EE2703: Applied Programming Lab
Assignment 9
Spectra of non-periodic signals

V. Ruban Vishnu Pandian
EE19B138

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Contents

1 Aim:

The aim of this assignment is to:

1. Learn how DFT is used to analyse spectra of non-periodic DT signals.
2. Analyse spectra of sinusoidal DT signals using and without using hamming window.
3. Estimate the frequency and initial phase of a single sinusoid by finding its DFT and analysing it.
4. FInd the DFT of a chirped signal and also plot DFTs of different segments of it to see how the frequency components change with time.

2 Theory:

2.1 DFT of signals with discontinuities:

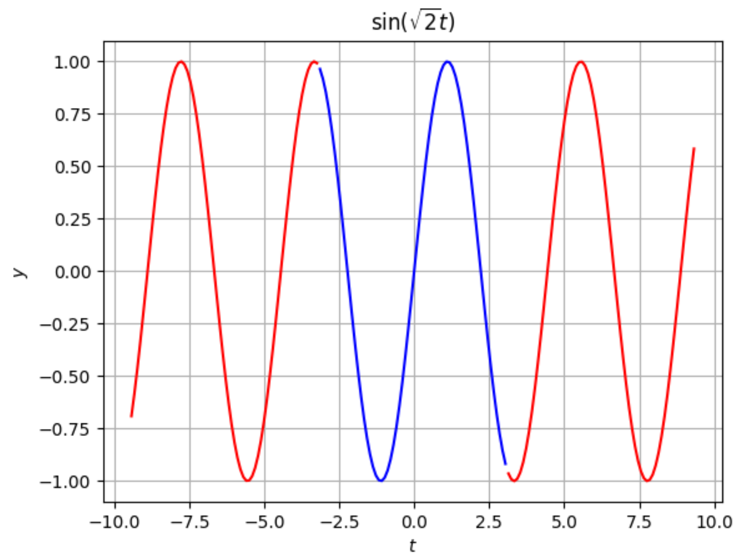
The DFT and inverse DFT equations are:

$$a[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$

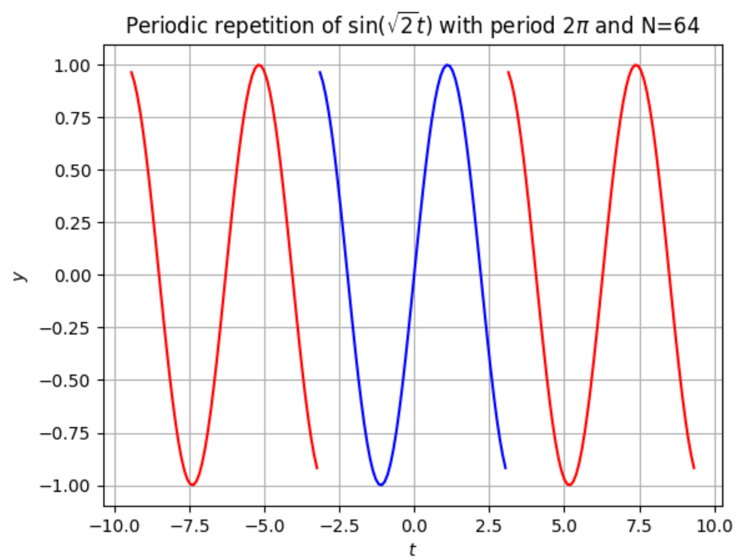
$$x[n] = \sum_{k=0}^{N-1} \frac{a[k]}{N} e^{j\frac{2\pi}{N}kn}$$

As we can see, both the equations are inherently periodic with period as N samples. Hence, both the functions $x[n]$ and $a[k]$ are periodic. Now, let's say the samples which are at the ends in one period of signal $x[n]$ have a big difference. Now, there is a discontinuity in the signal. Because of this discontinuity, the spectrum of the signal will contain fourier coefficients which will decay as $\frac{1}{\omega}$. So even if the samples are part of a sinusoid, the $\frac{1}{\omega}$ decay will be present if the samples at the end are not so close.

An example for this $x[n] = sin(\sqrt{2}t)$ with the period as 2π. Even though the signal is periodic, the time range we took is not a period of it. So, the periodic extension for period $T = 2\pi$ will not produce $sin(\sqrt{2}t)$ but a different signal. Hence, the DFT will also not have two impulses but a continuously decreasing spectrum.

