Solitons in Mathematics and Physics

Vladimir Zakharov

November 2, 2024

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Chapter 1

Hamiltonian Systems

1.1 Hamiltonian Systems

Let $\Omega \subset \mathbb{R}^{2N}$ be a domain in the Euclidean space of dimension $2N$. One can introduce coordinates in Ω - generalized momenta $p_1, ..., p_N$ (therefore simple momenta) and generalized coordinates $q_1, ..., q_N$ (therefore simple coordinates).

Let $H = H(p_1, ..., p_N, q_1, ..., q_N, t)$ be a real function defined on Ω. One can make p_i, q_i functions depending on time and introduce the following dynamical system.

$$\dot{q_n} = \frac{\partial H}{\partial p_n} \qquad \dot{p_n} = -\frac{\partial H}{\partial q_n} \tag{1.1}$$

for $i = 1, ..., N$, where $\dot{f} = \frac{d}{dt} f$ is the time derivative.

Let's calculate the time derivative of H.

$$\frac{dH}{dt} = \sum_{n=1}^{N} \Big(\frac{\partial H}{\partial p_n} \dot{p_n} + \frac{\partial H}{\partial q_n} \dot{q_n} \Big) + \frac{\partial H}{\partial t} \tag{1.2}$$

In virtue of (1.1), the expression in brackets is cancelled and

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \tag{1.3}$$

The system (1.1) is called a *Hamiltonian system*, and the function H is the *Hamiltonian*. If $\frac{\partial H}{\partial t} = 0$ (i.e. no explicit time dependence), the Hamiltonian is the *motion constant*.

Suppose $I = I(p_i, q_i)$ is some function on Ω. In virtue of (1.1), $I = I(t)$ is a function on time. If $\frac{dI}{dt} = 0$, I is the constant of motion. What conditions satisfy this?

$$\frac{dI}{dt} = \sum_{n=1}^{N} \Big(\frac{\partial I}{\partial q_n} \dot{q_n} + \frac{\partial I}{\partial p_n} \dot{p_n} \Big) \tag{1.4}$$

$$= \sum_{n=1}^{N} \Big(\frac{\partial I}{\partial q_n} \frac{\partial H}{\partial p_n} - \frac{\partial I}{\partial p_n} \frac{\partial H}{\partial q_n} \Big) = 0 \tag{1.5}$$

Let $A = A(p_1, ..., p_N, q_1, ..., q_N)$ and $B = B(p_1, ..., p_N, q_1, ..., q_N)$ be two functions on Ω. The third function, C, the *Poisson bracket* of A, B denoted by

$$C = \{A, B\}$$

is defined as follows

$$\{A, B\} = \sum_{n=1}^{N} \Big(\frac{\partial A}{\partial q_n} \frac{\partial B}{\partial p_n} - \frac{\partial A}{\partial p_n} \frac{\partial B}{\partial q_n} \Big) \tag{1.6}$$

Basic properties of the Poisson Bracket include

(1) *Skew-symmetry*

$$\{A, B\} = -\{B, A\} \tag{1.7}$$

(2) obeying the *Jacobi identity*

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0 \tag{1.8}$$

Proof: Exercise. □

According to (1.5)

$$\frac{dI}{dt} = \{I, H\} \tag{1.9}$$

Hence I is an *integral of motion* if and only if

$$\{I, H\} = 0 \tag{1.10}$$

Theorem. Let I_1, I_2 be integrals of motion. Then $I_3 = \{I_1, I_2\}$ is also an integral of motion.

In virtue of the Jacobi identity (1.8)

$$\{I_3, H\} = \{\{I_1, I_2\}, H\} = -\{H, \{I_1, I_2\}\}$$
$$\implies \{H, \{I_1, I_2\}\} + \{I_1, \{I_2, H\}\} + \{I_2, \{H, I_1\}\} = 0$$

□

Equation (1.10) means that the motion of integrals of any Hamiltonian system compose a *Lie Algebra.*

Example: Consider motion of a particle in a radially symmetric 3D force field.
The motion conserves three components of the angular moments.

$$M_1 = p_2x_3 - p_3x_2 \qquad M_2 = p_3x_1 - p_1x_3 \qquad M_3 = p_1x_2 - p_2x_3$$

The Poisson brackets are

$$\{M_1, M_2\} = M_3 \qquad \{M_3, M_1\} = M_2 \qquad \{M_2, M_3\} = M_1$$

This is the algebra of vector fields in $\mathbb{R}^3$ with respect to the operation of taking the vector product.

If $\{I_1, I_2\} = 0$, then the integrals I_1, I_2 are in *involution* of each other or, simply, *commute*. In a general case, the Lie algebra of the Hamiltonian system, L, contains some commutative sub-algebra $\tilde{L} \subset L$. All commuting integrals, including the Hamiltonian, belong to $\tilde{L}$.

Algebra $\tilde{L}$ is a linear space of dimension m. If $m = N$, the system is called *integrable*, and the following statement holds.

Theorem of integrability by Liouville and Arnold. An integrable system can be *integrated* explicitly. □

We explain the exact meaning of this term later on.

1.2 Examples of Integrable Systems

The following are examples of integrable systems.

Example: Suppose

$$H = \sum_{n=1}^{N} H_n, \qquad \text{where } H_n = H_n(p_n, q_n)$$

Then all particular Hamiltonians are motion constants which obviously commute.

Example: In particular, if

$$H_n = \frac{1}{2}\Big(\frac{p_n^2}{m_n} + k_n x_n^2\Big)$$

This is the system of linear oscillators with frequencies

$$w_n^2 = \frac{k_n}{m_n}$$

Example: Another example is mentioned above: motion of a particle in a radially-symmetric field. In this case, $\tilde{L}$ is contains three elements

$$\tilde{L} = \{H, M_i, M^2\} \qquad M^2 = M_1^2 + M_2^2 + M_3^2$$

where i is any index

Chapter 2

Variational Derivatives

2.1 Computing Variations

The technique of variational derivatives will be used throughout this book systematically. Let $f(x)$ be some smooth real function defined on the interval $a < x < b$. Let $H[f]$ be a linear functional defined as follows.

$$H[f] = \int_a^b g(x)f(x)dx \tag{2.1}$$

Where $g(x)$ is another function. Notice that for smooth f, g can be a generalized function. For instance, if $g(x) = A\delta(x - \lambda)$, then

$$H[f] = Af(\lambda) \tag{2.2}$$

By definition, $g(x)$ is a variational derivative of the functional H.
We use the notation:

$$\frac{\delta H}{\delta f} = g(x)$$

Let's consider a more general, nonlinear functional. Suppose H is defined as follows

$$H[f] = \int_a^b g(x)F(f)(x)dx \tag{2.3}$$

One can add a small perturbation to f, call it δf, which perturbs H to get H_ϵ

$$f \to f + \epsilon\delta f \qquad\qquad \Longrightarrow H \to H_\epsilon$$

If we calculate $H' = \lim_{\epsilon\to 0} \frac{H_\epsilon - H}{\epsilon}$, we get

$$H'[f] = \int_a^b g(x)F'(f(x))\delta f(x)dx \tag{2.4}$$

Then H' is a linear functional with respect to variation δf and hence

$$\frac{\delta H}{\delta f} = g(x)F'(f(x)) \tag{2.5}$$

A posterior, we assume that $\delta f\Big|_a = \delta f\Big|_b = 0$. i.e. The variations are 0 on the boundaries to avoid added contributions from using integration by parts.

Now suppose

$$H = \int_a^b F(x, f, f')dx$$

Then adding a perturbation to f means adding a perturbation to f', so to get

$$f \to \epsilon\delta f \qquad\qquad f' \to \epsilon\delta f'$$

To get rid of $\delta f'$ term, perform integration by parts and as a result,

$$\frac{\delta H}{\delta f} = \frac{\partial F}{\partial f} - \frac{\partial}{\partial x}\frac{\partial F}{\partial f'} \tag{2.6}$$

In the same way, if H is defined as

$$H = \int_a^b F(x, f, f', f'')dx \tag{2.7}$$

Then its variational derivative becomes

$$\frac{\delta H}{\delta f} = \frac{\partial F}{\partial f} - \frac{\partial}{\partial x}\frac{\partial F}{\partial f'} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial f''} \tag{2.8}$$

2.2 Euler equations

Suppose that I, a linear functional, depends on the two functions $\rho = \rho(x,t)$ and $\varphi = \varphi(x,t)$ and is defined as

$$I = \frac{1}{2}\int_0^L \rho\Big(\frac{\partial\varphi}{\partial x}\Big)^2 dx + \int_0^L \epsilon(\rho)dx \tag{2.9}$$

Then the variation with respect to each function is given by

$$\frac{\delta I}{\delta\rho} = \frac{1}{2}\Big(\frac{\partial\varphi}{\partial x}\Big)^2 + \epsilon'(\rho) \tag{2.10}$$

$$\frac{\delta I}{\delta\varphi} = -\frac{\partial}{\partial x}\rho\frac{\partial\varphi}{\partial x} \tag{2.11}$$

Now we consider the following Hamiltonian system

$$\frac{\partial\rho}{\partial t} = \frac{\delta H}{\delta\varphi} \qquad\qquad \frac{\partial\varphi}{\partial t} = -\frac{\delta H}{\delta\rho} \tag{2.12}$$

Substituting the variations in, we get

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \tag{2.13}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \epsilon''(\rho)\frac{\partial\rho}{\partial x} = 0 \tag{2.14}$$

Multiplying by ρ, equation (2.14) can be rewritten as follows

$$\rho\Big(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\Big) + \frac{\partial P}{\partial x} = 0 \tag{2.15}$$

where $P = P(\rho)$ and $\frac{\partial P}{\partial\rho} = \rho\epsilon''(\rho)$

Equations (2.13) and (2.14) make up the *Euler equations* for *compressible fluid* in one-dimensional geometry. Where ρ is density of the fluid, v is its horizontal velocity, and P is its pressure. If we assume that $P = P(\rho)$ (i.e. P a function of ρ only), then this fluid is called *barotropic*.

Chapter 3

Simple waves in Hydrodynamics

Let us consider the system of Euler equations for the compressible fluid

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\Big(\rho v\Big) = 0 \tag{3.1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \lambda(\rho)\frac{\partial \rho}{\partial x} = 0 \tag{3.2}$$

where $\lambda(\rho) = \frac{1}{\rho}\frac{\partial P}{\partial \rho}$

We assume that the fluid is *barotropic* so to presume that the pressure depends only on density, $P = P(\rho)$.
Note that

$$\frac{\partial P}{\partial \rho} = c^2(\rho) \tag{3.3}$$

where c is the velocity of sound and is a function of density.

We will study a special class of solutions of (3.1, 3.2) where the velocity is defined by the density

$$v = v(\rho) \tag{3.4}$$

According to (3.1), density satisfies the equation

$$\Big(\frac{\partial}{\partial t} + S\frac{\partial}{\partial x}\Big)\rho = 0 \tag{3.5}$$

where

$$S = \frac{\partial}{\partial \rho}(\rho v(\rho)) \tag{3.6}$$

and $v(\rho)$ is still unknown. To find it, we should study the second Euler equation (3.2), which takes the form of

$$\frac{\partial v}{\partial \rho}\Big(\frac{\partial \rho}{\partial t} + v(\rho)\frac{\partial \rho}{\partial t}\Big) + \lambda(\rho)\frac{\partial \rho}{\partial x} = 0 \tag{3.7}$$

Equations (3.5) and (3.7) must coincide. Hence

$$\frac{\partial}{\partial \rho}(v\rho) = v + \frac{\lambda}{\frac{\partial v}{\partial \rho}} \tag{3.8}$$

or

$$\Big(\frac{\partial v}{\partial \rho}\Big)^2 = \frac{1}{\rho}\lambda(\rho) = \frac{c^2}{\rho^2} \tag{3.9}$$

$$\implies \frac{\partial v}{\partial \rho} = \pm\frac{c}{\rho} \tag{3.10}$$

$$\implies v(\rho) = \pm\int_{\rho_0}^{\rho} \frac{c}{\tilde{\rho}} d\tilde{\rho} \tag{3.11}$$

where ρ_0 is some density.

So that

$$S_{\pm} = v(\rho) \pm c(\rho) \tag{3.12}$$

For the special case of a polytropic gas:

$$P = \frac{1}{\gamma} c_0^2 \rho_0 \Big(\frac{\rho}{\rho_0}\Big)^{\gamma} \tag{3.13}$$

and

$$c^2 = c_0^2 \Big(\frac{\rho}{\rho_0}\Big)^{\gamma-1} \tag{3.14}$$

$$\implies c = c_0 \Big(\frac{\rho}{\rho_0}\Big)^{\frac{\gamma-1}{2}} \tag{3.15}$$

where c_0 is the velocity of sound if $\rho = \rho_0$. Thus,

$$S_+ = \frac{\gamma+1}{\gamma-1} c_0 \Big(\frac{\rho}{\rho_0}\Big)^{\frac{\gamma-1}{2}} - \frac{2}{\gamma-1} c_0 \tag{3.16}$$

and

$$v(\rho) = \frac{2}{\gamma-1} c_0 \Big[\Big(\frac{\rho}{\rho_0}\Big)^{\frac{\gamma-1}{2}} - 1\Big] \tag{3.17}$$

Suppose that the density variation is small

$$\rho = \rho_0 + \delta\rho \tag{3.18}$$

$$S_+ = S_0 + S_1 \delta\rho \tag{3.19}$$

where $S_0 = c_0$ and $S_1 = \frac{\gamma+1}{2}\frac{c_0}{\rho_0}$.
For small deviations from the mean density ρ_0, equation (3.5) reads

$$\frac{\partial}{\partial t}(\delta\rho) + (S_0 + S_1\delta\rho)\frac{\partial}{\partial x}\delta\rho = 0 \tag{3.20}$$

The above is known as the *Hopf equation*. The coefficient S_1 changes sign if $\gamma < -1$. Note that

$$S_- = -c_0 + \frac{3-\gamma}{2}\frac{c_0}{\rho_0}\delta\rho \tag{3.21}$$

One can obtain the same result by another way. Let us try to find a function of two variables

$$A = A(\rho, v)$$

obeying the equation

$$\frac{\partial A}{\partial t} + S\frac{\partial A}{\partial x} = 0 \tag{3.22}$$

Using multi-variate chain rule, equation (3.22) can be rewritten as follows:

$$\frac{\partial A}{\partial \rho}\frac{\partial \rho}{\partial t} + \frac{\partial A}{\partial v}\frac{\partial v}{\partial t} + S\Big(\frac{\partial A}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial A}{\partial v}\frac{\partial v}{\partial x}\Big) = 0 \tag{3.23}$$

Substituting the time derivatives from (3.1, 3.2) into the above equation, we get

$$\Big(v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x}\Big)\frac{\partial A}{\partial \rho} + \Big(v\frac{\partial v}{\partial x} + \lambda\frac{\partial \rho}{\partial x}\Big)\frac{\partial A}{\partial v} + S\Big(\frac{\partial A}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial A}{\partial v}\frac{\partial v}{\partial x}\Big) = 0 \tag{3.24}$$

Rewrite the above equations by regrouping

$$\Big(\lambda\frac{\partial A}{\partial v} - (S-v)\frac{\partial A}{\partial \rho}\Big)\frac{\partial \rho}{\partial x} + \Big(\rho\frac{\partial A}{\partial \rho} - (S-v)\frac{\partial A}{\partial v}\Big)\frac{\partial v}{\partial x} = 0 \tag{3.25}$$

We require the coefficients of $\frac{\partial \rho}{\partial x}$ and $\frac{\partial v}{\partial x}$ to vanish, yielding the following system of equations

$$\lambda\frac{\partial A}{\partial v} = (S-v)\frac{\partial A}{\partial \rho} \tag{3.26}$$

$$\rho\frac{\partial A}{\partial \rho} = (S-v)\frac{\partial A}{\partial v} \tag{3.27}$$

We can rewrite the above system as follows

$$\begin{bmatrix} (S-v) & -\lambda \\ -\rho & (S-v) \end{bmatrix}\begin{bmatrix} A_\rho \\ A_v \end{bmatrix} = 0 \tag{3.28}$$

Compatibility conditions for the above system requires that $(S-v)^2 = \lambda\rho = c^2$ for a solution to exist. There are two solutions which are given by

$$S_\pm = v \pm c \tag{3.29}$$

$$\implies A_\pm = v \pm f(\rho) \tag{3.30}$$

with

$$f(\rho) = \int_{\rho_0}^{\rho} \frac{c}{\tilde{\rho}} d\tilde{\rho}$$

so that

$$A_\pm = v \pm \int_{\rho_0}^{\rho} \frac{c}{\tilde{\rho}} d\tilde{\rho} \tag{3.31}$$

Thus we have the following equations

$$\frac{\partial A_{\pm}}{\partial t} + S_{\pm}\frac{\partial A_{\pm}}{\partial x} = 0 \tag{3.32}$$

Equations (3.32) present another form of the initial system (3.1, 3.2), where functions $A_{\pm}$ are called *Riemann invariants.*

Suppose that $A_{-} = 0$, then by (3.31)

$$v = \int_{\rho_0}^{\rho} \frac{c}{\tilde{\rho}} d\tilde{\rho} \tag{3.33}$$

in accordance with (3.11). This solution is called a *simple wave.*

Example: Suppose that $\rho_0 = \rho^3$ and $c = \rho$ in (**??**), then

$$A_{\pm} = v \pm \rho \tag{3.34}$$

and

$$S_{\pm} = v \pm \rho \tag{3.35}$$

The systems splits into two independent systems

$$\frac{\partial A_{\pm}}{\partial t} + A_{\pm}\frac{\partial A_{\pm}}{\partial x} = 0 \tag{3.36}$$

This is the *superintegrable* case!

