EE6110: Project report

Paper: A Family of Robust M-Shaped Error Weighted Least Mean Square Algorithms: Performance Analysis and Echo Cancellation Application

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1 Introduction

In adaptive signal processing, the most popular algorithm used for various applications such as system identification, system tracking, adaptive equalization, etc. is the **Least mean squares (LMS)** algorithm. However, there are various variants proposed in the literature that enhance the performance of the standard LMS algorithm under different use cases. Based on the error weighing function, those variants can be classified into 3 categories namely:

- V-shaped
- Λ-shaped
- M-shaped

In the particular paper that is reviewed in this report, a robust M-shaped variant known as **NFRMS (New family of Robust M-shaped)** algorithm has been proposed.

2 Standard LMS algorithm: Brief overview

LMS algorithm is a recursive algorithm where the optimization metric is the squared error between the true input and its estimated value. If the statistics are known, we can find the optimal LMMSE solution using an analytical formula or recursively using the Steepest descent algorithm (SDA). But if only the training data is given, then SDA becomes LMS.

Let us assume we have an LTI system with input-output relationship as:

$$y_n = \overline{h}^T \overline{x}_n + v_n$$

where y_n is the system output, $\overline{x}_n = \begin{bmatrix} x_n & x_{n-1} & ... & x_{n-L} \end{bmatrix}^T$ is the system input, $\overline{h} = \begin{bmatrix} h_0 & h_1 & ... & h_L \end{bmatrix}^T$ is the system impulse response and v_n is AWGN noise. Let $\overline{w} = \begin{bmatrix} w_0 & w_1 & ... & w_L \end{bmatrix}^T$ be the adaptive filter. The estimation error at time index n is given as:

$$e_n = y_n - \overline{w}^T \overline{x}_n$$

The general LMS algorithm is then given as:

$$\overline{w}_{n+1} = \overline{w}_n + \mu f(e_n) e_n \overline{x}_n \tag{1}$$

The weights are recursively changed based on the estimation error. The function $f(e_n)$ is known as the error weighing function. For standard LMS it is unity. Based on the behaviour of this function, there are 3 categories of LMS variants:

2.1 V-shaped LMS

In this class of algorithms, the function $f(x)$ is always non-decreasing, i.e, $f^{'}(x) \geq 0$ for all values of x. These variants are particularly useful if the noise is sub-gaussian. The steady state MSE of V-shaped LMS algorithms is better for sub-gaussian noise than it is for super-gaussian. Most V-shaped algorithms are derived by optimizing higher order moments of the error.

2.2 Λ-shaped LMS

In this class of algorithms, the function $f(x)$ is always non-increasing, i.e, $f^{'}(x) \leq 0$ for all values of x. These variants are particularly useful if the noise is super-gaussian. These variants are mostly used to correct impulsive noise and outliers. They are derived by optimizing the lower order moments of the error.

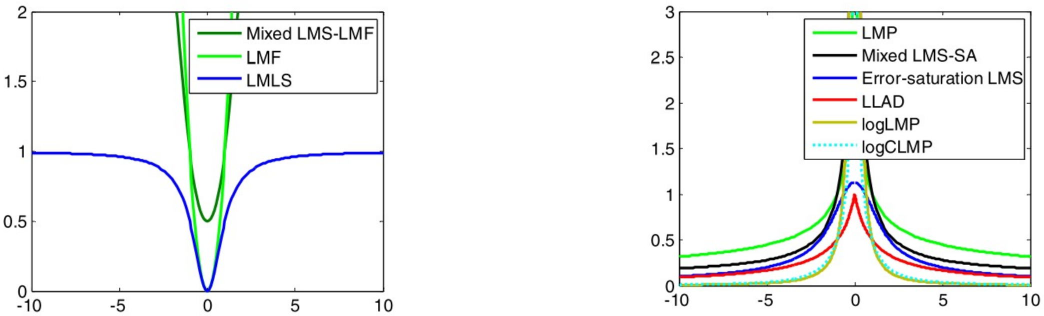


Figure 1: V-shaped and Λ-shaped error weighing functions

2.3 M-shaped LMS

In M-shaped LMS algorithms, there are two more sub-sections: Non-inverted M-shaped and inverted M-shaped algorithms. M-shaped error weighing functions have mixed characteristics, i.e, $f^{'}(x) \geq 0$ for certain values of x and $f^{'}(x) \leq 0$ for other values of x. Such a shape can capture both higher and lower order moments of the error. Hence, it can be used to provide both fast convergence and good steady state performance.

