Exercise 3 - Efficient Algorithms

Sonntag, 17. November 2024 23:17

Tim Suchan,Linus Palm,Nico Schum

Ex. 1

Let X be a random variable describing the number of l-sequences with $l = 1 + \log_2 n$. Let X_i be a indicator random variable describing whethe an l-sequence begins at index/toss i or not.

Since the coin gets tossed $n + \log_2 n$ times and an l-sequence is $1 + \log_2 n$ long, the highest index/toss, where an l-sequence can start is n. Therefore

$$X = \sum_{i=1}^{n} X_i \quad \text{and} \quad \mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}(X_i)$$

$$= \sum_{i=1}^{n} Pr(X_i)$$

The probability to get L consecutive of either heads or tails is

$$2 \cdot \left(\frac{1}{2}\right)^{L} = 2 \cdot \left(\frac{1}{2}\right)^{1+\log_2 n} = \left(\frac{1}{2}\right)^{\log_2 n} = \frac{1}{n}$$

head- or tail-sequence

$$\Rightarrow \quad \mathbb{E}(X) = \sum_{i=1}^{n} \frac{1}{n} = n \cdot \frac{1}{n} = 1$$

Ex. 2

Sonntag, 17. November 2024 15:23

a)

Let $m \in \mathbb{N}$ be the constant desired size of S.
Let $Pr: P \mapsto \mathbb{R}$ and $w: P \mapsto \mathbb{R}$ be functions.

```
weightedSet(P, d, C, OPT, m, Pr, w)
    S = ∅

    for every p ∈ P do
        Pr(p) = d(p, C) / OPT

    while |S| < m do
        s = Select a p ∈ P with probability
              Pr(p), without replacement
        w(s) = OPT / (m · d(s, C))
        S = S ∪ {s}

    return S, w
```

b) Line 1-2: $O(1)$
Line 3-4: $O(n)$
Line 5-9: $O(m)$ with constant $m \Rightarrow O(1)$
Line 10: $O(1)$

$\Rightarrow$ running time is $O(n)$

c) The bigger the distance $d(p,C)$ of a $p \in P$, the higher the probability

$$Pr(p) = \frac{d(p,C)}{\sum_{q \in P} d(q,C)} = \frac{d(p,C)}{OPT}$$

of being selected. The weight of a selected $s \in S$ behaves antiproportional with $d(s,C)$

$$w(s) = \frac{\sum_{p \in P} d(p,C)}{m \cdot d(s,C)} = \frac{OPT}{m \cdot d(s,C)}$$

Therefore the expected value of the contribution to the sum of a single point is

$$\mathbb{E}(w(s)d(s,C)) = \mathbb{E}\left(\frac{OPT}{|S| \cdot d(s,C)} \cdot d(s,C)\right)$$

$$= \mathbb{E}\left(\frac{OPT}{|S|}\right)$$

Since there is no more random variable contained

$$\mathbb{E}\left(\frac{OPT}{|S|}\right) = \frac{OPT}{|S|}$$

As a result we get

$$\mathbb{E}\left(\sum_{s\in S} w(s)d(s,C)\right) = \sum_{s\in S} \mathbb{E}(w(s)d(s,C))$$

$$= \sum_{s\in S} \frac{OPT}{|S|}$$

$$= OPT$$

$\square$

Ex. 3

Sonntag, 17. November 2024 23:56

a)
Let A be the given sequence and n its length.

```
1  randomPermutation(A, n)
2      B = []

3      for i in 1,...,n do
4          index = RANDOM(n-i+1)
5          B[i] = A[index]  # append A[index] to list B
6          swap(A[index], A[n-i+1])  # swap appended element with last element of A
7      return B
```

b) Line 1-2: $O(1)$
Line 3-6: $O(n)$
Line 7: $O(1)$

=> running time is $O(n)$

c) In each iteration an index of a nomber gets selected, that hasn't been appended to B yet. The number gets appended to B and in A it gets swapped with the last number, that hasn't been picked yet.

The probability for a number to be selected for the i-th position is $\frac{1}{n-i+1}$.

Therefore the probability for any output permutation is

$$\prod_{i=1}^{n} \frac{1}{n-i+1} = \frac{1}{n} \cdot \frac{1}{n-1} \cdot \ldots \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{n!}$$