Schramm-Löwner Evolution

Advanced Proability Exam

Professor Marco Barchiesi

Ruba Morsi

Università degli Studi di Trieste

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Introduction

- Several discrete random lattice models were predicted to have continuous conformally invariant limits.
- The first such result is the limit of simple random walks to the Brownian motion.
- Levy proved that the complex Brownian motion is Conformally invariant.
- Oded Schramm derived the Stochastic Lowner evolution when he was studying the scaling limit of some lattic models.
- The one-parameter family of random curves SLE_κ is related to Bronwian motion.

Motivation: Simple Random Walk Animation

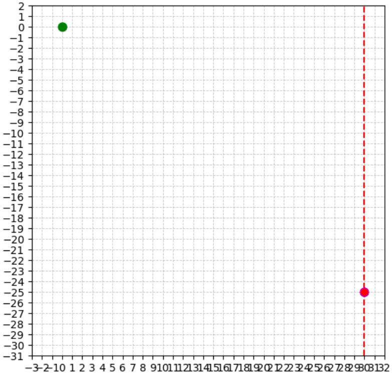


Figure: Simple Random walk

Donkser's Invariance Principle

Let $(S_n)_{n\geq 0}$ be a simple symmetric random walk in $\mathbb{R}^d$, where:

$$S_n = \sum_{i=1}^{n} X_i, \quad X_i \text{ are i.i.d. with } P(X_i = \pm e_j) = \frac{1}{2d}.$$

Define the rescaled process:

$$W_n(t) = \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}, \quad t \in [0,1].$$

Then, as $n \to \infty$,

$$W_n \Rightarrow B \text{ in } C([0,1], \mathbb{R}^d),$$

where B is standard d-dimensional Brownian motion.

Motivation: Loop-Erased Random Walk Animation

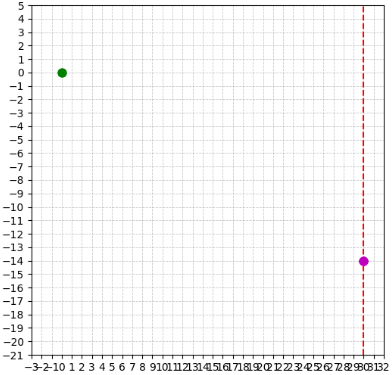


Figure: Loop Earsed Random walk

Motivation Percolation interface

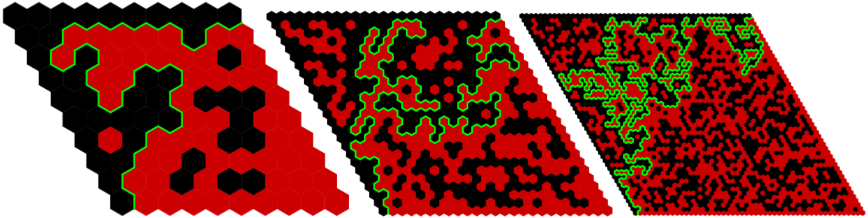


Figure: Percolation interface. In a domain of a hexagonal lattice grid, color each grid black and red with equal probabilities and a boundary condition on the side of the domain; the upper and left sides are colored black and the remaining two are colored in red. Trace the path from the corner so that a black tile is on its left and a red tile is on its right side.

Question: What is the large scale behavior of the interfaces between the red and the black sites?

Motivation The Uniform Spanning Tree

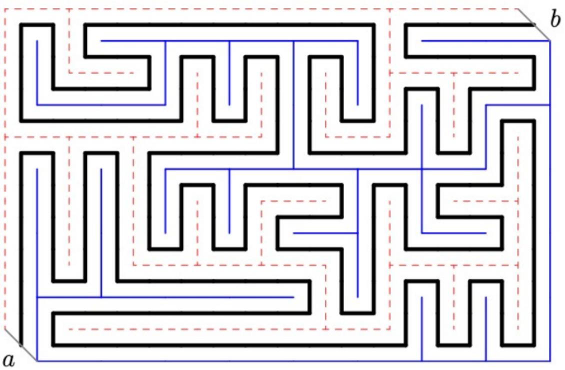


Figure: Percolation interface. Given a grid, choose a spanning tree at random, and define the curve that goes around the tree (the black curve). What happens to this curve as the grid spacing $\to 0$?