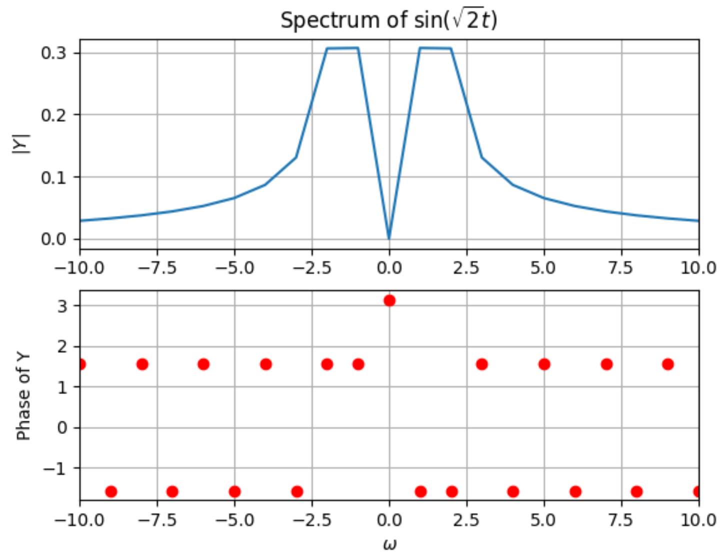


This is the original signal $sin(\sqrt{2}t)$



Signal whose first period (T=2π and $N = 64$) contains samples of $sin(\sqrt{2}t)$ and the remaining samples are periodic repetitions of the first-period samples

The DFT of this signal will be:



DFT of $sin(\sqrt{2}t)$ for $T = 2\pi$ and $N = 64$

As we can see, we have peaks around $\sqrt{2}$ but they are not impulses. Rather, we have a continuous spectrum which seems to have a $\frac{1}{\omega}$ decay. But, first of all where does this decay come from?

2.2 Gibbs phenomenon:

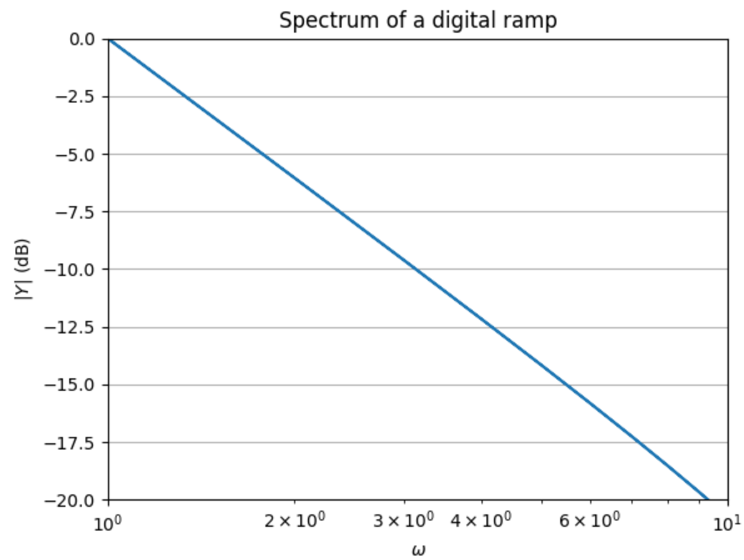
Gibbs phenomenon occurs when a signal contains discontinuites. We know sinusoids are continuous signals. Hence, when we try to model a discontinuous function as sum (or) integral of sinusoids (Definition of fourier transform), logically we will need more frequency components to achieve that discontinuity. Hence, the spectrum will decay slowly and won't even become zero after some frequency, i.e., spectrum won't be bandlimited. Let us take the example of a periodic linear ramp with period 2π. It's equation is defined as:

$$x(t) = t \forall t \in [-\pi, \pi)$$

and this segment is periodically repeated to get the periodic linear ramp. Clearly, we have discontinuities at $t = n\pi$ where n is an integer. Now, the fourier series representation of this signal would be:

$$x(t) = \frac{sin(t)}{1} - \frac{sin(2t)}{2} + \frac{sin(3t)}{3}...$$

The spectrum of this signal on a $dB - dec$ plot is as follows:



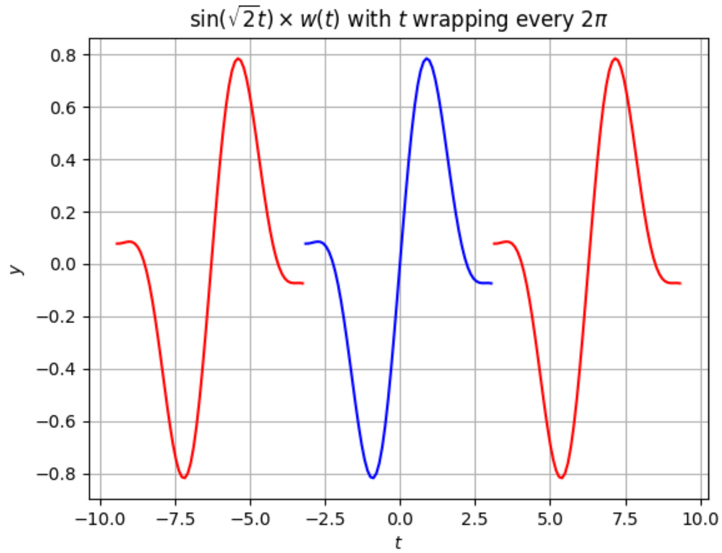
The spectrum decays linearly with respect to ω on the $dB - dec$ plot indicating $\frac{1}{\omega}$ delay as expected

2.3 Hamming window:

To avoid discontinuities, we use a hamming window. This hamming window is another DT signal which will be multiplied with the original signal. This window signal will be high inside the period and nearly zero at the edges. Hence, the edge discontinuities are dampened whereas the signal within the bulk of the period is preserved. An example for a hamming window is:

$$w[n] = 0.54 + 0.46cos(\frac{2\pi n}{N-1})\forall 0 \leq n < N$$

and this signal is also periodically repeated. The new refined signal after the multiplication with the window signal is plotted below:



The period is still maintained as 2π and the frequency as $\sqrt{2}$. But now the discontinuities are reduced because of the multiplication with the window signal

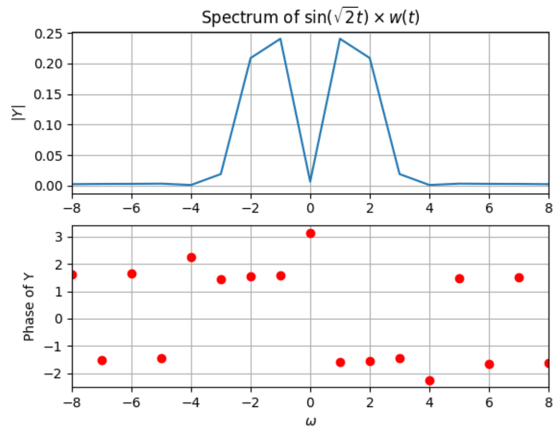
Now, let us plot the DFT of this refined signal. The DFT of this new signal will be circular convolution of the DFTs of the original signal and the window singal (Multiplication in time domain leads to convolution in the frequency domain and vice-vera):

$$G_k = \sum_{n=0}^{N} F_n W_{k-n}$$

Hence, we expect the $\frac{1}{\omega}$ decay to have vanished since the DFT of $w[n]$ only contains components at DC and $\omega = \pm 1$. Hence, we can write:

$$G_k = F_k W_o + F_{k+1} W_{-1} + F_{k-1} W_1$$

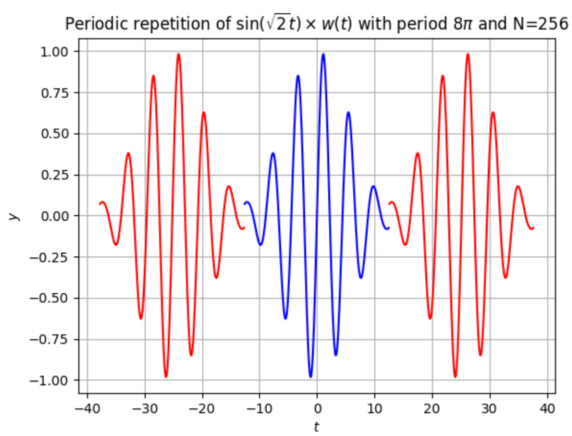
Clearly, this summation does not exist for all values of k but only at those values of k where the DFT of $x[n]$ is high. So, the DFT of the refined signal will be:



The DFT is not accurate but much better when compared to the DFT of the unrefined signal

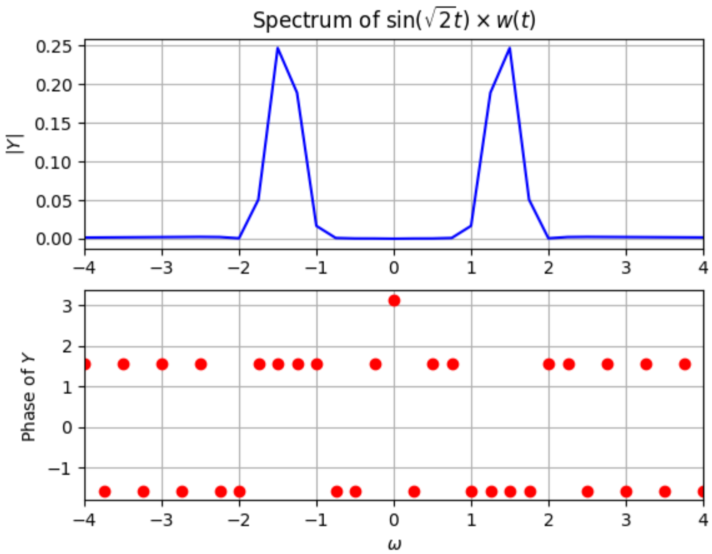
We got rid of the $\frac{1}{\omega}$ decay but still the peaks are broad. We really cannot do anything about it because it's due to the hamming window. The convolution sum doesn't decay instantly. It exists for k values which are around $\sqrt{2}$ and then only becomes zero. That's why we have a broad peak even after windowing is done.

One improvement which we can do to get an relatively accurate spectrum is to increase the resolution, i.e, increase the time range (or period) T for which it is calculated. Let's assume now the signal runs from $[-4\pi, 4\pi)$ with $N = 257$. The signal will look like this:



Here the period is 8π and $N = 257$. Again, we periodically repeat the first period to get the whole periodic signal

DFT of this signal is shown below:



The DFT is still broad but definitely better than the previous DFT plot

3 Assignment:

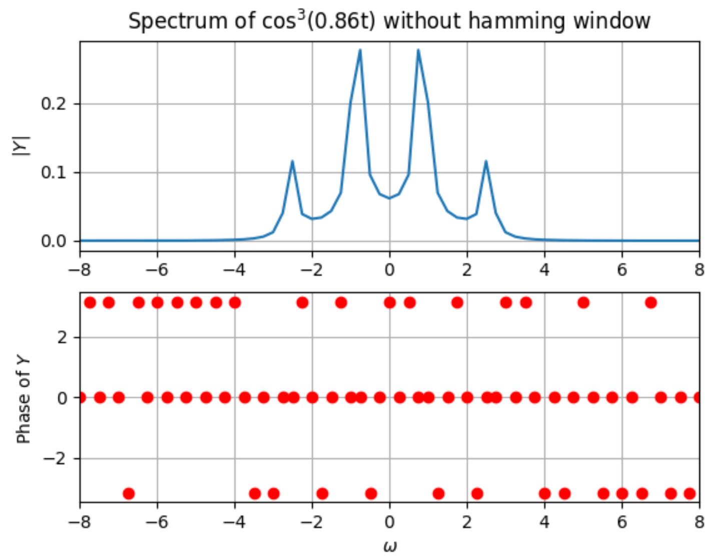
3.1 Spectrum of $cos^3(0.86t)$:

We know:

