EECS 204002
Data Structures 資料結構
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# CH. 1
# BASIC CONCEPTS


腦神經
重新繞線

# 1.5

# Algorithm

1.5

# What is an Algorithm?

An ***algorithm*** is a finite set of instructions that accomplishes a particular task (problem) and satisfies the following criteria:

- Input
  - Zero/more quantities are externally supplied.
- Output
  - At least one quantity is produced.
- Definiteness
  - Each instruction is clear and unambiguous.
- Finiteness
  - Terminate after a finite number of steps.
- Effectiveness:
  - Every instruction must be basic and easy to be computed.

1.5.1

# Representation of Algorithms

- Natural languages
  - English, …etc.
- Graphic representation
  - Flowchart.
  - Feasible only if the algorithm is small and simple.
- Programming language
  - C++
  - Concise and effective!

1.5.1

# Example: Binary Search

Problem statement: Assume we have $n \geq 1$ distinct integers that are ***sorted*** in array $A[0] \ldots A[n-1]$. Determine the existence of an integer $x$. If $x = A[j]$, return index $j$; otherwise return $-1$.

| | A[0] | A[1] | A[2] | A[3] | A[4] | A[5] | A[6] | A[7] |
|---|---|---|---|---|---|---|---|---|
| A | 1 | 3 | 5 | 8 | 9 | 17 | 32 | 50 |

Eg. For x=9, return index 4;
For x=10, return -1.

1.5.1

# BS in Plain English

1. Let *left* and *right* denote the left and right ends of the list with initial value 0 and n-1.
2. Let *middle* = *(left+right)* / 2 be the middle position in the list
3. Compare A[*middle*] with x and obtain three results:
   a. x < A[*middle*]: x must be somewhere between 0 and *middle-1*. We set *right* to *middle-1*
   b. x == A[*middle*]: We return *middle*
   c. x > A[*middle*]: x must be somewhere between *middle+1* and *n-1*. We set *left* to *middle+1*.
4. If x is not found and there are still integers to check, we recalculate *middle* and repeat the above comparison.

1.5.1

# BS in Pseudo C++ Code

```
int BinarySearch(int *A, const int x, const int n)
{ int left=0, right=n-1;

  while (left <= right)
  { // more integers to check
    int middle = (left+right)/2;
    if (x < A[middle])  right = middle-1;
    else if (x > A[middle])  left = middle+1;
    else return middle;
  } // end of while
  return -1; // not found
}
```

1.5.2

# Recursive Algorithm

- A powerful mechanism to make your algorithm or code more clear.
- Direct recursion :
  - Function calls itself directly.
  - E.g. funcA → funcA.
- Indirect recursion:
  - Function A calls other function B that invoke the function A itself.
  - E.g. funcA → funcB → funcA.

# A Recursively Defined Problem

The binomial coefficient

$$C(n,m) = \frac{n!}{m!(n-m)!}$$

can be computed by the recursive formula:

$$C(n,m) = C(n-1,m) + C(n-1,m-1)$$

where $C(0,0) = C(n,n) = 1$

# Principles for Feasible Recursive Algorithms

- Termination conditions:
  - The function should return a value or stop calling itself under certain conditions.
- Decreased Parameters
  - So that each call is one step closer to a termination condition.

# There is a "While" statement

- Replace with if-else and recursion
- In Binary Search problem…

```
int BinarySearch(int *A, const int x, const int n)
{ int left=0, right=n-1;

  while (left <= right)
  {
     …
  }
  return -1;
}
```