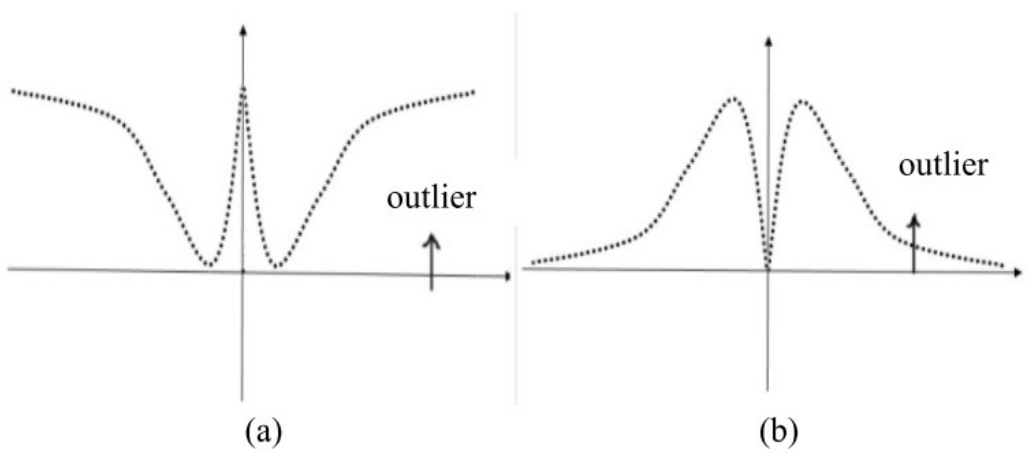


Figure 2: M-shaped and inverted M-shaped error weighing functions

3 Proposed model: Robust M-shaped LMS

The model proposed in the paper is an M-shaped LMS variant. The error weighing function is given by the equation:

$$f(e_n) = \frac{e_n^p}{\epsilon + e_n^{p+1}} \tag{2}$$

where ϵ and p are positive design parameters.

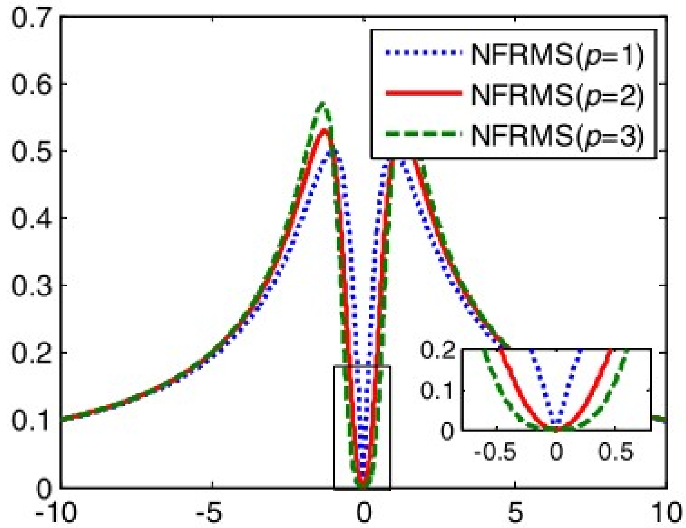


Figure 3: NFRMS error weighing function for $\epsilon = 1$ and different values of p

4 Steady State Analysis

In steady-state analysis, we assume that the LMS algorithm has converged. Let $v_n = w_{opt} - w_n$ be the convergence error. For white input signal with $R = \sigma_x^2 I$, the steady-state MSD is given by -

$$MSD = \frac{E\{f^2(v_n)v_n^2\}}{E\{f(v_n) + f'(v_n)v_n\}}\frac{uL}{2} \tag{3}$$