Chapter 4

General Solution of the Hydrodynamic Equation

In the last lecture we considered the polytropic gas in which pressure is the power function of density

$$P = \frac{1}{\gamma} c_0^2 \rho \Big(\frac{\rho}{\rho_0} \Big)^{\gamma} \tag{4.1}$$

where c_0 is the velocity of sound for $\rho = \rho_0$.

We found that the gas-dynamic equations has a special solution known as the *simple wave*. In this solution, the density satisfies the following equation

$$\frac{\partial \rho}{\partial t} + S(\rho) \frac{\partial \rho}{\partial x} = 0 \tag{4.2}$$

where

$$S(\rho) = \frac{\gamma + 1}{\gamma - 1} c_0 \Big(\frac{\rho}{\rho_0} \Big)^{\frac{\gamma - 1}{2}} - \frac{2}{\gamma - 1} c_0 \tag{4.3}$$

Note that $S(\rho_0) = c_0$.

Now suppose that $\rho = (1 + u)\rho_0$, with $u << 1$, a slight perturbation, then

$$S = c_0 \Big(1 + \frac{\gamma + 1}{2} u + ... \Big) \tag{4.4}$$

where the dimensionless quantity u satisfies the equation

$$\frac{\partial u}{\partial t} + c_0 \Big(1 + \frac{\gamma + 1}{2} u \Big) \frac{\partial u}{\partial x} = 0 \tag{4.5}$$

One can go to the frame moving with sound velocity c_0 and introduce the slow time variable, τ, such that

$$\frac{\partial u}{\partial t} + c_0 \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \tau} \tag{4.6}$$

Then

$$\frac{\partial u}{\partial \tau} + c_0 \frac{\gamma + 1}{2} u \frac{\partial u}{\partial x} = 0 \tag{4.7}$$

Let's denote

$$w = \frac{\gamma+1}{2} c_0 u$$

Notice that w has dimension of velocity and satisfies the equation

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = 0 \tag{4.8}$$

This equation is known as the *Hopf equation.* It is subject to the initial condition

$$w\Big|_{\tau=0} = F(x) \tag{4.9}$$

A solution of this initial value problem can be found in the implicit form as follows

$$w(x,\tau) = F(x - \tau w(x,\tau)) \tag{4.10}$$

Indeed

$$w_x = (1 - w_x\tau)F' \tag{4.11}$$

$$\implies w_x = \frac{F'}{1+\tau F'} \tag{4.12}$$

And

$$w_\tau = (x - w - \tau w_\tau)F' \tag{4.13}$$

$$\implies w_\tau = -\frac{wF'}{1+\tau F'} \tag{4.14}$$

Thus,

$$w_\tau + ww_x = 0 \tag{4.15}$$

Example: Suppose

$$F(x) = \frac{1}{1+x^2}$$

Then $w(x,\tau)$ is a solution of the cubic equation

$$(1 + (x - w\tau)^2)w = 1 \tag{4.16}$$

Then

$$F'(x) = -\frac{2x}{(1+x^2)^2} \qquad\qquad F''(x) = -2\frac{x^2-1}{(1+x^2)^3} \tag{4.17}$$

So at $x = 1$, $F'' = 0$, $F' = -\frac{1}{2}$, and $F = \frac{1}{2}$. We can see that F' reaches its minimum at $\tau = \frac{1}{2}$

$$w_x\Big|_{\tau=\frac{1}{2}} = \frac{F'}{1+\tau F'} = \frac{1}{2}\frac{1}{1-\tau/2} = \frac{1}{2-\tau} \tag{4.18}$$

so in the moment that $\tau = 2$, $w_x = \infty$

This is the *gradient catastrophe*

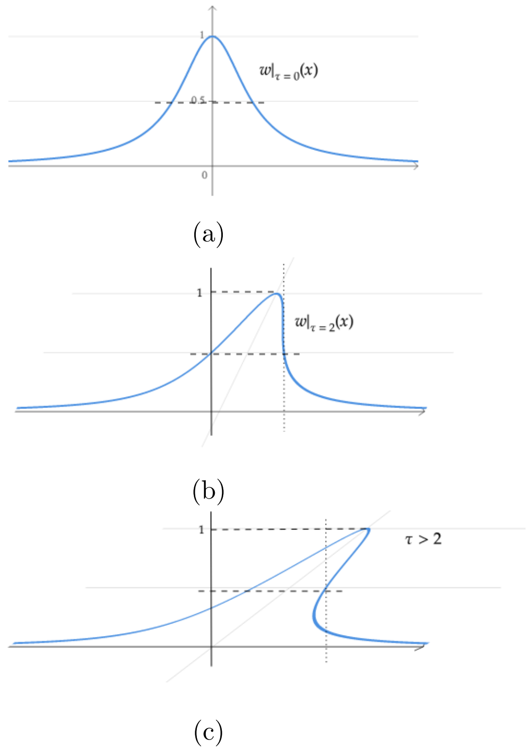

Figure 4.1: Time evolution of $F(x) = \frac{1}{1+x}$ under the Hopf equation

There are two ways to avoid the catastrophe.

1. In the dissipative medium, one can replace the Hopf equation by *Burgers equation*.

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = \epsilon \frac{\partial^2 w}{\partial x^2} \tag{4.19}$$

2. Another method requires replacing the Hopf equation with the *KdV equation*.

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = \epsilon \frac{\partial^3 w}{\partial x^3} \tag{4.20}$$

where ϵ is a small parameter sent to 0. This process is known as *regularization*.

One can seek a solution to Burgers equation in the form of a propagating wave.

$$w = w(x - st) \tag{4.21}$$

$$\implies -s\frac{\partial w}{\partial x} + \frac{1}{2}\frac{\partial}{\partial x}w^2 = \epsilon\frac{\partial^2 w}{\partial x^2} \tag{4.22}$$

We will integrate the above and assume the following boundary conditions

$$w \to 0 \qquad \text{as} \qquad x \to \infty$$

$$\implies -sw + \frac{1}{2}w^2 = \epsilon\frac{\partial w}{\partial x}$$

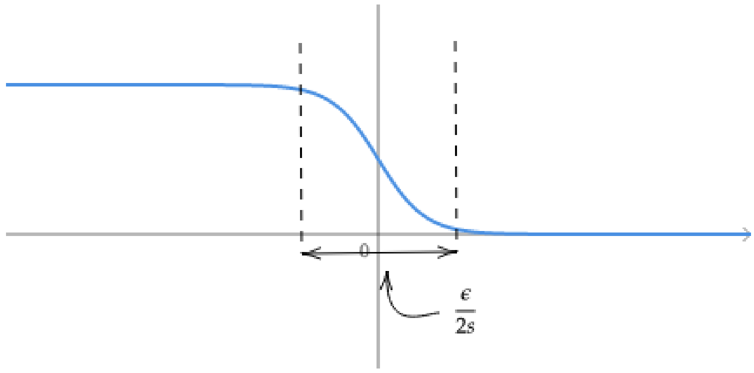

Figure 4.2: Shock wave

and assuming

$$\frac{\partial w}{\partial x} \to 0 \qquad \text{as} \qquad x \to \infty$$

and

$$w \to 2s \qquad \text{as} \qquad x \to -\infty$$

The equation has the solution

$$w = s\Big(1 - tanh\Big(\frac{sx}{2\epsilon}\Big)\Big) \tag{4.23}$$

The thickness of the shock wave is proportional to its intensity.

Now we visit the solution of the hydrodynamic equation using the Riemann invariants. Consider the initial equation to be

$$\rho_t + \Big(\rho v\Big)_x = 0 \tag{4.24}$$

$$v_t + v v_x + \frac{c^2}{\rho}\rho_x = 0 \tag{4.25}$$

The Riemann invariants are presented in the form

$$\frac{\partial A^{\pm}}{\partial t} + S^{\pm}(A^+, A^-)\frac{\partial A^{\pm}}{\partial x} = 0 \tag{4.26}$$

The solution can be found in the implicit form

$$A^+ = F(x - S^+(A^+, A^-)t) \tag{4.27}$$

$$A^- = G(x - S^-(A^+, A^-)t) \tag{4.28}$$

where $F = F(\xi)$ and $G = G(\eta)$ are arbitrary functions of a single variable.
Then,

$$A_t^+ = F'(\xi)\Big[- S^+ - \frac{\partial S^+}{\partial A^+}A_t^+ - \frac{\partial S^+}{\partial A^-}A_t^-\Big] \tag{4.29}$$

$$A_t^+ = G'(\xi)\Big[- S^- - \frac{\partial S^-}{\partial A^+}A_t^+ - \frac{\partial S^-}{\partial A^-}A_t^-\Big] \tag{4.30}$$

The column vector $\begin{bmatrix} A_t^+ \\ A_t^- \end{bmatrix}$ is a solution of the matrix problem

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} A_t^+ \\ A_t^- \end{bmatrix} = - \begin{bmatrix} F'(\xi)S^+ \\ G'(\eta)S^- \end{bmatrix} \tag{4.31}$$

Where the matrix M has components

$$M_{11} = 1 + \frac{\partial S^+}{\partial A^+}F' \qquad M_{12} = \frac{\partial S^+}{\partial A^-}F' \tag{4.32}$$

$$M_{21} = \frac{\partial S^-}{\partial A^+}G' \qquad M_{22} = 1 + \frac{\partial S^-}{\partial A^-}G' \tag{4.33}$$

The x derivatives A_x^+, A_x^- obey almost the same system

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} A_x^+ \\ A_x^- \end{bmatrix} = - \begin{bmatrix} F'(\xi) \\ G'(\eta) \end{bmatrix} \tag{4.34}$$

Comparing the above two systems of equations, we can see that (4.26) is satisfied. Hence we have found a general solution of the system hydrodynamic equation (4.2).

Chapter 5

The Dressing Method

Let $F = f + g$ be complex valued functions of two variables, u, v. The function F is analytic and the Cauchy-Riemann conditions hold

$$\frac{\partial f}{\partial u} - \frac{\partial g}{\partial v} = 0 \qquad\qquad \frac{\partial f}{\partial v} + \frac{\partial g}{\partial u} = 0 \tag{5.1}$$

by introducing complex notation $\lambda = u + iv$, one can rewrite the above condition (5.1) as follows

$$\frac{\partial F}{\partial \bar{\lambda}} = 0 \tag{5.2}$$

The function obeying this condition on the whole $\lambda, \bar{\lambda}$ plane is an *entire* function. Suppose that F is globally bounded. i.e.

$$|F| < C$$

for all λ. According to Liouville's Theorem, this function is identically constant.

Notice that if $F = P(\lambda)$, a polynomial, then $\frac{\partial F}{\partial \bar{\lambda}} = 0$.

Let's compute $\frac{\partial}{\partial \bar{\lambda}} \frac{1}{\lambda}$, where F is the simplest rational function.

$$\frac{\partial}{\partial \bar{\lambda}} \frac{1}{\lambda} = \frac{\partial}{\partial \bar{\lambda}} \lim_{\epsilon \to 0} \frac{\bar{\lambda}}{|\lambda \bar{\lambda}| + \epsilon} = \lim_{\epsilon \to 0} \frac{\epsilon}{(|\lambda|^2 + \epsilon)^2} = C\delta(u)\delta(v) \tag{5.3}$$

Where

$$C = \int \frac{\epsilon}{(r^2 + \epsilon^2)^2} r dr d\varphi = 2\pi\epsilon \int_0^\infty \frac{dy}{(y + \epsilon)^2} = \pi \tag{5.4}$$

Note that $r = \sqrt{u^2 + v^2}$ was the first substitution made and then $y = r^2$ to compute the above integral.

Finally, one obtains

$$\frac{\partial}{\partial \bar{\lambda}} \frac{1}{\lambda} = \pi\delta(u)\delta(v) \tag{5.5}$$

Or more generally,

$$\frac{\partial}{\partial \bar{\lambda}} \frac{1}{\lambda - \lambda_0} = \pi \delta(u - u_0)\delta(v - v_0) \tag{5.6}$$

This equality is known as the *Poincaré Formula.*

We call a function $F(\lambda, \bar{\lambda})$ *normalized quasi-analytic* if

$$\frac{\partial F}{\partial \bar{\lambda}} = f(\lambda, \bar{\lambda}) \qquad \text{and} \qquad F \to 1 \tag{5.7}$$

as $|\lambda| \to \infty$. By the use of the Poincaré formula, one can integrate equation (5.7) to get

$$F(\lambda) = 1 + \frac{1}{\pi} \int \frac{f(\xi, \bar{\xi})}{\lambda - \xi} d\xi \wedge d\bar{\xi} \tag{5.8}$$

Here $d\xi \wedge d\bar{\xi} = dudv$, and

$$\frac{1}{\lambda - \xi} = \lim_{\epsilon \to 0} \frac{\bar{\lambda} - \bar{\xi}}{|\lambda - \xi|^2 + \epsilon^2} \tag{5.9}$$

In virtue of (5.8), $F(\lambda, \bar{\lambda})$ has the following asymptotic expansion as $\lambda \to \infty$

$$F = 1 + \frac{F_0}{\lambda} + \frac{F_1}{\lambda^2} + ... \tag{5.10}$$

Where

$$\begin{aligned} F_0 &= \frac{1}{\pi} \int f(\xi, \bar{\xi}) d\xi \wedge d\bar{\xi} \\ F_1 &= \frac{1}{\pi} \int \xi f(\xi, \bar{\xi}) d\xi \wedge d\bar{\xi} \\ &\vdots \end{aligned}$$

and so on.