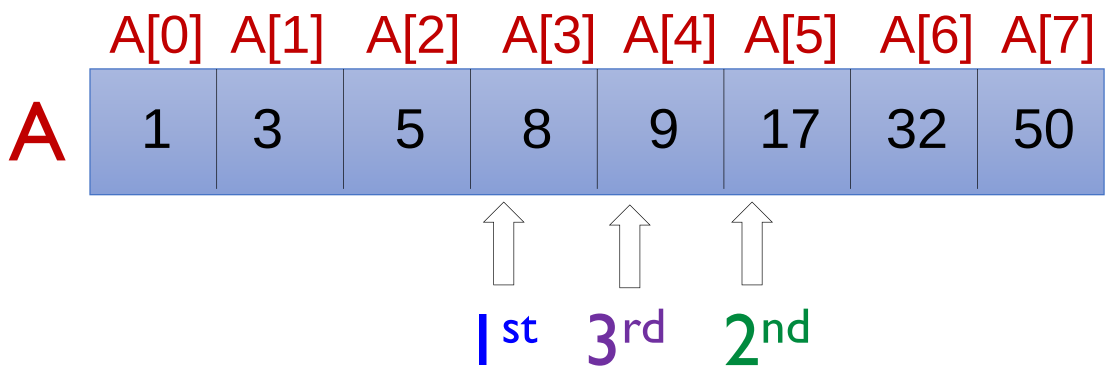
# Recursive Binary Search

```
int BinarySearch(int *A, const int x, const int
                    left, const int right )
{ // Search the A[left],...,A[right] for x
  if (left <= right) { // more integers to check
    int middle = (left+right)/2;
    if (x < A[middle])
       return BinarySearch(A, x, left, middle-1);
    else if (x > A[middle])
       return BinarySearch(A, x, middle+1, right);
    return middle;
  } // end of if
  return -1; // not found
}
```

1.5.2

# Example

- Search for x=9 in array A[0]…[7] :




- 1st call: `BinarySearch(A, 9, 0, 7)`

  2nd call: `BinarySearch(A, 9, 4, 7)`

  3rd call: `BinarySearch(A, 9, 4, 4)`

  return index 4.

# Quiz

Write down the recursive version of Binomial coefficient in

**Recursive form**

$C(n, m) = C(n-1, m) + C(n-1, m-1)$

**Termination conditions**

$C(0, 0) = C(n, n) = 1$

1.5.2

# Criteria of a "Good" Program

- Does it do what you want to do?
- Does it work correctly?
- Any documentation about how to use it?
- Are functions created logically?
- Is the code readable?
- However, the above questions are **HARD** to achieve (at least when only DS is taught).
- So, we focus on the "**Performance**" of the program

# 1.7

# Performance Analysis and Measurement

1.7

# Performance Evaluation

- Two aspects:
  - **Space Complexity**
    - How much memory space is used?
  - **Time Complexity**
    - How much execution time is needed?
- Two approaches:
  - **Performance Analysis**
    - machine independent
    - a prior estimate
  - **Performance Measurement**
    - machine dependent
    - a posterior measure

# 1.7.1 Uses Of Performance Analysis

- Determine practicality of algorithm
- Predict run time on large instance
- Compare algorithms with different complexity
  - e.g., $O(n)$ v.s. $O(n^2)$

1.7.1

# Performance Analysis

- Space complexity : $S(P) = C + S_P(I)$
- $C$ is a **fixed** part:
  - Independent of the size of input and output.
  - Space for instruction and static variables, fixed-size structured variables, constants.
- $S_P(I)$ is a **variable** part:
  - Depends on the specific problem instance.
  - Space of referenced variable and recursion stack space (**Instance Characteristics**).

# Instance Characteristics (I)

- Commonly used characteristics (I) include the size of the **input** and **output** of the problem.
- We shall concentrate solely on estimating the 2nd part, $S_P(I)$.

- Ex1. sorting(A[], n)
  Then I= number of integers = n.
- Ex 2. Summation of 1 to n, i.e., 1+2+3+…+n
  Then I= value of n = n.

