parameter to last element

**Post:** k-th largest value returned

# Selection Algorithms

```
if k > last + 1 then
    return 0
end
i = 1
originalSize = last + 1
while i < k do
    temp = heap[0]
    deleteHeap(heap, last, dataOut)
    heap[last + 1] = temp
    i = i + 1
end
```

## Selection Algorithms

```
// Desired element is now at top of heap
holdOut = heap[0]
// Reconstruct heap
while last < originalSize do
    last = last + 1
    reheapUp(heap, last)
end
return holdOut
End selectK
```

## Priority Queues

The heap is an excellent structure to use for a priority queue.

### Example

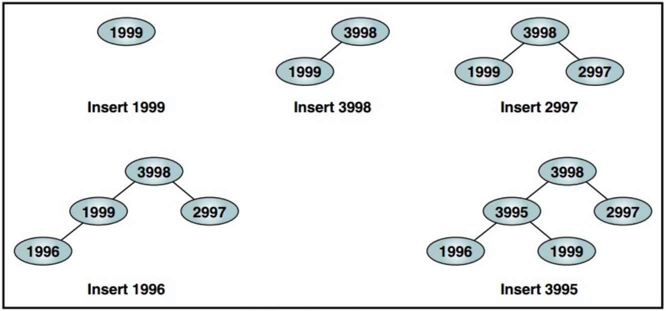
Assume that we have a priority queue with three priorities: high (3), medium (2), and low (1).
Of the first five customers who arrive, the second and the fifth are high-priority customers, the third is medium priority, and the first and the fourth are low priority.

| Arrival | Priority | Priority |
|---|---|---|
| 1 | low | 1999 (1 & (1000 - 1) |
| 2 | high | 3998 (3 & (1000 - 2) |
| 3 | medium | 2997 (2 & (1000 - 3) |
| 4 | low | 1996 (1 & (1000 - 4) |
| 5 | high | 3995 (3 & (1000 - 5) |

(Source: Data Structures - A Pseudocode Approach with C++)

# Priority Queues

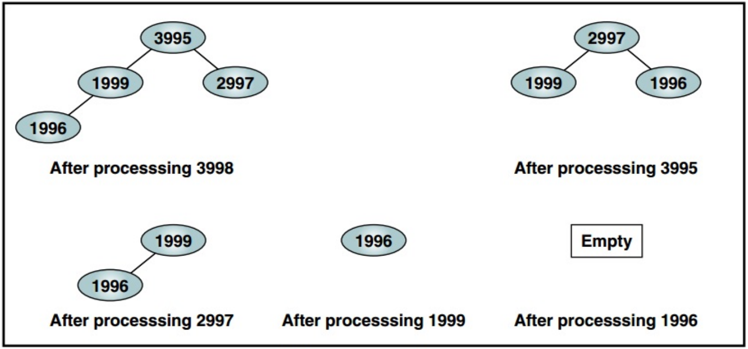
The customers are served according to their priority and within equal priorities, according to their arrival. Thus we see that customer 2 (3998) is served first, followed by customer 5 (3995), customer 3 (2997), customer 1 (1999), and customer 4 (1996).




(a) Insert customers

(Source: Data Structures - A Pseudocode Approach with C++)

# Priority Queues

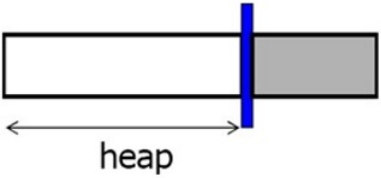



(b) Process customers

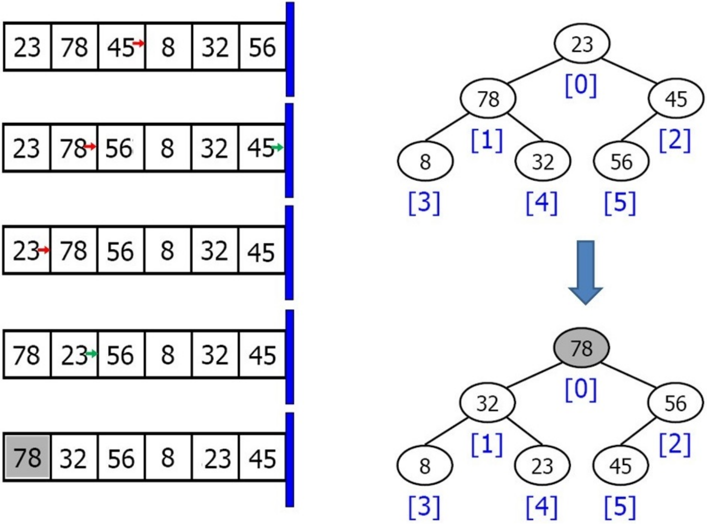
(Source: Data Structures - A Pseudocode Approach with C++)

## Heap Sort

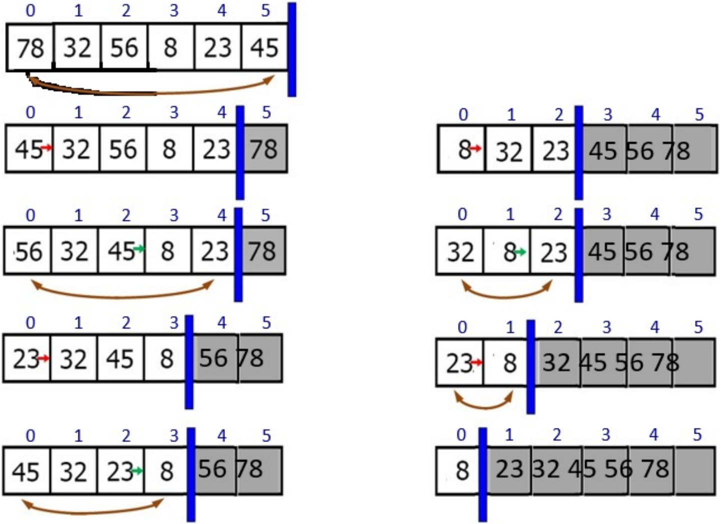
- The unsorted sublist is organized into a heap.
- In each pass, in the unsorted sublist, the largest element is selected and exchanged with the last element.
- The the heap is reheaped.



# Heap Sort


23 78 45 8 32 56
23 78 56 8 32 45
23 78 56 8 32 45
78 23 56 8 32 45
78 32 56 8 23 45
23
78
45
8
32
56
[0]
[1]
[2]
[3]
[4]
[5]
78
32
56
8
23
45
[0]
[1]
[2]
[3]
[4]
[5]

# Heap Sort


0 1 2 3 4 5
78 32 56 8 23 45
45 32 56 8 23 78
56 32 45 8 23 78
23 32 45 8 56 78
45 32 23 8 56 78
8 32 23 45 56 78
32 8 23 45 56 78
23 8 32 45 56 78
8 23 32 45 56 78

# Heap Sort

```
Algorithm HeapSort()
Sorts the contiguous list using heap sort.

position = count/2 - 1
while position >= 0 do
    ReheapDown(position, count - 1)
    position = position - 1
end
last = count - 1
while last > 0 do
    swap(0, last)
    last = last - 1
    ReheapDown(0, last - 1)
end
End HeapSort
```

# THANK YOU.