EE2703: Applied Programming Lab
Assignment 5
Laplace equation

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1 Aim:

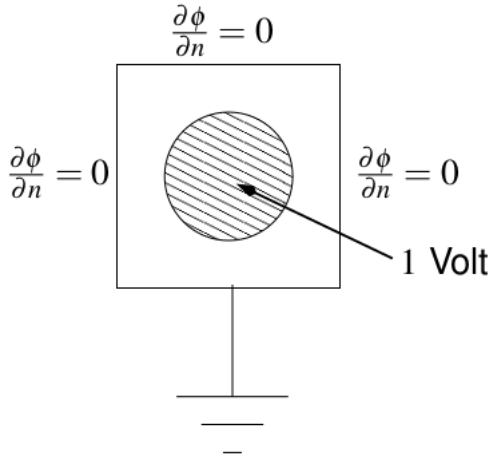
The aim of this assignment is to:

- Solve the Laplace equation of the electric potential (ϕ) in a resistor and hence, find the current density distribution.
- Analyse the potential and temperature profile of the resistor.
- Apply the "Average of neighbors" method to solve the Laplace equation and analyse the error present in the method.

2 Theory:

2.1 The resistor problem:

Let us consider a copper plate of dimensions 1 cm by 1 cm with a wire soldered in the middle portion of the plate. Also, the wire is maintained at a constant potential of 1 volt, the bottom edge of the plate is grounded and the other edges of the plate are floating.



We have the following equations:

1. $\vec{j} = \sigma \vec{E}$ (Ohm's law)
2. $\vec{E} = -\nabla \phi$ (Field-potential relation)
3. $\nabla . \vec{j} = -\frac{\partial \rho}{\partial t}$ (Continuity equation)

Clubbing up these equations, we have:

$$\nabla^2\phi = \frac{1}{\sigma}\frac{\partial\rho}{\partial t} \text{ (Assuming constant conductivity)}$$

For DC currents, we have:

$$\nabla^2\phi = 0 \text{ (Laplace equation)}$$

2.2 Solving the Laplace equation:

For a 2-D surface, the Laplace equation can be written as:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Assuming the neighbouring potential values are known, we can write:

$$\frac{\partial\phi}{\partial x}\Big|_{(x_i,y_j)} = \frac{\phi(x_{i+1/2}, y_j) - \phi(x_{i-1/2}, y_j)}{\Delta x}$$

and hence:

$$\frac{\partial^2\phi}{\partial x^2}\Big|_{(x_i,y_j)} = \frac{\phi(x_{i+1}, y_j) - 2\phi(x_i, y_j) + \phi(x_{i-1}, y_j)}{(\Delta x)^2}$$

Similarly, for the partial derivative with respect to y, we have:

$$\frac{\partial^2\phi}{\partial y^2}\Big|_{(x_i,y_j)} = \frac{\phi(x_i, y_{j+1}) - 2\phi(x_i, y_j) + \phi(x_i, y_{j-1})}{(\Delta y)^2}$$

Assuming same stepsize in both the axes (Δx=Δy), we have:

$$\phi_{i,j} = \frac{\phi_{i,j+1} + \phi_{i,j-1} + \phi_{i+1,j} + \phi_{i-1,j}}{4}$$

This solution is the called as the **Average of neighbours.** If there exists a discrete solution for the potential, it should satisfy this relationship. So, the method is clear. We have to guess a solution and carry out the averaging process until we reach a convergence point (Max. error $<$ Threshold).

2.3 Boundary conditions:

The following boundary conditions are given:

1. The soldered wire is always maintained at potential of 1 volt. (The circular region in the middle)

2. The bottom edge of the plate is grounded, which means potential is 0 volt there.

3. The other three edges are floating, meaning current can't flow out of them into the air. Hence, the normal derivative of potential ($\frac{\partial \phi}{\partial n}$) is 0 at those edges.

