Boundary Constrained Optimization Problems

AP Engelbrecht

Contents

See http://benchmarkfcns.xyz/benchmarkfcns/kealefcn.html

1 Boundary Constrained Benchmark Functions

For all of the functions below, n denotes the dimension, $\mathbf{x}$ refers to a n-dimensional vector, x_i refers to the i-th value of the vector $\mathbf{x}$, $\mathbf{x}^*$ refers to the global optimum.

1.1 Absolute Value

The absolute value function, also referred to as De Jong's function 3 and Schwefel 2.20, is defined as

$$f(\mathbf{x}) = \sum_{j=1}^{n} |x_i| \tag{1}$$

with $x_i \in [-100, 100]$, for which $f(\mathbf{x}^*) = 0.0$, and $\mathbf{x}^* = (0, \ldots, 0)$.

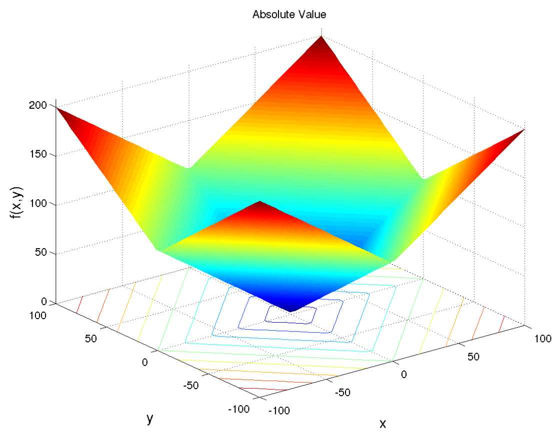


Figure 1: Absolute Value for $n = 2$

1.2 Ackley's Functions

A number of versions of the Ackley function are defined:

- Ackley 1:
$$f(\mathbf{x}) = -20e^{-0.2\sqrt{\frac{1}{n}\sum_{j=1}^{n} x_i^2}} - e^{\frac{1}{n}\sum_{j=1}^{n}\cos(2\pi x_i)} + 20 + e \tag{2}$$
with $x_i \in [-32, 32]$, for which $f(\mathbf{x}^*) = 0.0$, and $\mathbf{x}^* = (0, \ldots, 0)$.

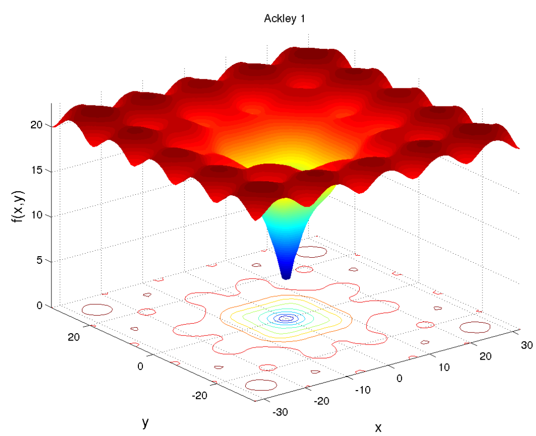
- Ackley 2:
$$f(\mathbf{x}) = -200e^{-0.2\sqrt{\frac{1}{n}\sum_{j=1}^{n} x_i^2}} \tag{3}$$
with $x_i \in [-32, 32]$, for which $f(\mathbf{x}^*) = -200.0$, and $\mathbf{x}^* = (0, \ldots, 0)$.

- Ackley 3 (defined for only $n = 2$):
$$f(\mathbf{x}) = -200e^{-0.02\sqrt{x_1^2 + x_2^2}} + 5e^{\cos(3\pi x_1) + \sin(3\pi x_2)} \tag{4}$$
with $x_i \in [-32, 32]$, for which $f(\mathbf{x}^*) \approx -219.1418$, and $\mathbf{x}^* = (0, \approx -0.4)$.

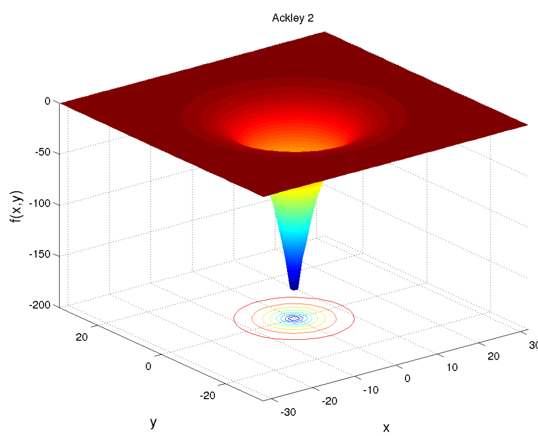
- Ackley 4 (Modified Ackley):

$$f(\mathbf{x}) = \sum_{i=1}^{n-1} \left(e^{-0.2\sqrt{x_i^2 + x_{i+1}^2}} + 3(\cos(2\pi x_i) + \sin(2\pi x_{i+1})) \right) \tag{5}$$

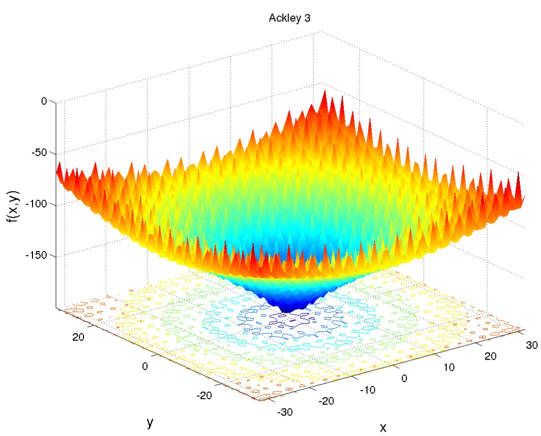
  with $x_i \in [-35, 35]$, for which $f(\mathbf{x}^*) = -3.917275$, and
  $\mathbf{x}^* = \{(-1.479252, -0.739807), (1.479252, -0.739807)\}$.



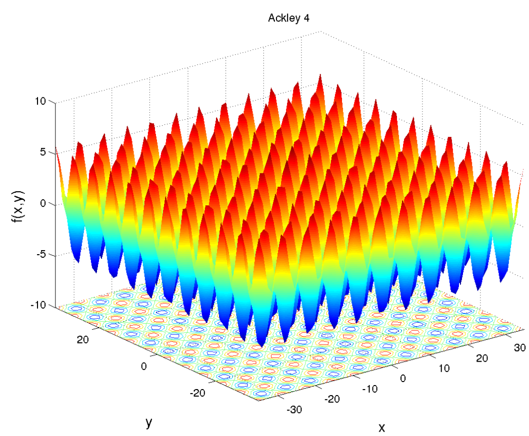
(a) Ackley 1



(b) Ackley 2



(c) Ackley 3



(d) Ackley 4

Figure 2: Ackley Functions for $n = 2$

1.3 Adjiman

The Adjiman function is defined as (in only two dimensions)

$$f(\mathbf{x}) = \cos(x_1)\sin(x_2) - \frac{x_1}{(x_2^2 + 1)} \tag{6}$$

with $x_i \in [-5, 5]$, for which $f(\mathbf{x}^*) = -5.02181$, and $\mathbf{x}^* = (5, 0.10578)$.

