EE2703: Applied Programming Lab
Assignment 6L
The Laplace Transform

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Contents

1 Aim:

The aim of this assignment is to:

1. Learn about the scipy.signal library and polynomial functions of the numpy library.

2. Use those tools to solve various LCCDEs using laplace transform techniques.

2 Theory: Laplace transform:

Laplace transform is a powerful technique used in engineering to solve complex mathematical differential equations. This technique helps us to solve various real-life problems such as:

1. Steady state response of an electrical circuit.

2. Steady state response of a mechanical spring-block system.

3. Analysing the behaviour of electrical/mechanical filters.

The formula for the unilateral laplace transform of a continuous time function x(t) is given as:

$$X(s) = \int_0^\infty x(t)e^{st}dt$$

where s is the complex frequency. One major use of laplace function is that if $x(t)$ gives $X(s)$,then $\frac{dx(t)}{dt}$ will give $sX(s)$ (provided the initial conditions are zero). Also, the transform is a linear transform. These two powerful properties will help us to solve differential equations easily since in the laplace domain, they will be algebraic equations, thanks to these two properties.

3 Assignment objectives:

3.1 Response of a spring-mass system:

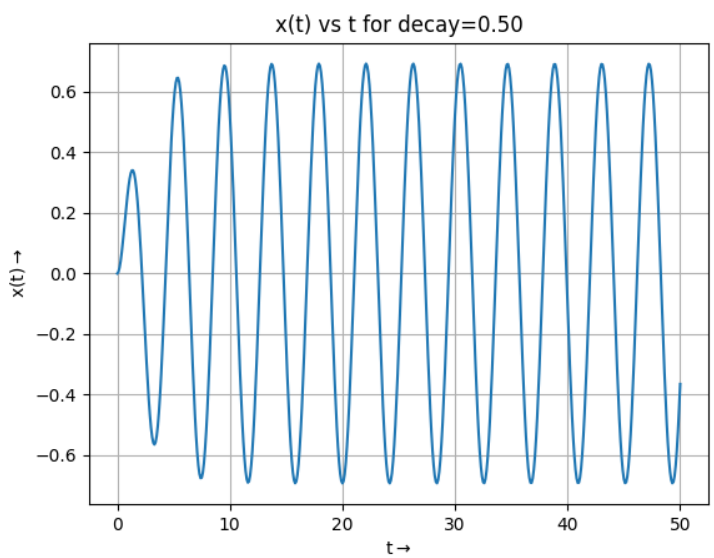
The differential equation of a spring-mass system is given as:

$$\ddot{x} + w_0^2 x = f(t)$$

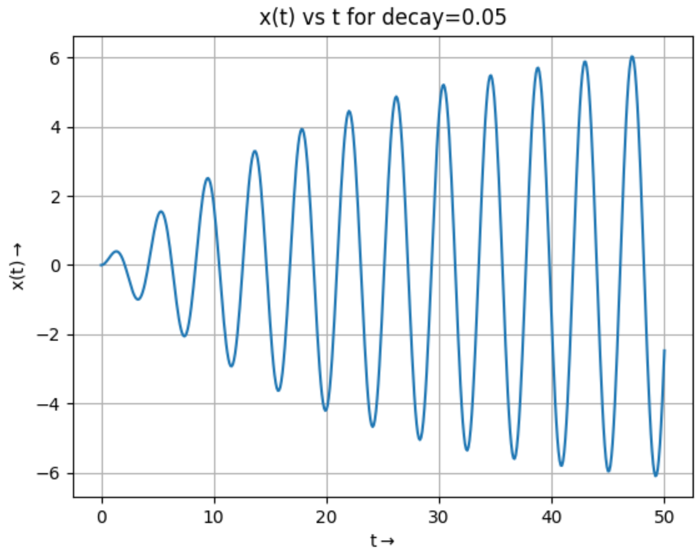
with $f(t) = cos(w_d t)e^{-at}u(t)$, where:

- a = Decay constant (Given as 0.5)
- w_d = Driving frequency in rad/s (Given as 1.5)
- w_0 = Natural frequency of the spring-mass system in rad/s (Given as 1.5)

The objective is to solve for the time response of x provided $x(0) = 0$ and $\dot{x}(t) = 0$. The plot for the response is given below:

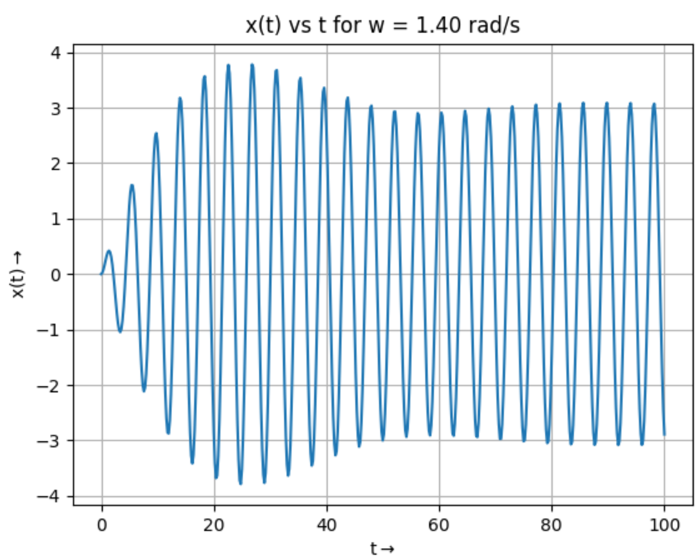


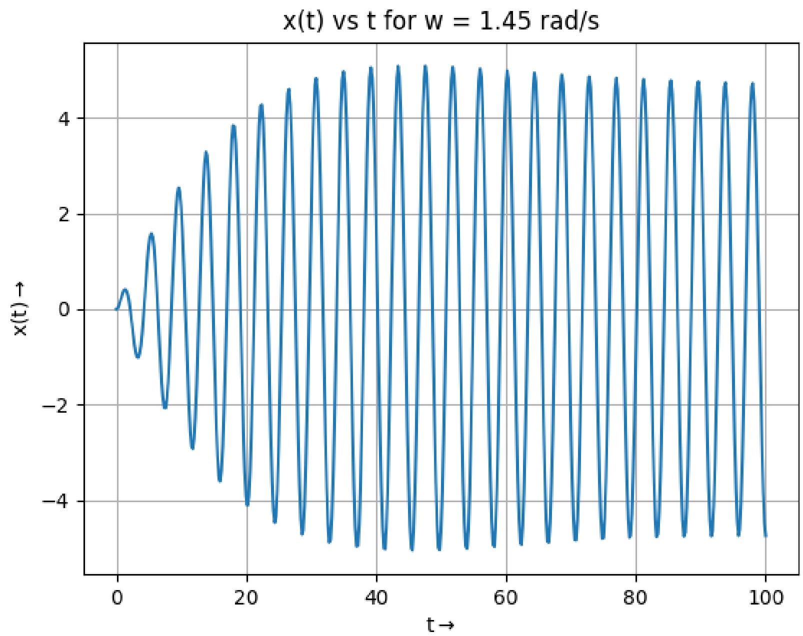
The decay in the response dies out after sometime and the response becomes purely sinusoidal