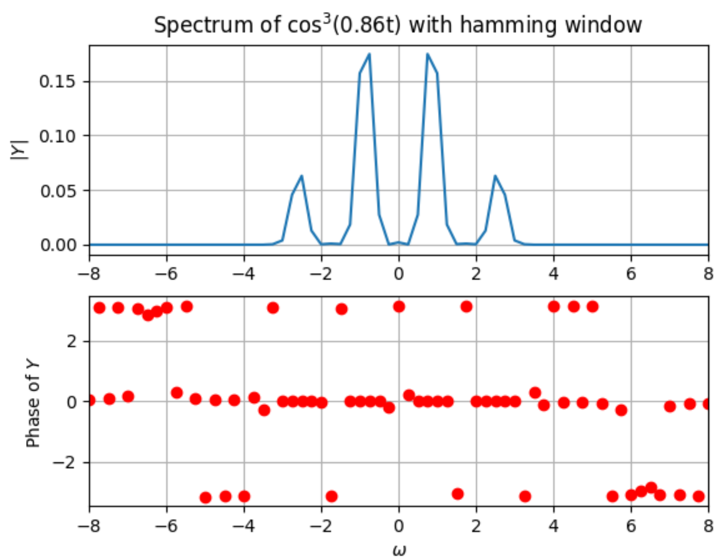
$$cos^3(w_0t) = 0.75cos(w_0t) + 0.25cos(3w_0t)$$

$$cos^3(w_0t) = 0.375e^{jw_0t} + 0.375e^{-jw_0t} + 0.125e^{j3w_0t} + 0.125e^{-j3w_0t}$$

Hence, we would expect 4 sharp peaks at the given frequencies. However, in this problem, our time range is a multiple of 2π which is clearly not a period since $w_o = 0.86$. Hence, the periodic repetition won't be $cos^3(0.86t)$ and we will be facing the same problems we had with $sin(\sqrt{2}t)$. So, our DFT plot will be continuous with broad peaks at the given frequencies. The DFT plot for period time range $[-4\pi, 4\pi)$ and $N = 512$ is given below:



This is the DFT plot without the hamming window



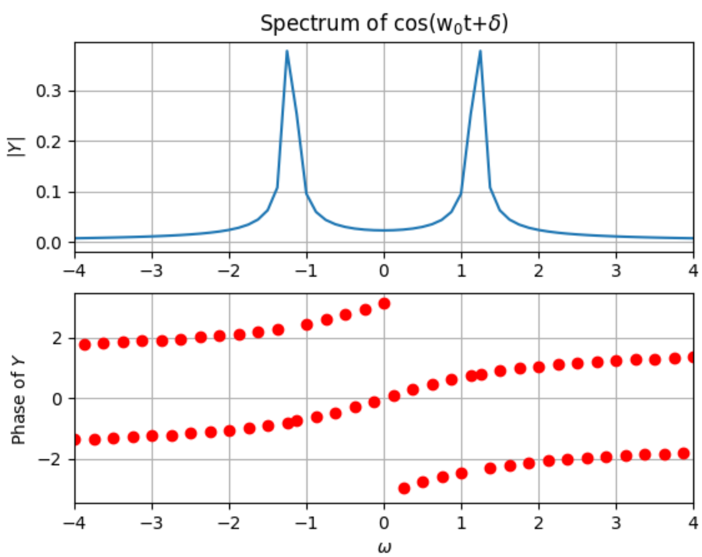
The DFT plot is better when the hamming window is introduced. However, the peaks are still broad because of the reasons discussed in section 2.3

3.2 Estimation of frequency and initial phase of a sinusoid using DFT:

DFT plotting could also be used to find the frequency and initial phase of a sinusoidal signal. If we know that the samples available to us belong to a sinusoid function, then we can use its DFT to find the frequency and phase. In this problem, it is given that:

$$x(t) = cos(w_0 t + \delta)$$

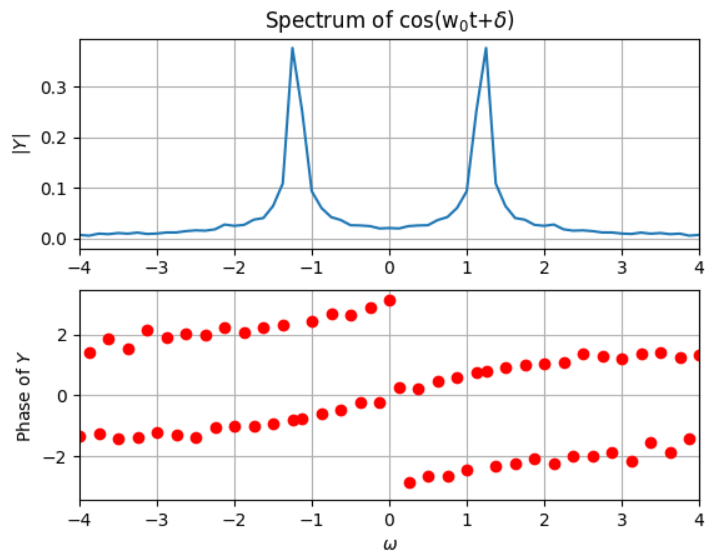
where $0.5 < w_0 < 1.5$ and δ is arbitrary. So, let's assume we have the samples ranging from $[-8\pi, 8\pi)$ with $N = 1024$. The DFT plot looks as follows:



This is the DFT for $w_0 = 1.2$ rad/s and $\delta = 0.78$ rad. The peaks are broad as expected

Now let's assume some white noise (Of $Amplitude = 0.1$) is present in the signal. This white noise is a gaussian distribution and hence, the magnitude doesn't get affected so much. There can be some significant changes in the phase spectrum but again, the phase values at frequencies $\pm w_0$ won't get affected very much.

The DFT plot of the noisy signal is:



The magnitude plot is a bit noisy but the peaks are still intact. Phase plot also got affected a bit but the overall structure of the plot is preserved

The reason for this behavior is because, the DFT of gaussian is another gaussian. Also, we assumed an amplitude of 0.1. So, that's the maximum value the DFT of gaussian can have. Hence, both the magnitude and phase of the original plots don't get affected to a noticeable extent.

Now, in order to estimate w_0 and δ,we need to find the value of w where the magnitude of DFT is maximum and find the phase there to find δ. The proof for this conclusion is given below:

$$a[k](DFT) = \sum_{n=0}^{N-1} cos(w_0 \frac{2T_0}{N} n + \delta) exp(-j\frac{2\pi}{N}kn)$$

$$a[k] = \sum_{n=0}^{N-1} 0.5[exp(w_0\frac{2T_0}{N}n+\delta)exp(-j\frac{2\pi}{N}kn)+exp(-w_0\frac{2T_0}{N}n-\delta)exp(-j\frac{2\pi}{N}kn)]$$

(**Note:** w_0 changed to $w_0\frac{2T_0}{N}$ in DT domain since CT domain frequencies will be scaled by $T_s = \frac{2T_0}{N}$ in DT domain.)

Let's assume we have a k_0 such that $w_0\frac{2T_0}{N} \approx \frac{2\pi}{N}k_0$. Now, the DFT summation becomes:

$$a[k] = \sum_{n=0}^{N-1} 0.5[exp(\delta)+exp(-2w_0\frac{2T_0}{N}n-\delta)] = 0.5Nexp(\delta)+\sum_{n=0}^{N-1} 0.5exp(-2w_0\frac{2T_0}{N}n-\delta) \approx 0.5N$$

We are able to discard the final summation because $2w_0\frac{2T_0}{N}n \approx \frac{4\pi}{N}k_0$ is N-periodic. Hence, whatever phase we find at estimated w_0,that would be almost equal to the actual phase. The results that were obtained by following this logic were:

Without white noise:

1. The estimated frequency w0 = 1.2
2. The estimated phase δ (in Rad) = 0.805

With white noise:

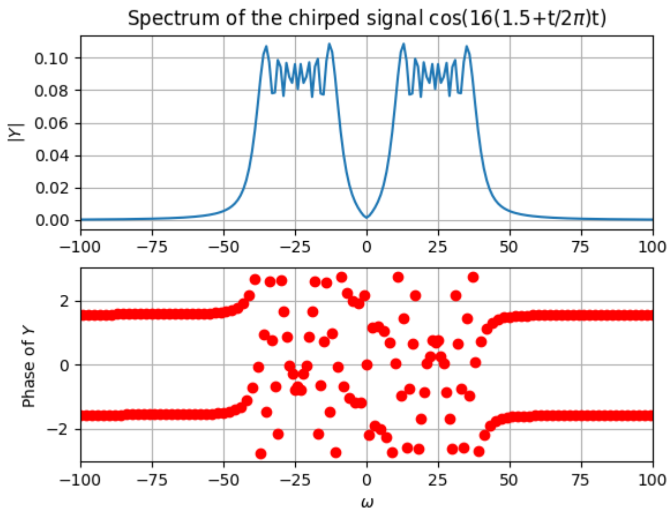
1. The estimated frequency w0 = 1.2
2. The estimated phase δ (in Rad) = 0.807