Let $\chi(\lambda, \bar{\lambda})$ be a quasi-analytic function satisfying the following equation (known as the *nonlocal* $\bar{\partial}$*-problem*)

$$\frac{\partial \chi}{\partial \bar{\lambda}} = \chi * T = \int \chi(\xi, \bar{\xi}) T(\xi, \bar{\xi}, \lambda, \bar{\lambda}) d\xi \wedge d\bar{\xi} \tag{5.11}$$

Here, $T(\xi, \bar{\xi}, \lambda, \bar{\lambda})$ is some kernel which is so far a free functional parameter.

This problem is *homogeneous*, and normalized by the condition $\chi \to 0$ as $|\lambda| \to \infty$. We will choose the kernel T such that the homogeneous $\bar{\partial}$-problem has only the trivial solution. In other words, asymptotes $\chi \sim \mathcal{O}(\frac{1}{\lambda})$ as $|\lambda| \to \infty$. This implies

$$\chi \equiv 0 \tag{5.12}$$

In this case, the $\bar{\partial}$-problem normalized by the condition $\chi \to 1$ as $\lambda \to \infty$ has a unique solution satisfying the integral equation

$$\chi = 1 + \frac{1}{\pi} \int \frac{\chi(\xi, \bar{\xi}) T(\xi, \bar{\xi}, \eta, \bar{\eta})}{\lambda - \eta} d\xi \wedge d\bar{\xi} d\eta \wedge d\bar{\eta} \tag{5.13}$$

In this case, χ has an expansion of the following form

$$\chi \to 1 + \frac{\chi_0}{\lambda} + \frac{\chi_1}{\lambda^2} + ... \tag{5.14}$$

where

$$\chi_0 = \frac{1}{\pi} \int \chi(\xi, \bar{\xi}) T(\xi, \bar{\xi}, \eta, \bar{\eta}) d\xi \wedge d\bar{\xi} d\eta \wedge d\bar{\eta} \tag{5.15}$$

Let's assume that χ and T depend on three additional parameters x, y, t and introduce three commuting differential operators.

$$D_1 \chi = \frac{\partial \chi}{\partial x} + i\lambda\chi \tag{5.16}$$

$$D_2 \chi = \frac{\partial \chi}{\partial y} + i\lambda^2\chi \tag{5.17}$$

$$D_3 \chi = \frac{\partial \chi}{\partial t} + 4i\lambda^3\chi \tag{5.18}$$

We will demand that the kernel T obeys the following equations.

$$\frac{\partial T}{\partial x} + i\lambda T = i\eta T \tag{5.19}$$

$$\frac{\partial T}{\partial y} + i\lambda^2 T = i\eta^2 T \tag{5.20}$$

$$\frac{\partial T}{\partial t} + 4i\lambda^3 T = 4i\eta^3 T \tag{5.21}$$

The above system of equations (5.19, 5.20, 5.21) can be resolved using the following

$$T = Re^{i(\phi(\eta) - \phi(\lambda))} \tag{5.22}$$

Where

$$\phi(\lambda) = \lambda x + \lambda^2 y + 4\lambda^3 t \tag{5.23}$$

and $R = R(\xi, \bar{\xi}, \eta, \bar{\eta})$ is a free functional parameter not depending on x, y, t. We will call it a *Dressing function.*

The main point of this construction is the following. Application of operator D_i to the $\bar{\partial}$-problem does not violate it. In other words,

$$\frac{\partial}{\partial \bar{\lambda}} D_i \chi = D_i \chi * T \tag{5.24}$$

We can that the $\bar{\partial}$-problem is invariant under application of the differential operators D_i. Notice that operators D_i and $\frac{\partial}{\partial \bar{\lambda}}$ commute.

Let $P(D_1, D_2, D_3)$ be some polynomial differential operator in D_i which includes multiplication of coefficients from the left side. The coefficients could be functions of x, y, t.

In a general case,

$$P(D_1, D_2, D_3)\chi \to P_0(\lambda) + o(\frac{1}{\lambda}) \tag{5.25}$$

Where $P_0(\lambda)$ is some polynomial.

However, one can choose coefficients of P in such a way that $P_0(\lambda) \equiv 0$ and $P\chi \to o(\frac{1}{\lambda})$ as $\lambda \to \infty$

In this case, $P\chi$ is a solution of the homogeneous $\bar{\partial}$-problem and we get

$$P\chi \equiv 0 \tag{5.26}$$

We will call an operator annihilating χ, the P-*operator*.

We will show that there are infinitely many P-operators and construct the simplest of them explicitly.

Let

$$P_1\chi = -iD_2\chi + D_1^2\chi + u\chi = -i\frac{\partial\chi}{\partial y} + \frac{\partial^2\chi}{\partial x^2} + 2i\lambda\frac{\partial\chi}{\partial x} + u\chi \tag{5.27}$$

Here, u is still an unknown function.
Remember that

$$\chi \to 1 + \frac{\chi_0}{\lambda} + \frac{\chi_1}{\lambda^2} + ... \tag{5.28}$$

as $|\lambda| \to \infty$

Plugging (5.28) into (5.27), one can see that

$$P_1\chi \to u + 2i\frac{\partial\chi_0}{\partial x} + o(\frac{1}{\lambda}) \tag{5.29}$$

as $|\lambda| \to \infty$

If we set

$$u = -2i\frac{\partial\chi_0}{\partial x} \tag{5.30}$$

Then we get our desired result

$$P_1\chi \to o(\frac{1}{\lambda}) \tag{5.31}$$

Hence P_1 is a P-operator

$$P_1\chi \equiv 0 \tag{5.32}$$

Now let

$$P_2\chi = \Big(D_3 + 4D_1^2 + vD_1 + w\Big)\chi \tag{5.33}$$

$$P_2\chi = -12\lambda^2\frac{\partial\chi}{\partial x} + i\lambda\Big(12\frac{\partial^2\chi}{\partial x^2} + v\chi\Big) + \frac{\partial\chi}{\partial t} + 4\frac{\partial^3\chi}{\partial x^3} + v\frac{\partial\chi}{\partial x} + w\chi \tag{5.34}$$

Let us send $|\lambda| \to \infty$. Cancellation of λ-terms leaves us with

$$v = -12i\frac{\partial \chi_0}{\partial x} = 6u \tag{5.35}$$

$$w = 12\frac{\partial \chi_1}{\partial x} + i\Big(12\frac{\partial^2 \chi_0}{\partial x^2} + v\chi_0\Big) = 12\frac{\partial \chi_1}{\partial x} + 6\Big(-\frac{\partial u}{\partial x} + iu\chi_0\Big) \tag{5.36}$$

After making these choices, we establish that

$$P_2\chi \equiv 0 \tag{5.37}$$

We will not need the rather complicated expression for w.

We introduce a new function

$$\varphi = \chi e^{i\phi(\lambda)} \qquad\qquad y = \varphi e^{-\phi(\lambda)} \tag{5.38}$$

Apparently,

$$D_i y = \partial_i \varphi e^{-\phi(\lambda)}$$

Moreover, for any polynomial differential operator P

$$P(D_1, D_2, D_3)\chi = P\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\Big)\varphi e^{-\phi(\lambda)} \tag{5.39}$$

In particular, equations (5.32) and (5.34) take forms of

$$i\frac{\partial \varphi}{\partial y} = L\varphi \qquad\qquad L = \frac{\partial^2}{\partial x^2} + u \tag{5.40}$$

$$\frac{\partial \psi}{\partial t} + M\psi = 0 \qquad\qquad M = 4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + w \tag{5.41}$$

$$\tag{5.42}$$

Equations (5.40) are compatible in virtue of the construction presented above. The compatibility conditions read

$$L_t - iM_y = [L, M] \tag{5.43}$$

$$\tag{5.44}$$

It is convenient to split w

$$w = 3u_x + r \tag{5.45}$$

$$M = M_0 + r \tag{5.46}$$

so that

$$M_0 = 4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3u_x$$

The commutator $[L, M_0]$ can be easily calculated

$$[L, M_0] = \Big(\frac{\partial^2}{\partial x^2} + u\Big)\Big(\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3u_x\Big) \tag{5.47}$$

$$-\Big(\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3u_x\Big)\Big(\frac{\partial^2}{\partial x^2} + u\Big) \tag{5.48}$$

Expanding the above expression leads to a very simple result. $[L, M_0]$ is a multiplicative operator to the scalar function. All derivatives are cancelled (check this!) and as a result

$$[L, M_0] = -6uu_x - u_{xxx} \tag{5.49}$$

and

$$[L, M] = [L, M_0] + \Big(\frac{\partial^2}{\partial x^2} + u\Big)r - r\Big(\frac{\partial^2}{\partial x^2} + u\Big) \tag{5.50}$$

$$= -6uu_x - u_{xxx} + r_{xx} + 2r_x\frac{\partial}{\partial x} \tag{5.51}$$

Then

$$iM_y = 6iu_y\frac{\partial}{\partial x} + 3iu_{xy} + ir_y \tag{5.52}$$

Gathering all above equations together, we find that these equations are equivalent to the system

$$u_t + 6uu_x + u_{xxx} = r_{xx} + 3iu_{xy} + ir_y \tag{5.53}$$

$$-6iu_y = 2r_x \tag{5.54}$$

After cancellation, one ends up with the semantically simple system of equations

$$u_t + 6uu_x + u_{xxx} = ir_y \tag{5.55}$$

$$r_x = -3iu_y \tag{5.56}$$

which is equivalent to the single equation

$$\frac{\partial}{\partial x}\Big(u_t + 6uu_x + u_{xxx}\Big) = 3u_{yy} \tag{5.57}$$

Equation (5.57) is known as the *KP-1* equation.

And replacing $y \to iy$, one obtains the *KP-2* equation.

$$\frac{\partial}{\partial x}\Big(u_t + 6uu_x + u_{xxx}\Big) = -3u_{yy} \tag{5.58}$$

Chapter 6

Lax Pairs

In the last chapter, we derived the KP equations (5.57) and (5.58). Thereafter we will assume $\frac{\partial u}{\partial y} = 0$ and $\frac{\partial \chi}{\partial y} = 0$ and simplify these equations to form the *Korteweg-de Vries (KdV)* equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{6.1}$$

As far as $\varphi = \chi e^{i\phi(x)}$ with $\varphi \simeq e^{i\lambda^2 y}\Psi(\lambda t)$, where Ψ satisfies the equations

$$L\Psi = -\lambda^2\Psi \tag{6.2}$$

$$\frac{\partial \Psi}{\partial t} + M\Psi = 0 \tag{6.3}$$

where

$$L\Psi = \frac{\partial^2\Psi}{\partial x^2} + u\Psi \tag{6.4}$$

$$M\Psi = 4\frac{\partial^3\Psi}{\partial x^3} + 6u\frac{\partial\Psi}{\partial x} + 3u_x\Psi \tag{6.5}$$

This is the traditional *Lax representation* of the KdV equation. The standard way to construct exact solutions leads to the development of the scattering theory for the Schrödinger equation (6.2). We will do this construction shortly, but make use of the Dressing method first, as it is a straightforward way to find a solution.
The condition $\frac{\partial T}{\partial y} = 0$ means that T satisfies the equation

$$(\lambda^2 - \eta^2)T(\lambda, \eta) = 0$$

Hence,

$$T(\lambda, \bar{\lambda}, \eta, \bar{\eta}) = \chi(-\lambda, -\bar{\lambda})R(\lambda, \bar{\lambda})e^{-2i\varphi} \tag{6.6}$$

Or,

$$\frac{\partial \chi}{\partial \bar{\lambda}} = \chi(-\lambda, -\bar{\lambda})R(\lambda, \bar{\lambda})e^{-2i\varphi} \tag{6.7}$$

where $\varphi = \lambda x + 4\lambda^3 t$, and $R(\lambda, \bar{\lambda})$ is the dressing function which is a free functional parameter.
Recall that

$$\chi \to 1 + \frac{\chi_0}{\lambda} + \frac{\chi_1}{\lambda^2} + ... \tag{6.8}$$

and

$$u = -2i\frac{\partial \chi_0}{\partial x}$$

Let $\lambda_n = i\kappa_n$ be complex numbers placed on the imaginary axis such that $\kappa_n + \kappa_m \neq 0$. We will seek a solution of the $\bar{\partial}$-problem (6.6) of the form (a meromorphic function)

$$\chi = 1 + i\sum_{m=1}^{N} \frac{\chi_m}{\lambda - i\kappa_m} \tag{6.9}$$

Then,

$$\chi_0 = i\sum_{n=1}^{N} \chi_n, \qquad u = 2\frac{\partial \chi_0}{\partial x} = 2\frac{\partial}{\partial x}\sum_{n=1}^{N} \chi_n \tag{6.10}$$

And

$$\chi(-\lambda) = 1 - i\sum_{m=1}^{N} \frac{\chi_m}{\lambda + i\kappa_m} \tag{6.11}$$

Evaluation of this function at the point $\lambda = i\kappa_n$ gives

$$\chi(-i\kappa_n) = 1 - \sum_{m=1}^{N} \frac{\kappa_m}{\kappa_n + \kappa_m} \tag{6.12}$$

and calculating the derivative with respect to $\bar{\lambda}$ gives

$$\frac{\partial \chi}{\partial \bar{\lambda}} = \pi i \sum_{m=1}^{N} \chi_m \delta(\lambda - i\kappa_m)\delta(\bar{\lambda} + i\kappa_m) \tag{6.13}$$

Matching (6.9) with (6.6), one must choose

$$R(\lambda, \bar{\lambda}) = \pi \sum_{m=1}^{N} c_m \delta(\lambda - i\kappa_m)\delta(\bar{\lambda} + i\kappa_m) \tag{6.14}$$

Collecting everything together, we end up with the following system of equations

$$\chi_n + c_n e^{-2\varphi_n} \sum_{m=1}^{N} \frac{\chi_m}{\kappa_n + \kappa_m} = c_n e^{2\varphi_n} \tag{6.15}$$

$$\varphi_n = \kappa_n x + 4\kappa_n^3 t \tag{6.16}$$

Now, we let

$$\chi_n = e^{-\varphi_n}\psi_n \tag{6.17}$$

$$\psi_n + c_n \sum_{m=1}^{N} \frac{\chi_m e^{-(\varphi_n + \varphi_m)}}{\kappa_n + \kappa_m} = c_n e^{\varphi_n} \tag{6.18}$$

and u becomes

$$u = 2\frac{\partial}{\partial x}\sum_{n=1}^{N} \psi_n e^{-\varphi_n} \tag{6.19}$$

Systems (6.15) and (6.17) have identical determinants, which we will call $\mathcal{A}$.