# Space Complexity: Simple Function

```
float Abc(float a, b, c)
{
  return a+b+b*c+(a+b-c)/(a+b)+4.0;
}
```

- I = a, b, c
- C = space for the program + space for variables a, b, c, Abc = constant
- $S_{Abc}(I) = 0$
- $S(Abc) = C + S_{Abc}(I) =$ constant

# Space Complexity : Iterative Summation

```
float Sum(float *A, const int n)
{ float s = 0;
  for(int i=0; i<n; i++)
      s += A[i];
  return s;
}
```

- $I = n$ (number of elements to be summed)
- $C$= constant
- $S_{Sum}(I) = 0$ (A stores only the address of array)
- $S(Sum) = C + S_{Sum}(I) =$ constant

# Space Complexity : Recursive Summation

```
float Rsum(float *A, const int n)
{
  if (n<=0) return A[0];
  else return (Rsum(A, n-1) + A[n-1]);
}
```

- $I = n$ (number of elements to be summed)
- $C$= constant
- Each recursive call "Rsum" requires $4(1 + 1 + 1) =$ 12 bytes.
- Number of calls: $Rsum(A, n) \rightarrow Rsum(A, n-1) \rightarrow \ldots \rightarrow Rsum(A, 0) \Rightarrow n + 1$ calls
- $S(Rsum) = C + S_{Rsum}(n) =$ const $+ 12\,(n + 1)$

1.7.1.2

# Time Complexity

$$T(P) = C + T_P(I)$$

- $C$ is a **constant**:
  - Compile time.
- $T_P(I)$ is **variable**:
  - Execution time.

# Performance Analysis

- How to evaluate $T_P(I)$ ?
  - Count every Add, Sub, Multiply, … etc.
  - Practically infeasible because each instruction takes different running time at different machine.
- Use "**program step**" to estimate $T_P(I)$
  - "program step" = a statement whose execution time is ***independent*** of instance characteristics(I).

abc=a+b+b*c; → one program step

a=2; → one program step

# Time Complexity : Iterative Summation

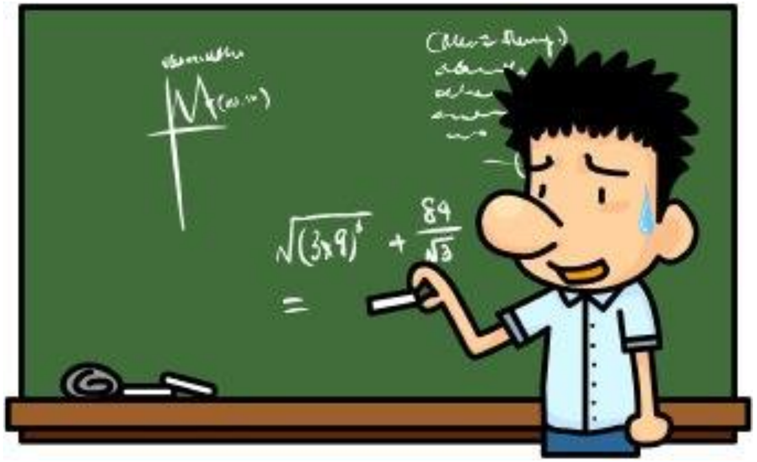
- $I = n$ (number of elements to be summed)
- $T_{Sum}(I) \ = 1 + n + 1 + n + 1 = 2n + 3$
- $T(Sum) = C + T_{Sum}(n)$ = const+$(2n + 3)$

```
float Sum(float *A, const int n)
{ float s = 0;                    // 1 step
  for(int i=0; i<n; i++)          // n+1 steps
      s += A[i];                  // n steps
  return s;                       // 1 step
}
```

# Time Complexity : Recursive Summation

```
float Rsum(float *A, const int n)
{
   if (n<=0)                          // 1 step
      return A[0];                    // 1 step
   else return(Rsum(A,n-1)+A[n-1]); // 1 step
}
```

- I = n (number of elements for summation)
- $T_{Rsum}(n)$ = ?



