MXB261 - A Simulation Project

Group Number: 16

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GitHub Repository: https://github.com/Ruben-Cooper/MXB261-Group16

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Introduction

The Parasite Model is a system of two Ordinary Differential Equations in a Predator Prey Relationship of the form:

$$\frac{dX_1}{dt} = k_1 X_1 X_2 - k_2 X_1$$

$$\frac{dX_2}{dt} = k_3 - k_4 X_2 - X_1$$

General aspects of note within this model include:

- Constants k are always positive
- It is an unconstrained growth model
- X1 is the parasite, representative of the predator
  - k_1 represents the parasites birth rate
  - k_2 represents the parasites death rate
- X2 is the food, representative of the prey
  - k_3 represents the growth rate of food
  - k_3 represents the decay rate of food
  - k_5 represents the rate that the parasites consume the food

In order to gather a deeper understanding into the inner workings of this model, numerous methods of analysis will be performed, with these methods including:

- Explicit Euler Method Simulation for Fixed parameters
- ODE45 Simulation for Fixed Parameters
- Parameter Sweep of K_3 for a given range
- Parameter Sweep of K_3 and K_4 as pairs for a given range
- Equilibrium Analysis
- Latin Hypercube Sampling for Parameter Sweeps of K_3 and K_4as pairs
- Latin Hypercube Sampling for Parameter Sweeps of K_3 and K_4 and K_5 as a trio
- Orthogonal Sampling for Parameter Sweeps of K_3 and K_4 and K_5 as a trio
- Spatial Stochastic Agent-Based Implementation

Methods

Task 1

In order to investigate a solution to the 2D parasite/food system, the long term behaviours of the parasite and food can be found through the use of both the explicit Euler method of simulation and Matlab's built in function ode45 for solving ordinary differential equations.

Use of Euler's method first requires its explicit derivation to be found which was done as follows.

From first principles we know that:
$y' = \frac{dy}{dt} = f(y(t)) = \frac{y(t+h)-y(t)}{h}$ as h approaches 0

From this we can rearrange to find that:
$y(t + h) = y(t) + h * f(y(t))$
and with the following assumptions that when $t(0)\ and\ y(t_0)$ that $y(t_0) = y_0$ we can substitute this in to find that:
$y_0(t_1) = y(t_0) + hf(y(t_0))$ where $t_1 = t_0 + h$
which simplifies to:
$y_1 = y_0 + hf(y_0)$

From this derivation we can conclude that generally:
$y_{n+1} = y_n + hf(y_n)\ where\ t_{n+1} = t_n + h = t_0 + h(n + 1)$

Applying this directly to the parasite equation of the parasite model ($\frac{dX_1}{dt} = k_1X_1X_2 - k_2X_1$) we can then find the explicit euler method formula:
$y_{n+1} = y_n + hf(y_n)$
$X_{1_{n+1}} = X_{1_n} + hf(X_{1_n})$
$X_{1_{n+1}} = X_{1_n} + h \times (k_1X_{1_n}X_{2_n} - k_2X_{1_n})$
$X_{1_{n+1}} = X_{1_n} + hk_1X_{1_n}X_{2_n} - hk_2X_{1_n}$
$X_{1_{n+1}} = X_{1_n}(1 + hk_1X_{2_n} - hk_2)$
Then with the assumptions that $h = 0.01, k_1 = 1, k_2 = 2$, we find the final equation of:

$X_{1_{n+1}} = X_{1_n}(0.98 + 0.01X_{2_n})$

Then applying this directly to the food equation of the parasite model ($\frac{dX_2}{dt} = k_3 - k_4X_2 - X_1$) we can then find the explicit euler method formula:

$y_{n+1} = y_n + hf(y_n)$

$X_{2_{n+1}} = X_{2_n} + hf(X_{2_n})$

$X_{2_{n+1}} = X_{2_n} + h \times (k_3 - k_4X_{2_n} - X_{1_n})$

$X_{2_{n+1}} = X_{2_n} + hk_3 - hk_4X_{2_n} - hX_{1_n}$

$X_{2_{n+1}} = X_{2_n}(1 - hk_4) + h(k_3 - k_5X_{1_n})$

Then with the assumptions that $h = 0.01, k_5 = 3$, we find the final equation of:

$X_{2_{n+1}} = X_{2_n}(1 - 0.01k_4) + 0.01(k_3 - 3X_{1_n})$

To summarise, the system of Euler equations to use are:

- $X_{1_{n+1}} = X_{1_n}(0.98 + 0.01X_{2_n})$
- $X_{2_{n+1}} = X_{2_n}(1 - 0.01k_4) + 0.01(k_3 - 3X_{1_n})$

Which will be simulated over the time range of 0 to 20 (2000 steps) with the final assumptions that $X_{1_0} = 1$ and $X_{2_0} = 1$ and that $k_3 = 20 \; and \; k_4 = 4$

Following this simulation, ODE45 will be used (with the same parameter assumptions) to compare results and the relative difference of the results of the two methods will be analyzed to view how different they are.

Then one of the simulation methods will be used to to perform a parameter sweep of k_3 (rate of food growth) over the range of even numbers between 0 and 50 to see how this affects the final size of both members of the parasite system. Noting that the assumptions mentioned above will remain the same except for $k_3 \; and \; k_4$ when they are altered.

The results will then be analysed under the assumption that only three results can occur:

1. That the system went negative at a point and hence the results need be disregarded as this is impossible in the parasite system
2. Scenario i: That the system converges to a level where the parasite dies out with a tolerance of 0.1 for variation ($0 \leq X_1(20) \leq 0.1$)
3. Scenario ii: That the system converges to a level where the parasite system becomes stable with a tolerance of 0.1 for variation ($1.9 \leq X_2(20) \leq 2.1$)

Then to further analyse the effect of deviation in the parameters driving the amount of food, a parameter sweep will be performed the same as previously but for all possible pairs of k_3 and k_4 (rate of food growth, rate of food decay). Noting that k_3 is over the range of even numbers between 0 and 50 and k_4 is the range of numbers between 0 and 50.

Task 2

Task 2 focuses on a comprehensive exploration of the parasite and food dynamics using Latin Hypercube Sampling in a 2D framework. The goal is to discern the relationships between parameter variations and the resulting equilibrium solutions, providing a holistic view of how the behaviour of both the parasites and food evolve overtime.

The first step for this task is to calculate the equilibriums of the parasite model, this can be completed as follows:

Equilibrium Solutions Analysis:

The equilibrium solutions are the values of X_1 and X_2 for which $\frac{dX_1}{dt} = 0$ and $\frac{dX_2}{dt} = 0$, i.e the rate of change of X_1 and X_2 should be zero.

Given:

$$\frac{dX_1}{dt} = k_1X_1X_2 - k_2X_1 \ (1)$$

$$\frac{dX_2}{dt} = k_3 - k_4X_2 - k_5X_1 \ (2)$$

Using the provided parasite equations, this translates to:

$$0 = k_1X_1X_2 - k_2X_1 \ (1)$$

$$0 = k_3 - k_4X_2 - k_5X_1 \ (2)$$

From equation (1):

$k_1X_1X_2 = k_2X_1$

If $X_1 \neq 0$, then:

divide both sides by X_1:

$k_1 X_2 = k_2$

$X_2 = \frac{k_2}{k_1}$

Now, substitute this equation into equation (2) to solve for X_1.

$0 = k_3 - k_4\left(\frac{k_2}{k_1}\right) - k_5 X_1$

Rearranging for X_1:

$X_1 = \frac{k_3 - k_4\left(\frac{k_2}{k_1}\right)}{k_5}$

Therefore, the non-trivial equilibrium solution (when $X_1 \neq 0$) is:

$X_1^* = \frac{k_3 - k_4\left(\frac{k_2}{k_1}\right)}{k_5}$

$X_2^* = \frac{k_2}{k_1}$

The other solution to consider is the trivial equilibrium solution from equation (1) (when $X_1 = 0$).

In this case:
Using equation (2):

$0 = k_3 - k_4 X_2$

$X_2 = \frac{k_3}{k_4}$

Thus, we have two equilibrium solutions:

$X_1^* = \frac{k_3 - k_4\left(\frac{k_2}{k_1}\right)}{k_5}, X_2^* = \frac{k_2}{k_1}$ or $X_1^* = 0, X_2^* = \frac{k_3}{k_4}$

Equilibrium Solutions with fixed parameters:

Given:

$$k_1 = 1, \quad k_2 = 2, \quad k_5 = 3$$

Substitute these values into our derived equilibrium expressions to find X_1^* and X_2^*:

Non-Trivial equilibrium:

$X_1^* = \frac{k_3 - k_4\left(\frac{2}{1}\right)}{3} = \frac{k_3 - 2k_4}{3}$

$X_2^* = \frac{2}{1} = 2$

Trivial equilibrium:

$$X_1^* = 0$$

$$X_2^* = \frac{k_3}{k_4}$$

Note: While not entirely necessary, including the trivial equilibrium details the equilibrium of the food source when the parasite has become extinct. This reveals the carrying capacity of the food source in the absence of parasites. Furthermore, it can also be used to compare to the non-trivial equilibrium, detailing how the existence of a parasite population affects the food population.

