Exercise 3 - Efficient Algorithms

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Ex. 1

Let X be a random variable describing the number of l-sequences with $l = 1 + \log_2 n$. Let X_i be a indicator random variable describing whether an l-sequence begins at index/toss i or not. ✓

Since the coin gets tossed $n + \log_2 n$ times and an l-sequence is $1 + \log_2 n$ long, the highest index/toss, where an l-sequence can start is n. ✓ Therefore

$$X = \sum_{i=1}^{n} X_i \quad \text{and} \quad \mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}(X_i)$$

$$= \sum_{i=1}^{n} Pr(X_i) \checkmark$$

The probability to get L consecutive of either heads or tails is

$$\underbrace{2}_{\text{head- or tail-sequence}} \cdot \left(\tfrac{1}{2}\right)^{L} = 2 \cdot \left(\tfrac{1}{2}\right)^{1+\log_2 n} = \left(\tfrac{1}{2}\right)^{\log_2 n} = \frac{1}{n} \checkmark$$

head- or tail-sequence ✓

$$\Rightarrow \mathbb{E}(X) = \sum_{i=1}^{n} \frac{1}{n} = n \cdot \frac{1}{n} = 1$$

✓

Ex. 2

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a) 1/4

Let $m \in \mathbb{N}$ be the constant desired size of S. ok

Let $Pr: P \mapsto R$ and $w: P \mapsto R$ be functions.

```
weightedSet(P, d, C, OPT, m, Pr, w)
    S = ∅

    for every p ∈ P do
        Pr(p) = d(p,C) / OPT     // Σ_p Pr(p) = 1

    while |S| < m do
        s = Select a p ∈ P with probability
            Pr(p), without replacement   -> changes Pr(p)!
        w(s) = OPT / (m · d(s,C))
        S = S ∪ {s}

    return S, w
```

unknown!

That's a lot more complicated than necessary :)

1/2 b) Line 1-2: $O(1)$ ✓
Line 3-4: $O(n)$ ok
Line 5-9: $O(m)$ with constant $m \Rightarrow O(1)$
→ How?
Line 10: $O(1)$ ✓

$\Rightarrow$ running time is $O(n)$ ✓

1/2 c) The bigger the distance $d(p,C)$ of a $p \in P$, the higher the probability

$$Pr(p) = \frac{d(p,C)}{\sum_{q \in P} d(q,C)} = \frac{d(p,C)}{OPT}$$ ✓

of being selected. The weight of a selected $s \in S$ behaves antiproportional with $d(s,C)$

$$w(s) = \frac{\sum_{p \in P} d(p,C)}{m \cdot d(s,C)} = \frac{OPT}{m \cdot d(s,C)}$$ ✓

Therefore the expected value of the contribution to the sum of a single point is

$$\mathbb{E}(w(s)d(s,C)) = \mathbb{E}\left(\frac{OPT}{|S| \cdot d(s,C)} \cdot d(s,C)\right)$$

$$= \mathbb{E}\left(\frac{OPT}{|S|}\right)$$ ✓

Since there is no more random variable contained

$$\mathbb{E}\left(\frac{OPT}{|S|}\right) = \frac{OPT}{|S|}$$ ✓

As a result we get

$$\mathbb{E}\left(\sum_{s \in S} w(s)d(s,C)\right) = \sum_{s \in S} \mathbb{E}(w(s)d(s,C))$$

$$= \sum_{s \in S} \frac{OPT}{|S|}$$

$$= OPT$$ ✓

$\square$

C & OPT known makes the proof easier...

Ex. 3

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4/4 a)

Let A be the given sequence and n its length.

```
1  randomPermutation(A, n)
2      B = []

3      for i in 1,...,n do
4          index = RANDOM(n-i+1)
5          B[i] = A[index]   # append A[index] to list B
6          swap(A[index], A[n-i+1])   # swap appended element with last element of A
7      return B
```

2/2 b) Line 1-2: $O(1)$

Line 3-6: $O(n)$

Line 7: $O(1)$

$\Rightarrow$ running time is $O(n)$ ✓

$\frac{1}{2}$ c) In each iteration an index of a number gets selected, that hasn't been appended to B yet. The number gets appended to B and in A it gets swapped with the last number, that hasn't been picked yet.

(*) The probability for a number to be selected for the i-th position is $\frac{1}{n-i+1}$. (f)

Therefore the probability for any output permutation is

$$\prod_{i=1}^{n} \frac{1}{n-i+1} = \frac{1}{n} \cdot \frac{1}{n-1} \cdot \ldots \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{n!}$$ ✓

(*) This statement is not what you wanted to say. It only holds under the assumption that $i-1$ other numbers are already fixed in position.