2.4 Current density distribution:

There will be current flowing from the soldered wire to the grounded edge because of a potential drop. However, the current distribution will be non-uniform because of the non-uniform potential profile. Current density (J) is given by:

$$\vec{J} = -\sigma(\nabla\phi)$$

Based on the formula we had for derivatives, we can write:

$$J_{x,ij} = \sigma\frac{\phi_{i,j-1} - \phi_{i,j+1}}{2(\Delta x)}$$

$$J_{y,ij} = \sigma\frac{\phi_{i-1,j} - \phi_{i+1,j}}{2(\Delta y)}$$

For simplicity, let's assume $\sigma = stepsize = \Delta x = \Delta y$. Hence, we will have:

$$J_{x,ij} = \frac{\phi_{i,j-1} - \phi_{i,j+1}}{2}$$

$$J_{y,ij} = \frac{\phi_{i-1,j} - \phi_{i+1,j}}{2}$$

2.5 Temperature profile:

Due to passage of currents, Joule's heating takes place and heat is liberated. This heat increases the temperature of the copper plate. However, this heating is not uniform and hence, the plate won't be in thermal equilibrium. Instead, it'll have a temperature profile. Temperature (T) will satisfy the following equation:

$$\nabla \cdot (k\nabla T) = -\frac{1}{\sigma}|J|^2$$

Using the Averaging technique, we will have:

$$\frac{T(x_{i+1}, y_j) - 2T(x_i, y_j) + T(x_{i-1}, y_j)}{(\Delta x)^2} + \frac{T(x_i, y_{j+1}) - 2T(x_i, y_j) + T(x_i, y_{j-1})}{(\Delta y)^2} = -\frac{1}{\sigma k}|J|^2$$

From this we will have:

$$T_{i,j} = \frac{T_{i,j+1} + T_{i,j-1} + T_{i+1,j} + T_{i-1,j}}{4} + \frac{(\Delta x)^2}{4\sigma k}|J|^2$$

We know from the previous section that $\sigma = stepsize = \Delta x = \Delta y$. To simplify the equation, let's also assume that $\sigma = k$ (Electrical conductivity = Thermal conductivity numerically). Hence, we have:

$$T_{i,j} = \frac{T_{i,j+1} + T_{i,j-1} + T_{i+1,j} + T_{i-1,j}}{4} + \frac{1}{4}|J|^2$$

with J obtained from the gradient of potential (ϕ).

The boundary conditions for temperature are:

1. The soldered wire is maintained at 300 K (Room temperature).
2. The grounded edge is maintained at 300 K (Room temp.).
3. The normal gradient of temperature at the other three edges is 0.
4. Initially, the plate was at thermal equilibrium (The temperature was 300 K initially all over the plate).

3 Procedure:

3.1 Variables used:

1. Nx = Number of X-axis divisions (No. of columns)
2. Ny = Number of Y-axis divisions (No. of rows)
3. rad = Radius of the central wire (In terms of the number of units)
4. Niter = Number of iterations till which the averaging process is done

In my code, the stepsize is assumed to be same in both the axes. Hence, the dimensions ratio of the plate is dictated by the values of Nx, Ny. So, let's say the user wants a plate of 50cm by 50 cm to be analysed using a stepsize of 0.7 cm, he should enter same values for Nx and Ny. And that value should satisfy:

$$\frac{50}{Nx} \leq 0.7$$

In general, the user should enter Nx, Ny values such that:

$$\frac{X - length}{Nx} \leq Stepsize$$

$$\frac{Y - length}{Ny} \leq Stepsize$$

The user cannot expect a square plate to be divided into a 5 x 7 grid in my code because that leads to different stepsizes in the axes and the code is not suited to work for different stepsizes. The default values are Nx=25, Ny=25, rad=8 and Niter=1500. All the plots which are present in this report are generated with these default values.