# Time Complexity : Recursive Summation

```
float Rsum(float *A, const int n)
{
  if (n<=0)                              // 1 step
    return A[0];                         // 1 step
  else return (Rsum(A, n-1) + A[n-1]);// 1 step
}
```

- $I = n$ (number of elements for summation)
- $T_{Rsum}(0) = 2$
- $T_{Rsum(n)} = 2 + T_{Rsum(n-1)}$

$$= 2 + (2 + T_{Rsum}(n-2))$$
$$= \dots$$
$$= 2n + T_{Rsum}(0) = 2n + 2$$

# Time Complexity : Matrix Addition

```
void Add(int **a, int **b, int **c, int m, int n)
{
  for(int i=0; i<m; i++)     // m+1 steps
   for(int j=0; j<n; j++)    // m*(n+1) steps
      c[i][j] = a[i][j]+b[i][j]; // m*n steps
}
```

- $I = m(rows), n\ (columns)$
- $T_{Add(I)} = (m+1) + m(n+1) + mn$
  $= 2mn + 2m + 1$
- $T(Add) = C + T_{Add}(I)$
  $= const + (2mn + 2m + 1)$

# Observation on Step Counts

- In the previous examples :
  $T_{Sum}(n) = 2n + 3$ steps
  $T_{Rsum}(n) = 2n + 2$ steps
- So, Rsum is faster than Sum?
  - **No!**
  - ∵The execution time of each step is different.
- "**Growth Rate**" is more critical
  - "*How the execution time changes in the instance characteristics?*"

# Program Growth Rate

- In the Sum program, $T_{Sum}(n) = 2n + 3$ means when $n$ is tenfold $(10X)$, the execution time $T_{Sum}(n)$ is tenfold $(10X)$.

- We say that Sum program runs in **linear** time.

- $T_{Rsum}(n) = 2n + 2$ also runs in **linear** time.

- We say $T_{Sum}(n)$ and $T_{Rsum}(n)$ have the same growth rate, and are equal in time complexity!

1.7.1.3

# Asymptotic Notation

- To make meaningful (but inexact) statements about the time and space complexities of a program.
  - Predict the growth rate.
- Two programs with time complexity
  - P1: $c_1n^2 + c_2n$
  - P2: $c_3n$
  - Which one runs faster?

1.7.1.3

# Asymptotic Notation

- Scenario 1: $c_1 = 1, c_2 = 2$, and $c_3 = 100$
  - $P1(n^2 + 2n) \leq P2(100n)$ for $n \leq 98$.
- Scenario 2: $c_1 = 1, c_2 = 2$, and $c_3 = 1000$
  - $P1(n^2 + 2n) \leq P2(1000n)$ for $n \leq 998$.
- No matter what values $c_1, c_2$ and $c_3$ are, there will be an n beyond which $c_1 n^2 + c_2 n > c_3 n$
- Therefore, we should compare the complexity for a ***sufficiently large value*** of $n$

# Notation: Big-O (O)

- Definition:
  $f(n) = O(g(n))$ iff there exist $\boldsymbol{c}$, $\boldsymbol{n_0} > \mathbf{0}$ such that $\boldsymbol{f(n)} \leq \boldsymbol{cg(n)}$ for **all** $\boldsymbol{n} \geq \boldsymbol{n_0}$.
- Ex1. $3n + 2 = O(n)$
  - $3n + 2 \leq 4n$ for all $n \geq 2$
- Ex2. $100n + 6 = O(n)$
  - $100n + 6 \leq 101n$ for all $n \geq 6$
- Ex3. $10n^2 + 4n + 2 = O(n^2)$
  - $10n^2 + 4n + 2 \leq 11\,n^2$ for all $n \geq 5$