Scaling Limits of Lattice Models

- SLE_2 is the scaling limit of the **loop-erased random walk**. *[Oded Schramm, 1999]*
- And SLE_8 is the scaling limit of the **uniform spanning tree**. *[Oded Schramm, 1999]*
- SLE_6 is the scaling limit of **the critical percolation model**. *[Stanislav Smiranov, 2001]*

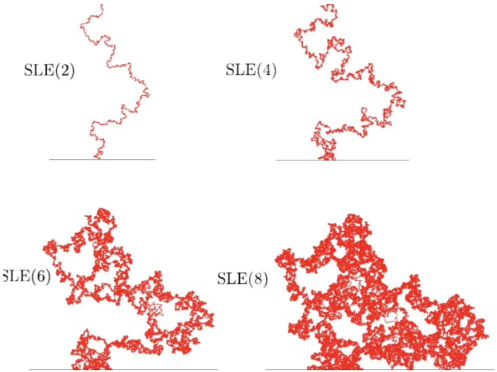
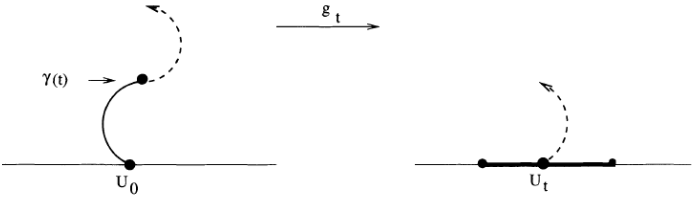


Figure: A simulation of SLE_κ curves for $= 2, 4, 6, 8$.

Löwner's Differential Equation



Löwner's evolution: A family of conformal maps from $\mathbb{H} \setminus \gamma([0,t])$ onto $\mathbb{H}$ such that $\gamma : [0,t] \to \mathbb{C}$ is a simple curve with $\gamma(0) \in \mathbb{R}$ and $\gamma(0,\infty) \in \mathbb{H}$.

- By Riemann's mapping thorem, the conformal maps exist for each $\gamma([0,t])$ and they have the following expansion $g_t(z) = b_{-1}z + b_0 + \sum_1^\infty \frac{b_i}{z^i}$.
- Using Schwarz's reflection princple, we can show that $b_0 = 0$ around ∞ and by normalising the series we get unique conformal maps of the form $g_t(z) = z + \frac{b(t)}{z} + O(\frac{1}{|z|^2})$

Löwner's Differential Equation (Cont'd)

Proposition Suppose γ_t is a simple curve as afore-described such that $b(t)$ is C^1 and $b(t) \to \infty$ as $t \to \infty$. Then for $z \in \mathbb{H}$, $g_t(z)$ is the solution of the initial value problem

$$\dot{g}_t(z) = \frac{b(t)}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{1}$$

where $U_t = g_t(\gamma(t))$.

Remarks:

- The real function U_t is unique for every γ_t.
- The function $b(t)$ is actually a quantity called the half-plane capacity of the curve γ_t. We can show that it is continuous in t and strictly increasing and thus can be parameterized. b(t) is conventionally parameterized as $2t$.

From Löwner's Equation Towards Scaling Limits.

- The Löwner equation (1923) was considered by Löwner in order to prove the Bieberbach conjecture(1916).
- Löwner was able to prove a special case and later in 1985 De Branges was able to prove it using the Löwner equation.
- Physicists have predicted that the lattice models have scaling limits and that such limits are conformally invariant.
- In 1999 Oded Schramm was able to prove the scalling limits of the loop-erased random walk and the uniform spanning tree models by considering the driving functions in Löwner's equations to be a scaled Brownian motion.

Deriving the Schramm Löwner Evolution

- Schramm proposed setting the driving function to **Brownian motion**:

$$U(t) = \sqrt{\kappa} B_t$$

  where B_t is standard Brownian motion.
- Along with Rhode, Schramm proved the existence of curves associated with $U_t = \sqrt{\kappa} B_t$.
- Assuming the existence of the limit of the LERW, Schramm showed that the limit has two key features: **the domain Markov property** and **Conformal invariance**.