Following these equilibrium calculations, the next section of the task involves utilising matlab to implement 2-dimensional latin hypercube sampling (LHS). This method ensures a thorough and even exploration of the parameter space for k_3 and k_4 with the focus on categorising the resulting samples into one of two different cases, helping to analyse the following two scenarios:

- The conditions (i.e., combinations of k_3 and k_4) under which the parasite population struggles to thrive.
- The conditions where the food source stabilised around an equilibrium value in the presence of the parasites.

This provides insights into the system's resilience or vulnerability when the rate of food growth (K_3) and food decay (K_4) are manipulated. Moreover, scatterplots can be utilised to better visualise these results and by grouping the two separate cases, a boundary between these two regions should the be calculated, revealing the transition point between X_1 nearing extinction and X_2 stabilising around an equilibrium value of 2.

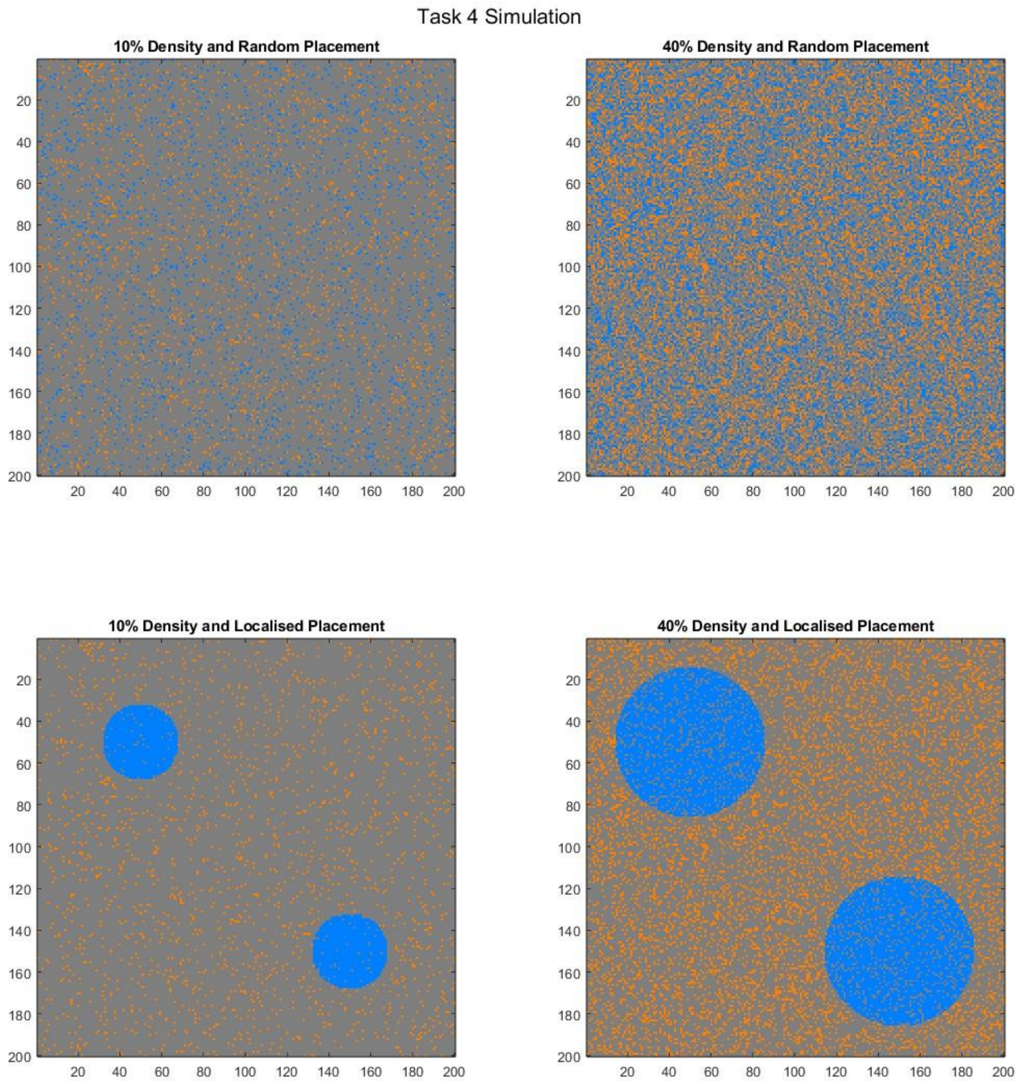
Task 3

The focus of Task 3 is to see the system dynamic over 3 parameters and the difference between 2 different sampling methods. The set parameters are k_1=1 and k_2=2 and the parameters are being generated are k_3, k_4 and k_5 . Like Task 2 the goal is to discern the relationship between parameter variation but instead over 3 parameters and see the difference between 2 different sampling methods. The 2 sampling methods are Latin Hypercube Sampling and Orthogonal Sampling but over a 3d parameter space.

Latin Hypercube Sampling is a near-random sampling method. The process splits the sample space into even columns and rows. An intersection of columns and rows is selected for a point within the sample space to be generated within the range of the column and row selected. Then the selected column and row are locked out of being able to have a point generated within them. This ensures that there are samples over the sampling space. The way that this is implemented is having an array of available sections for each dimension and when a section is sampled from the section is removed from the array. Since each dimension is independent of each other this allows each set of parameters to be generated over a loop with no additional checks.

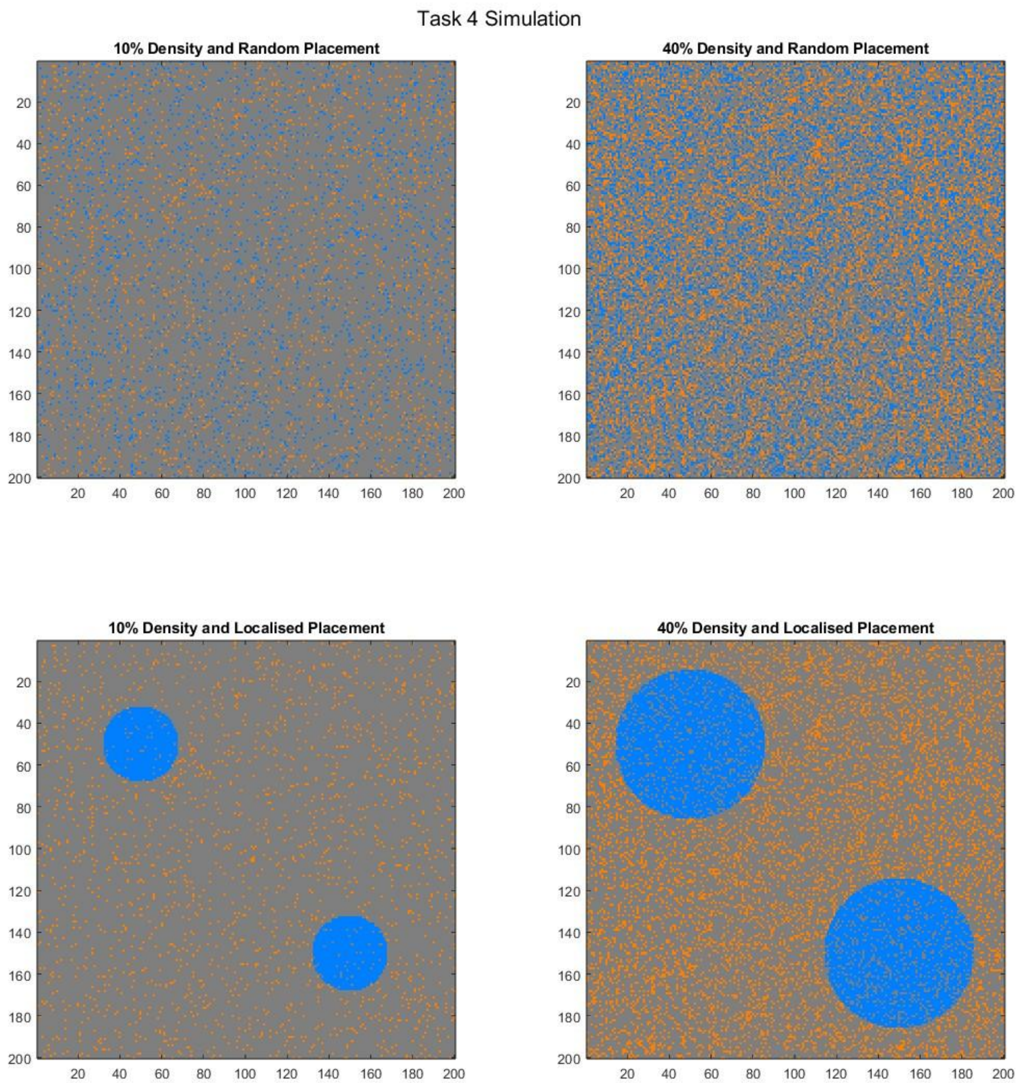
Orthogonal Sampling is a variation of Latin Hypercube Sampling which splits the sample space into equal sections that have an equal density of samples. This ensures that points are evenly distributed. The samplers made for Task 3 are in a 3d space making the Orthogonal Sampler have 8 subspaces that have the same density. The implementation of this involved sampling each subsection one at a time. Due to having these subspaces the dimensions become dependent on each other so that each subspace can have an equal density of values. The points are generated by having a point generated in subspace 1 then the next point in subspace 2 and so on to subspace 8. Then the next 8 points are sampled until all points are sampled. This solution can only have a multiple of 8 specified for the samples due to how it was implemented. The lock out of columns is similar to LHS with it being an array with the remaining sections to be sampled from.