The ***upper bound*** or ***worst-case running time***

# Notation: Omega ($\Omega$)

- Definition: $f(n) = \Omega(g(n))$ iff there exist $\boldsymbol{c, n_0 > 0}$ such that $\boldsymbol{f(n) \geq cg(n)}$ for all **all $\boldsymbol{n \geq n_0}$**.
- Ex1. $3n + 2 = \Omega(n)$
  - since $3n + 2 \geq 3n$ for all $n \geq 1$
- Ex2. $100n + 6 = \Omega(n)$
  - since $100n + 6 \geq 100\,n$ for all $n \geq 1$
- Ex3. $10n^2 + 4n + 2 = \Omega(n^2)$
  - since $10n^2 + 4n + 2 \geq n^2$ for all $n \geq 1$

The ***lower bound*** or ***best-case running time***

# Notation: Theta ($\Theta$)

- Definition: $f(n) = \Theta(g(n))$ iff $\boldsymbol{f(n) = O(g(n))}$ **and** $\boldsymbol{f(n) = \Omega(g(n))}$.
- Ex1. $3n + 2 = \Theta(n)$
- Ex2. $100n + 6 = \Theta(n)$
- Ex3. $10n^2 + 4n + 2 = \Theta(n^2)$

The ***tight bound*** or ***average-case running time***

# Theorem 1.2

If $f(n) = a_m n^m + \cdots + a_1 n + a_0, a_m > 0$, then $f(n) = O(n^m)$.

- $3n + 2 = O(n)$
- $100n + 6 = O(n)$
- $10n^2 + 4n + 2 = O(n^2)$
- $6n^4 + 1000\, n^3 + n^2 = O(n^4)$

- ***Leading constants*** and ***lower-order terms*** do not matter.

# Theorem 1.2 Proof

$$f(n) = a_m n^m + \ldots + a_1 n + a_0$$
$$\leq |a_m| n^m + \ldots + |a_1| n + |a_0|$$
$$\leq n^m (|a_m| + \ldots + |a_1| + |a_0|)$$
$$\leq n^m c \text{ for } n \geq 1$$

So, $f(n) = O(n^m)$

# Quiz

- $n^2 - 10n - 6 = O(?)$
- $n + \log n = O(?)$
- $n + n \log n = O(?)$
- $n^2 + \log n = O(?)$
- $2^n + n^{10000} = O(?)$
- $n^4 + 1000\, n^3 + n^2 = O(n^4)$, True or false?
- $n^4 + 1000\, n^3 + n^2 = O(n^5)$, True or false?