3.3 DFT of chirped signal:

A chirped signal is a signal whose frequency depends on the time instant. These type of signals will have different frequency components at different time instants. The example given in the assignment is:

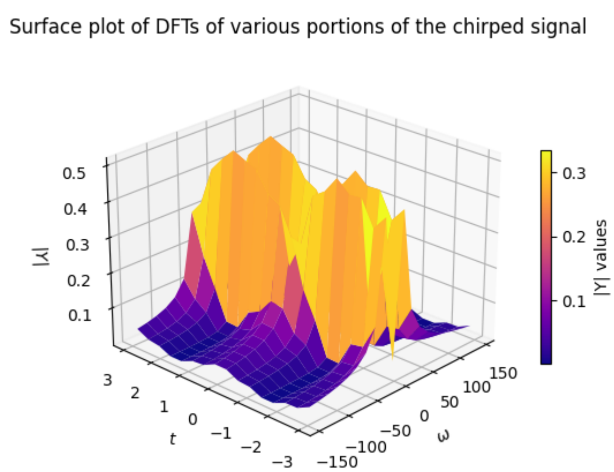
$$x(t) = cos(16(1.5 + \frac{t}{2\pi})t)$$

with t ranging from $[-\pi, \pi)$. Also, it is asked to assume $N = 1024$. The DFT of the periodic repetition of this signal is:

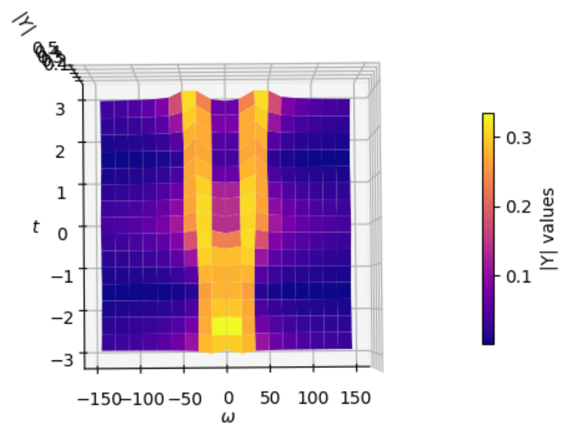


As expected, the signal contains frequency components from 16 rad/s to 32 rad/s and the spectrum decays quickly after that range

Now in order to analyse how the frequency components depend on the time range, let us divide the signal into groups of 64 samples. So, we would have 16 such groups. By finding the DFT for each group and plotting the DFT magnitude as a 3d plot with x-axis as time $'t'$ and y-axis as frequency $'w'$, we will have the following plot:



The DFT structure is still the same. However, the peaks are present at different locations depending on the time



When viewed from above, we can clearly see the peaks widen up indicating the frequencies present in the signal increase with time. Also, a linear widening of the peaks mean that the frequencies depend linearly on the time value which is also true

4 Conclusions:

1. When plotting DFT for signals with discontinuities, the spectrum is not merely a collection of distinct spikes but a continuous decreasing spectrum with $\frac{1}{\omega}$ decay.
2. To dampen the discontinuities, a hamming window is used. This signal is multiplied with the original signal so that the high frequency components get attenuated and hence, the discontinuities are dampened.
3. Signal $cos^3(0.86t)$ has a DFT which has broad peaks and continuous decay. However, when a hamming window is used, the continuous decay vanishes but still the peaks are broad.
4. To estimate the frequency and initial phase of a sinusoidal signal, its DFT is used. The results were fairly accurate when the period of the signal is high. White noise didn't affect the results much since its amplitude is very low.
5. The chirped signal has frequency components between 16 rad/s and 32 rad/s. Also, when the individual DFTs of the 64-samples signals were plotted as a function of w and t, the locations of the peaks widened linearly as t increased indicating linear dependence of frequency on t.