3.2 Solving the Laplace equation:

The potential and the temperature profiles are obtained using the Averaging the neighbors algorithm. The following code snippet does that:

```
for k in range(Niter):

    oldphi = phi.copy()
    phi[1:Ny-1,1:Nx-1] = 0.25*(Sum of neighbours)
    phi[0,:Nx]=0.0; phi[:Ny,0]=phi[:Ny,1];
    phi[:Ny,Nx-1]=phi[:Ny,Nx-2]; phi[Ny-1,:Nx]=phi[Ny-2,:Nx]
    phi[ii] = 1.0
    error[k]=np.max((np.abs(phi-oldphi)))

    Jx = 0.5*(phi[1:Ny-1,0:Nx-2]-phi[1:Ny-1,2:Nx]);
    Jy = 0.5*(phi[0:Ny-2,1:Nx-1]-phi[2:Ny,1:Nx-1])
    heats = np.multiply(Jx,Jx)+np.multiply(Jy,Jy)

    T[1:Ny-1,1:Nx-1] = 0.25*(Sum of neighbours+heats)
    T[0,:Nx]=300.0; T[:Ny,0]=T[:Ny,1];
    T[:Ny,Nx-1]=T[:Ny,Nx-2]; T[Ny-1,:Nx]=T[Ny-2,:Nx]
    T[ii] = 300.0

"""
In the actual code, the sum of neighbours is actually calculated
by adding the neighbour matrix slices. Here, it is just written
because of space constraints.
"""
```

3.3 Error analysis:

At every iteration, the maximum absolute error between the new potential values and old potential values is calculated and stored in an array named "error". The errors are plotted against the iteration number using "Semilogy" and "Loglog" commands. Also, the semilogy plot will almost become a straight line after 500 iterations implying the error is of the form:

$$log(error_k) = A + Bk$$

Now, we can find the fit (A, B) which satisfies this equation. We have to find two fits:

1. Fit 1: Training data: Data corresponding to all the iterations
2. Fit 2: Training data: Data corresponding to all the iterations excluding the first 500

Writing the equation in matrix form, we will have:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ ... & ... \\ 1 & Niter-2 \\ 1 & Niter-1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} log(error_0) \\ log(error_1) \\ log(error_2) \\ ... \\ log(error_{Niter-2}) \\ log(error_{Niter-1}) \end{pmatrix}$$

This equation is fo the form:

$$M.x = y$$

For fit 2, the matrix will start from 501, 502,...Niter-1. That's the only difference. Now, to solve for the x vector (A, B), let's use the lstsq() function of the linear algebra module provided by the Numpy library. The following code snippet does that:

```
import numpy as np

itervals = np.array([i for i in range(0,Niter)]); itervals.shape = (Niter,1)
A = np.c_[np.ones((Niter,1)),itervals]
b = np.log(error)
fit1 = np.linalg.lstsq(A,b,rcond=None)[0]
fit2 = np.linalg.lstsq(A[501:Niter],b[501:Niter],rcond=None)[0]
fit1_vals = np.dot(A,fit1)
fit2_vals = np.dot(A[501:Niter],fit2)
```

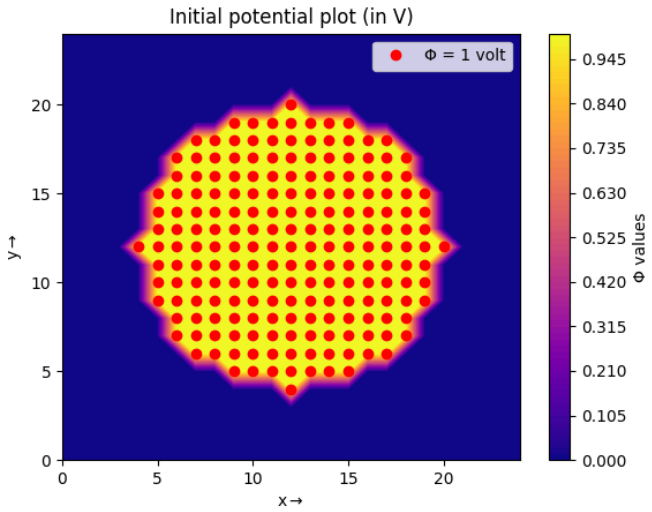
3.4 Plots:

The following plots are to be plotted:

1. Initial potential contour
2. Final potential contour
3. 3D surface plot of the potential
4. Semilogy plot of the error and the fits vs iteration number
5. Loglog plot of the error vs iteration number
6. Vector plot of the current density
7. Temperature profile contour