Let's denote $\xi_n = \frac{c_n}{2\kappa_n} e^{2\varphi_n}$.
For $N = 2$,

$$\mathcal{A} = 1 + \xi_1 + \xi_2 + \frac{(\kappa_1 - \kappa_2)^2}{(\kappa_1 + \kappa_2)^2} \xi_1 \xi_2 \tag{6.20}$$

For $N = 3$,

$$\mathcal{A} = 1 + \xi_1 + \xi_2 + \xi_3 + \frac{(\kappa_1 - \kappa_2)^2}{(\kappa_1 + \kappa_2)^2} \xi_1 \xi_2 + \frac{(\kappa_2 - \kappa_3)^2}{(\kappa_2 + \kappa_3)^2} \xi_2 \xi_3 + \frac{(\kappa_1 - \kappa_3)^2}{(\kappa_1 + \kappa_3)^2} \xi_1 \xi_3$$
$$+ \frac{(\kappa_1 - \kappa_2)^2}{(\kappa_1 + \kappa_2)^2} \frac{(\kappa_2 - \kappa_3)^2}{(\kappa_2 + \kappa_3)^2} \frac{(\kappa_1 - \kappa_3)^2}{(\kappa_1 + \kappa_3)^2} \xi_1 \xi_2 \xi_3$$

This hints at how to construct the determinant for a general case. It will consist of 2^N terms. The first term is 1 and the last term is

$$\xi_1 ... \xi_N \prod_{i \neq j} \frac{(\kappa_i - \kappa_j)^2}{(\kappa_i + \kappa_j)^2}$$

According to Kramer's rule, the solution of this system is

$$\psi_n = \frac{\mathcal{A}_n}{\mathcal{A}}$$

where $\mathcal{A}_n$ is the determinant of the system that has determinant $\mathcal{A}$, but whose n-th column is replaced with the column $\begin{pmatrix} c_1 e^{\varphi_1} \\ \vdots \\ c_N e^{\varphi_N} \end{pmatrix}$ According to (6.19)

$$u = 2 \frac{d}{dx} \sum_{n=1}^{N} e^{-\varphi_n} \frac{\mathcal{A}_n}{\mathcal{A}}$$

However, one can easily recognize that

$$\sum_{n=1}^{N} e^{-\varphi_n} \mathcal{A}_n = \frac{d\mathcal{A}}{dx}$$

Finally, we obtain the remarkable result

$$u = 2 \frac{d^2}{dx^2} ln(\mathcal{A}) \tag{6.21}$$

This formula conceals the following astonishing property: Suppose we replace $\mathcal{A}$ with $\tilde{\mathcal{A}}$

$$\mathcal{A} \to \tilde{\mathcal{A}} = \mathcal{A} e^{ax+bt+c} \tag{6.22}$$

with a, b, c being arbitrary constants. Then,

$$\frac{d^2}{dx^2} ln(\mathcal{A}) = \frac{d^2}{dx^2} ln(\tilde{\mathcal{A}}) \tag{6.23}$$

The above transformation does not change the solution of KdV! We will use this property generously.

Chapter 7

The Hirota Equation

We start with the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{7.1}$$

where

$$u = 2B_x = 2\frac{\partial^2}{\partial x^2} ln(\mathcal{A})$$

And after substituting $u = 2B_x$ into (7.1)

$$\frac{\partial}{\partial x}\Big(B_t + 6B_x^2 + B_{xxx}\Big) = 0$$

Which after integration becomes

$$B_t + 6B_x^2 + B_{xxx} = \mu \tag{7.2}$$

Now calculating each term in the above equation:

$$\begin{aligned}
B &= \frac{\mathcal{A}_x}{\mathcal{A}} \\
B_x &= \frac{\mathcal{A}_{xx}}{\mathcal{A}} - \frac{\mathcal{A}_x^2}{\mathcal{A}^2} \\
B_x^2 &= \frac{\mathcal{A}_{xx}^2}{\mathcal{A}^2} - 2\frac{\mathcal{A}_{xx}\mathcal{A}_x^2}{\mathcal{A}^3} + \frac{\mathcal{A}_x^4}{\mathcal{A}^4} \\
B_{xx} &= \frac{\mathcal{A}_{xxx}}{\mathcal{A}} - 3\frac{\mathcal{A}_x\mathcal{A}_{xx}}{\mathcal{A}^2} + 2\frac{\mathcal{A}_x^3}{\mathcal{A}^3} \\
B_{xxx} &= \frac{\mathcal{A}_{xxxx}}{\mathcal{A}} - 4\frac{\mathcal{A}_x\mathcal{A}_{xxx}}{\mathcal{A}^2} - 3\frac{\mathcal{A}_{xx}^2}{\mathcal{A}^2} + 12\frac{\mathcal{A}_x^2\mathcal{A}_{xx}}{\mathcal{A}^3} - 6\frac{\mathcal{A}_x^4}{\mathcal{A}^4} \\
B_t &= \frac{\mathcal{A}_{xt}\mathcal{A} - \mathcal{A}_t\mathcal{A}_x}{\mathcal{A}^2}
\end{aligned}$$

Substituting the above values into (7.2) gives

$$\mathcal{A}_{xt}\mathcal{A} - \mathcal{A}_t\mathcal{A}_x + \mathcal{A}\mathcal{A}_{xxxx} - 4\mathcal{A}_x\mathcal{A}_{xxx} + 3\mathcal{A}_{xx}^2 = \mu\mathcal{A}^2 \tag{7.3}$$

This is known as the *Hirota equation*.

From here onwards, we will mostly study the case $\mu = 0$ and assume $\mathcal{A} = 1 + C$. We obtain the following equation in terms of C

$$C_{xt} + C_{xxxx} + C_{xt}C - C_tC_x - 4C_xC_{xxxx} + 3C_{xx}^2 = 0 \tag{7.4}$$

If we let

$$C = e^{-2\kappa(x-4\kappa^2 t-p)} = \xi$$

then, by direct substitution, one can show that the function

$$\mathcal{A} = 1 + \xi \tag{7.5}$$

satisfies equation (7.4). The solution is the simple soliton

$$u = \frac{\partial^2}{\partial x^2} ln(1+\xi) = \frac{2\kappa^2}{cosh^2[\kappa(x-4\kappa^2 t-p)]} \tag{7.6}$$

It is easy to check by direct calculation that the function

$$\mathcal{A} = 1 + \xi_1 + \xi_2 + \frac{(\kappa_1-\kappa_2)^2}{(\kappa_1+\kappa_2)^2}\xi_1\xi_2 \tag{7.7}$$

for $\kappa_1 + \kappa_2 \neq 0$ is also a solution of the Hirota equation if

$$\xi_1 = e^{-2\kappa_1(x-4\kappa_1^2 t-p_1)}$$

and

$$\xi_2 = e^{-2\kappa_2(x-4\kappa_2^2 t-p_2)}$$

Let's study the asymptotic behavior of this solution as $t \to \pm\infty$ assuming that $\kappa_2 > \kappa_1 > 0$. To do this, we establish the following simple fact. If some function $\mathcal{A}$ generates a solution of the KdV equation by the standard formula

$$u = 2\frac{d^2}{dx^2} ln(\mathcal{A})$$

, then the function $\tilde{\mathcal{A}} = \mathcal{A}e^{Ax+Bt+C}$, for arbitrary constants A, B, C generate the same solution.
Let's denote

$$c_1 = -2\kappa_1(x - 4\kappa_1^2 t - p_1) \tag{7.8}$$

$$c_2 = -2\kappa_2(x - 4\kappa_2^2 t - p_2) \tag{7.9}$$

Then for

$$\Delta := c_2 - c_1 = 2(\kappa_1 - \kappa_2)x - 8(\kappa_1^3 - \kappa_2^3)t + const \tag{7.10}$$

So when $\kappa_2 > \kappa_1$

$$\Delta \to \begin{cases} -\infty & \text{as } t \to -\infty \\ \infty & \text{as } t \to \infty \end{cases}$$

Now we consider solution $\mathcal{A}$ on the characteristic straight line $c_1 = const$. When $t \to -\infty$,

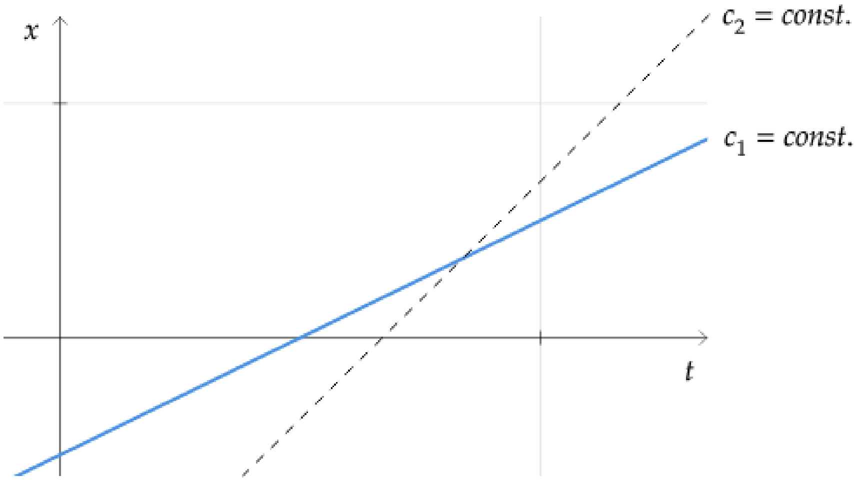

Figure 7.1

$\xi_2 \to 0$ and the solitonic solution becomes

$$\mathcal{A} = 1 + \xi_1 \tag{7.11}$$

At $t \to -\infty$,

$$\mathcal{A} \to \xi_2 \left[1 + \frac{(\kappa_1 - \kappa_2)^2}{(\kappa_1 + \kappa_2)^2} \xi_1 \right] \simeq 1 + \frac{(\kappa_1 - \kappa_2)^2}{(\kappa_1 + \kappa_2)^2} \xi_1 \tag{7.12}$$

ξ_2 is pure exponent and can be cancelled. Comparing (7.11) and (7.12) shows that the "slow" soliton generated by the exponent ξ_1, after interaction with the "fast" soliton generated by the exponent ξ_2, acquires the negative shift

$$\delta x_1 = \frac{1}{2\kappa_1} \ln \left[\left(\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \right)^2 \right] < 0 \tag{7.13}$$

Consider the similar phenomenon on the characteristic $c_2 = const.$

$$\mathcal{A} \to \xi_1 \left[1 + \frac{(\kappa_1 - \kappa_2)^2}{(\kappa_1 + \kappa_2)^2} \xi_2 \right] \simeq 1 + \frac{(\kappa_1 - \kappa_2)^2}{(\kappa_1 + \kappa_2)^2} \xi_2 \tag{7.14}$$

as $t \to \infty$ and

$$\mathcal{A} \to 1 + \xi_2 \tag{7.15}$$

as $t \to \infty$
This means that the fast soliton acquires the positive shift

$$\delta x_2 = -\frac{1}{2\kappa_2} ln \left[\left(\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \right)^2 \right] > 0 \tag{7.16}$$

In fact, the slow soliton turns into the fast soliton. This effect could be treated as "cannibalism" of solitons. As far as shifts of solitons do not depend on their initial positions, the total soliton shift is an algebraic sum of its shifts due to double interactions. If solitons are parameterized by

$$0 < \kappa_1 < ... < \kappa_N$$

then the total shift of the soliton of index k is

$$\delta x_k = \frac{1}{2\kappa_k}\left\{\sum_{l=1}^{k-1} ln\Big[\Big(\frac{\kappa_l+\kappa_k}{\kappa_l-\kappa_k}\Big)^2\Big] - \sum_{l=k+1}^{N} ln\Big[\Big(\frac{\kappa_l-\kappa_k}{\kappa_l+\kappa_k}\Big)^2\Big]\right\}$$

Chapter 8

Hirota Derivatives

In Chapter 7, we showed that the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{8.1}$$

under the transformation

$$u = 2\frac{\partial^2}{\partial x^2} ln(\mathcal{A}) \tag{8.2}$$

turns into the Hirota bilinear equation

$$\mathcal{A}_{xt}\mathcal{A} - \mathcal{A}_t\mathcal{A}_x + \mathcal{A}\mathcal{A}_{xxxx} - 4\mathcal{A}_x\mathcal{A}_{xxx} + 3\mathcal{A}_{xx}^2 = \mu\mathcal{A}^2 \tag{8.3}$$

Now we will present this equation in a more elegant form.

Let $f(x), g(x)$ be a pair of smooth functions. We define the n-th order *Hirota derivative* for this pair as follows

$$D_x^n f \cdot g = \frac{\partial^n}{\partial y^n} f(x+y)g(x-y)\Bigg|_{y=0} \tag{8.4}$$

In particular,

$$\begin{aligned}
D_x^1 f \cdot g &= f_x g - f g_x \\
D_x^2 f \cdot g &= f_{xx} g - 2f_x g_X + f g_{xx} \\
D_x^3 f \cdot g &= f_{xxx} - 3f_{xx} g_x + 3f_x g_{xx} - g_{xxx} \\
D_x^4 f \cdot g &= f_{xxxx} - 4f_{xxx} g_x + 6f_{xx} g_{xx} - 4f_x g_{xxx} + f g_{xxxx}
\end{aligned}$$

One can verify that

$$D_x^1 f \cdot f = 0 \tag{8.5}$$

and more generally,

$$D_x^{2n+1} f \cdot f = 0 \tag{8.6}$$

$$D_x^2 f \cdot f = 2(f_{xx} f - f_x^2) \tag{8.7}$$

$$D_x^4 f \cdot f = 2(f_{xxxx} f - 4f_{xxx} f_x + 3f_{xx}^2) \tag{8.8}$$

In the same way, one can define mixed derivatives of two or more variables

$$D_x^m D_t^n f \cdot g = \frac{\partial^m}{\partial y^m} \frac{\partial^n}{\partial \tau^n} f(x+y, t+\tau) g(x-y, t-\tau) \Big|_{\substack{y=0 \\ \tau=0}} \tag{8.9}$$

In particular,

$$D_t^1 D_x^1 f \cdot f =: D_{xt}^2 = 2(f_{xt} f - f_x f_t) \tag{8.10}$$

One can see that the Hirota equation can be presented as follows

$$(D_x D_t + D_x^2)\mathcal{A} \cdot \mathcal{A} = \frac{\mu}{2} \mathcal{A}^2 \tag{8.11}$$

The central point of the Hirota theory is the theorem about exponents.

$$f = e^{P_1 x + \Omega_1 t + y_1} \qquad\qquad g = e^{P_2 x + \Omega_2 t + y_2} \tag{8.12, 8.13}$$

Then

$$D_x^m D_t^n f \cdot g = (P_1 - P_2)^m (\Omega_1 - \Omega_2)^n f \cdot g \tag{8.14}$$

and in particular

$$D_x^m D_x^n f \cdot f = 0 \tag{8.15}$$

Corollary:
Let f, g, h be three arbitrary exponents. Then

$$D_x^m D_t^n hf \cdot hg = h^2 D_x^m D_t^n f \cdot g \tag{8.16}$$

An even stronger statement holds too.
Let F, G be two arbitrary functions while h is a exponential. Then

$$D_x^m D_t^n hF \cdot hG = h^2 D_x^m D_t^n F \cdot G \tag{8.17}$$

In other words, the Hirota derivative treats multiplication by an exponential as multiplication by a constant. Using this fact makes it possible to prove the following important theorem.

Let u be an n-solitonic solution of the KdV equation. It can be presented as

$$u = 2\frac{d^2}{dx^2} ln(\mathcal{A}) \tag{8.18}$$

where $\mathcal{A}$ is the determinant of the system (6.15) found in Lecture 6. Then $\mathcal{A}$ is a solution of the Hirota equation is $\mu = 0$.

