Exercise 3 - Efficient Algorithms

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Ex. 1

Let X be a random variable describing the number of l-sequences with $l = 1 + \log_2 n$. Let X_i be a indicator random variable describing whethe an l-sequence begins at index/toss i or not.

Since the coin gets tossed $n + \log_2 n$ times and an l-sequence is $1 + \log_2 n$ long, the highest index/toss, where an l-sequence can start is n. Therefore

$$X = \sum_{i=1}^{n} X_i \quad \text{and} \quad \mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}(X_i)$$

$$= \sum_{i=1}^{n} Pr(X_i)$$

The probability to get L consecutive of either heads or tails is

$$\underbrace{2}_{\text{head- or tail-sequence}} \cdot \left(\frac{1}{2}\right)^{L} = 2 \cdot \left(\frac{1}{2}\right)^{1+\log_2 n} = \left(\frac{1}{2}\right)^{\log_2 n} = \frac{1}{n}$$

$$\Rightarrow \mathbb{E}(X) = \sum_{i=1}^{n} \frac{1}{n} = n \cdot \frac{1}{n} = 1$$

Ex. 2

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a)

Let $m \in \mathbb{N}$ be the constant desired size of S.
Let $Pr: P \mapsto \mathbb{R}$ and $w: P \mapsto \mathbb{R}$ be functions.

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weightedSet(P, d, C, OPT, m, Pr, w)
    S = ∅

    for every p ∈ P do
        Pr(p) = d(p,C) / OPT

    while |S| < m do
        s = Select a p ∈ P with probability
              Pr(p), without replacement
        w(s) = OPT / (m · d(s,C))
        S = S ∪ {s}

    return S, w
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b) Line 1-2: $O(1)$
Line 3-4: $O(n)$
Line 5-9: $O(m)$ with constant $m \Rightarrow O(1)$
Line 10: $O(1)$

$\Rightarrow$ running time is $O(n)$

c) The bigger the distance $d(p,C)$ of a $p \in P$, the higher the probability

$$Pr(p) = \frac{d(p,C)}{\sum_{q \in P} d(q,C)} = \frac{d(p,C)}{OPT}$$

of being selected. The weight of a selected $s \in S$ behaves antiproportional with $d(s,C)$

$$\omega(s) = \frac{\sum_{p \in P} d(p,C)}{m \cdot d(s,C)} = \frac{OPT}{m \cdot d(s,C)}$$

Therefore the expected value of the contribution to the sum of a single point is

$$\mathbb{E}(\omega(s) d(s,C)) = \mathbb{E}\left(\frac{OPT}{|S| \cdot d(s,C)} \cdot d(s,C)\right)$$

$$= \mathbb{E}\left(\frac{OPT}{|S|}\right)$$

Since there is no more random variable contained

$$\mathbb{E}\left(\frac{OPT}{|S|}\right) = \frac{OPT}{|S|}$$

As a result we get

$$\mathbb{E}\left(\sum_{s \in S} w(s) d(s,C)\right) = \sum_{s \in S} \mathbb{E}(w(s) d(s,C))$$

$$= \sum_{s \in S} \frac{OPT}{|S|}$$

$$= OPT$$

$\square$