# Naming Common Functions

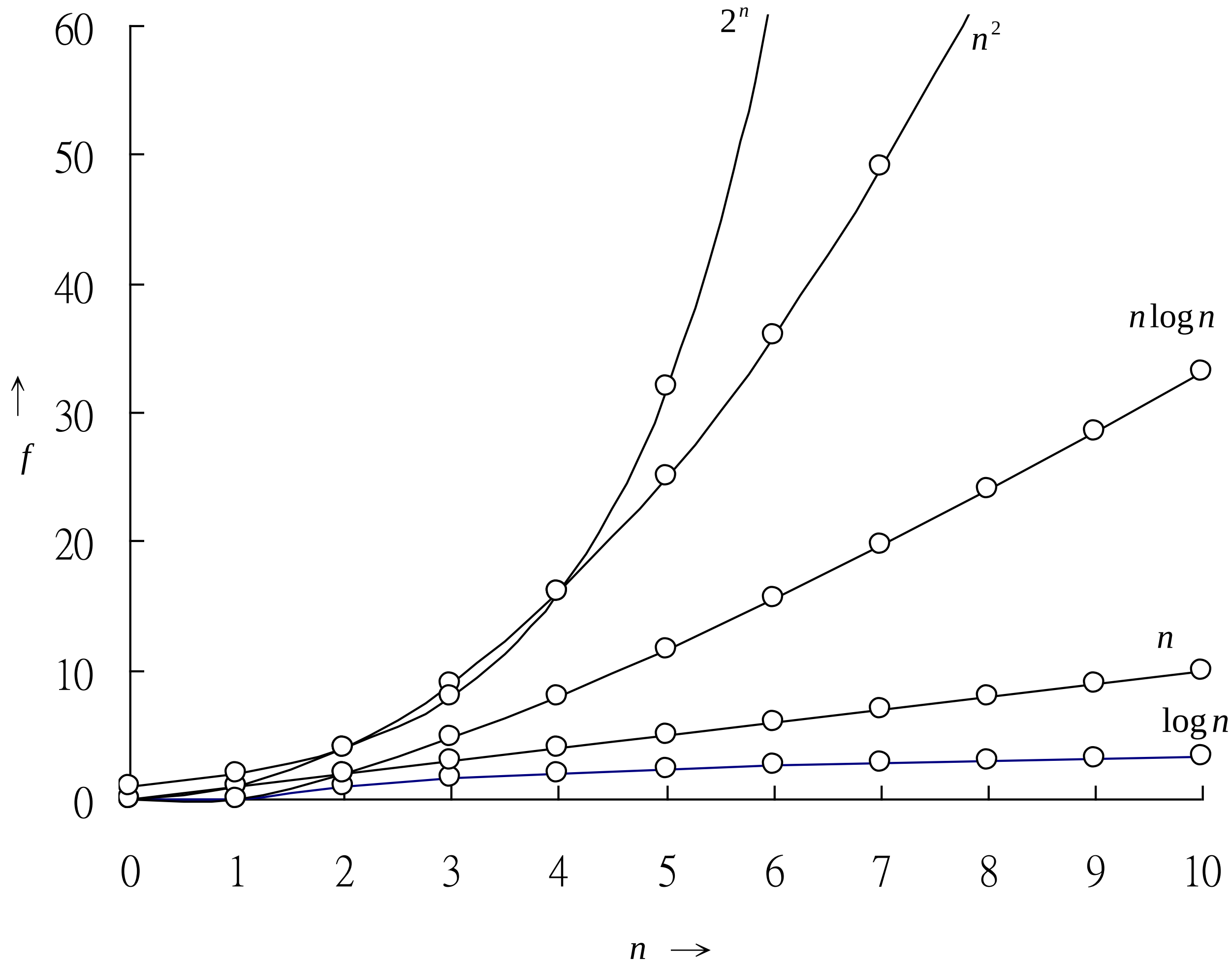
| Complexity | Naming |
|---|---|
| $O(1)$ | Constant time |
| $O(\log n)$ | Logarithmic time |
| $O(n \log n)$ | $O(\log n) \leq . \leq O(n^2)$ |
| $O(n^2)$ | Quadratic time |
| $O(n^3)$ | Cubic time |
| $O(n^{100})$ | Polynomial time |
| $O(2^n)$ | Exponential time |

When n is large enough, the latter terms take more time than the former ones.

1.7.1

F1.4

# Plot of Common Function Values



1.7.1

T1.8

# Execution Time Comparison

| | f (n) | | | | | | |
|---|---|---|---|---|---|---|---|
| n | $n$ | $n \log_2 n$ | $n^2$ | $n^3$ | $n^4$ | $n^{10}$ | $2^n$ |
| 10 | .01 µs | .03 µs | .1 µs | 1 µs | 10 µs | 10s | 1 µs |
| 20 | .02 µs | .09 µs | .4 µs | 8 µs | 160 µs | 2.84h | 1ms |
| 30 | .03 µs | .15 µs | .9 µs | 27 µs | 810 µs | 6.83d | 1s |
| 40 | .04 µs | .21 µs | 1.6 µs | 64 µs | 2.56ms | 121d | 18m |
| 50 | .05 µs | .28 µs | 2.5 µs | 125 µs | 6.25ms | 3.1y | 13d |
| 100 | .10 µs | .66 µs | 10 µs | 1ms | 100ms | 3171y | $4*10^{13}$y |
| $10^3$ | 1 µs | 9.96 µs | 1 ms | 1s | 16.67m | $3.17*10^{13}$y | $32*10^{283}$y |
| $10^4$ | 10 µs | 130 µs | 100 ms | 16.67m | 115.7d | $3.17*10^{23}$y | ... |
| $10^5$ | 100 µs | 1.66 ms | 10s | 11.57d | 3171y | $3.17*10^{33}$y | ... |
| $10^6$ | 1ms | 19.92ms | 16.67m | 31.71y | $3.17*10^7$y | $3.17*10^{43}$y | ... |