1.4 Alpine Functions

A number of versions of the Alpine function are defined:

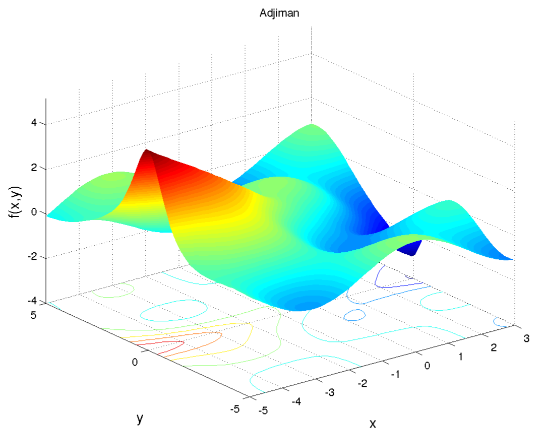


Figure 3: Adjiman for $n = 2$

- Alpine 1:

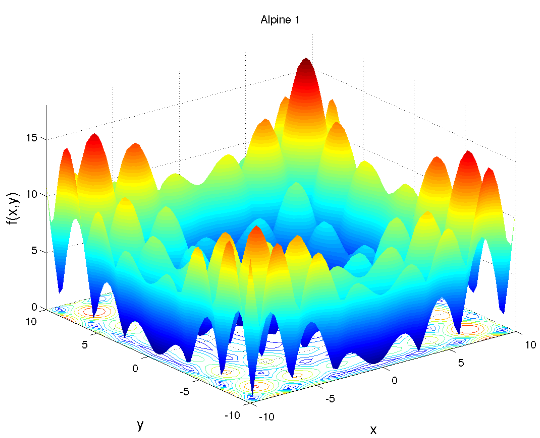
$$f(\mathbf{x}) = \sum_{i=1}^{n} |x_i \sin(x_i) + 0.1x_i| \tag{7}$$

with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = 0.0$, and $\mathbf{x}^* = (0, \ldots, 0)$.

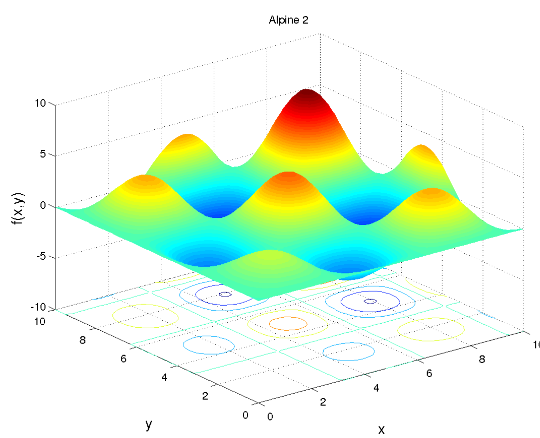
- Alpine 2:

$$f(\mathbf{x}) = \prod_{i=1}^{n} \sqrt{x_i} \sin(x_i) \tag{8}$$

with $x_i \in [0, 10]$, for which $f(\mathbf{x}^*) = -6.1295$, and $\mathbf{x}^* = (7.917, \ldots, 7.917)$.



(a) Alpine 1



(b) Alpine 2

Figure 4: Alpine Functions for $n = 2$

1.5 Arithmetic Mean

The arithmetic mean function is defined as

$$f(\mathbf{x}) = \left(\frac{1}{n} \sum_{i=1}^{n} x_i - \sqrt[n]{\prod_{i=1}^{n} x_i} \right)^2 \tag{9}$$

with $x_i \in [0, 10]$, for which $f(\mathbf{x}^*) = 0.0$ for $x_1 = x_2 = \dots x_n$.

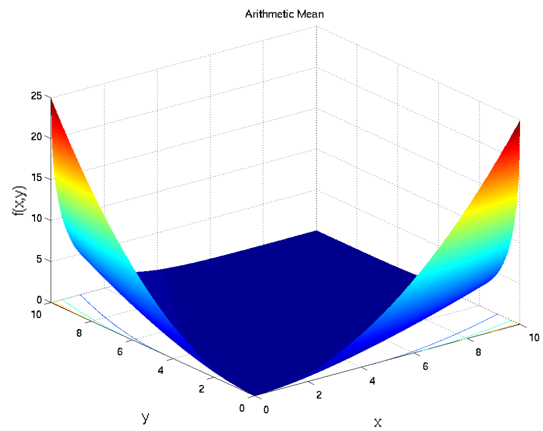


Figure 5: Arithmetic Mean for $n = 2$

1.6 Bartels-Conn

The Bartels-Conn function is defined as

$$f(\mathbf{x}) = |x_1^2 + x_2^2 + x_1 x_2| + |\sin(x_1)| + |\cos(x_2)| \tag{10}$$

with $x_i \in [-50, 50]$, for which $f(\mathbf{x}^*) = 1.0$ for $\mathbf{x}^* = (0, 0)$.

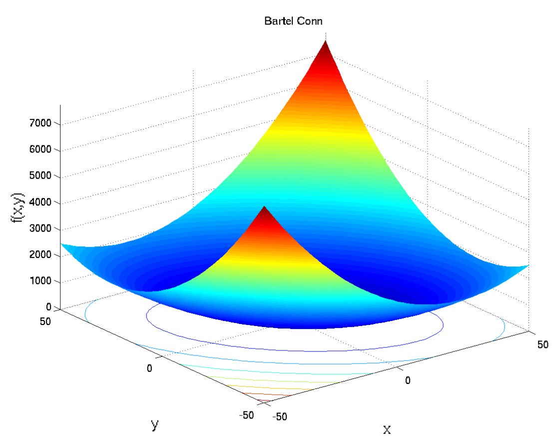


Figure 6: Bartels-Conn for $n = 2$

1.7 Beale

The Beale function is defined as

$$f(\mathbf{x}) = (1.5 - x_1 + x_1 x_2)^2 + (2.25 - x_1 + x_1 x_2^2)^2 + (2.625 - x_1 + x_1 x_2^3)^2 \tag{11}$$

with $x_i \in [-4.5, 4.5]$, for which $f(\mathbf{x}^*) = 0.0$ for $\mathbf{x}^* = (3, 0.5)$.

The Beale function has very sharp peaks at the corners of the search space.