Task 4



Subplots 1 and 2 for Task 4 were constructed by generating a matrix of random values the same size of the simulation grid, then setting all values below a threshold to be equal to food particles, and all values above a threshold to be equal to parasite particles, this was then plotted with imagesc() using a custom colour map constructed from the colour values used in the task sheet. Subplots 3 and 4 were constructed by calculating the distance between every cell and the two given locations of (50, 50) and (150, 150), sorting them in order of least to most distance from those locations, then placing food in the top n positions, where n is the amount of food to add, calculated by multiplying the desired population density by the area of the grid,

e.g. 5% * 200 squares * 200 squares = 2000 pieces of food.
Note that I have assumed that the population density is split between both the parasite population and the food population, rather than them both having an equal 10% of the grid to occupy.



The second simulation video has a very similar starting layout, as none of the parameters that were modified are related to the starting layout. The only difference is in the random variation inherent from using rand() a second time.

Results and Discussions

Task 1

From simulating the system of explicit Euler equations derived in the Task 1 methods section with the fixed parameter values, Figure 1 below demonstrates the behaviour of the Parasite system over the time interval $[0, 20]$ with h increments of 0.01. The Parasite and Food overtime diagram in this figure highlights that around t=4, the system converges to a point of equilibrium for both the Parasite and food. This equilibrium denotes a stable system with the number of parasites being approximately 4 and the number of food being approximately 2.

The process followed by the system to reach this equilibrium position is further highlighted by the phase graph in Figure 1 which shows the number of Parasites against the number of food at each time following the pattern of the number of food increasing to a certain level and then the number of food following subsequently. This being because the Parasite obviously requires a given amount of Food in order for it to grow and once there are more Parasites, the more food that will be consumed and hence the less that will remain at a given time. This continues until it reaches the equilibrium point where the phase chart no longer changes.

Figure 1:

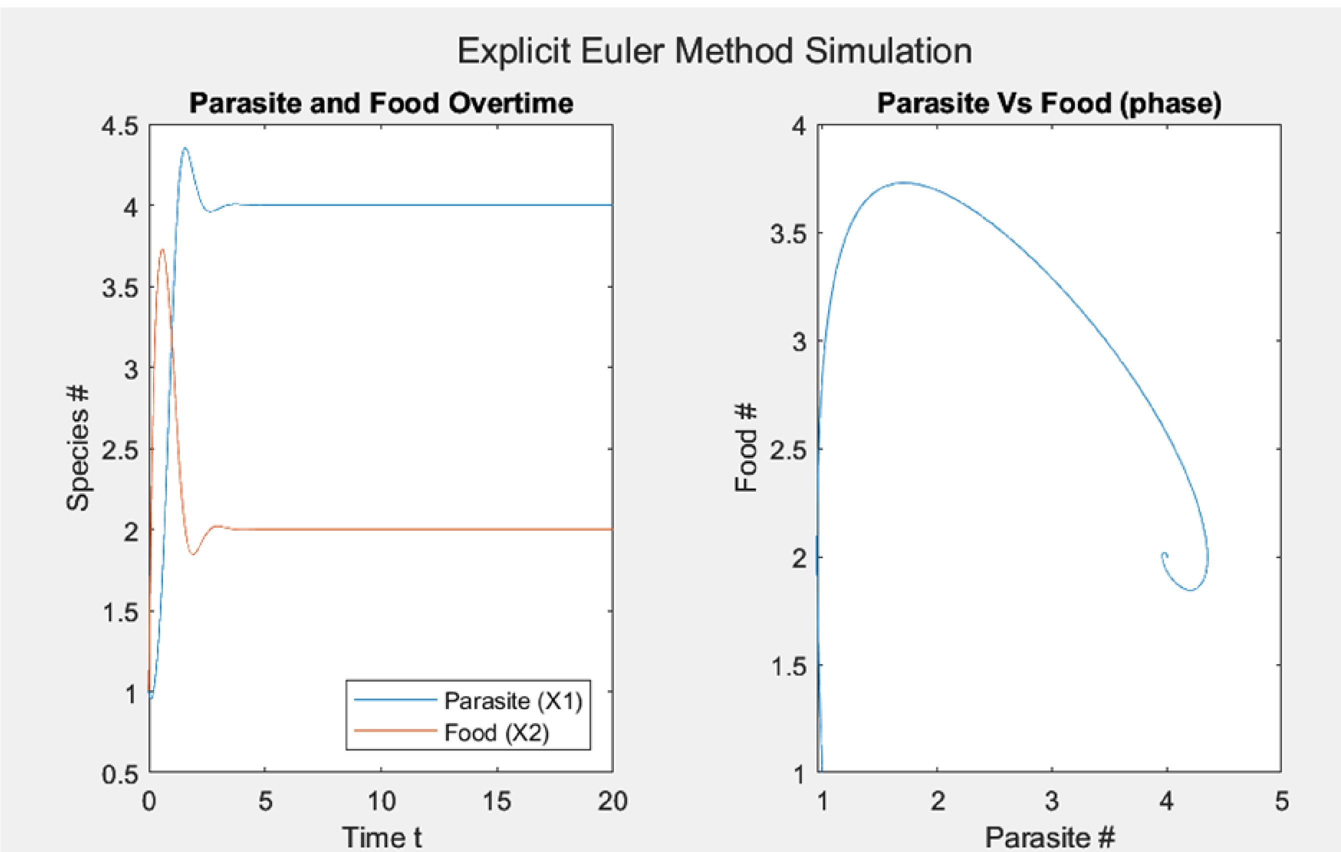
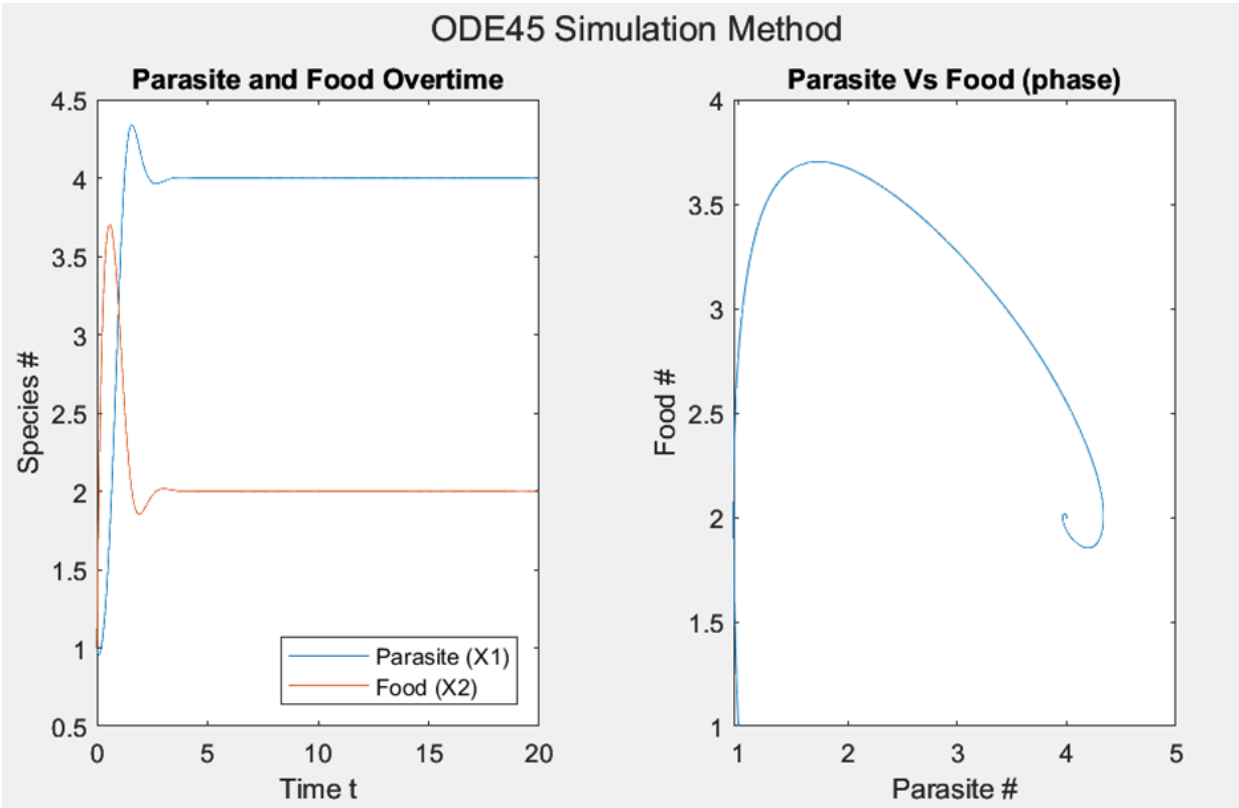
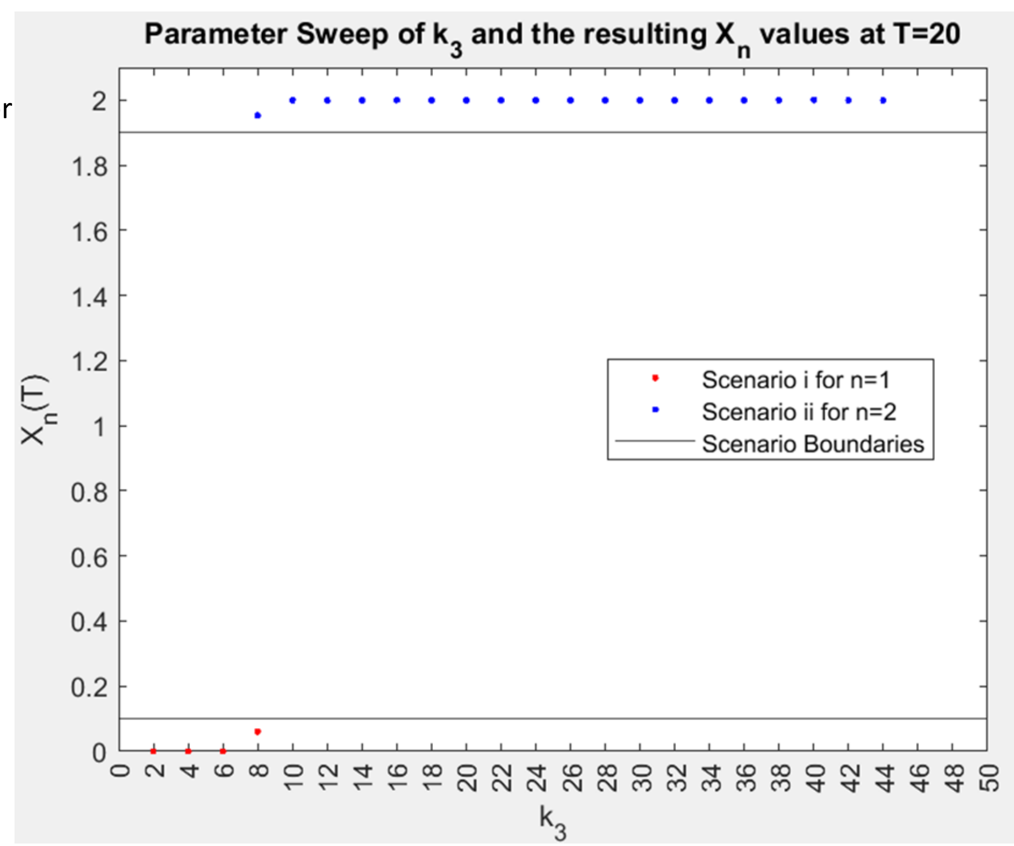


