Dr. Alexander Munteanu
Dennis Köhn, Dr. Vibha Sahlot

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## 4. Home Exercises

# Efficient Algorithms

**Ex. 1** *(2 + 2 points)*

Suppose you shuffle $n$ pieces of paper labeled with $1, \ldots, n$ in an urn and draw one after another without replacement.

a) What is the probability that the number drawn in the $i$-th round is larger than all previous numbers? Explain your answer. *(Hint: You can also use backward analysis. Fix the largest number to be the one drawn in the i-th iteration. Then, calculate the probability that all the numbers drawn before it are smaller than this.)*

b) Compute the expectation of the number of rounds in which you draw a number that is larger than all previous numbers.

**Ex. 2** *(4 points)*

In the analysis of the edge contraction algorithm it was used that

$$\Pr\left(\bigcap_{j=1}^{i}\{e_j \notin C\}\right) = \prod_{j=1}^{i} \Pr(e_j \notin C \mid C \subseteq E_{j-1}),$$

see Section 2.4.3. This is a consequence of the chain rule of probability theory, which states that for random variables $X_1, X_2, \ldots, X_n$ we have that

$$\Pr(X_1 \cap X_2 \cap \ldots \cap X_n) = \Pr(X_1) \Pr(X_2 \mid X_1) \cdots \Pr(X_n \mid X_1, \ldots, X_{n-1}).$$

Prove the chain rule by induction.

**Ex. 3** *(4 points)*

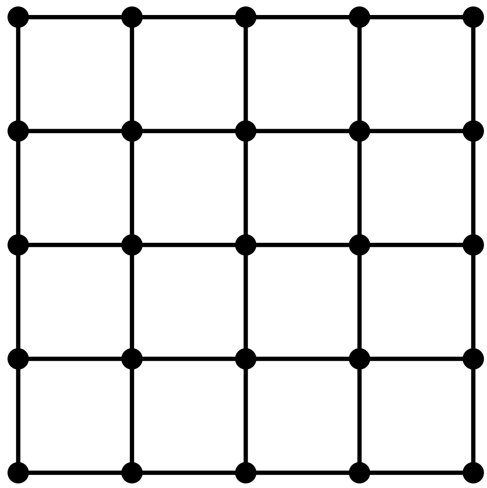
Recall that a spanning tree of a graph $G = (V, E)$ is a subgraph without cycles that connects all vertices of $G$. In this exercise, we call a spanning tree $\alpha$-*good* if it is an $\alpha$-spanner of $G$.

Give an example of a graph for which there exists no $\alpha$-good spanning tree with $\alpha \leq |V|/2$. Explain your answer.

**Ex. 4** *(4 + 2 + 2 points)*

In this exercise we will look at *grid graphs*. The vertex set of a grid graph $G$ of size $n^2$ is the set of points in the plane with coordinates $(i, j)$ for $1 \leq i \leq n$ and $1 \leq j \leq n$. The edges of $G$ are given by $\{(i, j), (i + 1, j)\}$ and $\{(i, j), (i, j + 1)\}$. Note that the number of edges of a grid graph with $n^2$ vertices is $(2 - o(1))n^2$, where the $o(1)$ factor is due to the vertices in the outermost layer of the graph having degree less than 4.

An example of a grid graph with $5^2$ vertices is given in the figure below.



a) Design an algorithm that computes a spanner of a grid graph. The number of edges of the spanner should be at most $(\frac{3}{2} + o(1))n^2$ (even fewer is of course also okay) and the stretch factor should be constant, i.e., it should not depend on $n$.

b) Show that the number of edges of your spanner is indeed at most $(\frac{3}{2} + o(1))n^2$.

c) Show that the stretch factor of your spanner is constant.