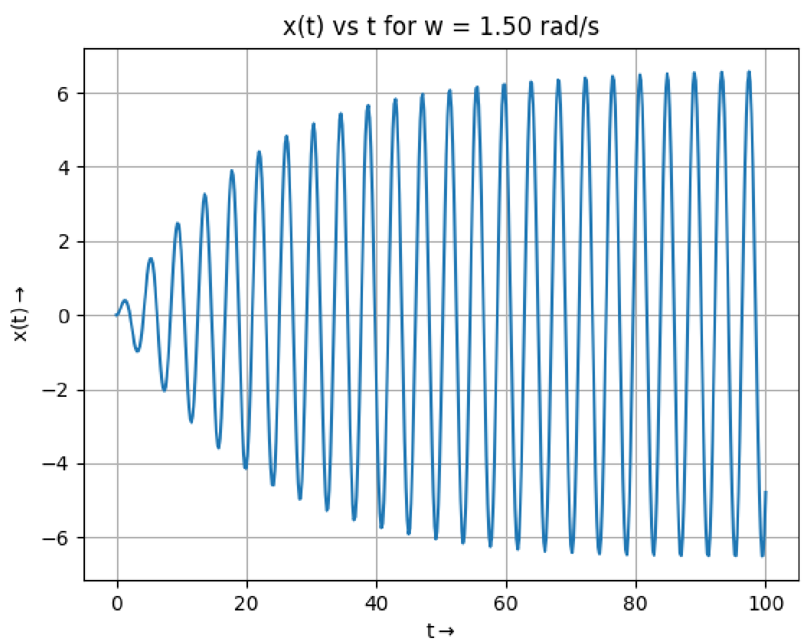


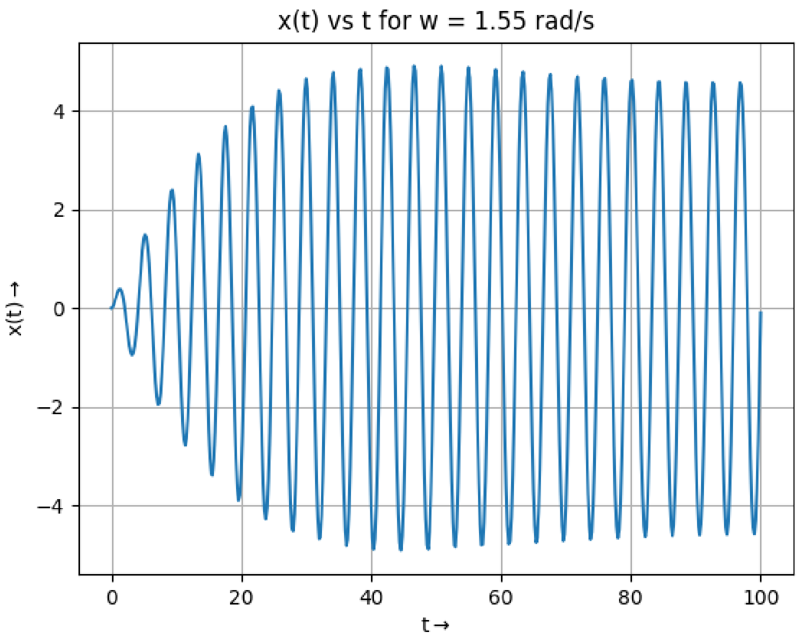
The decay dies in this graph as well but takes more time to vanish since the decay constant is low compared to the previous plot

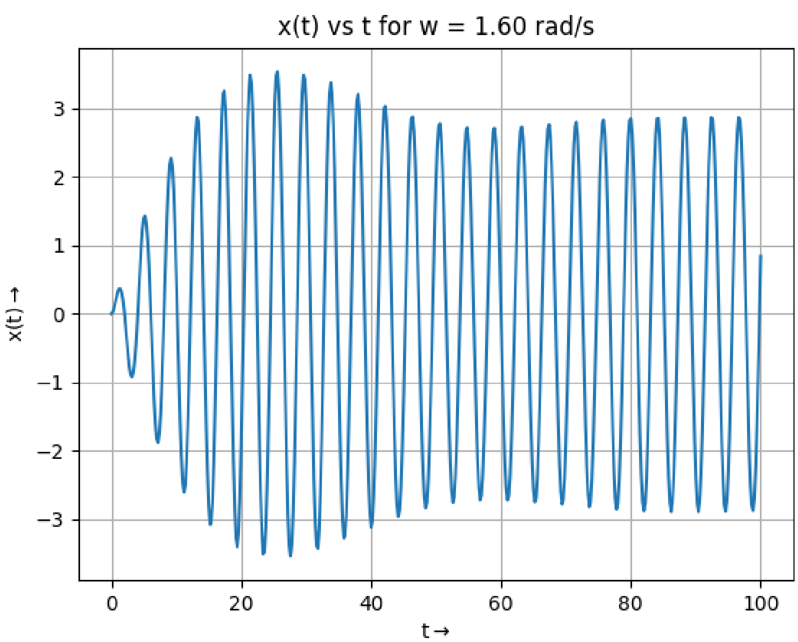
3.2 Response of the spring-mass system for different frequencies:




x(t) vs t for w = 1.45 rad/s
x(t) →
t →
4
2
0
−2
−4
0
20
40
60
80
100


x(t) vs t for w = 1.50 rad/s
x(t) →
t →
6
4
2
0
−2
−4
−6
0
20
40
60
80
100


x(t) vs t for w = 1.55 rad/s
x(t) →
t →
4
2
0
−2
−4
0
20
40
60
80
100


x(t) vs t for w = 1.60 rad/s
x(t) →
t →
3
2
1
0
−1
−2
−3
0
20
40
60
80
100

Clearly, the response is having maximum amplitude for w = 1.5 rad/s. It is obvious since the resonant frequency of the spring-mass system is $\sqrt{2.25} = 1.5$ rad/s and that is equal to the driving frequency as well. Hence, the response is maximum at w = 1.5 rad/s and starts to attenuate for frequencies around it.