Figure 2:



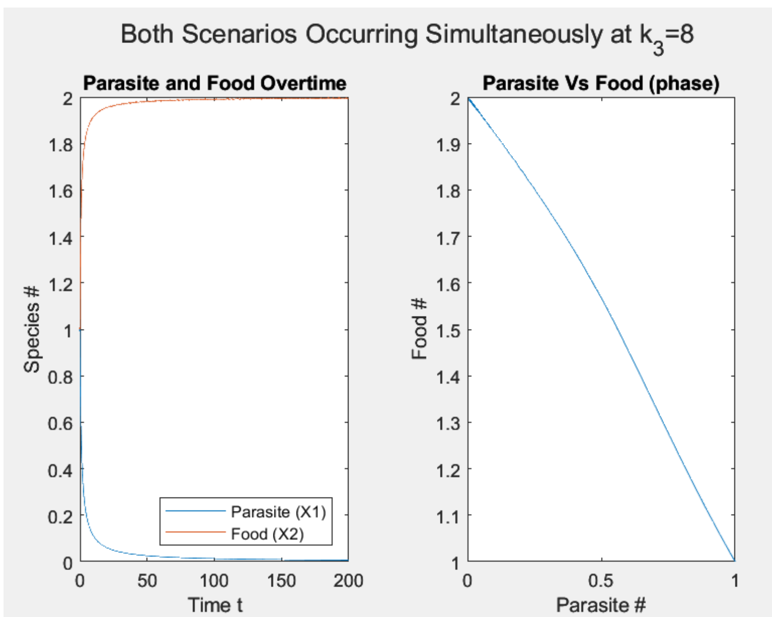
The system was then simulated using ODE45 and as Figure 2 above shows, the same process and outcome occurred as did for the Euler method. Due to the results being similar, comparison was performed with it being found that on average the relative differences for X1 between the Eurler method and ODE45 method was 0.0186% less and for X2 it was 0.0247% more. This shows that the difference between them is negligible which is to be expected as the methods follow similar structures.

Figure 3:



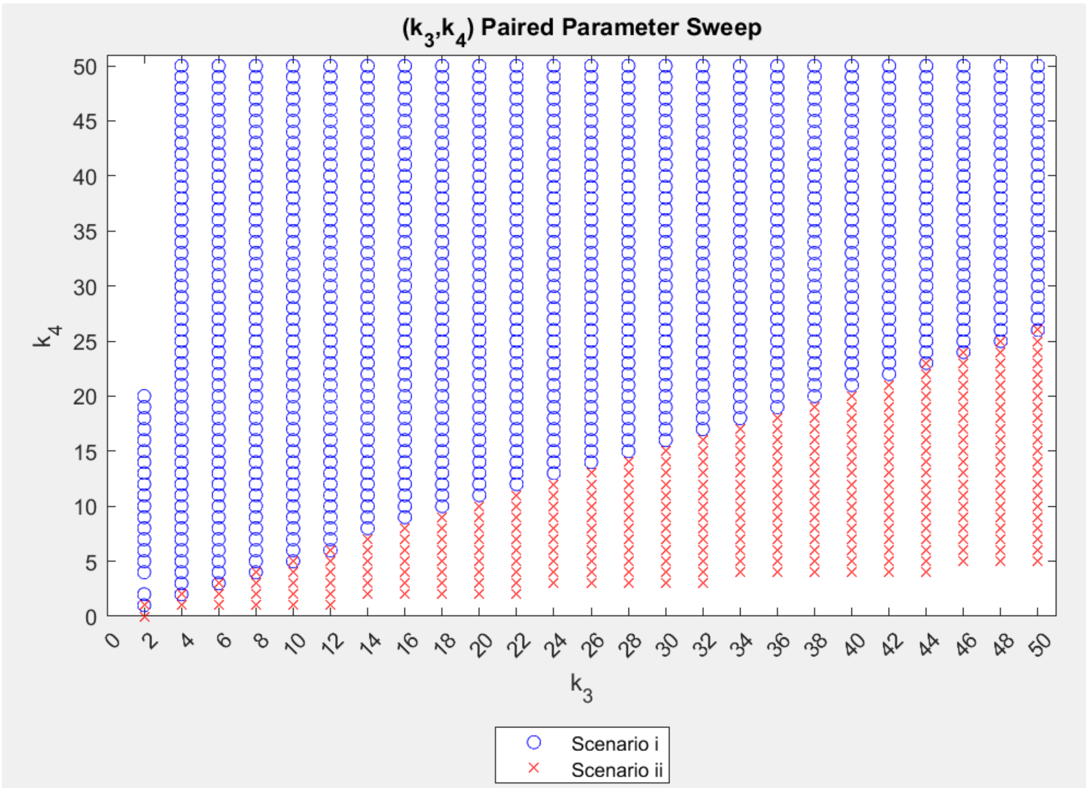
From performing a parameter sweep of the rate of growth of the Food within the Parasite Model (k_3), as shown in Figure 3 below, it was discovered that specific regions of the range of values used fall within each of the given scenarios. That being that for the range of 2 to 6, the system converges to a level where the Parasite does not thrive and the range of 10 to 44 the system converges to a level where the parasite model becomes stable with both Parasite and Food reaching an equilibrium level. An outlier exists for the value 8 because it is the only value whereby both scenarios occur, Figure 4 below shows the way the system behaves over the time period and highlights how the food reaches equilibrium even though the Parasite also dies out. This value of 8 acting as the barrier between each scenario and as for this simulation the value of food decay rate(k_4) was 4, this highlights that this outlier of meeting both scenarios criteria only occurs when $k_3 = 2k_4$.

Figure 4:



This aspect of the Parasite model can be further viewed through the parameter sweep of both k_3 and k_4 with the results being visible in figure 5 below. In which it is visible that a boundary exists between each scenario occurring and from viewing the line between where both scenarios are occurring, it is visible that $k_3 = 2k_4$ is the boundary between the system thriving or failing. Whilst a large section of parameter pairs along the boundary do not meet both scenarios, it does not take away from this finding as if the tolerance value were to be increased it would be found that this property holds true.