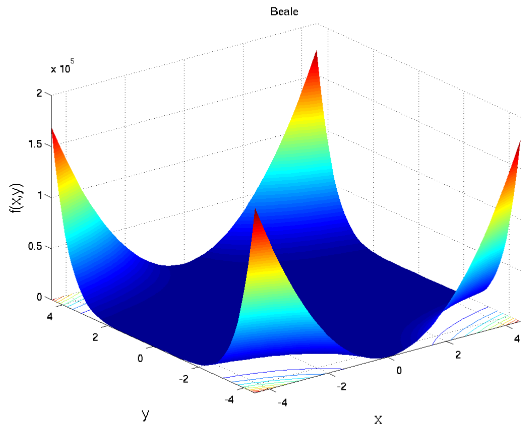


Figure 7: Beale for $n = 2$

1.8 Biggs EXP Functions

The following are five of the Biggs EXP functions, starting in two dimensions up to six dimensions:

- Biggs EXP 2:
$$f(\mathbf{x}) = \sum_{i=1}^{10} (e^{-t_i x_1} - 5e^{-t_i x_2} - y_i)^2 \tag{12}$$
where $t_i = 0.1i$ and $y_j = e^{-t_i} - 5e^{10t_i}$. Here $x_i \in [0, 20]$, for which $f(\mathbf{x}^*) = 0.0$ for $\mathbf{x}^* = (1, 10)$.

- Biggs EXP 3:
$$f(\mathbf{x}) = \sum_{i=1}^{10} (e^{-t_i x_1} - x_3 e^{-t_i x_2} - y_i)^2 \tag{13}$$
where $t_i = 0.1i$ and $y_j = e^{-t_i} - 5e^{10t_i}$. Here $x_i \in [0, 20]$, for which $f(\mathbf{x}^*) = 0.0$ for $\mathbf{x}^* = (1, 10, 5)$.

- Biggs EXP 4:
$$f(\mathbf{x}) = \sum_{i=1}^{10} (x_3 e^{-t_i x_1} - x_4 e^{-t_i x_2} - y_i)^2 \tag{14}$$
where $t_i = 0.1i$ and $y_j = e^{-t_i} - 5e^{10t_i}$. Here $x_i \in [0, 20]$, for which $f(\mathbf{x}^*) = 0.0$ for $\mathbf{x}^* = (1, 10, 1, 5)$.

- Biggs EXP 5:
$$f(\mathbf{x}) = \sum_{i=1}^{11} (x_3 e^{-t_i x_1} - x_4 e^{-t_i x_2} + 3e^{-t_i x_5} - y_i)^2 \tag{15}$$
where $t_i = 0.1i$ and $y_j = e^{-t_i} - 5e^{10t_i} + 3e^{-4t_i}$. Here $x_i \in [0, 20]$, for which $f(\mathbf{x}^*) = 0.0$ for $\mathbf{x}^* = (1, 10, 1, 5, 4)$.

- Biggs EXP 6:
$$f(\mathbf{x}) = \sum_{i=1}^{11} (x_3 e^{-t_i x_1} - x_4 e^{-t_i x_2} + 6e^{-t_i x_5} - y_i)^2 \tag{16}$$
where $t_i = 0.1i$ and $y_j = e^{-t_i} - 5e^{10t_i} + 3e^{-4t_i}$. Here $x_i \in [0, 20]$, for which $f(\mathbf{x}^*) = 0.0$ for $\mathbf{x}^* = (1, 10, 1, 5, 4)$.

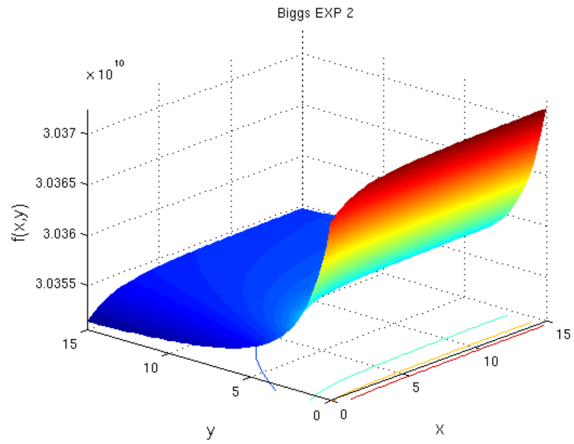


Figure 8: Biggs EXP 2

1.9 Bird

The Bird function is defined as:

$$f(\mathbf{x}) = \sin(x_1)e^{(1-\cos(x_2))^2} + \cos(x_2)e^{(1-\sin(x_1))^2} + (x_1 - x_2)^2 \tag{17}$$

with $x_i \in [-2\pi, 2\pi]$, for which $f(\mathbf{x}^*) = -106.764537$ for $\mathbf{x}^* = \{(4.70104, 3.15294), (-1.58214, -3.13024)\}$.

1.10 Bohachevsky Functions

Three versions of the Bohachevsky function are provided below. Note that these are generalized versions of the orginal two-dimensional versions.

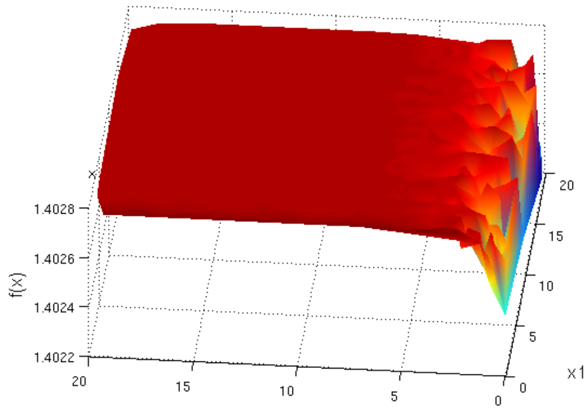
- Bohachevsky 1:
$$f(\mathbf{x}) = \sum_{i=1}^{n-1}(x_i^2 + 2x_{i+1}^2 - 0.3\cos(3\pi x_i) - 0.4\cos(4\pi x_{i+1}) + 0.7) \tag{18}$$
with $x_i \in [-15, 15]$, for which $f(\mathbf{x}^*) = 0$ for $\mathbf{x}^* = (0, \ldots, 0)$.
- Bohachevsky 2:
$$f(\mathbf{x}) = \sum_{i=1}^{n-1}(x_i^2 + 2x_{i+1}^2 - 0.3\cos(3\pi x_i) \times 0.4\cos(4\pi x_{i+1}) + 0.3) \tag{19}$$
with $x_i \in [-15, 15]$, for which $f(\mathbf{x}^*) = 0$ for $\mathbf{x}^* = (0, \ldots, 0)$.
- Bohachevsky 3:
$$f(\mathbf{x}) = \sum_{i=1}^{n-1}(x_i^2 + 2x_{i+1}^2 - 0.3\cos(3\pi x_i + 4\pi x_{i+1}) + 0.3) \tag{20}$$
with $x_i \in [-15, 15]$, for which $f(\mathbf{x}^*) = 0$ for $\mathbf{x}^* = (0, \ldots, 0)$.