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Definition (Conformal Markov Property)

Suppose that (A_t) is a random family of compact hulls, encoded with the Lowner driving function U. (A_t) satisfies the *Markov property* if the following is true. For each $t \geq 0$, let $\mathcal{F}_t = \sigma(U_s : s \leq t)$. Then:

(i) The conditional law of $(g_t(A_{t+s}) - U_t)_{s \geq 0}$ given $\mathcal{F}_t$ is equal to that of $(A_s)_{s \geq 0}$. (*Markov property*)

(ii) For each $r > 0$, $(rA_t/r^2) \stackrel{d}{=} (A_t)$.

Characterization of the Löwner Driving Function

From the Loewner equation, the growth process is fully encoded by its driving function:

$$\hat{U}_t = U_{s+t} - U_s.$$

Key implications:

- The conformal Markov property implies that $\{U_t\}_{t\geq 0}$ is a **stationary process with independent increments**.
- Since U_t is pathwise continuous, the only possibility is a **drifted Brownian motion**:

$$U_t = \sqrt{\kappa} B_t + at, \quad \text{for some } \kappa > 0, a \in \mathbb{R}.$$

- Imposing **scaling invariance** leads to $a = 0$, so:

$$U_t = \sqrt{\kappa} B_t.$$

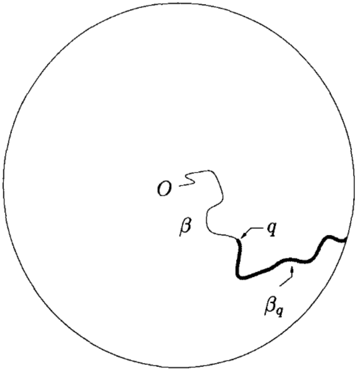

Figure: Domain Markov Property

Phases of the Chordal Schram Lowner Evolution

Theorem. Let $\gamma(t)$ be the random curve generated by the chordal Schramm-Loewner Evolution (SLE_κ) in $\mathbb{H}$ with driving function $\sqrt{\kappa}B_t$, where B_t is standard Brownian motion. Then:

- $\mathbf{0} < \kappa \leq \mathbf{4}$: $\gamma(t)$ is a **simple curve** (non-self-intersecting).
- $\mathbf{4} < \kappa < \mathbf{8}$: $\gamma(t)$ has **self-intersections** but is not space-filling.
- $\kappa \geq \mathbf{8}$: $\gamma(t)$ is **space-filling**.

To prove this, we need to introduce the **Bessel Stochastic Equation**.

Stochastic Differential Equations (SDEs)

Definition. A process X_t satisfies the SDE:

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

where B_t is standard Brownian motion. If b, σ are Lipschitz, then the SDE has a unique solution.

Square Bessel Process: BESQ_d

Definition. Let $B_t^1, \ldots, B_t^d$ be independent Brownian motions, and define:

$$Z_t = (B_t^1)^2 + \cdots + (B_t^d)^2.$$

Applying Itô's formula:

$$Z_t = (B_t^1)^2 + \cdots + (B_t^d)^2 = Z_0 + 2\int_0^t B_s^1 \, dB_s^1 + \cdots + 2\int_0^t B_s^d \, dB_s^d + t.$$

Define:

$$Y_t = \int_0^t \frac{B_s^1}{Z_s^{1/2}} dB_s^1 + \cdots + \int_0^t \frac{B_s^d}{Z_s^{1/2}} dB_s^d.$$

We can show that the **quadratic variation** of Y_t is t. By Levy's Characterisation Theorem, Y_t is a brownian motion $\bar{B}_t$.

$$dZ_t = 2Z_t^{1/2} d\bar{B}_t + d.dt.$$

This defines the **square Bessel process** of dimension d, denoted $Z_t \sim \mathrm{BESQ}_d$.

Bessel Process: BES_d

Definition. Define $U_t = Z_t^{1/2}$. Then:

$$dU_t = \left(\frac{d-1}{2}\right)\frac{1}{U_t}dt + dB_t.$$

This is the **Bessel process** of dimension d, denoted $U_t \sim \mathrm{BES}_d$.

Proposition: Behavior of $\mathrm{BES_d}$ at Zero

We will later show the connection between the Bessel process and the Schramm Lowner Evolution. The following proposition characterizes the behavior of the Bessel equation given the dimension d. This is equivalent to the SLE_κ phases.