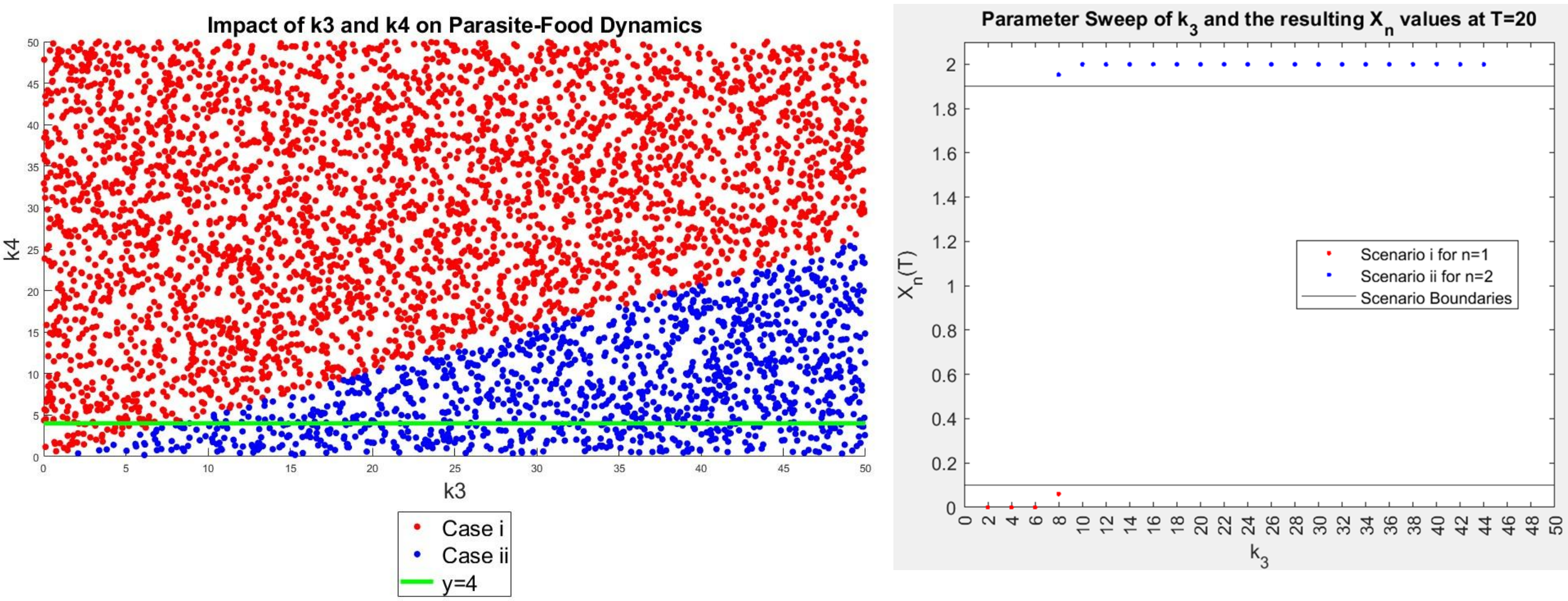
Figure 5:



Task 2

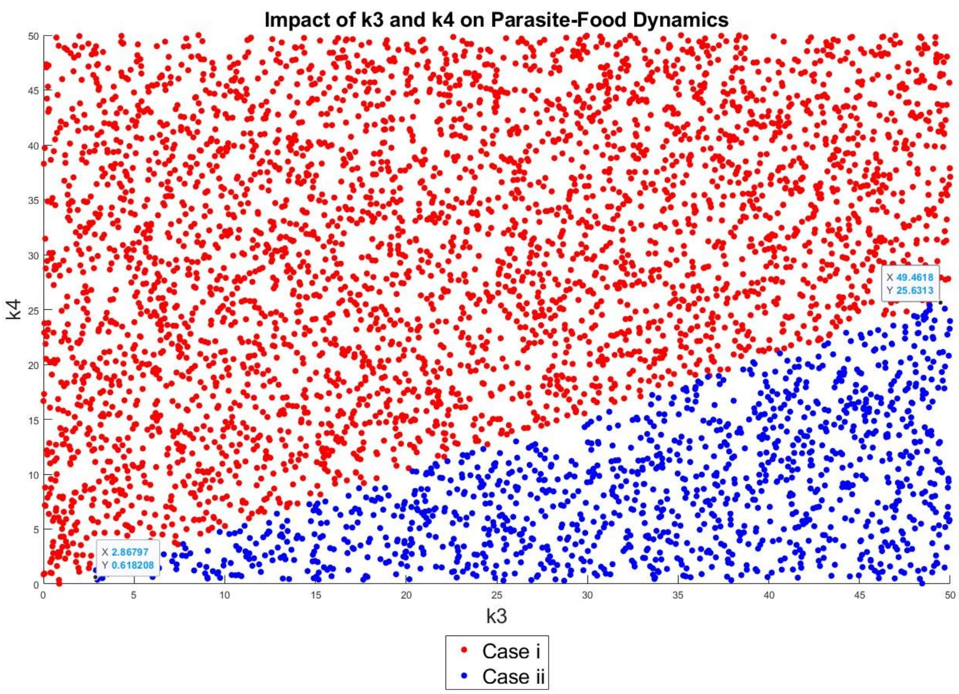
While calculating the equilibrium solutions for the parasite model, the resulting values can be used to better understand Part c in Task 1. The equilibrium solutions serve as benchmarks or reference points for the dynamic behaviour of the system and as visualised in Figure 3, the trends and patterns observed in the parameter sweep directly correspond to these equilibrium points. Regions in the parameter space where the system closely approaches these equilibria are clearly highlighted, allowing for a deeper interpretation of system stability under different conditions. Furthermore, it can be noted that as the parameter k_3 is adjusted there are distinct regions where one steady state appears to dominate or where the system's dynamics converge more consistently to that state. This suggests that the stability of a particular steady state is influenced by the magnitude of k_3.

Figure 6:



The behaviour of the parameter sweep versus the LHS can be further realised in Figure 6 where both the LHS and parameter sweep are presented, recalling that in the parameter sweep k_4 = 4. In this figure, the differences between the LHS and the parameter sweep becomes particularly evident. The LHS showcases a more diverse and evenly distributed exploration of the parameter space, capturing nuances and potentially overlooked regions that the traditional parameter sweep misses. On the other hand, the parameter sweep provides a systematic, grid-based exploration, offering a clear depiction of trends and patterns across the parameter spectrum. However, by comparing these two methods, and utilising a line at y=4 on the LHS (representing k_4 = 4) the similarities of these two models can be realised, where the intercept of the y=4 line with the boundary in the LHS underlines the points of agreement between the two approaches. This is represented as the line intercepts when approximately K_3 is equal to 8, similarly in the parameter sweep it was noted that this is the transition point between the two cases.

Figure 7:



In our analysis, it was found to be crucial to represent relationships between variables using mathematical formulations. One such relationship can be visually observed in Figure 7, which graphically displays two distinct points: (x1,y1) as (2.87, 0.62) and (x2,y2) as (49.47, 25.63). These points are selected at both the lower and upper value of the boundary. By employing these points, we can derive the slope equation of the boundary between case i and case ii as follows:

Boundary Line Slope Formula Estimate:

Given:

$x_1 = 2.87,\ y_1 = 0.62$

$x_2 = 49.47,\ y_2 = 25.63$

The formula to be used is:

$y - y_1 = m(x - x_1)$

Where:

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{25.63 - 0.62}{49.47 - 2.87}$

$m \approx 0.5367$

Now, plug the slope value and the point (2.87, 0.62) into the equation:

$y - 0.62 = 0.5367(x - 2.87)$

Expand:

$y = 0.5367x - 0.5367(2.87) + 0.62$

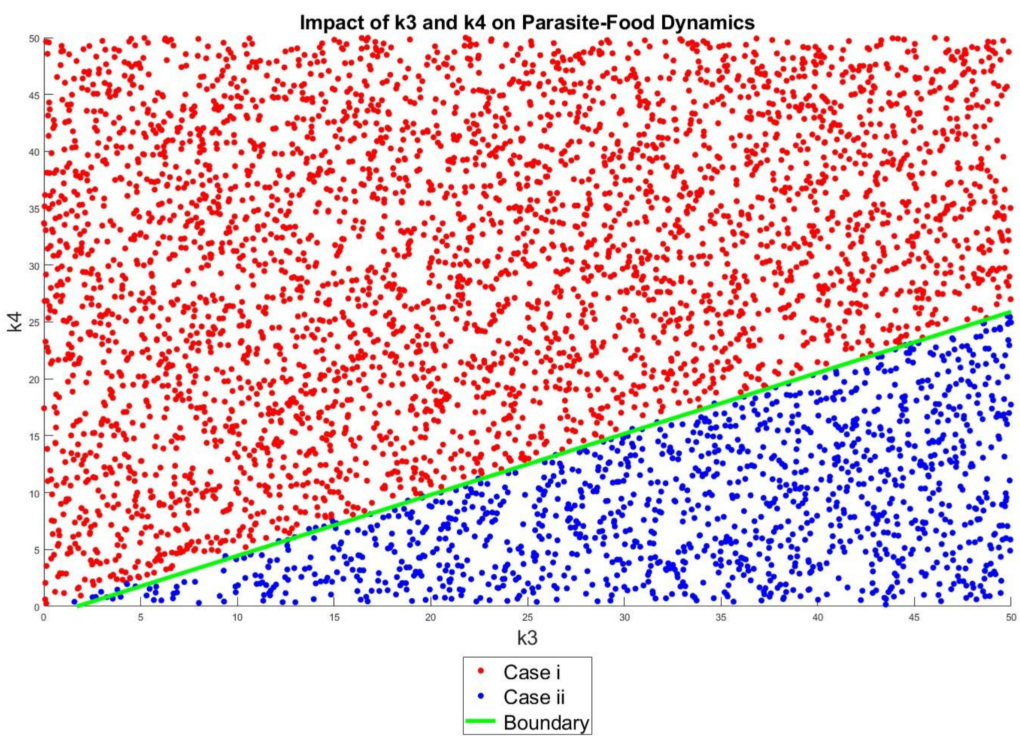
$y = 0.5367x - 0.9190$

So the approximate equation for the boundary line is:

$y \approx 0.5367x - 0.9190$

Subsequently, based on this equation figure 8 was produced:

Figure 8:



The boundary between Case i and Case ii provides insights into the dynamic behaviour of the parasite model under various food growth and food decay combinations. It signifies a transition between the different end states of the system after a certain period (T=20 time units). In Case i, the parasite population, X_1, approaches zero, suggesting that the parasites have been unable to sustain themselves, likely due to inadequate food supply or overconsumption. On the other hand, in Case ii, the food population X_2, hovers around a value of 2, suggesting a stable and sustainable food source of the parasite over the given period. In comparison to the equilibrium, which represents a static, balanced point in the system, the boundary illustrates how the system behaves or where it ends up after a specific time. Moreover, the fact that the system approaches these specific states as time progresses indicate that the equilibrium is in fact stable.

Task 3

Adding a third parameter in the form of k5 further shows how the equilibrium can be affected with the change of k5. Since k5 represents the food consumption by the parasites it is safe to assume it will have an impact on the equilibrium of the system. So if k5 is greater than k3 which is the rate of food growth the state of X2 can slip into the negative. This is similar to the relationship of k3 to k4 instead in this case it is the food decaying and this relationship determines which equilibrium state the parameters are in.

Figure 9:

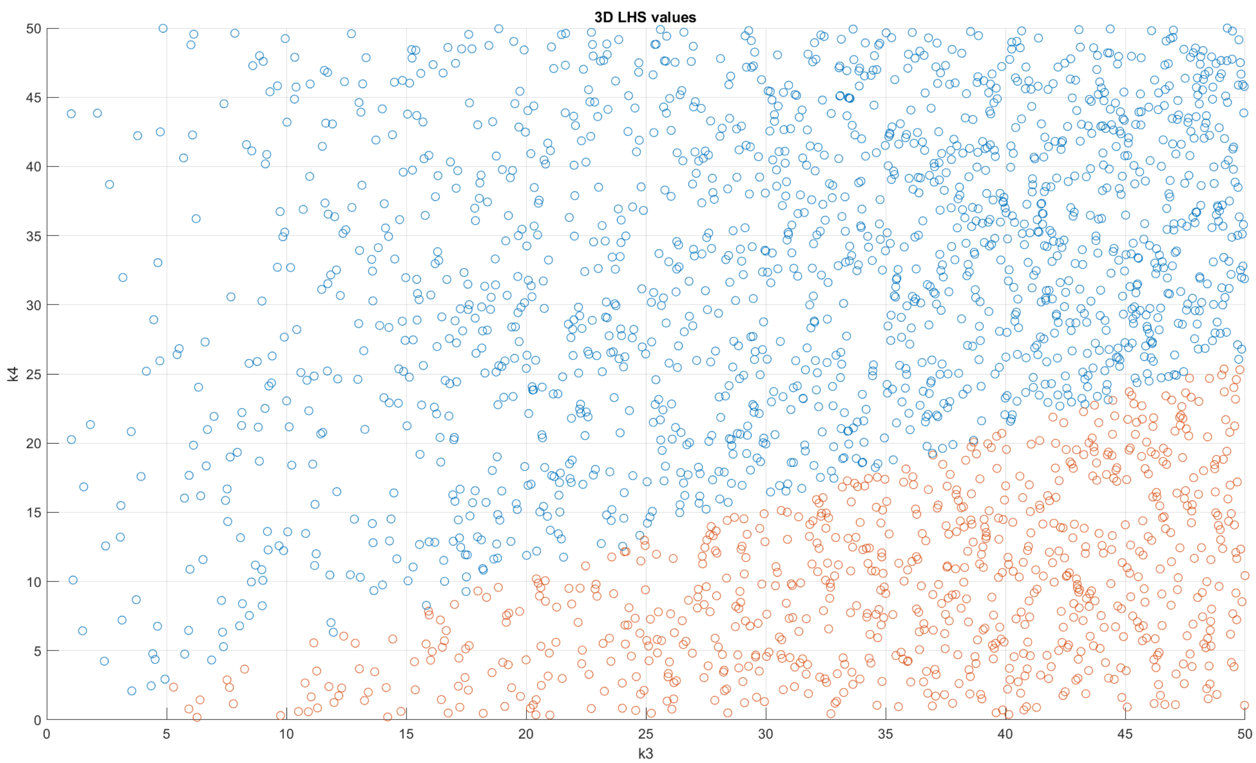
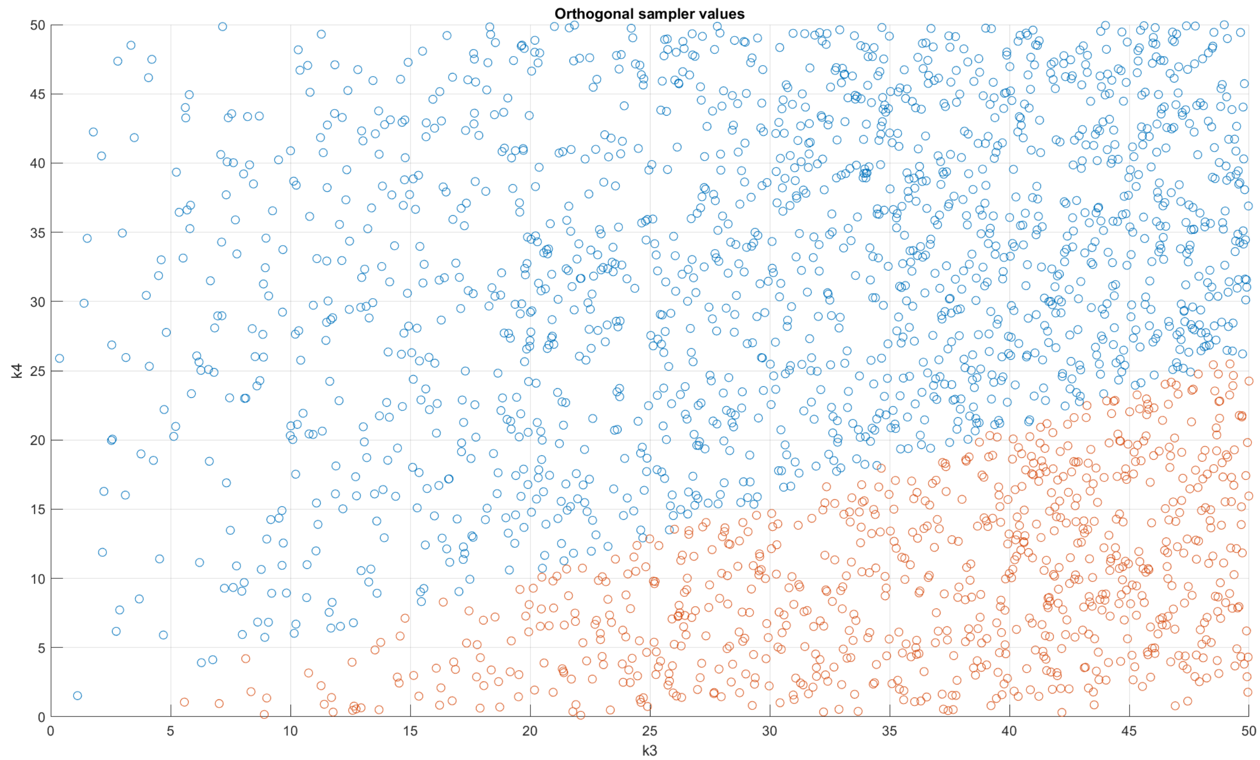


Figure 10:



As stated k3 and k4 determine the equilibrium state this is also shown for the 2d LHS in Task 2. There is a greater density of parameter groups when k3 increases. This is shown in figures 9 and 10. Density of values is due to different k5 values being generated alongside k3 and k4 since if the growth of food is greater than the consumption there is a greater amount of parameters meeting equilibrium.