1.11 Bonyadi-Michalewicz

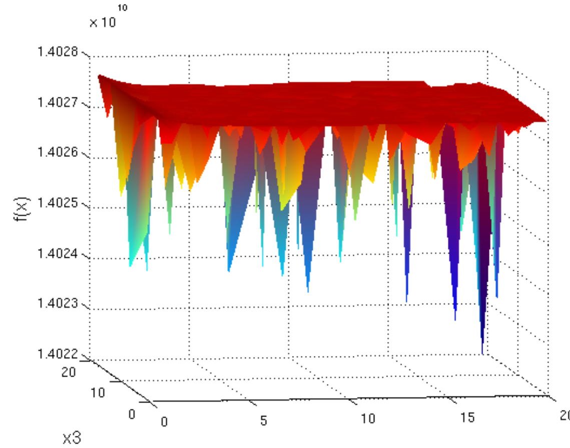
The Bonyadi-Michalewicz function is generalized from its original two dimensional form:

$$f(\mathbf{x}) = \frac{\prod_{i=1}^{n}(x_i + 1)}{\prod_{i=1}^{n}((x_i + 1)^2 + 1)} \tag{21}$$

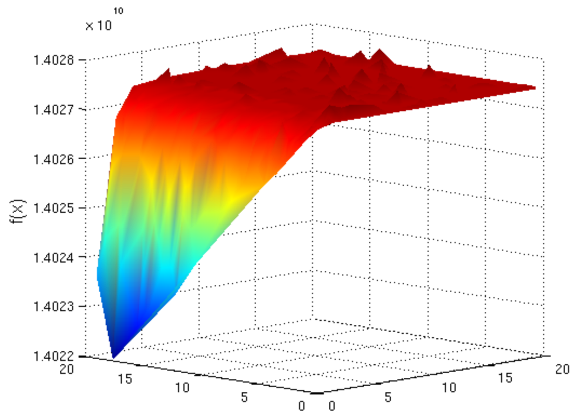
with $x_i \in [-5, 5]$, for which $f(\mathbf{x}^*) = -0.25$ for $\mathbf{x}^* = (-2, 0)$, $n = 2$.



(a) x_1 versus x_2



(b) x_1 versus x_3



(c) x_2 versus x_3

Figure 9: Biggs EXP 2 for $n = 2$

1.12 Booth

The Booth function is defined as

$$f(\mathbf{x}) = (x_1 + 2x_2 - 7)^2 + (2x_1 + x_2 - 5)^2 \tag{22}$$

with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = 0$ for $\mathbf{x}^* = (1, 3)$.

1.13 Box-Betts Quadratic Sum

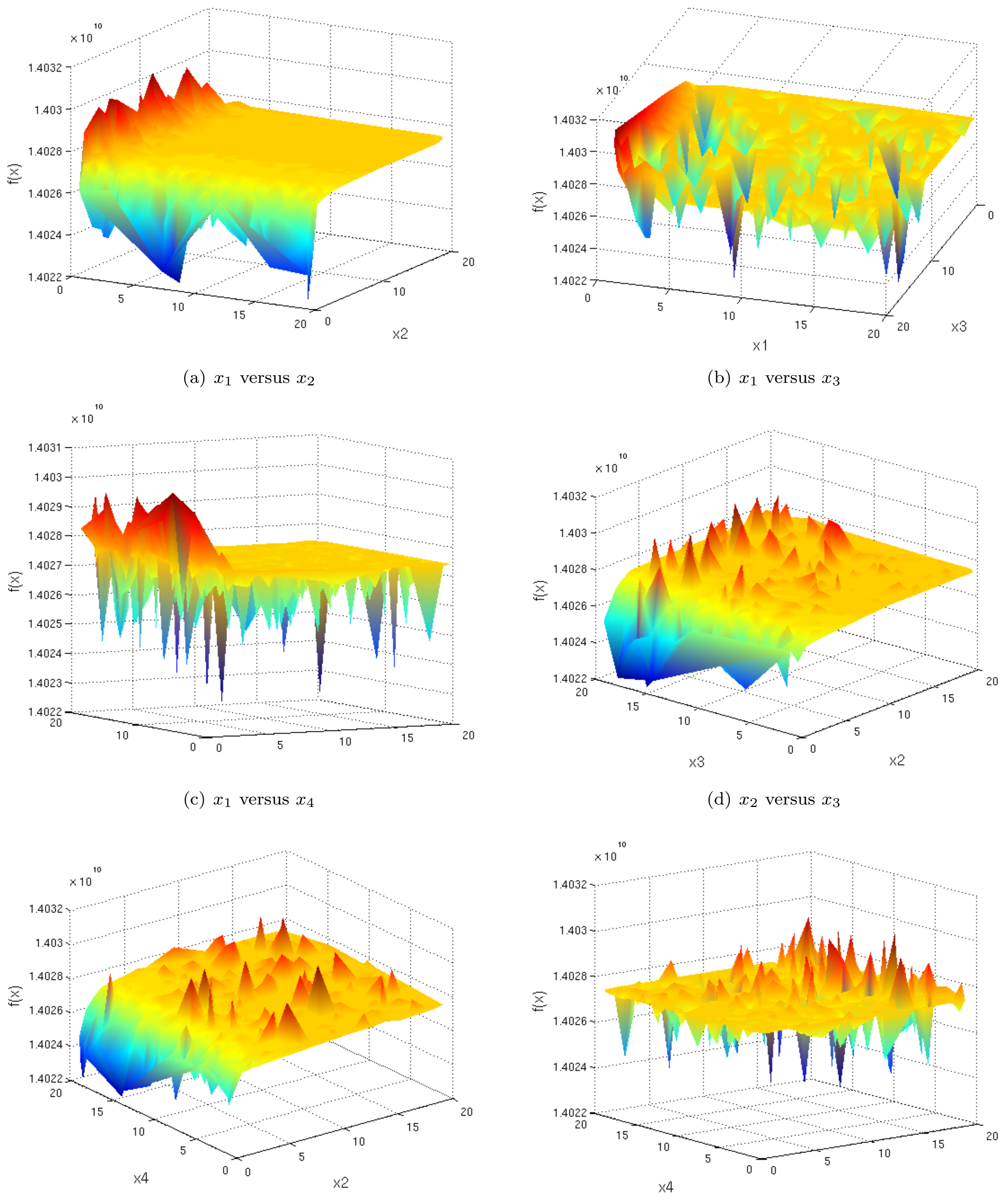
The Box-Betts Quadratic Sum function is defined, in three dimensions, as

$$f(\mathbf{x}) = \sum_{i=1}^{k} g(x_i)^2 \tag{23}$$

where

$$g(x) = e^{-0.1(i+1)x_1} - e^{-0.1(i+1)x_2} - e^{[(-0.1(i+1))-e^{-(i+1)}]x_3}$$

with $x_1, x_3 \in [0.9, 1.2]$ and $x_2 \in [9, 11.2]$, for which $f(\mathbf{x}^*) = 0$ for $\mathbf{x}^* = (1, 10, 1)$ when $k = 10$.



