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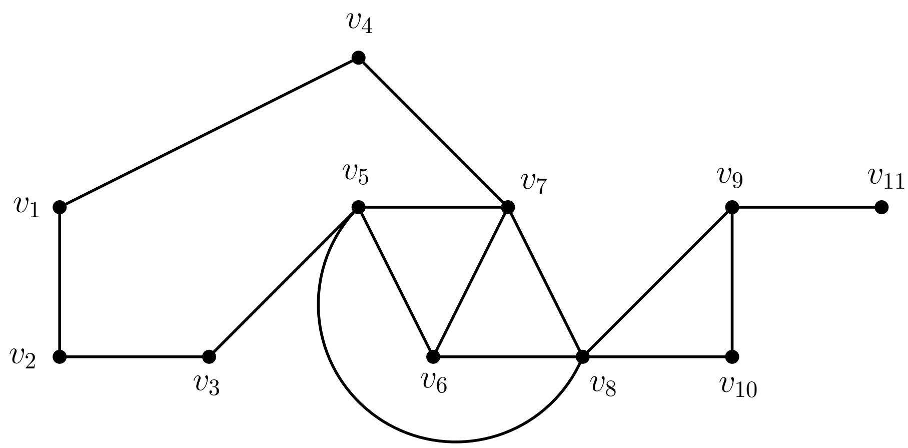
2. Home Exercises

Efficient Algorithms

Ex. 1 *(4 points)*

Consider the **Densest Subgraph** problem and the **DensestSubgraph** algorithm introduced in the lecture. Apply the algorithm to the input instance in the following figure.



As tie-breaking rule you may use that when multiple vertices have minimal degree, the vertex with the smallest index is chosen.

For each iteration of the **for**-loop, explain which vertex is deleted, draw the remaining graph $G[U]$ and explain whether S and D are updated or not.

Ex. 2 *(2 + 2 + 2 + 2 points)*

Consider a fair 6-sided die, i.e., every side has probability $1/6$ of being rolled. We will roll the die until we obtain a 6. Let X be the random variable that denotes the number of times the die is rolled until a 6 is obtained. If, for example, we roll consecutively 1,3,2 and 6, then $X = 4$.

a) Compute the probability that $X = 3$.

b) For an arbitrary natural number k, compute the probability that $X = k$. Explain your answer.

In the next two questions, suppose that we repeat the procedure described above 10 times. In other words, we roll the die until we obtain a 6, then we start over and roll again until we obtain a 6 and so on, until we obtain a 6 ten times.

c) Compute the probability that in each of the ten runs there was a 6 within the first three rolls of that run.

d) What is the expected number of runs that roll a 6 within the first three rolls? Explain your answer.

Ex. 3 *(3 + 2 + 3 Points)*

In this exercise, we will look at a relation between the k-**Center** and **Set Cover** problems. We start by considering the decision versions of both problems.

The decision version of **Set Cover** is: given a ground set S of m items, a collection $\mathcal{T} = \{T_1, \ldots, T_n\}$ of n subsets of S and a natural number N, decide whether there exists a set I of N indices such that $\cup_{i \in I} T_i = S$.

The decision version of k-**Center** is: given a metric space (P, d) and a real number r, decide whether there exists a set C of k centers such that $\text{cost}(P, C) = \max_{p \in P} d(p, C)$ is at most r.

a) Suppose that we have a blackbox algorithm that solves the decision version of **Set Cover**. Explain how this algorithm can be applied to solve the decision version of k-**Center**. The number of calls of the blackbox algorithm should be at most $O(|P|^2)$.

   Hint: given an instance of the k-**Center** problem, what is the corresponding instance of the **Set Cover** problem?

b) We now consider the original problems, so not the decision versions. Suppose that we have an α-approximation algorithm for the **Set Cover** problem. Can we use this algorithm to obtain an α-approximation for the k-**Center** problem in the same way as in a)? Explain why or why not.

c) Suppose we are given the optimal cost OPT of the k-**Center** problem in advance. Describe an algorithm that finds a set of centers C such that $\text{cost}(P, C) \leq 2\,\text{OPT}$ in time $O(|P|^2)$.

   Note: algorithms based on Gonzales' algorithm will not be awarded any points.