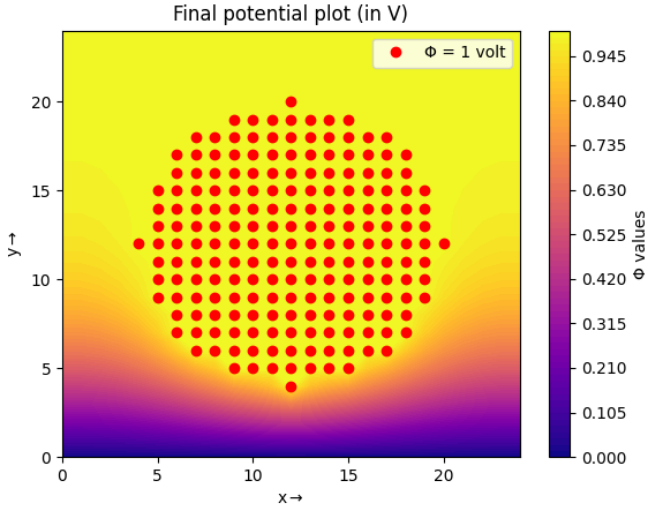
4 Observations and plots:

4.1 Initial potential contour:



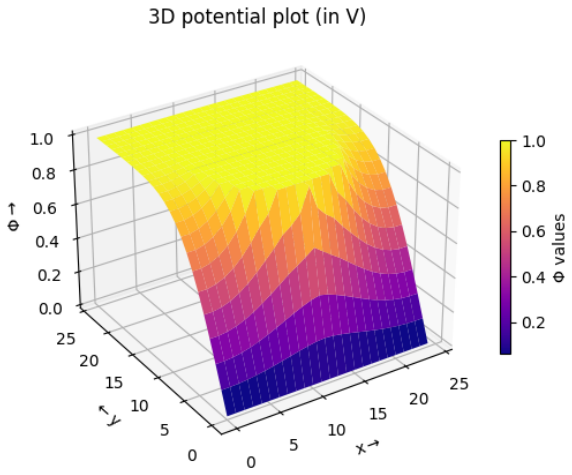
Initially, the potential is zero at all points except the wire. The wire is at 1 volt

4.2 Final potential contour:



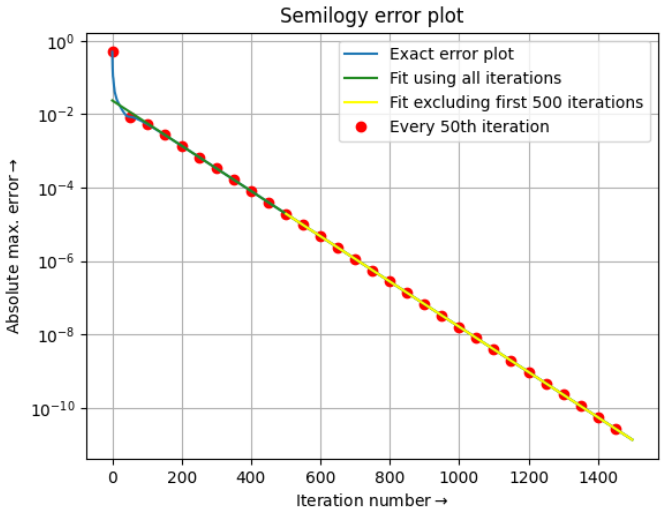
After the averaging is done and the boundary conditions are satisfied, the potential becomes much more smoother

4.3 3D surface potential plot:



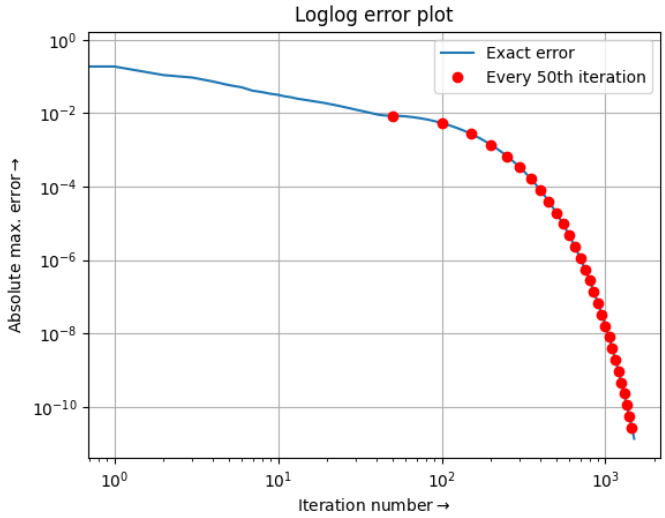
The 2D contour matches with the 3D plot exactly and both the contours follow the boundary conditions as well