(a) x_1 versus x_2

(b) x_1 versus x_3

(c) x_1 versus x_4

(d) x_2 versus x_3

(e) x_2 versus x_4

(f) x_3 versus x_4

Figure 10: Biggs EXP 3

1.14 Brad

The Brad function is defined as

$$f(\mathbf{x}) = \sum_{i=1}^{15} \left[\frac{y_i - x_1 - u_i}{v_i x_2 + w_i x_3} \right]^2 \tag{24}$$

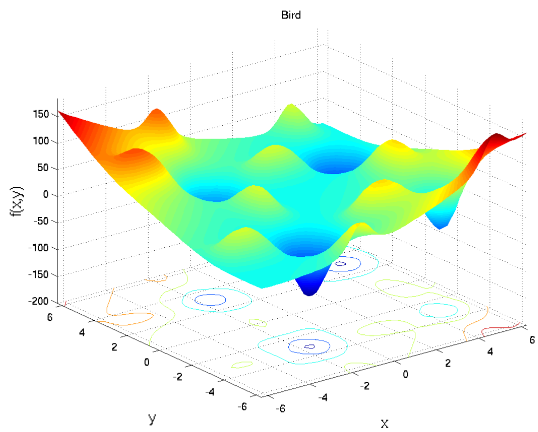
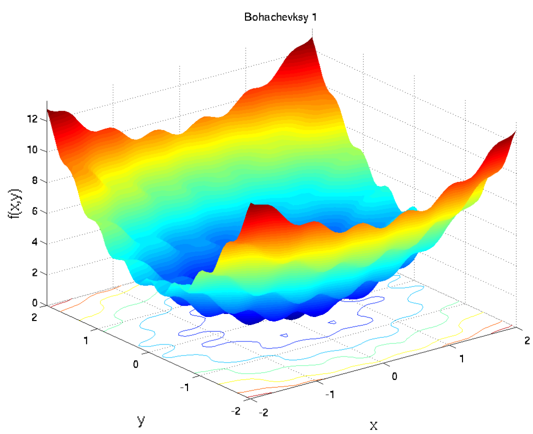
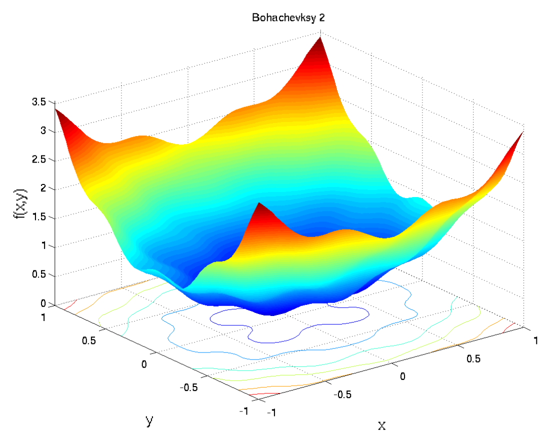


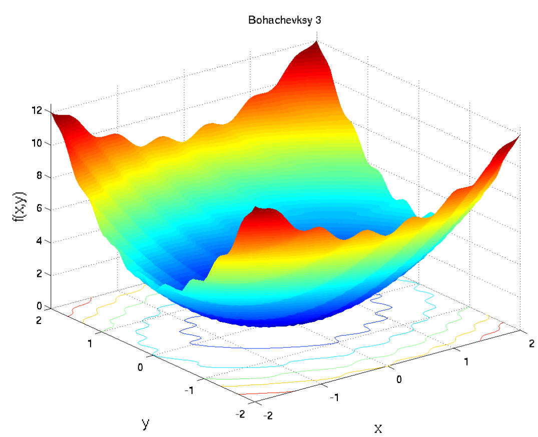
Figure 11: Bird for $n = 2$



(a) Bohachevsky 1



(b) Bohachevsky 2



(c) Bohachevsky 3

Figure 12: Bohachevsky Functions for $n = 2$

where $u_i = i, v_i = 16 - i, w_i = \min\{u_i, v_i\}$ and

$$\mathbf{y} = [0.14, 0.18, 0.22, 0.25, 0.29, 0.32, 0.35, 0.39, 0.37, 0.58, 0.73, 0.96, 1.34, 2.10, 4.39]^T$$

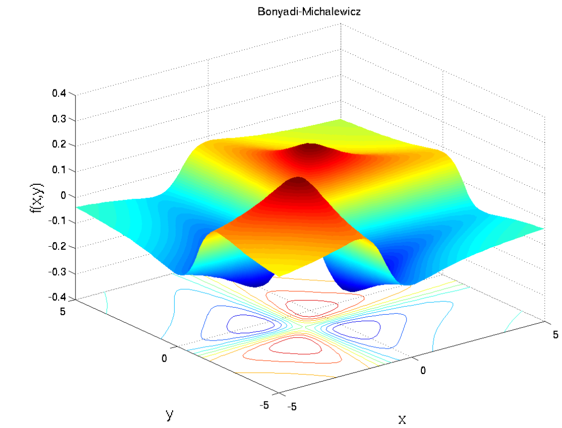


Figure 13: Bonyadi-Michalewicz for $n = 2$

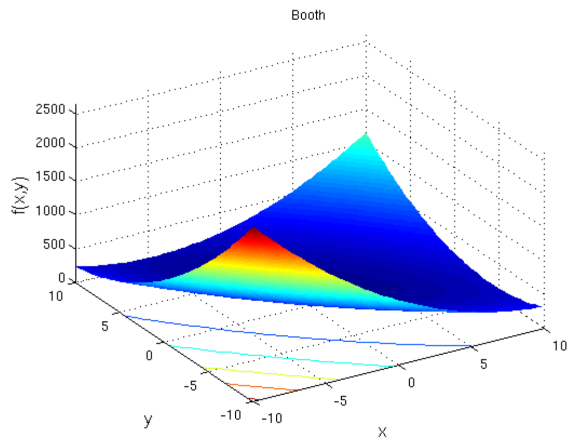


Figure 14: Booth for $n = 2$

. Here, $x_1 \in [-0.25, 0.25]$ and $x_2, x_3 \in [0.01, 2.5]$, for which $f(\mathbf{x}^*) = 0.00821487$ for $\mathbf{x}^* = (0.0824, 1.133, 2.3437)$.