For three-solitonic solution, it could be checked by direct calculation. For a general case, the proof is more tedious, but later on we will present a really simple proof of this fact. Let's now study the KP-2 equation.

$$\frac{\partial}{\partial x}\Big(u_t + 6uu_x + u_{xxx}\Big) + 3_{yy} = 0 \tag{8.19}$$

One can easily check (we omit the details here) that u is presented by (8.18), then $\mathcal{A}$ satisfies the $2+1$ version of the Hirota equation.

$$D_x\Big(D_t + D_{xxx}\Big)\mathcal{A}\cdot A + 3D_y^2\mathcal{A}\mathcal{A} = 0 \tag{8.20}$$

Now suppose that

$$\mathcal{A} = 1 + f \tag{8.21}$$

where f is an arbitrary solution of two compatible equations

$$f_y = f_{xx} \tag{8.22}$$
$$f_t = f_{xxx} \tag{8.23}$$

Then $\mathcal{A}$ is a solution of the Hirota equation (8.20).

Chapter 9

The Knoidal Wave

We consider again the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{9.1}$$

where we look for a solution which presents the propagating wave

$$u = u(x - ct) = u_\xi \tag{9.2}$$

with $\xi = x - ct$, so that $u_t = -cu_\xi$ and $u_x = u_\xi$.
Equation (9.1) gives

$$\frac{\partial}{\partial \xi}\Big(- cu + 3u^2 + u_{\xi\xi}\Big) = 0 \tag{9.3}$$

and integration of the above equation gives

$$-cu + 3u^2 + u_{\xi\xi} = -cp + 3p^2 \tag{9.4}$$

Here p is some constant. Using the transformation $u = p + v$ leads to the following equation

$$v_{\xi\xi} + 3v^2 - \tilde{c}v = 0 \tag{9.5}$$

with $\tilde{c} = c - 6p$.
One can see that the constant p can be removed by renormalization of the velocity c. Thus we can put $p = 0$ without loss of generality. Our resulting equation becomes

$$v_{\xi\xi} + 3v^2 - cv = 0 \tag{9.6}$$

This equation can integrated as follows

$$\frac{1}{2}v_\xi^2 + v^3 - \frac{c}{2}v^2 = E \tag{9.7}$$

Thereafter we assume $c > 0$. E is a parameter of integration. If $E = 0$, then equation (9.7) describes a soliton. However, if $E < 0$, the solution of (9.7) is a periodic function.
This equation can be easily integrated in terms of elliptic functions, but we like to solve it by the use of the Hirota equation. Now it takes the form:

$$H\mathcal{A}\cdot\mathcal{A} = \Big(- cD_\xi^2 + D_\xi^4\Big)\mathcal{A}\cdot\mathcal{A} - \mu\mathcal{A}^2 \tag{9.8}$$

We will consider that the solution is an even function $u(x) = u(-x)$. Hence, $\mathcal{A}(x) = \mathcal{A}(-x)$. We will look for a solution of equation (9.7) in the form of an infinite series

$$\mathcal{A} = \sum_{n=-\infty}^{\infty} \varphi_n \tag{9.9}$$

$$\varphi_n = e^{-4\lambda n^2 + 2kn\xi} \tag{9.10}$$

equation (9.8) means that

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} L_{nm} = 0 \tag{9.11}$$

Here, $L_{nm} = H\varphi_n \cdot \varphi_m = e^{-4\lambda(n^2+m^2)+2k(n+m)\xi} \Big[16k^4(n-m)^4 - 4ck^2(n-m)^2 - \mu \Big]$ Now we present the following variable substitutions

$$2n = p + q \tag{9.12}$$

$$2m = p - q \tag{9.13}$$

$$p = n + m \tag{9.14}$$

$$q = n - m \tag{9.15}$$

$$p^2 + q^2 = 2(n^2 + m^2) \tag{9.16}$$

for p, q in the integers. Now, $L_{nm} \to L_{pq}$ with

$$L_{pq} = e^{2\lambda p^2 + 2kp\xi} e^{-2\lambda q^2} \Big[16k^4 q^4 - 4ck^2 q^2 - \mu \Big] \tag{9.17}$$

So that equation (9.11) turns into

$$\sum_p \sum_q L_{pq} = 0 \tag{9.18}$$

But

$$L_{pq} = A(p, \xi) B(q) \tag{9.19}$$

Let's fix p and demand

$$\sum_q B(q) = 0 \tag{9.20}$$

It seems that L_{pq} is factorized, but there is a delicate subtlety. As far as n, m are integers, p, q must be integers of the same parity - both even (if n, m are of the same parity) or both odd (if n, m are of different parities). As a result, equation (9.18) is equivalent to the two following equations

$$\sum_{\substack{q=2l \\ \text{(all even terms)}}} e^{-2\lambda q^2} \Big[16k^4 q^4 - 4ck^2 q^2 - \mu \Big] = 0 \tag{9.21}$$

$$\sum_{\substack{q=2l+1 \\ \text{(all odd terms)}}} e^{-2\lambda q^2} \Big[16k^4 q^4 - 4ck^2 q^2 - \mu \Big] = 0 \tag{9.22}$$

The above equations impose two restrictions on three parameters λ, k, mu. One of them, namely λ, can be chosen as an arbitrary positive parameter. If $\lambda \to \infty$, the solutions tend towards the solitonic solution. If $\lambda \to 0$, the solution is a weakly nonlinear wave of small amplitude on constant base.

In equation (9.21), one can replace $k \to ik$ and $c \to -c$, and they preserve their form.

Now we should explain why solution (9.9) leads to a periodic v. Let's make a shift

$$\xi \to \xi + \frac{4\lambda}{k}$$

Then $\varphi_n \to e^{-4\lambda(n-1)^2+2kn\xi} = e^{-4\lambda(n-1)+2k(n-1)\xi}e^{2k\xi}$ Where n takes on any integer, and so does $(n-1)$. As a result, $\mathcal{A} \to \mathcal{A}e^{2k\xi}$, while v remains unchanged.

In the same way, one can construct two-periodic solutions of the KdV. This is a good subject for individual research.

Chapter 10

Scattering in the Schrödinger equation

We start with the equation

$$\frac{d^2}{dx^2}\Psi + k^2\Psi = u(x)\Psi \quad -\infty < x < \infty \tag{10.1}$$

where $u(x)$ is a real-valued function satisfying the condition

$$\int_{-\infty}^{\infty}(1+|x|)|u(x)|dx < \infty \tag{10.2}$$

$k = k_n$ is an eigenvalue if the solution f_n of equation (10.1) tends to zero as $|x| \to \infty$. It is well known that this solution is unique. Indeed, if Ψ_1, Ψ_2 are two solutions of (10.1), then

$$\{\Psi_1, \Psi_2\} = const = C \tag{10.3}$$

Here, $\{\Psi_1, \Psi_2\} = \Psi_{1x}\Psi_2 - \Psi_{2x}\Psi_1$, which is the Wronskian of the functions Ψ_1 and Ψ_2. If Ψ_1 and Ψ_2 are eigen-functions, they tend to zero as $|x| \to \infty$. Hence, $C = 0$ and Ψ_1, Ψ_2 are proportional to each other.

Eigenvalue k_n must be purely imaginary. Indeed, if k_n is complex

$$\begin{cases} \frac{d^2 f_n}{dx^2} + k_n^2 f_n & = u f_n \\ \frac{d^2 \bar{f_n}}{dx^2} + \bar{k_n}^2 \bar{f_n} & = u\bar{f_n} \end{cases} \tag{10.4}$$

From the above system, (10.4), one gets

$$\frac{d}{dx}\{f, \bar{f_n}\} = (k_n^2 - \bar{k_n}^2)|f_n|^2 \tag{10.5}$$

and after integrating, one obtains

$$\bar{k}_n^2 = k_n^2 \tag{10.6}$$

Apparently, $\{f, f_n\} = 0$, and eigenfunction F can be made real. Next, we introduce the Jost functions Ψ, Φ - solutions to equation (10.1) - defined by the boundary conditions

$$\begin{array}{ll} \Psi \to e^{ikx} & \Phi \to e^{-ikx} \\ x \to +\infty & x \to -\infty \end{array} \tag{10.7}$$

Jost functions satisfy certain integral equations. One can present Ψ in the form

$$\Psi(k,x) = c_1(x)e^{ikx} + c_2(x)e^{-ikx}$$

with the additional condition

$$c_1' e^{ikx} + c_2' e^{-ikx} = 0 \tag{10.8}$$

Hence,

$$\Psi' = ik\Big(c_1(x)e^{ikx} - c_2(x)e^{-ikx}\Big) \tag{10.9}$$

$$\Psi'' + k^2\Psi = ik\Big(c_1'(x)e^{ikx} - c_2'(x)e^{-ikx}\Big) = u\Psi \tag{10.10}$$

Combining (10.8) and (10.10), one gets

$$c_1' = \tfrac{1}{2ik}u\Psi e^{-ikx} \quad c_2' = -\tfrac{1}{2ik}u\Psi e^{ikx} \tag{10.11}$$

Integrating (10.11), we take into account boundary conditions

$$c_1 = 1 - \frac{1}{2ik}\int_x^\infty u\Psi e^{-iky}dy \tag{10.12}$$

$$c_2 = \frac{1}{2ik}\int_x^\infty u\Psi e^{iky}dy \tag{10.13}$$

One can introduce a new function $A = \Psi e^{-ikx} = c_1 + c_2 e^{-2ikx}$. And from the above equations, we conclude that B satisfies the integral equation

$$B(x,k) = 1 - \frac{1}{2ik}\int_x^\infty u(y)(1 - e^{2ik(y-x)})A(k,y)dy \tag{10.14}$$

The same operations can be performed with function Φ.

$$c_1 = \frac{1}{2ik}\int_x^\infty u\Phi e^{-iky}dy \tag{10.15}$$

$$c_2 = 1 - \frac{1}{2ik}\int_x^\infty u\Phi e^{iky}dy \tag{10.16}$$

We will denote $B = \Phi e^{ikx}$. Then this function satisfies the integral equation

$$A(x,k) = 1 - \frac{1}{2ik}\int_x^\infty u(y)(1 - e^{2ik(x-y)})B(k,y)dy \tag{10.17}$$

Suppose now that $k = \xi_i\eta$, $\eta > 0$. Then

$$|e^{2ik(y-x)}| = e^{-2\eta(y-x)}$$

In (10.15), $y > x$, so this exponent tends to zero as $y \to \infty$. In (10.17), $|e^{2ik(x-y)}| = e^{-\eta(x-y)}$. As far as $y < x$, this exponent also tends to zero as $\eta \to \infty$.
Hence, both function A, B could be analytically continued to the upper-half plane. They have the following asymptotic expansions.

$$B \to 1 - \frac{1}{2ik}\int_x^\infty u(y)dy \tag{10.18}$$

$$A \to 1 - \frac{1}{2ik}\int_{-\infty}^x u(y)dy \tag{10.19}$$

as $k \to \infty$ and $\Im k > 0$ and

$$\Psi \to e^{ikx}\left(1 - \frac{1}{2ik}\int_x^\infty u(y)dy\right) \tag{10.20}$$

$$\Phi \to e^{ikx}\left(1 - \frac{1}{2ik}\int_{-\infty}^x u(y)dy\right) \tag{10.21}$$

let $k = i\aleph_n$. Then

$$\Psi\Big|_{k=i\aleph_n} \to e^{-\aleph_n x} \qquad x \to \infty \tag{10.22}$$

$$\Phi\Big|_{k=i\aleph_n} \to e^{\aleph_n x} \qquad x \to -\infty \tag{10.23}$$

They represent the same eigen-function f_n and can differ only by some factor. Suppose that f_n is designed by asymptotic expansion

$$f_n \to e^{\aleph_n x} \qquad x \to -\infty \tag{10.24}$$

$$f_n \to b_n e^{\aleph_n x} \qquad x \to \infty \tag{10.25}$$

Hence,

$$f_n = \Phi\Big|_{k=i\aleph_n} = b_n \Phi\Big|_{k=i\aleph_n} \tag{10.26}$$

At this point, Ψ and Φ are proportional to each other. $\bar{\Psi}(k,x) = \Psi(-k,x)$ and $\bar{\Phi}(k,x) = \Phi(-k,x)$ are also solutions of equation (10.1). Thus, they are analytic on the lower-half plane. Solutions $\Psi, \bar{\Psi}$ comprise a fundamental system. Then one can put

$$\Phi(k,x) = a(k)\Psi(-k,x) + b(k)\Psi(k,x) \tag{10.27}$$

$$\Phi(-k,x) = b(-k)\Psi(-k,x) + a(-k)\Psi(k,x) \tag{10.28}$$

So,

$$a(-k) = \bar{a}(k) \qquad\qquad b(-k) = \bar{b}(k) \tag{10.29}$$

Note that

$$\{\Psi(k), \Psi(-k)\} = 2ik \qquad\qquad \{\Phi(k), \Phi(-k)\} = -2ik \tag{10.30}$$

Calculating $\{\Phi(k), \Phi(-k)\}$ by the use of (10.27), one finds

$$|a(k)|^2 - |b(k)|^2 = 1 \tag{10.31}$$

We will call $\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}$ a monodromy matrix. According to (10.31), this matrix is unimodular. Now from (10.27) and (10.30), we get

$$a(k) = \frac{1}{2ik}\{\Psi, \Phi\} \qquad\qquad b(k) = \frac{1}{2ik}\{\bar{\Psi}, \Phi\} \tag{10.32}$$

Hence, $a(k)$ is analytic in the upper-half plane. By using (10.26) into (10.32) one gets

$$a \to \frac{1}{2ik}\Big\{ik\Big(1 - \frac{1}{2ik}\int_{-\infty}^x u(y)dy + ...\Big) + \Big(1 - \frac{1}{2ik}\int_x^\infty u(y)dy\Big)\Big\} = 1 - \frac{1}{4k}\int_{-\infty}^\infty u(y)dy + ... \tag{10.33}$$

The scattering amplitude $c(k)$ is defined as follows

$$c(k) = \frac{b(k)}{a(k)} \tag{10.34}$$

Also, we define $d(k) = \frac{1}{a(k)}$ as the amplitude of penetration through the potential barrier. From (10.31), we obtain

$$|c(k)|^2 + |d(k)|^2 = 1 \tag{10.35}$$

This is the "unitary condition". By definition, the potential $u(x)$ is reflectionless if $b(k) \equiv 0$. In this case, $a(k)$ can be found explicitly form the conditions $|a(k)| = 1$ for real k, $a(-k) = \bar{a}(k)$ and $a(k) \to 1$ as $k \to \infty$; $a(k)$ is also analytic on the upper-half plane.
If $a(k)$ has no zeros in the upper-half plane, then $a(k) \equiv 1$. In virtue of the condition $a(-k) = \bar{a}(k)$, all zeros are located on the imaginary axis and they are exact eigenvalues $\aleph_n$. The function $a(k)$ can be presented as the product

$$a(k) = \prod_{m=1}^{n} \frac{k - i\aleph_n}{k + i\aleph_n} \tag{10.36}$$

For the reflectionless potential function

$$\frac{A(k,x)}{a(k)} = B(-k,x) \tag{10.37}$$

Chapter 11

Lecture 11

Consider the Non-linear Schrödinger equation

$$\Psi_{xx} + (k^2 - u)\Psi = 0 \tag{11.1}$$

and the Jost functions

$$\Phi \to e^{-ikx} \qquad x \to -\infty \tag{11.2}$$

$$\Psi \to e^{ikx} \qquad x \to \infty \tag{11.3}$$

Φ and Ψ are analytic in the upper-half plane.

$$\Phi = a(k)\Psi(-k) + b(k)\Psi(k) \tag{11.4}$$

where

$$a(k) = \frac{1}{2ik}\{\Psi, \Phi\} = -\frac{1}{2ik}\{\Phi, \Psi\} \tag{11.5}$$

is also analytic in the upper-half plane.