4.4 Semilogy plot of the Error vs Iteration number:



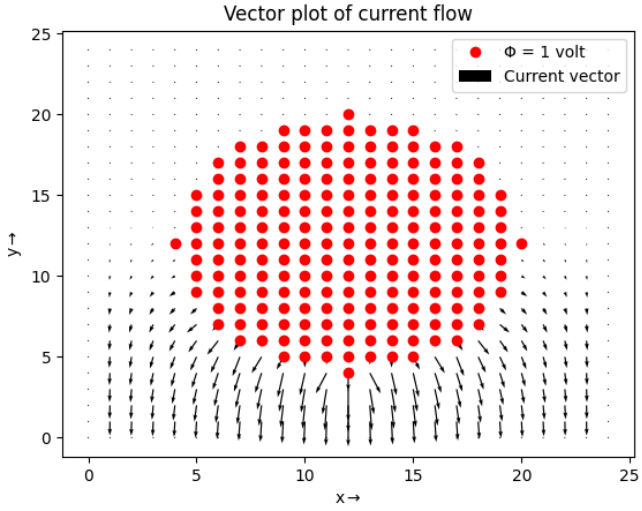
The exact error is different from the fit for low values of iteration. However, the fit is somewhat accurate for higher values of iteration

4.5 Loglog plot of the Error vs Iteration number:



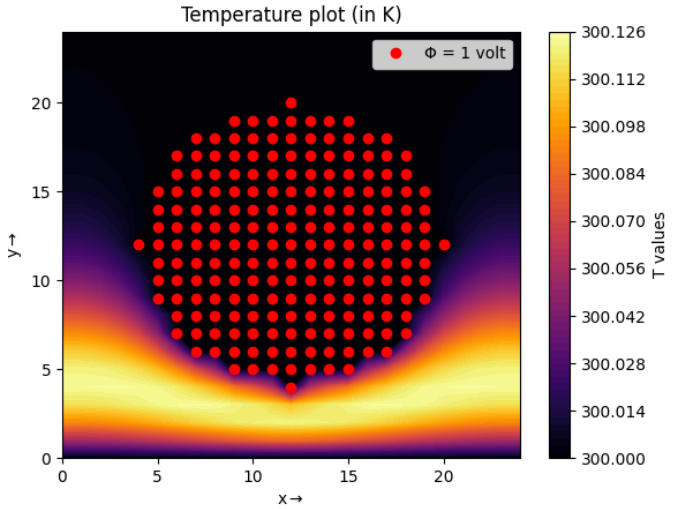
On the loglog plot, the error is intially decreasing linearly with respect to the iteration number. However, the plot reaches an exponential regime for higher values of iteration

4.6 Vector plot of the current density:



As expected, the currents are only existing in the region between the wire and the grounded edge. The current density values in the floating regions are practically negligible

4.7 Temperature profile contour:



The temperature is high in the region where the currents primarily exist and is almost 300 K in the region where the currents are near zero. Hence, the Laplace equation of temperature is satisfied properly

5 Results and conclusions:

1. The potential contour follows the boundary conditions properly and the plot is also smooth.
2. The currents are primarily concentrated near the grounded edge. It makes sense as that is where the potential gradients exist in significant values.
3. The temperature is higher than room temperature near the grounded region because the currents primarily exist there and hence, most of the Joule's heating take place there.
4. The error initially is reducing steeply but reduces linearly after 500th iteration on the Semilogy plot.
5. Judging by the time constant (Time constant = 1/B = 70.33), we can say this averaging method is not so preferable as the time taken by the error to settle is very high.