1.15 Branin RCOS Functions

Below are versions of the Branin RCOS function:

- Branin RCOS 1:

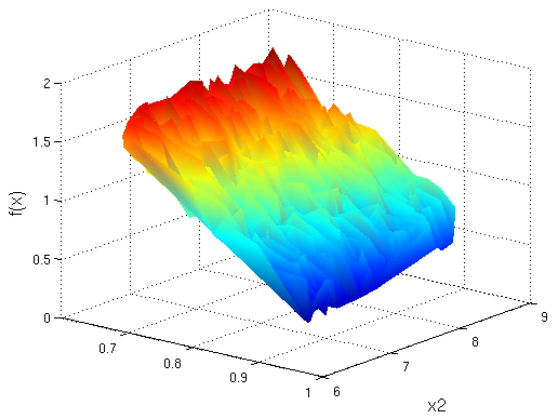
$$f(\mathbf{x}) = \left(x_2 - \frac{5.1}{4\pi^2}x_1^2 + \frac{5}{\pi}x_1 - 6\right)^2 + 10\left(1 - \frac{1}{8\pi}\right)\cos(x_1) + 10 \tag{25}$$

with $x_1 \in [-5, 10]$ and $x_2 \in [0, 15]$, for which $f(\mathbf{x}^*) = 0.3978873$ for $\mathbf{x}^* = \{(-\pi, 12.275), (\pi, 2.275), (3\pi, 2.425)\}$.

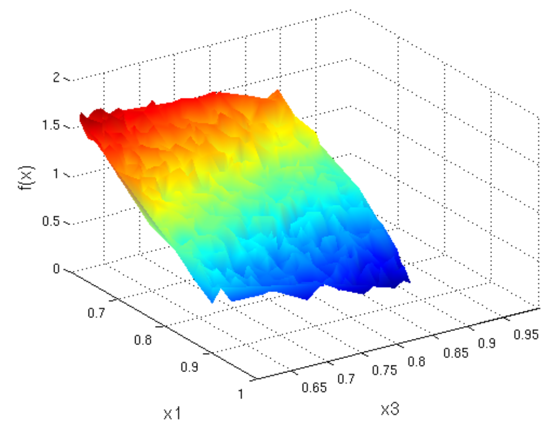
- Branin RCOS 2:

$$f(\mathbf{x}) = \left(x_2 - \frac{5.1}{4\pi^2}x_1^2 + \frac{5}{\pi}x_1 - 6\right)^2 + 10\left(1 - \frac{1}{8\pi}\right)\cos(x_1)\cos(x_2)\ln(x_1^2 + x_2^2 + 1) + 10 \tag{26}$$

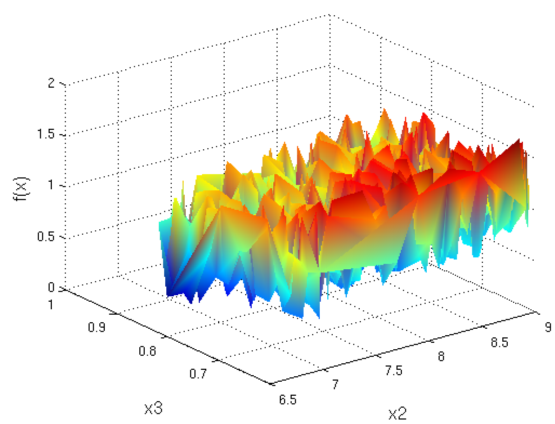
with $x_1 \in [-5, 10]$ and $x_2 \in [0, 15]$, for which $f(\mathbf{x}^*) = 5.559037$ for $\mathbf{x}^* = (-3.2, 12.53)$.



(a) x_1 versus x_2



(b) x_1 versus x_3



(c) x_2 versus x_3

Figure 15: Box-Betts Quadratic Sum for $n = 2$

1.16 Brent

The Brent function is defined below, as a generalization of the original two-dimensional function:

$$f(\mathbf{x}) = \sum_{i=1}^{n}(x_i + 10)^2 + e^{-\sum_{i=1}^{n} x_i^2} \tag{27}$$

with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = 0.0$ for $\mathbf{x}^* = (0, \ldots, 0)$.

1.17 Brown

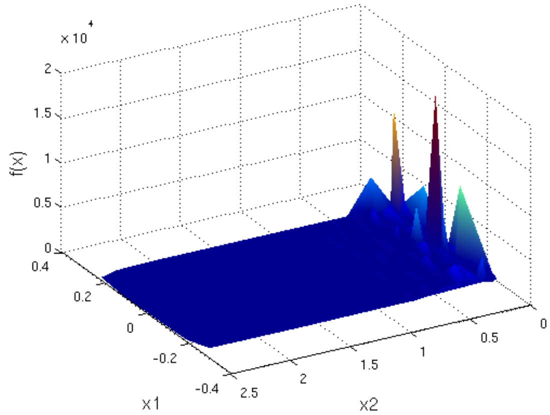
The Brown function is defined below:

$$f(\mathbf{x}) = \sum_{i=1}^{n-1}(x_i^2)^{(x_{i+1}^2+1)} + (x_{i+1}^2)^{(x_i^2+1)} \tag{28}$$

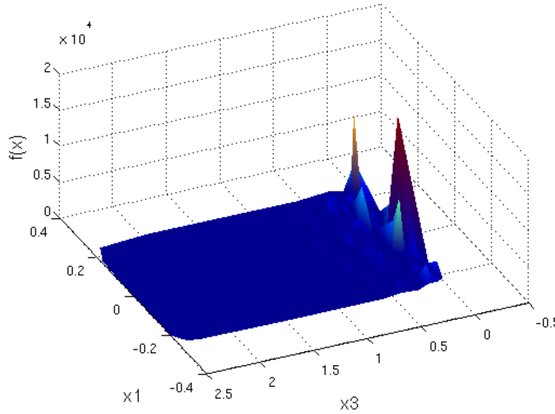
with $x_i \in [-1, 1]$ for which $f(\mathbf{x}^*) = 0.0$ for $\mathbf{x}^* = (0, \ldots, 0)$.