As we can see, the steady-state MSD of the LMS-based algorithm with error non-linearity is not only controlled by the step-size µ and the filter length L, but is also dependent on the distribution of the noise signal, the weighted function f(.) and its first-order derivative f'(.). Substituting f = 1 and f' = 0 for the LMS algorithm with small step-size condition in the above equation yields MSD = $\mu L\sigma_o^2/2$. This is different from the performance of the LMS-based algorithm with error non-linearity as the MSD function is independent of the noise distribution.

4.1 Proposed NFRMS-LMS algorithm

For the proposed NFRMS-LMS algorithm, the function f(.) and it's derivative f'(.) can be substituted by -

$$f(v_n) = \frac{|v_n|^p}{\epsilon + |v_n|^p} \tag{4}$$

$$f'(v_n) = \frac{\epsilon p sign(v_n)|v_n|^{(p-1)} - sign(v_n)|v_n|^{2p}}{(\epsilon + |v_n|^{p+1})^2} \tag{5}$$

We substitute the above two equations in (3) and consider 2 cases to simplify the steady-state MSD :

- Case 1 : If $|v_n|^3$ is sufficiently small such that $|v_n|^3 << \epsilon$, the steady state MSD simplifies to -

$$MSD = \frac{\mu L E\{|v_n|^6\}}{6\epsilon E\{|v_n|^2\}} \tag{6}$$

- Case 2 : If $|v_n|^3$ is sufficiently large such that $|v_n|^3 >> \epsilon$, the steady state MSD simplifies to -

$$MSD = \mu L(E\{|v_n|^{-2}\} + 2\epsilon E\{|v_n|^{-4}\})^{-1} \tag{7}$$

From Case 1 and Case 2, we observe that the steady-state MSD of the proposed NFRMS-LMS depends on the magnitude of the noise. For large noise, the steady-state MSD depends on the negative order moments of noise. However, small noise activates the the higher-order moments of steady-state MSD.
Hence, we can conclude that **the steady-state MSD acts like a V-shaped or Λ-shaped function depending on the magnitude of noise**.

5 Acoustic Echo Cancellation

The paper proposes the Proportionate Normalized Robust M-shaped (PNRMS) algorithm for Acoustic Echo Cancellation. The stochastic cost function from M-shaped variant is given by -

$$J(e_n) = \int_0^{e_n} \frac{x|x|^p}{\epsilon + |x|^{p+1}} dx \tag{8}$$

The PNRMS algorithm modifies this stochastic cost function with a posteriori error e_p and a proportionate matrix G_n such that -

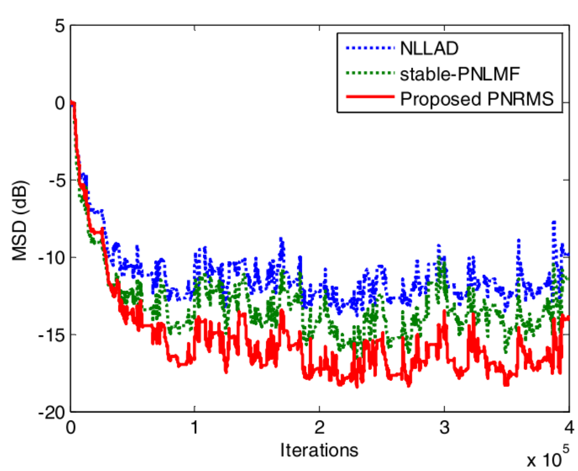
$$J_{AC} = ||w_{n+1} - w_n||^2_{G_n} + \mu \int_0^{\frac{e_{p,n}}{||x_n||_{G_n}}} \frac{x|x|^p}{\epsilon + |x|^{p+1}} dx \tag{9}$$

where $\mu > 0$ is a small positive parameter which acts as step-size in the proposed PNRMS algorithm , G_n is a proportionate matrix with diagonal elements -