$$\Phi = e^{ikx}A(k,x) \qquad \Psi = e^{ikx}B(k,x) \tag{11.6}$$

and

$$A(k,x) = a(k) + B(k,x) \cdot b(k)e^{2ikx} \tag{11.7}$$

If $k = k_R + ik_I$ with $k_I > 0$, then

$$a(k) = \lim_{x\to\infty} A(k,x) \tag{11.8}$$

And

$$A_{xx} - 2ikA_x - uA = 0 \tag{11.9}$$

with boundary conditions

$$A \to 1 \qquad x \to -\infty \tag{11.10}$$

We now introduce the following substitution

$$A(k,x) = \exp\Big[\int_{-\infty}^{x} \chi(x', k)dx'\Big] \tag{11.11}$$

Which implies

$$\ln\left[a(k)\right] = \int_{-\infty}^{\infty} \chi(x,k)dx \tag{11.12}$$

where χ solves the following equation

$$\chi_x + \chi^2 - u = 2ik\chi \tag{11.13}$$

We then assume that the solution to the above equation takes on the form of a power series

$$\chi = \sum_{n=1}^{\infty} \frac{\chi_n}{(2ik)^n} \tag{11.14}$$

which after substitution we find

$$\chi_1 = -u \tag{11.15}$$

$$\chi_2 = -u_x \tag{11.16}$$

$$\chi_3 = -u_{xx} + u^2 \tag{11.17}$$

$$\chi_4 = -u_{xxx} + 2\frac{d}{dx}u^2 \tag{11.18}$$

$$\chi_5 = -u_{xxxx} + \frac{d^2}{dx^2}u^2 + 2u_{xx}u - 2u^3 \tag{11.19}$$

and in general

$$\chi_{n+1}(x) = \frac{d}{dx}\chi_n(x) + \sum_{l=1}^{n-1} \chi_l \chi_{n-l} \tag{11.20}$$

where all χ_n are real. However, χ can be separated in into its real and imaginary parts.

$$\chi = \chi_R + i\chi_I \tag{11.21}$$

and from (11.13) one obtains

$$\chi_{Ix} + 2\chi_I\chi_R - 2k\chi_R = 0 \tag{11.22}$$

$$\chi_R = \frac{1}{2}\frac{1}{\chi_I - k}\frac{d\chi_I}{dx} = \frac{1}{2}\frac{d}{dx}\ln\chi_I \tag{11.23}$$

We assume χ_R takes the form of a power series

$$\chi_R = \sum_{l=1}^{\infty} \frac{\chi_{2l}}{(2ik)^{2l}} \tag{11.24}$$

As far as χ_R is a complete derivative,

$$\int_{-\infty}^{\infty} \chi_{2l} dx = 0 \tag{11.25}$$

and we define I_n as follows

$$I_n = \int_{-\infty}^{\infty} \chi_{2n+1} dx \tag{11.26}$$

Then

$$I_1 = -\int_{-\infty}^{\infty} u dx = -N \tag{11.27}$$

$$I_2 = \int_{-\infty}^{\infty} u^2 dx = P \tag{11.28}$$

$$I_3 = -\int_{-\infty}^{\infty} (u_x^2 + 2u^3) dx = -2H \tag{11.29}$$

where

$$H = \int (\frac{1}{2}u_x^2 + u^3) dx \tag{11.30}$$

and

$$\frac{\delta H}{\delta u} = -u_{xx} + 3u^2 \tag{11.31}$$

By considering the following equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\frac{\delta H}{\delta u} \tag{11.32}$$

we obtain

$$u_t - 6uu_x + u_{xxx} = 0 \tag{11.33}$$

All I_n are motion integrals of the KdV equation.
Let's consider the pure solitonic (reflectionless) case

$$|a(k)|^2 - |b(k)|^2 = 1 \qquad\qquad b(k) = 0 \tag{11.34}$$

$$a(k) = \prod_n \frac{k - i\kappa_n}{k + i\kappa_n} \tag{11.35}$$

So,

$$ln\big[a(k)\big] = \sum_n ln(k - i\kappa_n) - ln(k + i\kappa_n) \tag{11.36}$$

$$= \sum_n \Big[ln(1 - i\frac{\kappa_n}{k}) - ln(1 + i\frac{\kappa_n}{k})\Big] \tag{11.37}$$

$$\tag{11.38}$$

recalling the series expansion for $ln(1+x)$, for each n, the above becomes

$$= 2\sum_l \frac{1}{2l+1}\Big(\frac{\kappa_n}{ik}\Big)^{2l+1} \tag{11.39}$$

$$= 2\sum_l \frac{2^{2l+1}}{2l+1}\Big(\frac{\kappa_n}{2ik}\Big)^{2l+1} \tag{11.40}$$

comparing the last equation above with (11.14) one obtains

$$\int_{-\infty}^{\infty} \chi_{2l+1}(x)dx = \frac{2^{2l+1}}{2l+1} \sum_{l} \kappa^{2l+1} \tag{11.41}$$

In particular,

$$N = \int_{-\infty}^{\infty} u dx = -4 \sum \kappa_n < 0 \tag{11.42}$$

$$P = \int_{-\infty}^{\infty} u^2 dx = 8 \sum \kappa_n^3 \tag{11.43}$$

$$H = \int_{-\infty}^{\infty} (\frac{1}{2} u_x^2 + u^3) dx = -\frac{32}{5} \sum \kappa_n^5 \tag{11.44}$$

in the pure solitonic (reflectionless) case, $N < 0$ and $H < 0$

Chapter 12

Lecture 12

In the next two lectures the following basic lemma will play an important role.

Theorem 1 (Basic lemma)**.** *Let $f_1(x), f_2(x)$ be two solutions of the Schrödinger equation with different values of k.*

$$f_{1xx} + (k_1^2 - u)f_1 = 0 \tag{12.1}$$

$$f_{2xx} + (k_2^2 - u)f_2 = 0 \tag{12.2}$$

Then

$$f_1 f_2 = \frac{1}{k_2^2 - k_1^2}\frac{d}{dx}\{f_1, f_2\} \tag{12.3}$$

where we recall that the Wronskian is

$$\{f_1, f_2\} = f_2\frac{df_1}{dx} - f_1\frac{df_2}{dx} \tag{12.4}$$

A proof of this lemma is straightforward. The basic lemma makes it possible to calculate Poisson brackets between different elements of scattering data. Let φ_1, φ_2 be to eigenfunctions of the Schrödinger operator.

$$\frac{\partial^2\varphi_1}{\partial x^2} - E_1\varphi_1 = u\varphi_1 \tag{12.5}$$

$$\frac{\partial^2\varphi_2}{\partial x^2} - E_2\varphi_2 = u\varphi_2 \tag{12.6}$$

for $\varphi_{1,2} \to 0$ as $|x| \to \infty$ Let's calculate the Possion bracket

$$\{E_1, E_2\} = \frac{1}{2}\int\Big(\frac{\delta E_1}{\delta u}\frac{\partial}{\partial x}\frac{\delta E_2}{\delta u} - \frac{\delta E_2}{\delta u}\frac{\partial}{\partial x}\frac{\delta E_1}{\delta u}\Big)dx \tag{12.7}$$

We start with the equation

$$\frac{d^2\varphi}{dx^2} - E\varphi(x) = u(x)\varphi(x) \tag{12.8}$$

And by calculating the variational derivative

$$G(x, z) = \frac{\delta\varphi(x)}{\delta u(z)} \tag{12.9}$$

Remember that $\frac{\delta u(x)}{\delta u(z)} = \delta(x-z)$. Then G satisfies the equation

$$\frac{d^2 G}{dx^2} - EG - uG = \frac{\delta E}{\delta u(z)}\varphi(x) + \delta(x-z)\varphi(x) \tag{12.10}$$

Then we multiply (12.8) by $G(x,z)$ and (12.10) by $\varphi(x)$ and subtract the two to obtain

$$\frac{\delta E}{\delta u(z)}\varphi^2(x) + \delta(x-z)\varphi^2(x) = \varphi\frac{\partial^2 G}{\partial x^2} - G\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial}{\partial x}\{G,\varphi\} \tag{12.11}$$

Integration of the above equation gives the result known from the theory of perturbation.

$$\frac{\delta E}{\delta u(z)} = -\varphi^2(z) \tag{12.12}$$

Hence,

$$\{E_1, E_2\} = \frac{1}{2}\int \Big[\varphi_2^2(x)\frac{\partial}{\partial x}\frac{\varphi_1^2(x)}{\partial x} - \varphi_1^2(x)\frac{\partial}{\partial x}\frac{\varphi_2^2(x)}{\partial x}\Big]dx \tag{12.13}$$

$$= \int \varphi_1\varphi_2\{\varphi_1,\varphi_2\}dx \tag{12.14}$$

$$= \frac{1}{E_1 - E_2}\int \frac{d}{dx}\{\varphi_1,\varphi_2\}\cdot\{\varphi_1,\varphi_2\}dx \tag{12.15}$$

$$= \frac{1}{E_1 - E_2}\frac{1}{2}\int \frac{d}{dx}\{\varphi_1,\varphi_2\}^2 dx = 0 \tag{12.16}$$

So that finally, we obtain

$$\{E_1, E_2\} = 0 \tag{12.17}$$

In other words, discrete eigenvalues of the Schrödinger equation commute.
Let's return to the Jost function $\Phi(x,k)$ satisfying

$$\frac{d^2\Phi}{dx^2} + k^2\Phi = u(x)\Phi \tag{12.18}$$

with the boundary condition

$$\Phi \to e^{-ikx} \tag{12.19}$$

as $x \to -\infty$. We denote

$$G(x,z) = \frac{\delta\Phi(x)}{\delta u(z)} \tag{12.20}$$

Then $G(x,z) = 0$ for $x < z$. It then satisfies the equation

$$\Big(\frac{d^2}{dx^2} + k^2\Big)G(x,z) = u(x)G(x,z) + \delta(x-z)\Phi(x) \tag{12.21}$$

But $\delta(x-z)\Phi(x) = \delta(x-z)\Phi(z)$. Hence,

$$G(x,z) = \Phi(z)K(x,z) \tag{12.22}$$

where $K(x, z)$ is the greens function of the Schrödinger operator that satisfies the equation

$$\Big(\frac{d^2}{dx^2} + k^2\Big)K(x, z) = u(x)K(x, z) + \delta(x - z) \tag{12.23}$$

$$K(x, x) = 0 \tag{12.24}$$

$$\frac{\partial K}{\partial x}\Big|_{x=z+\epsilon} = 1 \tag{12.25}$$

For $x > z$, K is a solution of the homogeneous Schrödinger equation, this solution must satisfy the above equations. That is

$$K(x, z) = \frac{1}{2ik}\Big[\Psi(x, k)\bar{\Psi}(z, x) - \bar{\Psi}(x, z)\Psi(z, x)\Big] \tag{12.26}$$

Hence,

$$G(x, z) = \frac{\Phi(z, x)}{2ik}\Big\{\Psi(x, k)\bar{\Psi}(z, x) - \bar{\Psi}(x, z)\Psi(z, x)\Big\} \tag{12.27}$$

$$G(x, z) \to \frac{1}{2ik}\Big\{ - \Psi(x, k)\bar{\Psi}(z, x)e^{-ikx} + \bar{\Psi}(x, z)\Psi(z, x)e^{ikx}\Big\} \tag{12.28}$$

On the other hand,

$$G(x, z) \to \frac{\delta a(k)}{\delta u(z)}e^{-ikx} + \frac{\delta b(k)}{\delta u(z)}e^{ikx} \tag{12.29}$$

We end up on with the remarkable result

$$\frac{\delta a(k)}{\delta u(z)} = -\frac{1}{2ik}\Phi(z, k)\Psi(z, k) \tag{12.30}$$

$$\frac{\delta b(k)}{\delta u(z)} = \frac{1}{2ik}\Phi(z, k)\bar{\Psi}(z, k) \tag{12.31}$$

And as expected, $\frac{\delta a(k)}{\delta u(z)}$ is analytic on the upper-half plane.

Chapter 13

Derivation of the Marchenko equation

Let's return to the Schrödinger equation

$$-\Psi'' + k^2\Psi = u\Psi \tag{13.1}$$

or

$$-\Psi'' + u\Psi = E\Psi$$

Consider the Jost functions

$$\begin{array}{ll} \Phi \to e^{-ikx} & x \to -\infty \\ \Psi \to e^{ikx} & x \to +\infty \end{array}$$

that are connected by the following relation

$$\Phi(k,x) = a(k,x)\Psi(x,-k) + b(k,x)\Psi(x,k) \tag{13.2}$$

where the functions Φ and Ψ are analytic in the upper half-plane. Where $a(k)$ is defined as

$$a(k) = \frac{1}{-2ik}[\Phi,\Psi]$$

and is also analytic. For each point in the discrete spectrum $a(k)$ has the zeros

$$a(i\kappa_n) = 0 \qquad\qquad \text{if } E_n = -\kappa_n^2 \tag{13.3}$$

$a(i\kappa) = 0$ and functions Ψ and Φ are also proportional to each other.

$$\Phi_n = c_n\Psi_n \qquad \text{where } \Phi_n = \Phi(i\kappa_n) \text{ and } \Psi_n = \Psi(i\kappa_n) \tag{13.4}$$

Now, Φ_n and Ψ_n are real eigen-functions with c_n which are real coefficients (positive or negative). The eigen-function Φ_n has the asymptotics

$$\begin{array}{ll} \Phi_n \to e^{\kappa_n x} & x \to -\infty \\ \Phi_n \to c_n e^{-\kappa_n x} & x \to +\infty \end{array}$$

And has a finite norm:

$$\int_{-\infty}^{\infty} |\Phi_n|^2 dx = \|\Phi_n\|^2 \tag{13.5}$$

We consider the following important question: how are c_n and $\|\Phi_n\|^2$ connected?
Let's differentiate equation (13.1) to obtain

$$\Phi_k'' + k^2\Phi_k = u\Phi_k - 2k\Phi \tag{13.6}$$

Let $J = \{\Phi, \Phi_k\}$ be a Wronskian. From equations (13.1) and (13.6), one obtains

$$\frac{d}{dx}\{\Phi, \Phi_k\} = -2k\Phi^2 \tag{13.7}$$

If $k = i\kappa_n$, this equation can be integrated.

$$-2i\kappa_n\|\Phi_n\|^2 = \lim_{x\to\infty}\{\Phi_n, \Phi_k\} \tag{13.8}$$

Indeed, $\{\Phi, \Phi_k\} \to 0$ as $x \to -\infty$. We also know that $\Phi_n \to c_n e^{-\kappa_n x}$ so that equation (13.8) means that

$$\Phi_k\Big|_{k=i\kappa} = \alpha e^{\kappa_n x}$$

Our problem is now in determining the value of α. To do this, we return to equation (2 - undefined) and will analytically continue this equation to the upper half-plane. In a general case, $\Psi(k, x)$ can be continued only to the lower half-plane, while $b(k)$ does not admit an analytic continuation from the real axis at all, but we will use a special trick. We will replace the potential u in (13.1) with $\tilde{u}$ in the following way.

$$u \to \tilde{u} = \begin{cases} u & |x| < L \\ 0 & |x| > L \end{cases} \tag{13.9}$$

Eventually, we will take the limit $L \to \infty$. However, for any finite value of L, all functions Φ, Ψ are entire functions having no singularities on the whole complex plane. The same is true for coefficients $a(k), b(k)$. If L is large enough, $a(\kappa_n)$ has the same number of zeros, but are slightly shifted points. i.e.

