Names: Tim Suchan, Linus Palm, Nico Schum

Ex. 1

Donnerstag, 5. Dezember 2024 23:07

given: random permutation $x_1, x_2, \dots, x_n$ of the numbers $\{1, 2, \dots, n\}$

a) There are $i!$ possible permutations of the first i numbers. Fixing the largest one at the end results in $(i-1)!$ permutations where $x_i > x_1, x_2, \dots, x_{i-1}$ holds. Therefore

$$Pr(x_i > x_1, x_2, \dots, x_{i-1}) = \frac{(i-1)!}{i!} = \frac{1}{i}$$

b) Let S be the number of rounds in which we draw a number, that is larger than all previous numbers:

$$S = \sum_{i=1}^{n} X_i \quad \text{with} \quad X_i = \begin{cases} 1, \text{ if } x_i > x_1, x_2, \dots, x_{i-1} \\ 0, \text{ else} \end{cases}$$

$$\mathbb{E}(S) = \mathbb{E}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathbb{E}(X_i)$$

$$= \sum_{i=1}^{n} \frac{1}{i} = H_n \quad (\text{Harmonic number})$$

Ex. 2

Donnerstag, 5. Dezember 2024 23:08

- Assumption:

$$Pr(X_1 \cap X_2 \cap \ldots \cap X_n) = Pr(X_1) \cdot Pr(X_2 | X_1) \cdot \ldots \cdot Pr(X_n | X_1, \ldots, X_{n-1})$$

- Basis step with $n = 1$:

$$Pr(X_1) = Pr(X_1) \quad \checkmark$$

- Induction step $(n \to n+1)$:

Suppose $Pr(X_1 \cap X_2 \cap \ldots \cap X_n) = Pr(X_1) \cdot Pr(X_2 | X_1) \cdot \ldots \cdot Pr(X_n | X_1, \ldots, X_{n-1})$ (*)

$$Pr(A | B) = \frac{Pr(A \cap B)}{P(B)}$$

$$\Leftrightarrow \; Pr(A \cap B) = Pr(B) Pr(A | B) \quad (*)$$

$$\Rightarrow \Pr(X_1 \cap X_2 \cap \ldots \cap X_{n+1})$$

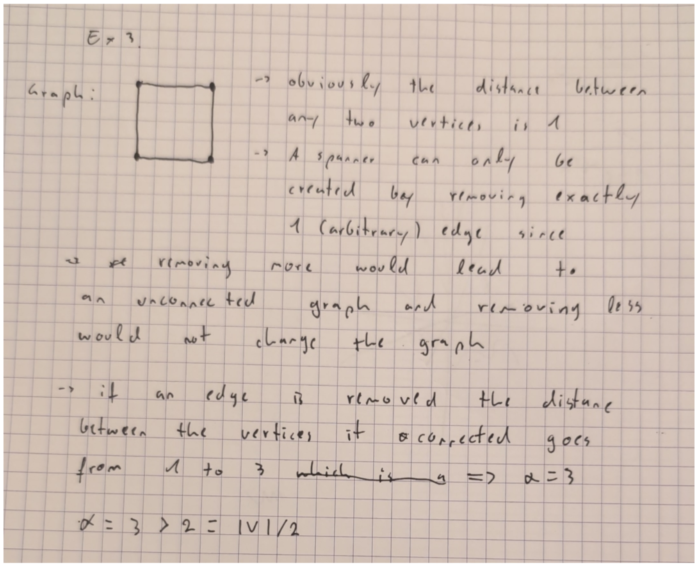
$$= \Pr((X_1 \cap X_2 \cap \ldots \cap X_n) \cap X_{n+1})$$

$$\overset{(*)}{=} \Pr(X_1 \cap X_2 \cap \ldots \cap X_n) \cdot \Pr(X_{n+1} \mid X_1, X_2, \ldots, X_n)$$

$$\overset{(*)}{=} \Pr(X_1)\Pr(X_2 \mid X_1) \cdot \ldots \cdot \Pr(X_n \mid X_1, X_2, \ldots, X_{n-1}) \cdot \Pr(X_{n+1} \mid X_1, X_2, \ldots, X_n)$$

↳ chain rule for $n+1$ $\square$

Ex 3:



Ex 3

Graph:

-> obviously the distance between any two vertices is 1

-> A spanner can only be created by removing exactly 1 (arbitrary) edge since

-> removing more would lead to an unconnected graph and removing less would not change the graph

-> if an edge is removed the distance between the vertices it connected goes from 1 to 3 ~~which is~~ => $\alpha = 3$

$\alpha = 3 > 2 = |V|/2$

ex 4

a)

Algorithm 1 Compute spanner

$E_s \leftarrow E$
for each pair $((i_1, j_1), (i_2, j_2)) \in E_s$ **do**
  if $i_1 = i_2$ **and** $i_1 \bmod 2 \neq 0$ **then**
    $E_s \leftarrow E_s \setminus \{((i_1, j_1), (i_2, j_2))\}$
  end if
end for
return E_s

b)

b) First we observe that only edges where both vertices share the i coordinate are removed, so all edges ~~j~~ included in $\{((i,j),(i+1,j))\}$ stay in the graph. Since there are n different j and $(n-1)$ different i that can be input into this pattern while statisfying that no i or j index of a coordinate exceeds n, there are $(n-1)\cdot n$ such edges

The same number of edges with constant ~~j~~ i exists in the original graph. Out of these all edges with an uneven i are removed from the graph in the algorithm, all ~~th~~ edges where i is even remain. From $1 \leq i \leq n$ we know there are at most $\frac{1}{2}n$ even i. The number of edges ~~therefore is~~ that remain therefore is $\frac{1}{2}n(n-1)$.

Combining the two observations for edges with constant i or j, which, per definition, includes all edges there are

$$(n-1)\cdot n + (n-1)\cdot \frac{1}{2}n \leq n^2 + \frac{1}{2}n^2 = \frac{3}{2}n^2 \leq \left(\frac{3}{2}+O(1)\right)n^2$$

edges in the spanner

c)

Let $(i_1, j_1), (i_2, j_2)$ be two vertices in the G. Now let $((i_t, j_t), (i_t, j_t+1))$ meaning $i_t) has to be uneven. Then in the spanner G_s this edge can be avoided by instead moving along the edges.:

- $((i_t, j_t), (i_t + 1, j_t))$ -> different i coordinates
- $((i_t + 1, j_t), (i_t + 1, j_t + 1))$ -> $i_t + 1 $ is obviously even
- $((i_t + 1, j_t + 1), (i_t, j_t + 1))$ -> different i coordinates

As pointed out none of these edges meet the removal criteria of the algorithm, they therefore still exist in G_s. If $i_t = n$ then an equivalent path can be choosen with $i_t - 1$ instead of $i_t + 1$ Therefore the stretch factor is at most 3 since any shortest path can be adapted accordingly. It is also at least three since the direct path between two vertices can only be replaced at least three steps through the neighboring vertices.

=> $\alpha = 3$