1. (a) Assume that the rectangle is aligned with y-axis and x-axis.

Set the (x_{min}, y_{min}) point of the rectangle, and the length (parallel to the y-axis) is l, width is w (parallel to the x-asis)

Therefore we are going to minimize lw.

$\because$ The rectangle contains all of the point

$\therefore$ if $\vec{y}^i = [y_1^i, y_2^i]^T$,

then $x_{min} \le y_1^i$ and $y_{min} \le y_2^i$, $\forall i \in \{1, \dots, k\}$

$x_{min} + w \ge y_1^i$ and $y_{min} + l \ge y_2^i$, $\forall i \in \{1, \dots, k\}$

Therefore the problem will be:

$$
\begin{aligned}
\text{minimize} \quad & lw \\
\text{subject to} \quad & x_{min} \le y_1^i, \quad \forall i \in \{1, \dots, k\} \\
& y_{min} \le y_2^i, \quad \forall i \in \{1, \dots, k\} \\
& y_1^i \le x_{min} + w, \quad \forall i \in \{1, \dots, k\} \\
& y_2^i \le y_{min} + l, \quad \forall i \in \{1, \dots, k\} \\
& \vec{y}^i = [y_1^i, y_2^i]^T \\
& \vec{y}^i \in R^2 \\
& x_{min}, y_{min}, y_1^i, y_2^i, l, w \in R
\end{aligned}
$$

This problem is Nonlinear optimization because the objective function is nonlinear.

(b) This will change the objective function lw to $l+w$

The problem will become:

$$
\begin{aligned}
\text{minimize} \quad & 2(l+w) \\
\text{subject to} \quad & x_{min} \le y_1^i, \quad \forall i \in \{1, \dots, k\} \\
& y_{min} \le y_2^i, \quad \forall i \in \{1, \dots, k\} \\
& y_1^i \le x_{min} + w, \quad \forall i \in \{1, \dots, k\} \\
& y_2^i \le y_{min} + l, \quad \forall i \in \{1, \dots, k\} \\
& \vec{y}^i = [y_1^i, y_2^i]^T \\
& \vec{y}^i \in R^2 \\
& x_{min}, y_{min}, y_1^i, y_2^i, l, w \in R
\end{aligned}
$$

let's take $x_{min}=(y_1^i)_{min}$ and $y_{min}=(y_2^i)_{min}$, $x_{min}+w=(y_1^i)_{max}$ and $y_{min}+l=(y_2^i)_{max}$

$\therefore w=(y_1^i)_{max}-(y_1^i)_{min}$ and $l=(y_2^i)_{max}-(y_2^i)_{min}$

Now prove this is the optimal solution:

if $x_{min}\neq(y_1^i)_{min}$, then x_{min} must $< y_1^i{}_{min}$,

$\because x_{min}+w'\geq y_1^i$ for all i, so $x_{min}+w'\geq(y_1^i)_{max}$

$\therefore x_{min}+w'-x_{min}>(y_1^i)_{max}-(y_1^i)_{min}$

$\therefore w'>(y_1^i)_{max}-(y_1^i)_{min}$

$\therefore w'>w$

$\therefore w'$ is not a better solution, $w=(y_1^i)_{max}-(y_1^i)_{min}$ is the optimal solution

In a similar way, we can prove that $l=(y_2^i)_{max}-(y_2^i)_{min}$ is the optimal solution

So for this problem, the optimal value is $2[(y_2^i)_{max}-(y_2^i)_{min}+(y_1^i)_{max}-(y_1^i)_{min}]$,

and the optimal solutions are $w=(y_1^i)_{max}-(y_1^i)_{min}$ and $l=(y_2^i)_{max}-(y_2^i)_{min}$

This solution will be the SAME one in problem (a) because we assume that the rectangle is aligned with the x-axis and y-axis. And the constrains of two problem are the same. Also, the w and l we found in (b) are the minimum of w and l respectively. They are the smallest parameter. Therefore, using the solution of w and l to compute $l\cdot w$ in problem (a) will also be the minimum of $l\cdot w$. So they have the same solution.

2. (a) Assume the company produces a product I, b product II, c product III

$\therefore$ the daily profit will be

$$(70-13)a+(62-8)b+(110-30)c=57a+54b+80c$$

the constrains will be
$$\frac{1}{3}a+\frac{1}{4}b+\frac{1}{2}c\le 150$$
$$\frac{1}{5}a+\frac{1}{4}b+\frac{1}{4}c\le 80$$

therefore, $\vec{c}^T=[-57,-54,-80]$ $\quad \vec{x}^T=[a,b,c]$

$$\vec{A}=\begin{bmatrix}\frac{1}{3} & \frac{1}{4} & \frac{1}{2}\\ \frac{1}{5} & \frac{1}{4} & \frac{1}{4}\end{bmatrix},\quad \vec{b}=\begin{bmatrix}150\\80\end{bmatrix}$$

So the problem will be

$$\begin{aligned}\text{minimize}\quad & \vec{c}^T\vec{x}\\ \text{s.t.}\quad & \vec{A}\vec{x}\le \vec{b}\\ & \vec{x}\geq 0\end{aligned}$$