$$\kappa_n(L) \to \kappa_n \qquad\qquad \text{as } L \to \infty$$

Then we denote $b_L(i\kappa_{nL})$.
The function $b_L(k)$ might acquire singularities if $L \to \infty$, but the value of $b_L(i\kappa)$ becomes finite. Moreover,

$$\lim_{L\to\infty} b_L(i\kappa_n(L)) = c_n \tag{13.10}$$

Since $a(i\kappa_n(L)) = 0$, we obtain the result

$$\Phi_k(x)\Big|_{k=i\kappa} = a_k' e^{\kappa_n(L)x} + \frac{d}{dk}b(k)e^{-\kappa_n(L)x} \tag{13.11}$$

Then we take into account that in this approximation (for $|x| > L$)

$$\Phi_n(x) = b(i\kappa_n(L))e^{\kappa_n(L)x} \tag{13.12}$$

The second term in (13.11) vanishes as $x \to \infty$. Hence, we get

$$\lim_{x\to\infty}\{\Psi, \Phi\} = a'(i\kappa)b(i\kappa)(2\kappa_n) \tag{13.13}$$

Now we let $L \to \infty$ and end up with the answer

$$\|\Phi_n\|^2 = \int_{-\infty}^{\infty} |\Phi|^2 dx = +ia'(i\kappa_n)c_n \tag{13.14}$$

The above equation proves that $ia'(\kappa_n)$ is a real number. Let's check this fact for the pure solitonic case.
Let the eigenvalues be ordered as follows

$$\kappa_1 > \kappa_2 > ... > \kappa_N$$

Then

$$a(k) = \prod \frac{k - i\kappa_n}{k + i\kappa_n} \tag{13.15}$$

$$\implies ia'(i\kappa_n) = \frac{1}{2\kappa_n} \prod_{m \neq n} \frac{\kappa_n - \kappa_m}{\kappa_n + \kappa_m} \tag{13.16}$$

Thus, one can see that $ia'(i\kappa_1) > 0$, $ia'(i\kappa_2) < 0$, while $c_1 > 0$ and $c_2 > 0$. $ia'(i\kappa_n)$ and c_n are affinely intermittent real numbers and their product is always positive.
Now mentioning that $-a'(i\kappa_n)^2 < 0$ is a negative real number. Then the quantity

$$\frac{ic_n}{a'(i\kappa)} = -M_n^2 = \frac{1}{(a'(i\kappa))^2}\|\Phi_n\|^2 \tag{13.17}$$

is a set of negative real numbers.

Chapter 14

The Marchenko Equation (continued)

Remember that

$$\Phi(x,k) = A(x,k)e^{-ikx} \tag{14.1}$$
$$\Psi(x,k) = B(x,k)e^{ikx} \tag{14.2}$$

where the functions $A(x,k)$ and $B(x,k)$ are analytic in the upper half-plane and are connected by the following relation

$$A(x,k) = a(k)B(x,-k) + e^{-2ikx}b(k)B(x,k) \tag{14.3}$$

dividing both sides of (14.3) gives

$$\frac{A(x,k)}{a(k)} = B(x,-k) + e^{-2ikx}c(k)B(x,k) \tag{14.4}$$

Here, $c(k)$ is the *reflection coefficient*. In virtue of the relation $|a|^2 - |b|^2 = 1$, we get

$$|c|^2 \leq 1 \tag{14.5}$$

Let's introduce the function χ as follows

$$\chi = \begin{cases} \frac{A(k)}{a(k)} & \Im k > 0 \\ B(-k) & \Im k < 0 \end{cases} \tag{14.6}$$

Then the function χ is analytic on the whole complex plane except for the real axis. Moreover, it has simple poles on the positive imaginary axis.
One can denote

$$\chi^+(k,x) = \lim_{\epsilon\to 0} \frac{A(x,k+i\epsilon)}{a(k+i\epsilon)} \tag{14.7}$$

and similarly,

$$\chi^-(k,x) = B(-k+i\epsilon) \tag{14.8}$$
$$\chi^+(-k,x) = B(x,k) \tag{14.9}$$

Then the relation

$$\frac{A(x,k)}{a(k)} - B(x,-k) = e^{-2ikx}c(k)B(x,k) \tag{14.10}$$

can be written as follows. The function χ satisfies the *Riemann-Hilbert problem*

$$\chi^{+}(k) - \chi^{-}(k) = e^{-2ikx}c(k)\chi^{+}(-k,x) \tag{14.11}$$

and the function χ is a solution of the following equation

$$\frac{\partial^2 \chi}{\partial x^2} + 2ik\frac{\partial \chi}{\partial x} = u\chi \tag{14.12}$$

It has the following asymptotic expansion at infinity

$$\chi = 1 + \frac{\chi_0}{ik} + \frac{\chi_1}{(ik)^2} + ... \tag{14.13}$$

After substitution of the above expansion into (14.12), the constant order equation reveals

$$u = 2\frac{\partial \chi_0}{\partial x} \tag{14.14}$$

Now remember that

$$B(k) \to 1 - \frac{1}{2ik}\int_x^{\infty} u(y)dy \tag{14.15}$$

Let's calculate the following integral

$$K(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(B(k) - 1\Big)e^{ik(x-y)}dk \tag{14.16}$$

so $B(k) - 1 \to 0$ when $k \to \infty$ and is analytic in the upper half-plane. Then if $x > y$, the integral can be closed over the upper half-plane. Hence,

$$K(x,y) \equiv 0 \quad \text{if } x > y \tag{14.17}$$

(This is known as the *Paley-Wiener Theorem*).
Returning to Jost function Ψ, we realize that it has the following (famous) triangle representation

$$\Psi(x,k) = e^{ikx} + \int_x^{\infty} K(x,y)e^{iky}dy \tag{14.18}$$

After setting $k = i\kappa_n$ in the above equation, we find that the eigen-function

$$\Psi_n = \Psi(x, i\kappa_n) = e^{-\kappa_n x} + \int_x^{\infty} K(x,y)e^{-\kappa_n y}dy \tag{14.19}$$

Hence,

$$\Phi_n = c_n\Big(e^{-\kappa_n x} + \int_x^{\infty} K(x,y)e^{-\kappa_n y}dy\Big) \tag{14.20}$$

Also notice that

$$\Psi(x,-k) = e^{-ikx} + \int_x^\infty K(x,y)e^{-iky}dy \tag{14.21}$$

So that the basic relationship between Φ and Ψ can be rewritten as follows

$$\frac{\Phi(x,k)}{a(k)} - e^{-ikx} = \int_x^\infty K(x,y)e^{-iky}dy + c(k)\Big[e^{ikx} + \int_x^\infty K(x,y)e^{iky}dy\Big] \tag{14.22}$$

Let's multiply this equation on both sides by $\frac{1}{2\pi}e^{ikz}$ for $z > x$ and integrate in k. Since

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(z-x)}dx = \delta(x-z)$$

, the right hand side of the integral is easily evaluated

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\frac{\Phi(x,k)}{a(k)} - e^{-ikx}\Big]e^{ikz}dk = K(x,z) + \tilde{F}(x+z) + \int_x^\infty K(x,s)\tilde{F}(s+z)ds \tag{14.23}$$

The left hand side is the sum of residues

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\frac{\Phi(x,k)}{a(k)} - e^{-ikx}\Big]e^{ikz}dk = i\sum\frac{\Phi_n(x)}{a'(i\kappa_n)} \tag{14.24}$$

Now we use equation (14.20) and equation (13.17) from the previous lecture to derive the famous *Marchenko equation*

$$K(x,z) + F(x+z) + \int_x^\infty K(x,s)F(s+z)dz = 0 \tag{14.25}$$

Here,

$$F(\xi) = \frac{1}{2}\int_{-\infty}^{\infty} c(k)e^{ik\xi}dk + \sum M_n^2 e^{-\kappa_n\xi} \tag{14.26}$$

This equation is completely equivalent to the Riemann-Hilbert problem presented in Chapter 6.

$$\frac{\partial\chi}{\partial\bar{k}} = R(k)\chi(-k)e^{-2ikx} \tag{14.27}$$

Where the dressing function R consists of two parts

$$R(k) = \sum M_n^2\delta(k-i\kappa_n) + c(k_R)\delta(k_I)\theta(-k_I) \tag{14.28}$$

where we denote

$$k = k_R + ik_I \tag{14.29}$$

$$\theta(-k_I) = \begin{cases} 1 & k_I > 0 \\ 0 & k_I < 0 \end{cases} \tag{14.30}$$

Chapter 15

The NLS equation

Let Ψ be a complex-valued $N \times N$ matrix satisfying the following system of compatible equations

$$\Psi_x = \hat{U}\Psi \tag{15.1}$$
$$i\Psi_t = \hat{V}\Psi \tag{15.2}$$

Here, $\hat{U} = \hat{U}(x,t,\lambda)$ and $\hat{V} = \hat{V}(x,t,\lambda)$ are rational functions of λ.

The compatibility conditions for the above system is

$$i\hat{U}_t - \hat{V}_x + \left[\hat{U}, \hat{V}\right] = 0 \tag{15.3}$$

Suppose,

$$\hat{U} = I\lambda + u \tag{15.4}$$
$$\hat{V} = \lambda\hat{U} + v = I\lambda^2 + u\lambda + v \tag{15.5}$$

where u, v do not depend on λ. Then equation (15.3) reads

$$v_x = \left[I, u\right] \tag{15.6}$$
$$u_t - v_x + \left[u, v\right] = 0 \tag{15.7}$$

Thereafter, we assume, $N = 2$. and we define I and u in the following way.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad u = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} \tag{15.8}$$

From equation (15.7) one gets

$$\begin{aligned} &\hat{V} \begin{bmatrix} A & p \\ q & -A \end{bmatrix} \qquad v = 0 \\ &\qquad\qquad w = \begin{bmatrix} B & w_{12} \\ w_{21} & -B \end{bmatrix} \end{aligned} \tag{15.9}$$

Where A, B are some unknown functions. Notice then that

$$\left[u, v\right] = -2A \begin{bmatrix} 0 & p \\ -q & 0 \end{bmatrix} \tag{15.10}$$

From equation (UNKNOWN) one obtains $A_x = 0$. One can put $A = 0$, $v = u$. Now,

$$\hat{V} = \lambda \hat{U} + w \tag{15.11}$$

Equation (7) is

$$\big[I, w\big] = u_x \tag{15.12}$$

Hence,

$$w_{12} = \frac{1}{2} p_x \tag{15.13}$$

$$w_{21} = -\frac{1}{2} q_x \tag{15.14}$$

Then

$$\big[u, v\big] = -2B \begin{bmatrix} 0 & p \\ -q & 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}(pq)_x & 00 & \frac{1}{2}(pq)_x \end{bmatrix} \tag{15.15}$$

Hence,

$$-B_x - \frac{1}{2}(pq)_x = 0 \tag{15.16}$$

$$\implies B = -\frac{1}{2} pq \tag{15.17}$$

Then equation (8 - UNKNOWN) is reduced to the following PDE

$$ip_t - \frac{1}{2} p_{xx} + p^2 q = 0 \tag{15.18}$$

$$iq_t + \frac{1}{2} q_{xx} - pq^2 = 0 \tag{15.19}$$

Equations (15.18) and (15.19) admit the following basic reductions

1. $q = \bar{p}$
   Now p obeys the "defocusing nonlinear Schrödinger Equation" (defocusing NLSE)

$$ip_t + \frac{1}{2} p_{xx} + |p|^2 p = 0 \tag{15.20}$$

2. $q = -\bar{p}$
   In this case, p obeys the focusing NLSE

$$ip_t - \frac{1}{2} p_{xx} - |p|^2 p = 0 \tag{15.21}$$

Both equations are Hamiltonian systems. They can be written in the form of

$$ip_t + \frac{\delta H}{\delta \bar{p}} = 0 \tag{15.22}$$

In the defocusing case,

$$H = \frac{1}{2} \int |p_x|^2 dx + \frac{1}{2} \int |p|^4 dx \tag{15.23}$$

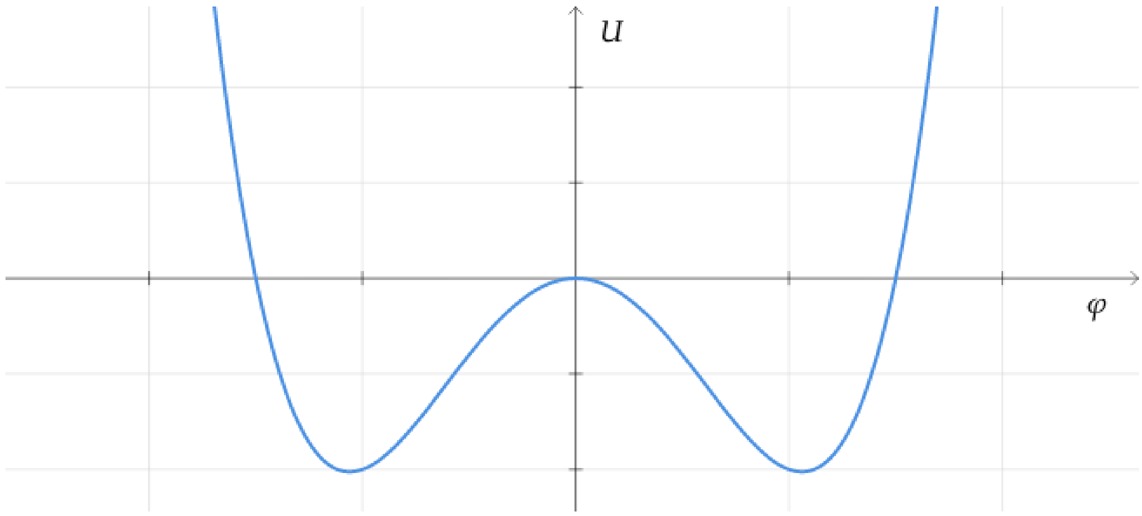

Figure 15.1: Potential well given by U(φ)

And in the focusing case,

$$H = \frac{1}{2}\int |p_x|^2 dx - \frac{1}{2}\int |p|^4 dx \tag{15.24}$$

the focusing equation (15.21) has a rich family of solitonic solutions. The simplest ones are resting solitons

$$p = \varphi(x) e^{-\frac{i}{2}\lambda^2 t} \tag{15.25}$$

Note that $\varphi(x)$ satisfies the equation

$$\varphi'' - \lambda^2\varphi + 2\varphi^3 = 0 \tag{15.26}$$

This equation (15.26) describes motion of a particle inside of the potential well

$$\varphi'' + \frac{\partial U}{\partial p} = 0 \tag{15.27}$$

where

$$U = -\frac{1}{2}\lambda^2\varphi^2 + \frac{1}{2}\varphi^4$$

Equation (15.27) has the motion integral

$$\frac{1}{2}\varphi_x^2 + U = E$$

We will set $E = 0$. Then the equation

$$\varphi_x^2 - \lambda^2\varphi + \varphi^4 = 0$$

has the following exact solution

$$\varphi = \frac{\lambda}{cosh(\lambda x)}$$

This solution can be generalized

$$p = \frac{\lambda}{cosh(\lambda(x - x_0))} e^{i\xi}$$

where x_0 and ξ are arbitrary real numbers.
An even more general solution can be found in the form of

$$\Psi = e^{i(ax+bt)} \varphi(x - vt)$$

The real function $\varphi(y)$ satisfies the equation

$$(-b + ia^2)\varphi - i(v + a)\varphi_y = \frac{1}{2}\varphi_{yy} + \varphi^3$$

We will set

$$a = -v \qquad \frac{1}{2}a^2 - b = \frac{1}{2}\lambda^2 \implies b = \frac{1}{2}(v^2 - \lambda^2)$$

Then the expression

$$\Psi = \frac{\lambda e^{-ivx + \frac{i}{2}(v^2 - ^2)t + i\xi}}{cosh(\lambda(x - vt - x_0))}$$

is a general solitonic solution of the focusing NLSE. It depends on four arbitrary parameters λ, v, x_0, ξ. A soliton can move with an arbitrary velocity in both directions.