1.18 Bukin Functions

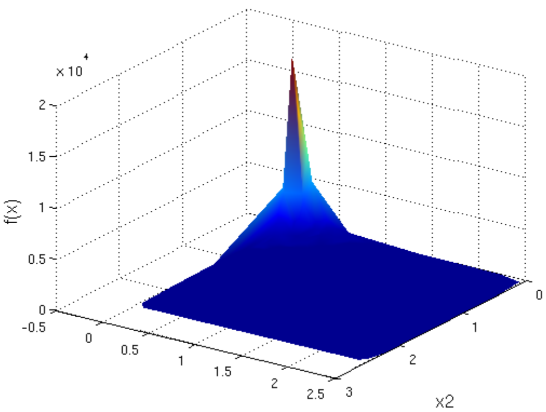
The following Bukin functions are defined, all in two dimensions:



(a) x_1 versus x_2

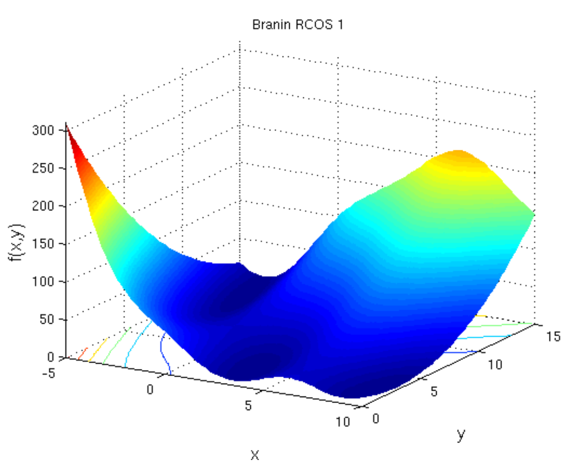


(b) x_1 versus x_3

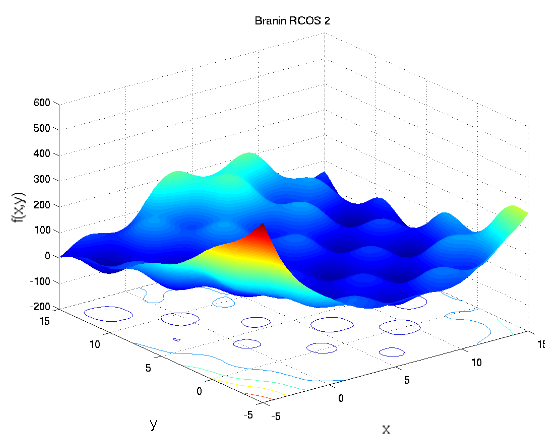


(c) x_2 versus x_3

Figure 16: Brad



(a) Branin RCOS 1



(b) Branin RCOS 2

Figure 17: Branin RCOS 1 for $n = 2$

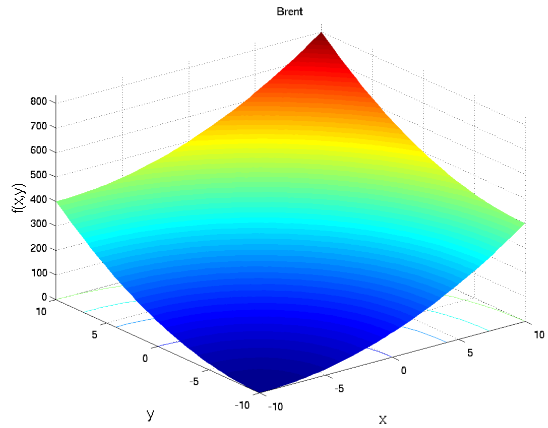


Figure 18: Brent for $n = 2$

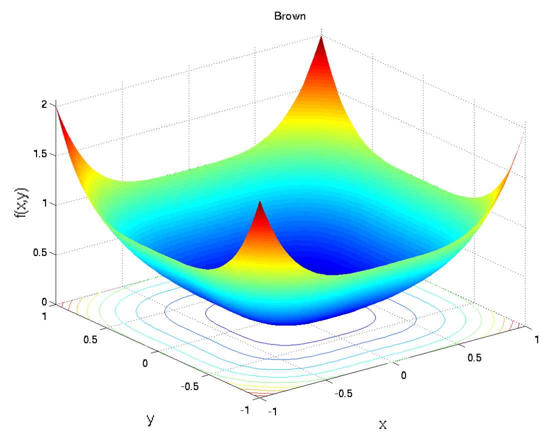


Figure 19: Brown for $n = 2$

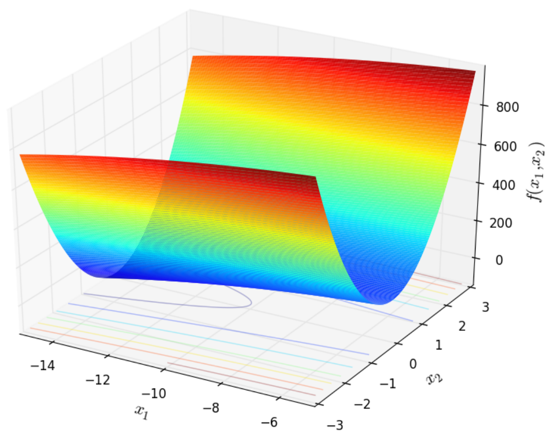
- Bukin 2:
$$f(\mathbf{x}) = 100(x_2 - 0.01x_1^2 + 1) + 0.01(x_1 + 10)^2 \tag{29}$$
with $x_1 \in [-15, -5]$ and $x_2 \in [-3, 3]$ for which $f(\mathbf{x}^*) = 0.0$ for $\mathbf{x}^* = (-10, 0)$.

- Bukin 2 Adapted:
$$f(\mathbf{x}) = (100(x_2 - 0.01x_1^2 + 1) + 0.01(x_1 + 10)^2)^2 \tag{30}$$
with $x_1 \in [-15, -5]$ and $x_2 \in [-3, 3]$ for which $f(\mathbf{x}^*) = 0.0$ for $\mathbf{x}^* = (-?, ?)$.

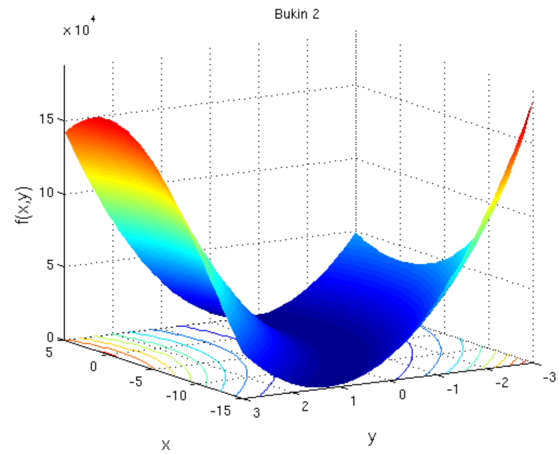
- Bukin 4:
$$f(\mathbf{x}) = 100x_2^2 + 0.01|x_1 + 10| \tag{31}$$
with $x_1 \in [-15, -5]$ and $x_2 \in [-3, 3]$ for which $f(\mathbf{x}^*) = 0.0$ for $\mathbf{x}^* = (-10, 0)$.