$$g_{i,n} = \frac{1-\rho}{2L} + (1+\rho)\frac{|w_{i,n}|}{2||w_{i,n}||_1 + \delta} \tag{10}$$

where i = 1, 2,..., L, -1 $\leq \rho \leq$ 1, and δ is a small positive number.
The PNRMS update equation is formulated by taking the derivative of (9). Finally, by taking suitable approximations, the update equation is governed by -

$$w_{n+1} = w_n + \mu \frac{|e_n|^p}{\epsilon ||x_n||^{p+1}_{G_n} + |e_n|^{p+1}} \frac{e_n G_n x_n}{||x_n||_{G_n}} \tag{11}$$



The performance of the proposed PNRMS algorithm is compared with NLLAD and stable-PNLMF. The inputs are speech signal in a double-talk scenario. The measurement noise is assumed to the zero-mean Gaussian signal with SNR = 5 dB. With suitable step-sizes, a plot of MSD is shown for all the aforementioned algorithms.

We can see that, for the same initial convergence speed, **the PNRMS algorithm reaches a lower steady-state MSD than the NLLAD and stable PNLMF algorithms**.

6 MATLAB Simulations

MATLAB simulations were performed to understand how the proposed M-shaped algorithm works under different test cases. The theory and results are explained in the following sub-sections:

6.1 System identification

In this model, a classic LTI system with impulse response [1, 2, 3] was considered. It was driven with a white gaussian signal of variance 0.5 and the measurement noise was assumed to be an IID, Laplace noise with variance 5 (It is a super-gaussian noise). The adaptive filter is initialized with zeros and the rate of convergence (μ) is 0.1. For the M-shaped error weighing function, $p = 2$ and $\epsilon = 1e^{-4}$.
MSD ,i.e, mean squared deviation of the adaptive weights from the actual system weights are plotted as a function of number of iterations for LMS and NFRMS algorithms:

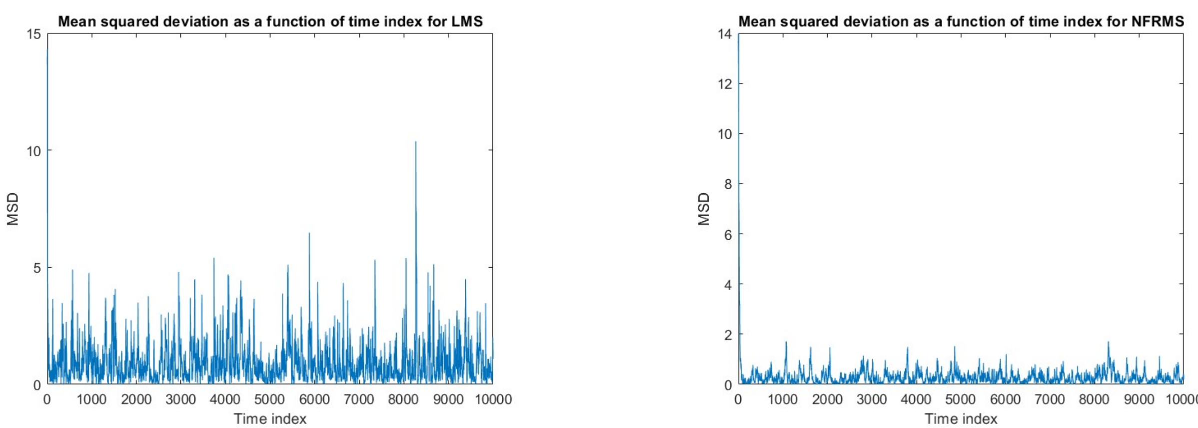
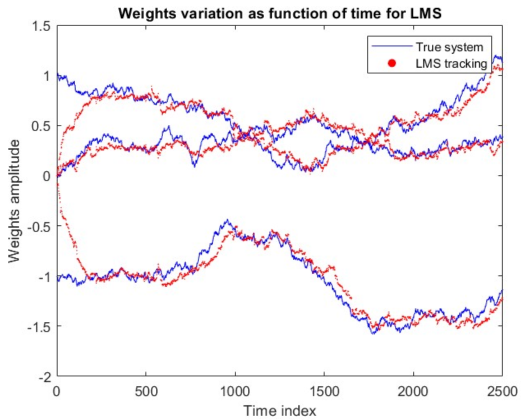


Figure 4: MSD plots for LMS and NFRMS

Clearly, NFRMS algorithm provides better performance than the standard LMS algorithm. Both algorithms converge to steady state pretty fast but the MSD at steady state is lower in case of NFRMS.