$\because$ here is minimization, therefore variable in $\vec{c}$ should be negative

where $\vec{c}=\begin{bmatrix}-57\\-54\\-80\end{bmatrix}$, $\vec{x}=\begin{bmatrix}a\\b\\c\end{bmatrix}$, $\vec{A}=\begin{bmatrix}\frac{1}{3} & \frac{1}{4} & \frac{1}{2}\\ \frac{1}{5} & \frac{1}{4} & \frac{1}{4}\end{bmatrix}$, $\vec{b}=\begin{bmatrix}150\\80\end{bmatrix}$

(b) Assume schedule t hours of overtime assembly labor.

therefore the daily profit will be $57a+54b+80c-50t$

and one of the constrains will be $\frac{1}{3}a+\frac{1}{4}b+\frac{1}{2}c\le 150+t$

which is $\frac{1}{3}a+\frac{1}{4}b+\frac{1}{2}c-t\le 150$

So the problem becomes

$$\begin{aligned}\text{minimize}\quad & \vec{c}^T\vec{x}\\ \text{s.t.}\quad & \vec{A}\vec{x}\le \vec{b}\\ & \vec{x}\geq \vec{0}\end{aligned}$$

where $\vec{c}=\begin{bmatrix}-57\\-54\\-80\\50\end{bmatrix}$, $\vec{x}=\begin{bmatrix}a\\b\\c\\t\end{bmatrix}$, $\vec{A}=\begin{bmatrix}\frac{1}{3} & \frac{1}{4} & \frac{1}{2} & -1\\ \frac{1}{5} & \frac{1}{4} & \frac{1}{4} & 0\end{bmatrix}$, $\vec{b}=\begin{bmatrix}150\\80\end{bmatrix}$

(c) By using python, we get the result that $\vec{x}_{op}=[0,40,280]^T$,

produce 0 product I, 40 product II, and 280 product III everyday.

and the maximum daily profit is 24560 RMB

The code is attached.

3. Assume x_{ij} is the number of iron rods moved from building i to j.

$\therefore$ We got 6 variable. $x_{A1}, x_{A2}, x_{B1}, x_{B2}, x_{C1}, x_{C2}$

$\therefore \vec{x} = [x_{A1}, x_{A2}, x_{B1}, x_{B2}, x_{C1}, x_{C2}]^T$

Therefore we are going to minimize

$$(9x_{A1} + 6x_{A2} + 7x_{B1} + 4x_{B2} + 4x_{C1} + 6x_{C2}) \cdot 0.1$$

$\therefore \vec{c} = [0.9, 0.6, 0.7, 0.4, 0.4, 0.6]^T$

However, from the cost table, we can find that sometimes moving iron rods through a middle building will cost less than moving iron rods directly from one to another.

We found that when moving iron rods from A to 1, it cost less from A via C to 1 than directly from A to 1. Therefore, the cost from moving iron rods from A to 1 will be $0.1 \cdot (3+4) = 0.7$

So, $\vec{c} = [0.7, 0.6, 0.7, 0.4, 0.4, 0.6]^T$

Now consider the constrains:

The iron rods moved to building 1 and 2 need to meet the requirement. So $\sum_i x_{i1} \geq 8000$, $\sum_i x_{i2} \geq 6000 - 500 = 5500$, $i \in \{A, B, C\}$

The iron rods left in initial building should more than it needs. which is $\sum_j x_{Aj} \leq 7000$, $\sum_j x_{Bj} \leq 6000 - 2000 = 4000$, $\sum_j x_{Cj} \leq 3500$, $j \in \{1, 2\}$

So the list of inequality:

$x_{A1} + x_{B1} + x_{C1} \geq 8000$, which is $-x_{A1} - x_{B1} - x_{C1} \leq -8000$

$x_{A2} + x_{B2} + x_{C2} \geq 5500$, which is $-x_{A2} - x_{B2} - x_{C2} \leq -5500$

$x_{A1} + x_{A2} \leq 7000$, $x_{B1} + x_{B2} \leq 4000$, $x_{C1} + x_{C2} \leq 3500$

therefore we can let $\vec{A} = \begin{bmatrix} -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -8000 \\ -5500 \\ 7000 \\ 4000 \\ 3500 \end{bmatrix}$

Therefore, the linear optimization problem will be:

minimize $\vec{c}^T \vec{x}$

subject to $\vec{A}\vec{x} \leq \vec{b}$

$\vec{x} \geq \vec{0}$

where $\vec{A} = \begin{bmatrix} -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -8000 \\ -5500 \\ 7000 \\ 4000 \\ 3500 \end{bmatrix}$

$\vec{c} = [0.7, 0.6, 0.7, 0.4, 0.4, 0.6]^T$

$\vec{x} = [x_{A1}, x_{A2}, x_{B1}, x_{B2}, x_{C1}, x_{C2}]^T$

Using Pyton, we get the solution:

$\vec{x} = [4500, 1500, 0, 4000, 3500, 0]$

$A \to C \to 1$: 4500

$A \to 2$: 1500

$B \to 1$: 0

$B \to 2$: 4000

$C \to 1$: 3500

$C \to 2$: 0

the cost is 7050 RMB

4. (a) Assume that $x_{ij} = \begin{cases} 1, & \text{if the ith item is allocated to the jth person} \\ 0, & \text{if the ith item isn't allocated to the jth person, or } i=j \end{cases}$

$\because$ the dogs cannot be seperated, so we see two dogs as one item

w_i implies the value of the i-th item

therefore we are going to minimize $\sum_{i=1}^{12} x_{i1} w_i - \sum_{i=1}^{12} x_{i3} w_i$

the constrains will be $\sum_{j=1}^{3} x_{ij} = 1 \quad i \in \{1, \dots, 12\}$ This makes sure one item will only be allocated to one person

$x_{ij} \in \{0, 1\}$

$\sum x_{i1} w_i \geq \sum x_{i2} w_i. \quad i \in \{1, \dots, 12\}$ This makes sure that

$\sum x_{i2} w_i \geq \sum x_{i3} w_i, \quad i \in \{1, \dots, 12\}$ Cassandra > Daniela > Bela

therefore the problem will be:

$$\text{minimize} \quad \sum_{i=1}^{12} x_{i1} w_i - \sum_{i=1}^{12} x_{i3} w_i$$

$$\text{subject to} \quad \sum_{j=1}^{3} x_{ij} = 1 \quad i \in \{1, \dots, 12\}$$

$$x_{ij} \in \{0, 1\}, \quad i \in \{1, \dots, 12\}, \quad j \in \{1, 2, 3\}$$

$$\sum_i x_{i1} w_i \geq \sum x_{i2} w_i. \quad i \in \{1, \dots, 12\},$$

$$\sum_i x_{i2} w_i \geq \sum x_{i3} w_i, \quad i \in \{1, \dots, 12\}$$

where the item list is:

	ith	
Sketch by Mondrian	1	w_1 : 8
Bust of Alexander the Great	2	w_2 : 0.5
Yuan Dynasty Chinese Vase	3	w_3 : 3.5
911 Porsche	4	w_4 = 6
Diamonds	5, 6, 7	w_5 : 1.2 = w_6 = w_7
Louis XV Sofa	8	w_8 : 0.3
Jack Russell Race Dogs	9	w_9 = 0.3 × 2 = 0.6
Ancient Sculpture	10	w_{10} = 1
Sailing Boat	11	w_{11} = 2
Harley Davidson Motorbike	12	w_{12} = 1

(b) Using Python, we found that it's impossible to TOTALLY equally distributed, but there is an optimal solution:

Sketch by Mondrian	1×	8	Daniela
Bust of Alexander the Great	1×	0.5	Daniela
Yuan Dynasty Chinese Vase	1×	3.5	Cassandra
911 Porsche	1×	6	Bela
Diamonds	3×	1.2	Bela ×1, Cassandra ×2
Louis XV Sofa	1×	0.3	Daniela
Jack Russell Race Dogs	2×	0.3 (×2=0.6)	The will mentions that the dogs can not be separated. Bela
Ancient Sculpture	1×	1	Bela
Sailing Boat	1×	2	Cassandra
Harley Davidson Motorbike	1×	1	Cassandra

Therefore Cassandra gets value of 8.9 (10000 \$)

Daniela gets value of 8.8 (10000 \$)

Bela gets value of 8.8 (10000 \$)

the difference of value between Cassandra and Bela is 0.1 (10000 \$)

The code is attached

5. Define $f(i,j) = \begin{cases} (i,j), & \text{if room } i \text{ and room } j \text{ are connected} \\ (0,0), & \text{if room } i \text{ and room } j \text{ are not connected OR } i=j \end{cases}$

Define $x_{(i,j)} = \begin{cases} 1, & \text{if there is a guard between the } i\text{-th and } j\text{-th room} \\ 0, & \text{if there is no guard between the } i\text{-th and } j\text{-th room OR } (i,j)=(0,0) \end{cases}$

Therefore we are going to minimize $\sum x_{f(i,j)}$ $\quad i,j \in \{1,\dots,10\}$ because we want to minimize the total number of guards

Now we consider the constrain. All room must be supervised by at least one guard, so for one specific i, $\sum_{i=1}^{10} x_{f(i,j)} \geq 2$. We use 2 because if $x_{(i,j)}=1$, then $x_{(j,i)}=1$ too.

Therefore, the optimization problem can be formulated as:

$$\text{minimize} \quad \sum_{i,j} x_{f(i,j)}, \quad \forall i,j \in \{1,2,3,\dots,10\}$$

$$\text{subject to} \quad \sum_{j=1}^{10} x_{f(i,j)} \geq 2, \quad \forall i \in \{1,2,3,\dots,10\}$$

$$x_{(i,j)} \in \{0,1\}$$

$$f(i,j) = \begin{cases} (i,j), & \text{if room } i \text{ and room } j \text{ are connected} \\ (0,0), & \text{if room } i \text{ and room } j \text{ are not connected OR } i=j \end{cases}$$

room list:

1st	A	6th	F
2nd	B	7th	G
3rd	C	8th	H
4th	D	9th	I
5th	E	10th	J