3.3 Coupled spring problem:

The coupled differential equations corresponding to the responses of two springs are given as follows:

$$\ddot{x}x + x - y = 0$$

$$\ddot{y} + 2(y - x) = 0$$

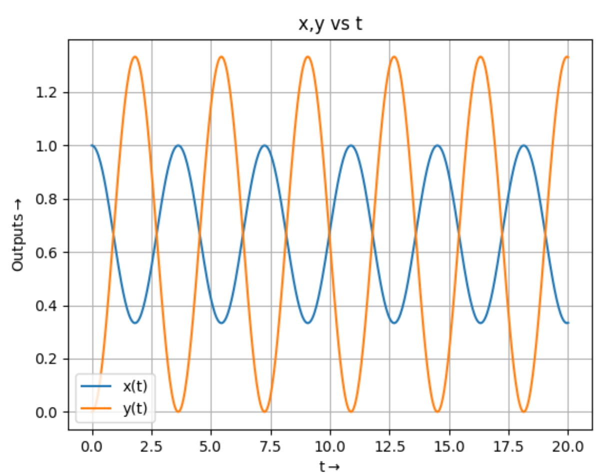
with $x(0) = 1, \dot{x}(0) = y(0) = \dot{y}(0) = 0$.

Solving for X(s) and Y(s), we have:

$$X(s) = \frac{s^2 + 2}{s^3 + 3s}$$

$$Y(s) = \frac{2}{s^3 + 3s}$$

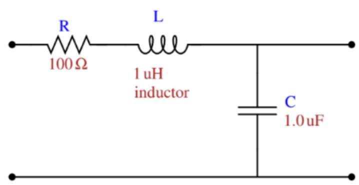
These laplace transform expressions are again converted back to time domain and the time responses are obtained in the plot given below:



Both the springs have sinusoidal responses with the response of y being more in amplitude than that of x. The reason is simple: y(t) = 2(u(t)-x(t))

3.4 RLC filter:

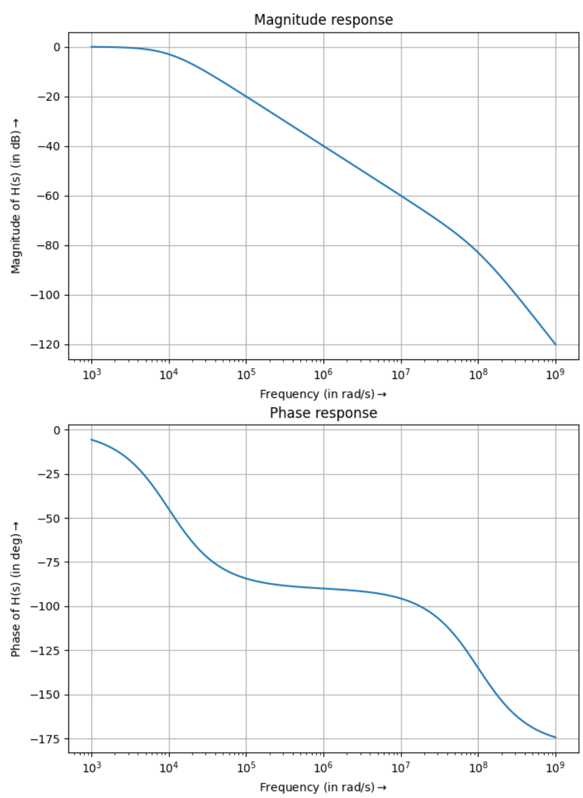
This is an electrical engineering example. The objective is to analyse the frequency domain gain and phase of the transfer function of the system shown below:



The transfer function of this system is:

$$H(s) = \frac{1}{s^2LC + sRC + 1}$$

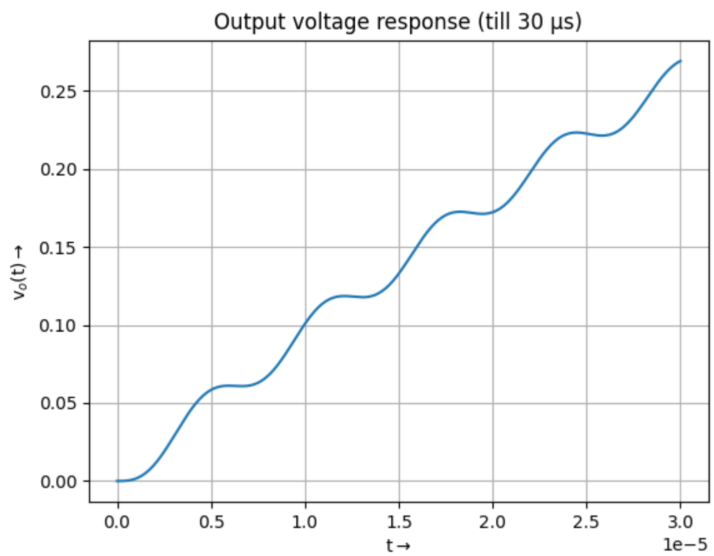
where R $= 100\Omega$, L $= 1\mu$H and C $= 1\mu$F. The magnitude and phase response plots are shown below:



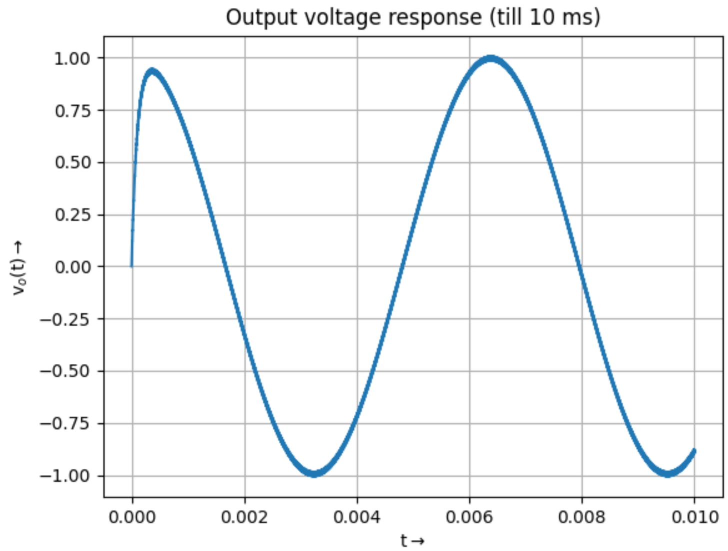
Now an input voltage is applied to this system. The input voltage expression is given as:

$$v_i(t) = cos(10^3 t) - cos(10^6 t)$$

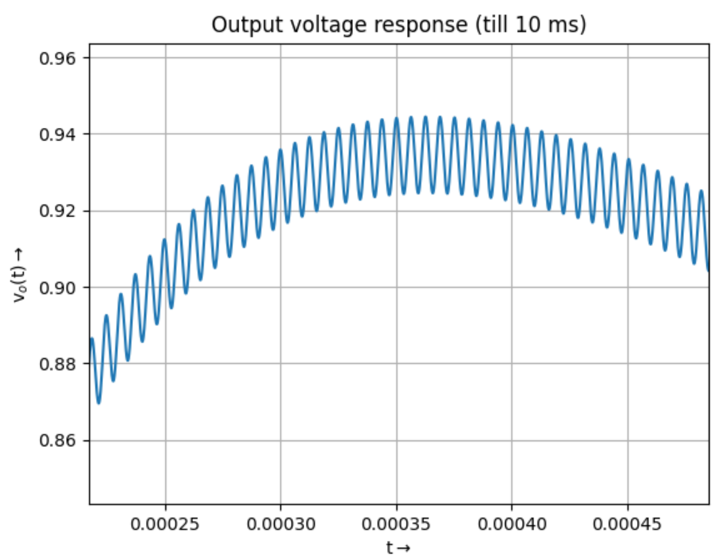
Converting this to the laplace form, multiplying it with the transfer function and then inverting the result back to time domain, we have the following graphs:



The response is steadily increasing with slight sinusoidal variations



The response looks almost sinusoidal with frequency of 1000 rad/s



On zooming, we can see that there are small sinusoidal variations on top of the sinusoidal graph obtained in the previous figure.

4 Conclusions:

1. The time responses of the spring-mass system in Q1 and Q2 reach steady state and have a sinusoidal variation. The exponential components decay after some time and the time taken to reach steady state is less in case of $a = 0.5$ as expected.

2. In Q3, the response has maximum amplitude at $w = 1.5$ rad/s since that is equal to the resonant frequency of the spring-mass system. Frequencies other than 1.5 rad/s have lower amplitudes.

3. In Q4, the response of spring Y is having more amplitude than that of spring X since y(t) = 2(u(t)-x(t)). Both the responses are sinusoidal with $w = \sqrt{3} = 1.717$ rad/s.

4. In Q5, from the magnitude and phase responses, it is evident that the filter is a second order low-pass filter with its pole frequencies as $p1 = 10^4$ rad/s and $p2 = 10^8$ rad/s. Hence, it is a real pole system and hence, the magnitude response is a strictly decreasing function.

5. In Q6, the output voltage clearly is dominated by the component of $w = 10^3$ rad/s and this is obvious since 10^3 is less than the first pole frequency, 10^4. However, the component of $w = 10^6$ rad/s is very low and it just acts like a noise to the output.