6.2 System tracking

For this simulation, a 3-tap random walk model is considered as the system. The initial taps are [-1 0 1]. The measurement noise is once again an IID, Laplace noise of variance 0.1. $\mu = 0.034, p = 3$ and $\epsilon = 1e^{-4}$. System taps and adaptive filter taps are plotted against the time indices below:



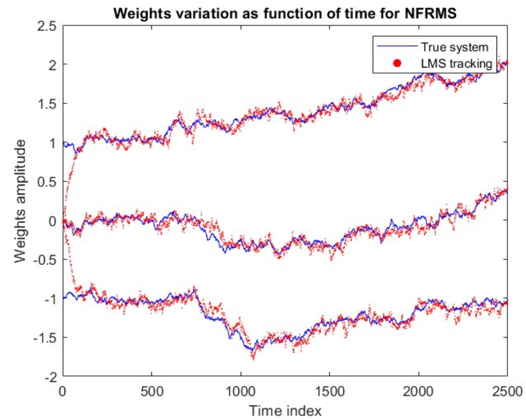
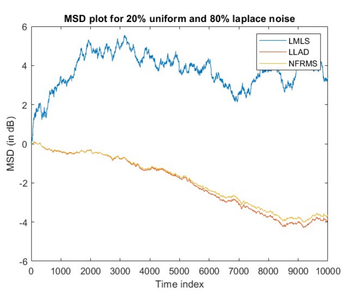


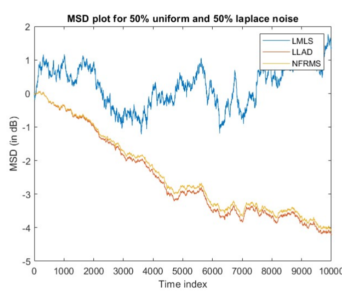
Figure 5: Filter taps against time for LMS and NFRMS

It is evident from the plots that NFRMS can track changing systems better than standard LMS. The steady state error between the adaptive filter and the system taps seems to be much lower for NFRMS than it is for LMS.

6.3 Performance against different mixtures of noise

NFRMS algorithm was also tested against other algorithms (V-shaped LMLS and Λ-shaped LLAD) to see how it performs under different mixtures of sub-gaussian and super-gaussian noises (Sub-gaussian is uniform, Super-gaussian is Laplace). The model used is an LTI system whose weights are chosen randomly and normalized to unit L-2 norm. 40 taps are considered and the input signal is a white gaussian signal with variance 0.1. MSD plots for different noise mixtures are attached below: Clearly, LLAD and NFRMS provide better steady state response





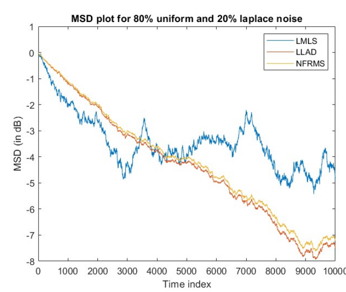


Figure 6: MSD plots for different noise mixtures

since they have a Λ-shaped component in their error weighing functions. LMLS suffers from poor steady state performance since it's not able to provide low weight for outliers. Within LLAD and NFRMS, the performance seems to be almost the same.

7 Conclusion

The proposed NFRMS algorithm performs better than standard LMS algorithm under different test cases since it is able to weigh errors of different magnitudes cleverly. It is utilizing both higher order and lower order moments of the noise and hence, has better convergence rate and steady state performance.