Figure 11:

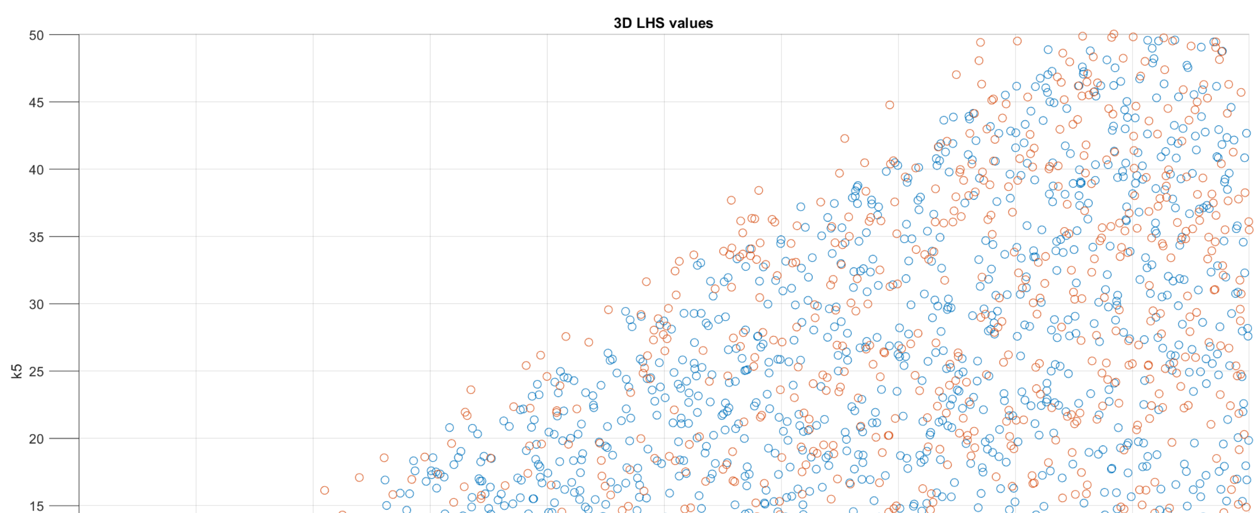
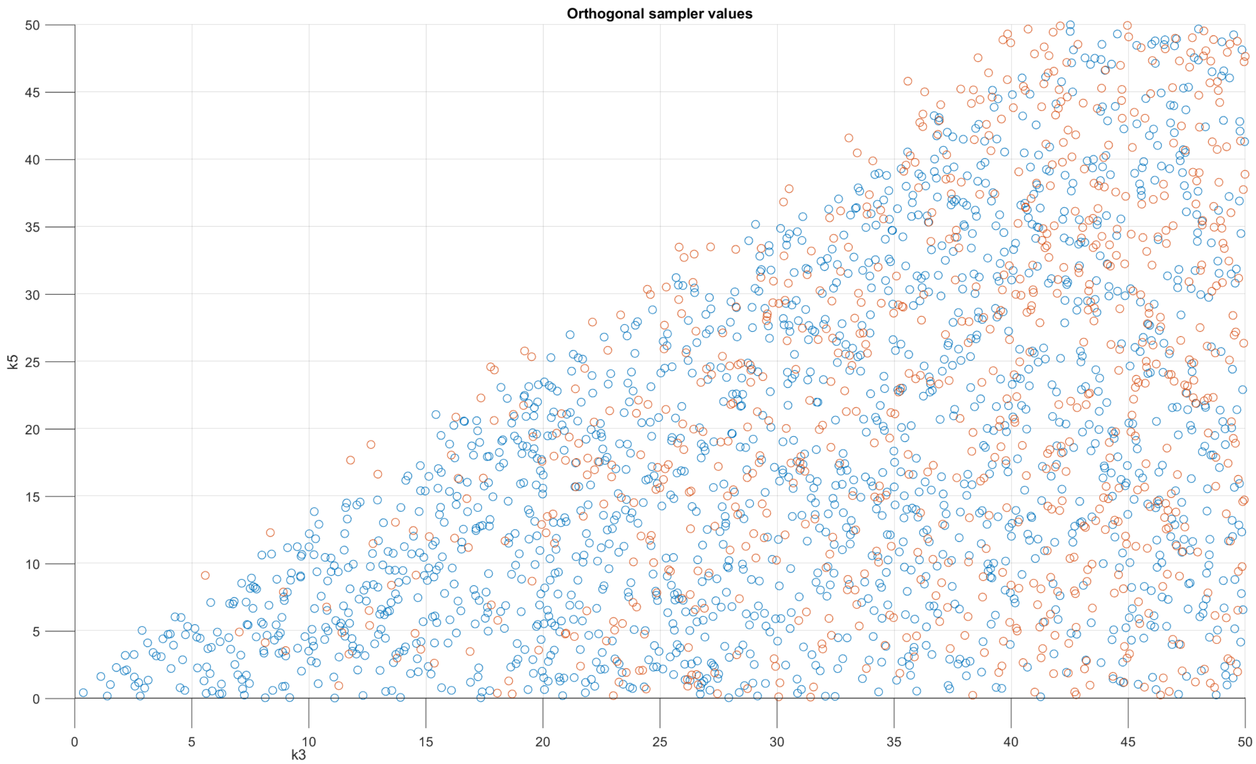


Figure 12:



Figures 11 and 12 show this relationship between k3 and k5 a little better with the region that has no values due X2 dipping into the negatives. An interesting occurrence is the fact that when k3 is around 40 and k5 is around 50 the system doesn't slip into the negatives and out of an equilibrium. As shown the k5 is the higher k3 needs to be to meet an equilibrium.

Figure 13:

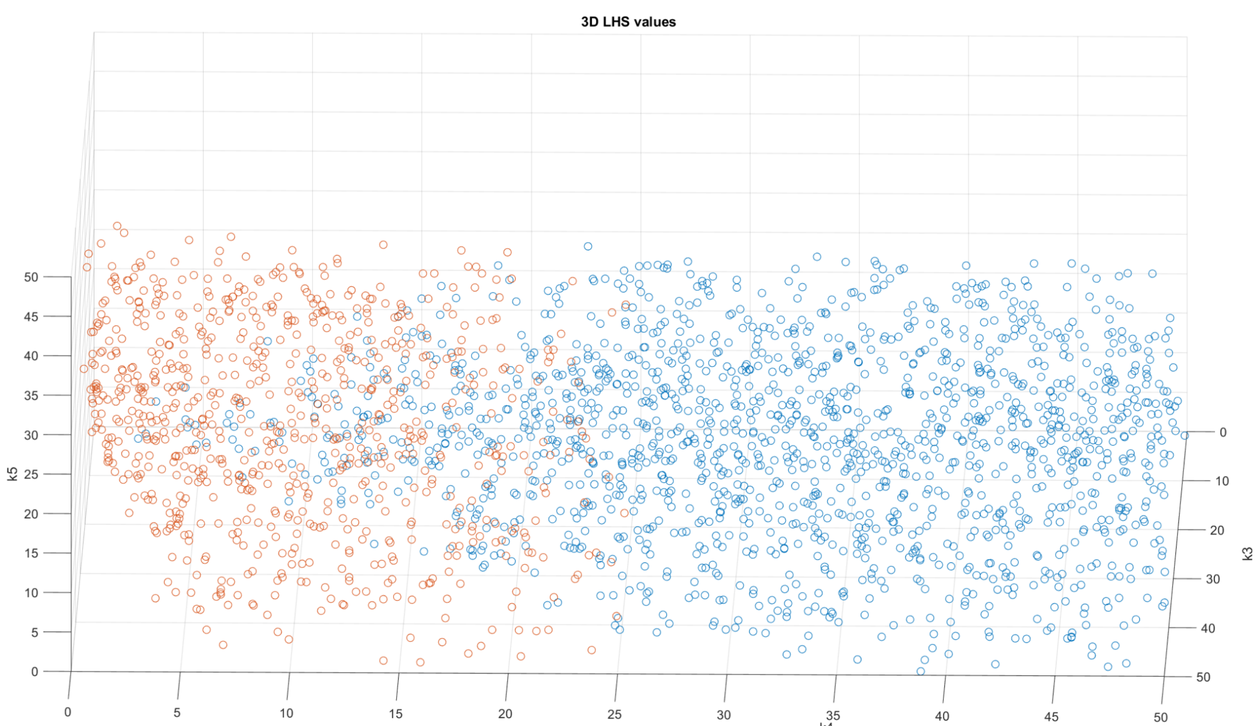
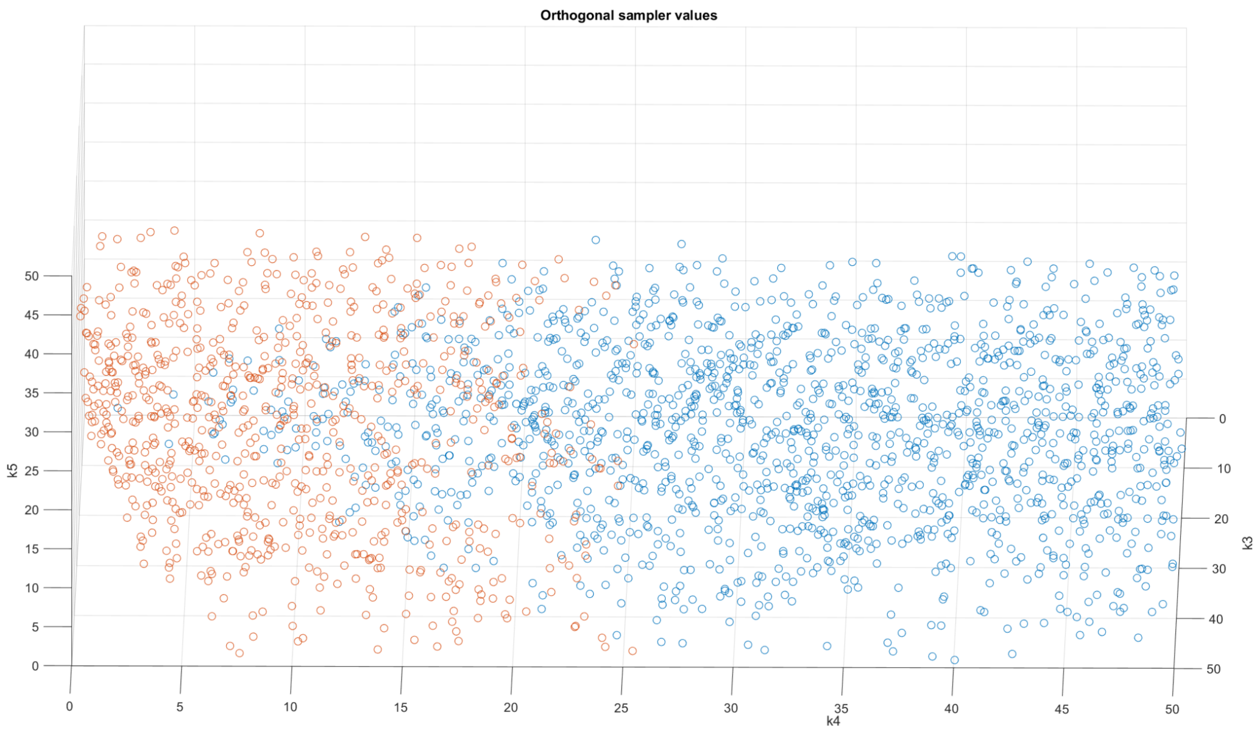