**µs = microsecond = $10^{-6}$ second; ms = milliseconds = $10^{-3}$ seconds**
**s = seconds; m = minutes; h = hours; d = days; y = years;**

# Compute Execution Time in Big-O

- Two approaches to compute the time complexity of a program in big-O
- Approach 1:
  Step1: Compute the total step-count.
  Step2: Take big-O using theorem 1.2.
- Approach 2:
  Step1: Take big-O on each step.
  Step2: Sum up the big-O of all steps.

# Rule of Sum

- If $f_1(n) = O(g_1(n))$, and $f_2(n)=O(g_2(n))$, then **$f_1(n) + f_2(n) = O(\max(g_1(n), g_2(n)))$.**
  - Ex. $f_1(n) = O(n)$, $f_2(n) = O(n^2)$
    Then $f_1(n) + f_2(n) = O(n^2)$.
  - Ex. $f_1(n) = O(n)$, $f_2(n) = O(n)$
    Then $f_1(n) + f_2(n) = O(n)$.
- Good for computing the time complexity of a sequential program.

# Rule of Product

```
for (i=0; i<n; i++) {            // O(n)
    for (j=0; j<n; j++)          // O(n)
        sum := sum + 1;          // O(1)
}
```

$f(n) = O(n \cdot n \cdot 1) = O(n^2)$.

- If $f_1(n) = O(g_1(n))$, and $f_2(n)=O(g_2(n))$, then $\mathbf{f_1(n) \cdot f_2(n) = O(g_1(n) \cdot g_2(n))}$.
  - Ex. $f_1(n) = O(n)$, $f_2(n) = O(n)$
    Then $f_1(n) \cdot f_2(n) = O(n^2)$.
- Applicable to **nested loops.**

# Complexity of Binary Search

```
int BinarySearch(int *A, const int x, const int n)
{ int left=0, right=n-1;

  while (left <= right)                        → O(?)
  { // more integers to check
    int middle = (left+right)/2;               → O(1)

    if (x < A[middle])  right = middle-1;      → O(1)

    else if (x > A[middle])  left = middle+1;  → O(1)

    else return middle;                        → O(1)
  } // end of while
  return -1; // not found
}
```

# Complexity of Binary Search

- Analysis of the while loop:
  - Iteration 1: $n$ values to be searched
  - Iteration 2: $n/2$ left for searching
  - Iteration 3: $n/4$ left for searching
  - …
  - Iteration k+1: $n/(2^k)$ left for searching

  When $n/(2^k) = 1$, searching **must** finish.

  i.e. $n = 2^k \Rightarrow k = \log_2 n$
- Hence, **worst-case exe time** of binary search is $O(\log_2 n)$.

1.7.2

# Performance Measurement

- Obtain actual space and time requirement when running a program.
- How to do time measurement in code?
  - Method 1: Use clock(), measured in clock ticks
  - Method 2: Use time(), measured in seconds
- To time a short program, it is necessary to repeat it many times, and then take the average.

# Performance Measurement

Method 1: Use clock(), measured in clock ticks

```
#include <time.h>

void main()
{
  clock_t start = clock();
  // main body of program comes here!
  clock_t stop = clock();
  double duration = ((double) (stop-start))
                    / CLOCKS_PER_SEC;
}
```

# Performance Measurement

Method 2: Use time(), measured in seconds

```
#include <time.h>

void main()
{
  time_t start = time(NULL);

  // main body of program comes here!

  time_t stop = time(NULL);

  double duration = (double) difftime(stop,start);
}
```