Chapter 16

Condensate and its stability

Both focusing and defocusing equations can be written in the unified form

$$i\Psi_t = \frac{1}{2}\Psi_{xx} + s|\Psi|^2\Psi \tag{16.1}$$

with $s = \pm 1$.
Let us separate the amplitude and phase. i.e.

$$\Psi = Ae^{-i\phi} \tag{16.2}$$
$$\Psi_t = (A_t - iA\phi_t)e^{-i\phi} \tag{16.3}$$
$$\Psi_x = (A_x - iA\phi_x)e^{-i\phi} \tag{16.4}$$
$$\Psi_{xx} = (A_{xx} - 2iA\phi_x - iA\phi_{xx} - A\phi_x^2)e^{-i\phi} \tag{16.5}$$

Now vector equation (16.1) is split into two scalar equations

$$A_t + A_x\phi_x + \frac{1}{2}A\phi_{xx} = 0 \tag{16.6}$$
$$A(\phi_t + \frac{1}{2}\phi_x^2) - sA^3 - \frac{1}{2}A_{xx} = 0 \tag{16.7}$$

One can introduce the substitution

$$\rho = A^2$$

so that equation (16.6) takes form of the continuity equation

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}\rho\phi_x = 0 \tag{16.8}$$

and equation (16.7) becomes the Bernoulli equation with an additional dispersive term

$$\phi_t + \frac{1}{2}\phi_x^2 - s\rho - \frac{1}{2\sqrt{\rho}}(\sqrt{\rho})_{xx} = 0 \tag{16.9}$$

Equations (16.8) and (16.9) are Hamiltonian

$$\frac{\partial\rho}{\partial t} = \frac{\delta H}{\delta\phi} \tag{16.10}$$
$$\frac{\partial\phi}{\partial t} = \frac{\delta H}{\delta\rho} \tag{16.11}$$

With

$$H = \frac{1}{2}\int (\sqrt{\rho})_x^2 dx - \frac{1}{2}s\int \rho^2 dx + \frac{1}{2}\int \rho\phi_x^2 dx \tag{16.12}$$

The simplest solution of equation (16.1) is the condensate

$$\Phi = A_0 e^{-is|A_0|^2 t} \tag{16.13}$$

where A_0 is an arbitrary complex number $A_0 = |A_0|e^{i\xi}$. Thereafter we will assume A_0 to be real. In the case where $\phi = sA_0^2$, one can replace $\phi \to sA_0^2 + \phi$, then equation (16.7) transforms into

$$A(\phi_t + \frac{1}{2}\phi_x^2) - sA(A^2 - A_0^2) - \frac{1}{2}A_{xx} = 0 \tag{16.14}$$

Let's suppose $A = A_0(1+u)$, where $u << 1$ and we will consider $\phi_x << 1$. Then equations (16.6) and (16.14) simplify to the following linear system

$$u_t + \frac{1}{2}\phi_{xx} = 0 \tag{16.15}$$

$$\phi_t - 2sA_0^2 u - \frac{1}{2}u_{xx} = 0 \tag{16.16}$$

Assuming plane wave solutions,

$$u, \phi \simeq e^{-iwt+ipx}$$

one can easily find the dispersion relation.

$$w^2 = (-sA_0^2 + \frac{1}{4}p^2)p^2 \tag{16.17}$$

This is the famous *Bogolyubov* formula.
In the defocusing case, $s = -1$,

$$w^2 = (A_0^2 + \frac{1}{4}p^2)p^2 \tag{16.18}$$

with asymptotics $w \to \frac{1}{2}p^2$ as $p \to \infty$ and $w \to A_0 p$ as $p \to 0$.
In the focusing case, $s = 1$

$$w^2 = (-A_0^2 + \frac{1}{4}p^2)p^2$$

The condensate is unstable if $p^2 < 4A_0^2$. This is the modulational instability. The growth-rate maximum is achieved at $p^2 = 2A_0^2$. And at this point, $\Im w = A_0^2$.

Equation (16.1) describes the one-dimensional Bose-gas in the classical wave limit. The interaction between particles is extremely short-scale. If $s = -1$, the interaction is repelling in the Bose-condensate and it is a stable interaction. If $s = 1$, the interaction is attraction and the Bose-condensate becomes unstable.

I recommend to my class the following paper. "Super solitonic solutions: a novel scenario for the nonlinear shape of modulational instability" A.A. Gelash and V.E. Zakharov. Nonlinearity 27 (2014) R1-R39.

Chapter 17

Dark Solitons

In this lecture, we study solitons in the defocusing NLSE. They exist only on a nontrivial condensate. We start with the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho\phi_x = 0 \tag{17.1}$$

and we will assume that $\rho = \rho(x - vt)$, $\phi_x = \phi_x(x - vt)$ and $\phi_t = -v\phi_x$.

$$\frac{\partial \rho}{\partial t} = -v\frac{\partial \rho}{\partial x} \tag{17.2}$$

where $\rho \to \rho_0$ at $x \to \pm\infty$. Thereafter, we denote $u := \phi_x$.
Now we assume that $u \to 0$ as $x \to \pm\infty$. The continuity equation gives

$$\frac{\partial}{\partial x}\rho u = v\frac{\partial \rho}{\partial x} \tag{17.3}$$

Or

$$\rho u = v(\rho - \rho_0) \tag{17.4}$$

$$\implies u = v(1 - \frac{\rho_0}{\rho}) = v(1 - \frac{A_0^2}{A^2}) \tag{17.5}$$

The Bernoulli equation gives

$$A\Big(-vu + \frac{1}{2}u^2\Big) + A\Big(A^2 - A_0^2\Big) - \frac{1}{2}A_{xx} = 0 \tag{17.6}$$

And after some simple calculations, we end up with

$$A_{xx} - 2A^3 + \Big(v^2 + 2A_0^2\Big)A - \frac{v^2A_0^4}{A^2} = 0 \tag{17.7}$$

This equation can be rewritten as follows

$$A_{xx} + \frac{\partial V}{\partial A} = 0 \tag{17.8}$$

So that

$$V = \Big(\frac{1}{2}v^2 + A_0^2\Big)A^2 - \frac{1}{2}A^4 + \frac{1}{2}\frac{v^2A_0^4}{A^2} \tag{17.9}$$

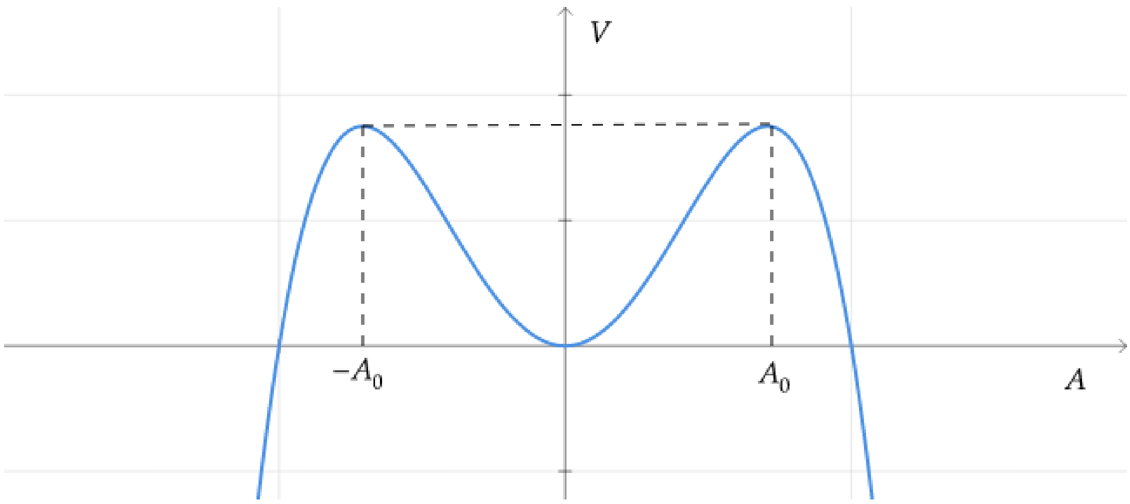

Figure 17.1: Potential well for $v = 0$.

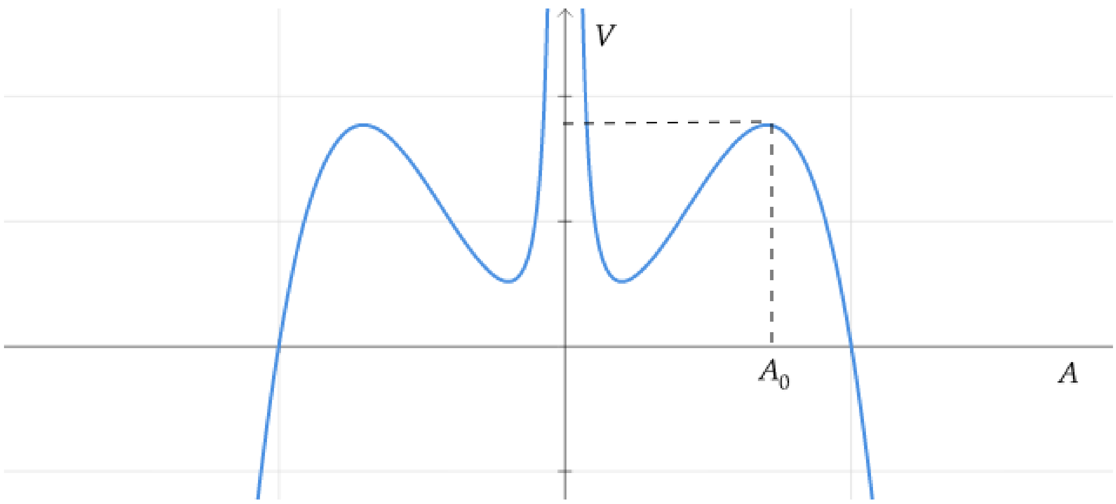

Figure 17.2: Potential well for arbitrary v. This describes the motion of a particle in a radially symmetric field.

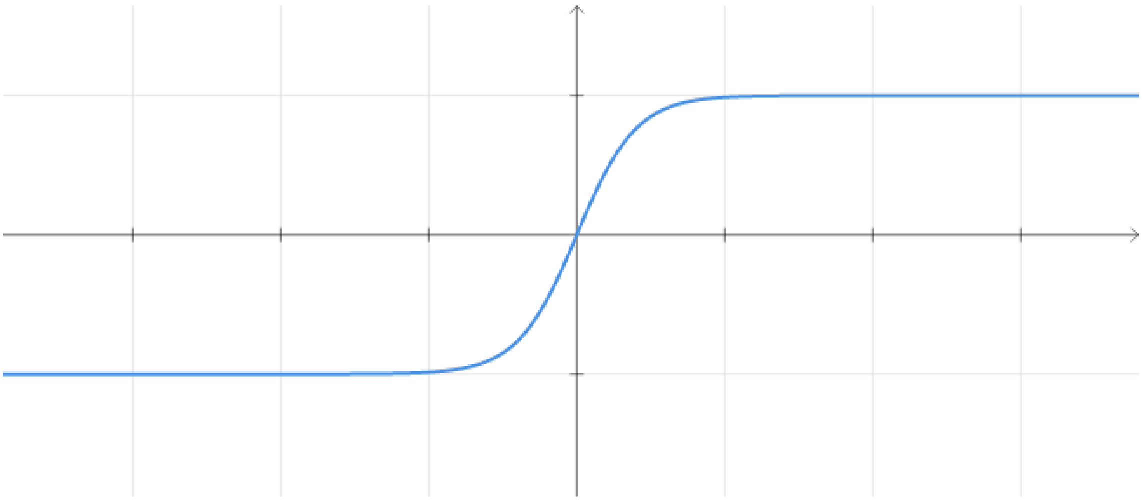

Figure 17.3: a Kink solution

Energy of the system can be written in the form

$$\frac{1}{2}A_x^2 + V(A) = E \tag{17.10}$$

for the solitonic solution, we seek $E = V(A_0)$

$$E = \Big(\frac{1}{2}v^2 + A_0^2\Big)A_0^2 - \frac{1}{2}A_0^4 + \frac{1}{2}v^2 A_0^2 = v^2 A_0^2 + \frac{1}{2}A_0^2$$

We end up with the following first order ODE

$$\frac{1}{2}A_x^2 + \frac{1}{2}v^2\Big(A^2 + \frac{A_0^4}{A^2} - 2A_0^2\Big) - \frac{1}{2}(A^- A_0^2)^2 = 0 \tag{17.11}$$

This equation can be exactly integrated and describes dark solitons.
In the case that $v = 0$, one obtains

$$A_x^2 = (A_0^2 - A^2)^2 \tag{17.12}$$

This equation has a solution

$$A = A_0 tanh\Big[A_0(x - x_0)\Big] \tag{17.13}$$

This is known as a kink (resting) But in a general case, the dark soliton has a form of It could propagate with an arbitrary velocity less than the sound velocity $v < A_0$. When $v \to A_0$, the soliton is described by the KdV equation.

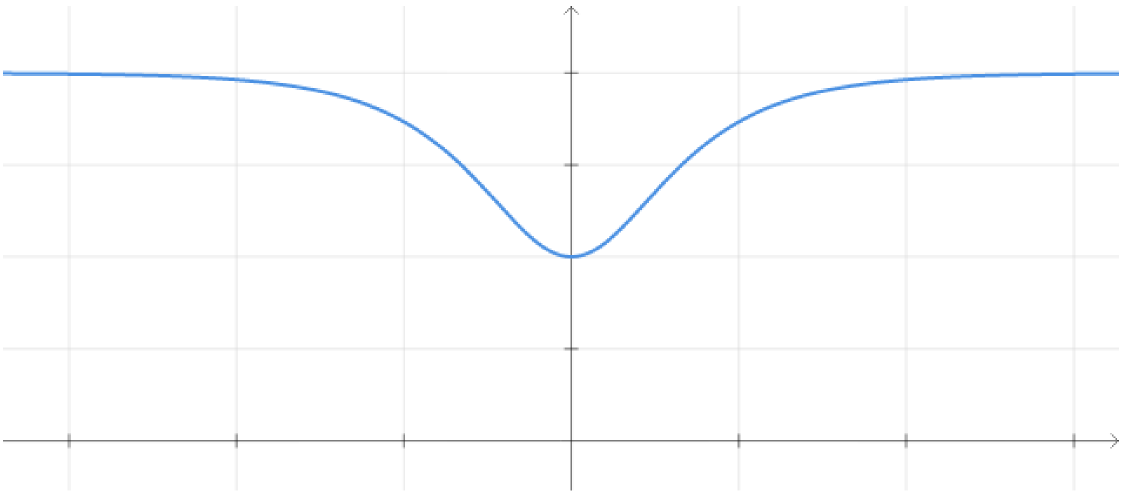

Figure 17.4: General form of a dark soliton. This profile propagates to the right at constant velocity