Figure 14:

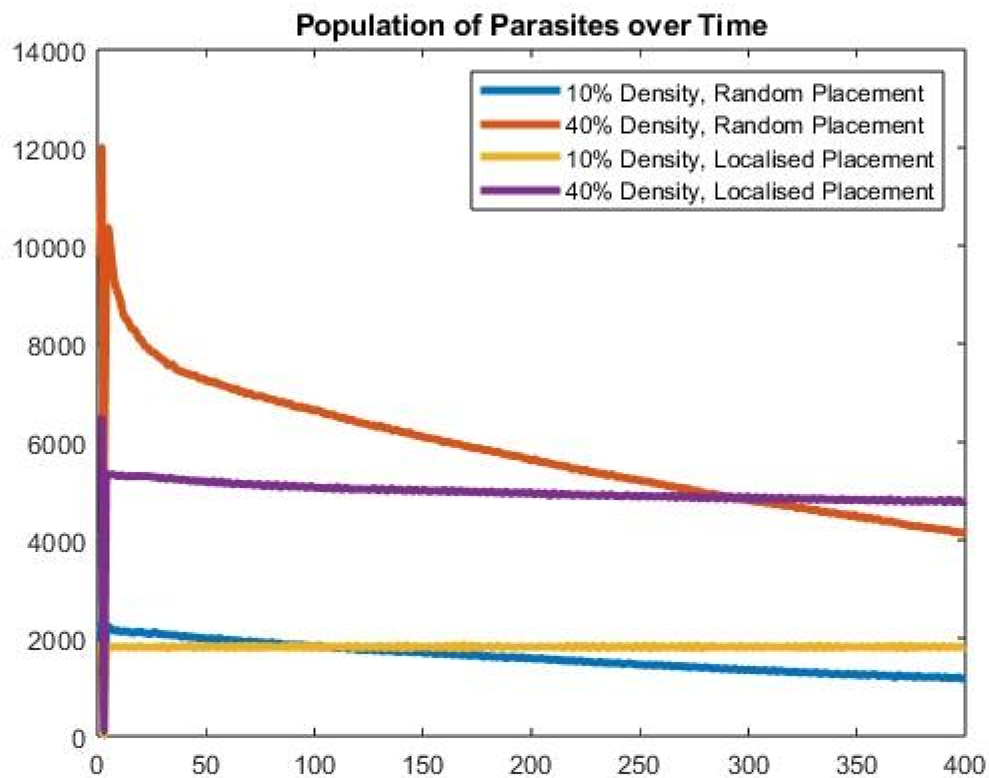


Figures 13 and 14 show the relationship that all 3 parameters share, being a region of no values in the corner at the high end of k3 and the low ends of k4 and k5. This can be explained by the relationship between the parameters in the equilibrium equations. With a high k3 value the population of parasites and the amount of food increases rapidly resulting in the low k4 and k5 being able to decrease the food amount to negatives.

With larger sample sizes the difference between the two sampling methods is not as noticeable although a more even distribution of points can be seen in the orthogonal sampler for figures 9 and 10. This being on the left side of the plots.

Task 4

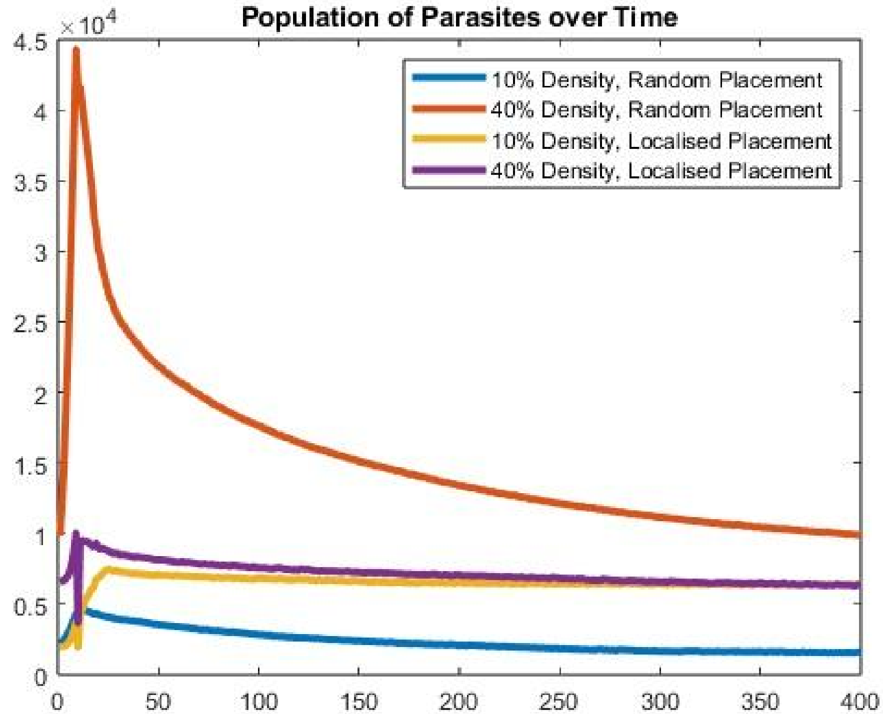
Scenario 1



f1 = 3, f2 = 0.1, f3 = 100

The populations of all the starting scenarios spike heavily at the beginning of the simulation, due to all the parasites having a beginning age of zero. This spike is more apparent in figure 2 where the parasites are given over three times the amount of time to reproduce before they start to die off heavily and the population stabilises.

Scenario 2



1 = 10, f2 = 0.01, f3 = 300

There is an apparent error in my code that has caused the population of the 40% density with random placement to peak nearly 5000 parasites over the maximum allowable population of the grid size. The longer time that the parasites are alive for in scenario 2 helps them reproduce significantly more before appearing to stabilise at a given population nearly double that of scenario 1 where the parasites died after 3 steps instead of 10.

Conclusions

Overall, the intricate relationship between parameters and system dynamics in the parasite-food model was revealed through a variety of simulation and sampling methods. Both Euler's method and ode45 simulations proved to be robust models and when combined with Latin Hypercube and Orthogonal Samplings, they helped to highlight distinct regions in the parameter space that influenced the system's stability and equilibrium. The spatial agent-based model revealed that initial population densities and food-placement strategies significantly influences outcomes, with some scenarios promoting parasite survival and others diminishing it. These insights emphasise the critical role of nuanced parameter interactions and initial conditions in determining the fate of such ecological systems.