Proposition Suppose that $U_t \sim \mathrm{BES}_d$, then:

(i) If $d < 2$, then U_t hits 0 almost surely.

(ii) If $d \geq 2$, then U_t does not hit 0 almost surely.

Proof of the Proposition

Consider the process U_t^{2-d}. By Itô's formula, we obtain:

$$U_t^{2-d} = U_0^{2-d} + \int_0^t (2-d) U_s^{1-d} dU_s + \frac{1}{2} \int_0^t (2-d)(1-d) U_s^{-d} d[U]_s$$

$$= \mathrm{U}_0^{2-d} + \int_0^t (2-d) U_s^{1-d} d\tilde{B}_s + \int_0^t \frac{(d-2)(d-1)}{2U_s} U_s^{1-d} ds + \frac{1}{2} \int_0^t (2-d)(1-d) U_s^{-d} ds$$

$$= U_0^{2-d} + \int_0^t (2-d) U_s^{1-d} d\tilde{B}_s.$$

, this simplifies to:

$$U_t^{2-d} = U_0^{2-d} + \int_0^t (2-d) U_s^{1-d} d\tilde{B}_s.$$

Thus, U_t^{2-d} is a continuous local martingale.

Proof of the Proposition (Cont'd)

Define the stopping times $\tau_a = \inf\{t \geq 0 : U_t = a\}$. For $0 \leq a < U_0 < b < \infty$, the process $U^{2-d}_{t\wedge\tau_a\wedge\tau_b}$ is a bounded, continuous martingale.
By the Optional Sampling Theorem:

$$U_0^{2-d} = \mathbb{E}[U^{2-d}_{\tau_a\wedge\tau_b}]$$

which implies:

$$U_0^{2-d} = a^{2-d}\mathbb{P}(\tau_a < \tau_b) + b^{2-d}\mathbb{P}(\tau_b < \tau_a).$$

If $d < 2$ we can take $a = 0$, then

$$\mathbb{P}(\tau_b < \tau_0) = (\frac{U_0}{b})^{2-d}$$

Taking $b \to \infty$ we get the first assertion of the proposition.

Chordal Lowner Equation and Cut-off Time

Given chordal Lowner equation $\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$

For each $x \in \mathbb{R}$, define:
$V_t^x = g_t(x) - U_t, \quad \tau_x = \inf\{t \geq 0 : V_t^x = 0\}.$
Note that τ_x is the first time x is cut off from ∞ by γ.

Stochastic Equation for V_t^x:

$$dV_t^x = \frac{2}{V_t^x} dt - \sqrt{\kappa} dB_t$$

Setting $\tilde{B}_t = -B_t$, we obtain: $d\left(\frac{V_t^x}{\sqrt{\kappa}}\right) = \frac{2}{\kappa} \frac{1}{V_t^x/\sqrt{\kappa}} dt + d\tilde{B}_t.$
Thus, $V_t^x/\sqrt{\kappa}$ is $\sim \mathrm{BES}_d$ with:

$$\frac{d-1}{2} = \frac{2}{\kappa} \quad \Rightarrow \quad d = 1 + \frac{4}{\kappa}.$$

- If $d \geq 2$, then $\kappa \leq 4$ and $\tau_x = \infty$ (no cut-off).
- If $d < 2$, then $\kappa > 4$ and $\tau_x < \infty$ (cut-off occurs).

Phase Transition in SLE_κ Curves

Proposition

SLE_κ corresponds to a simple curve for $\kappa \leq 4$. It is self-intersecting for $\kappa > 4$.

- Previous discussion implies that SLE_κ intersects the real line if and only if $\kappa > 4$.
- Fix $t > 0$. Then $s \mapsto g_t(\gamma(s+t)) - U_t$ is an SLE_κ curve.
- For each $t \geq 0$, intersection points between $\gamma|_{[t,\infty)}$ and $\gamma|_{[0,t]}$ correspond to points where the curve $s \mapsto g_t(\gamma(s+t)) - U_t$ hits the boundary
- Therefore, self-intersections occur precisely when $\kappa > 4$

Bibliography

- Conformally Invariant Process, Gregory F. Lawler.
- Schramm Loewner Evolutions, John Miller.
- Scaling Limits of Erased Loop Random Walks and Uniform Spanning Trees, Oded Schramm.