- Bukin 6:
$$f(\mathbf{x}) = 100\sqrt{|x_2 - 0.01x_1^2|} + 0.01|x_1 + 10| \tag{32}$$
with $x_1 \in [-15, -5]$ and $x_2 \in [-3, 3]$ for which $f(\mathbf{x}^*) = 0.0$ for $\mathbf{x}^* = (-10, 1)$.

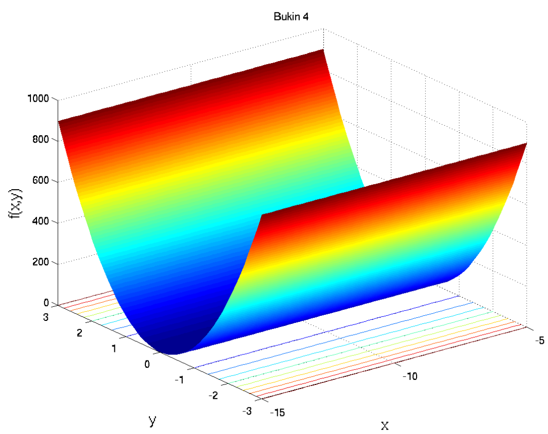
The Bukin functions are almost fractal in the surroundings of the minima. This makes the Bukin problems very difficult to solve.



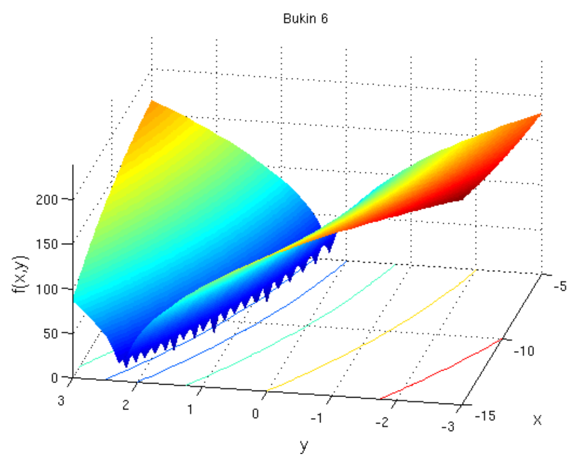
(a) Bukin 2



(b) Bukin 2 Adapted



(c) Bukin 4



(d) Bukin 6

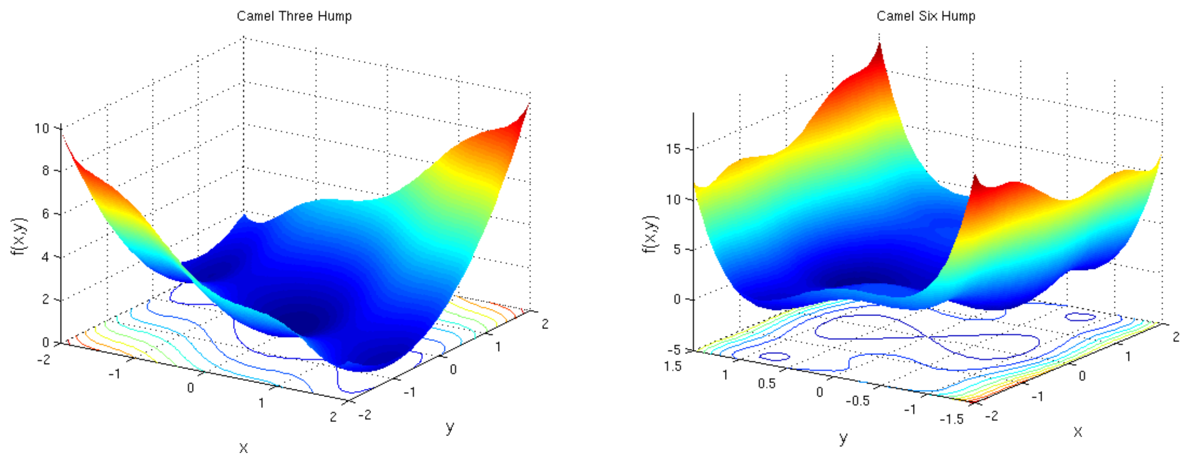
Figure 20: Bukin Functions for $n = 2$

1.19 Camel Functions

The following versions of the Camel functions:

- Three Hump Camel:
$$f(\mathbf{x}) = 2x_1^2 - 1.05x_1^4 + x_1^6/6 + x_1x_2 + x_2^2 \tag{33}$$
with $x_i \in [-5, 5]$, for which $f(\mathbf{x}^*) = 0.0$ for $\mathbf{x}^* = (0, 0)$. In addition, there are two local minima.

- Six Hump Camel:
$$f(\mathbf{x}) = 4x_1^2 - 2.1x_1^4 + \frac{1}{3}x_1^6 + x_1x_2 - 4x_2^2 + 4x_2^4 \tag{34}$$
with $x_i \in [-5, 5]$, for which $f(\mathbf{x}^*) = -1.0316285$ for $\mathbf{x}^* = \{(-0.08983, 0.7126), (0.0898, -0.7126)\}$. The function is symmetric about the origin.



(a) Three Hump Camel (b) Six Hump Camel

Figure 21: Camel Functions for $n = 2$

1.20 Centreal Two Peak Trap

The following one-dimensional function has its global minimum on the boundary and one local minimum:

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ -160/10 * x & \text{if } x \leq 10 \\ -160/5 * (15 - x) & \text{if } x \leq 15 \\ -200/5 * (x - 15) & \text{if } x \leq 20 \\ -200 & \text{otherwise} \end{cases} \tag{35}$$

with $x \in [0, 20]$, for which $f(x^*) = -200$, with $x^* = 20$.

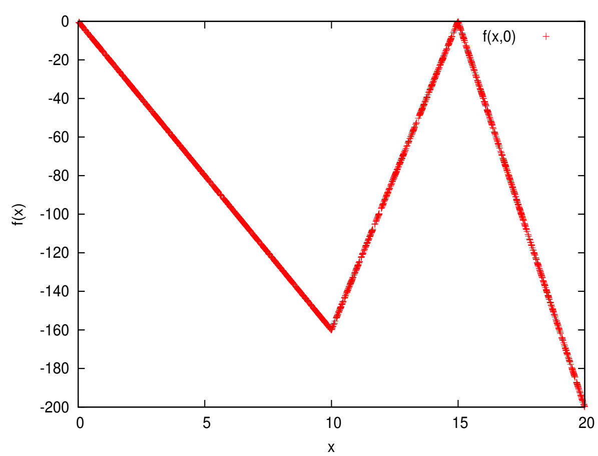


Figure 22: Central Two Peak Trap for $n = 1$

1.21 Chen Functions

The Chen functions are defined in two dimensions as:

- Bird Function:

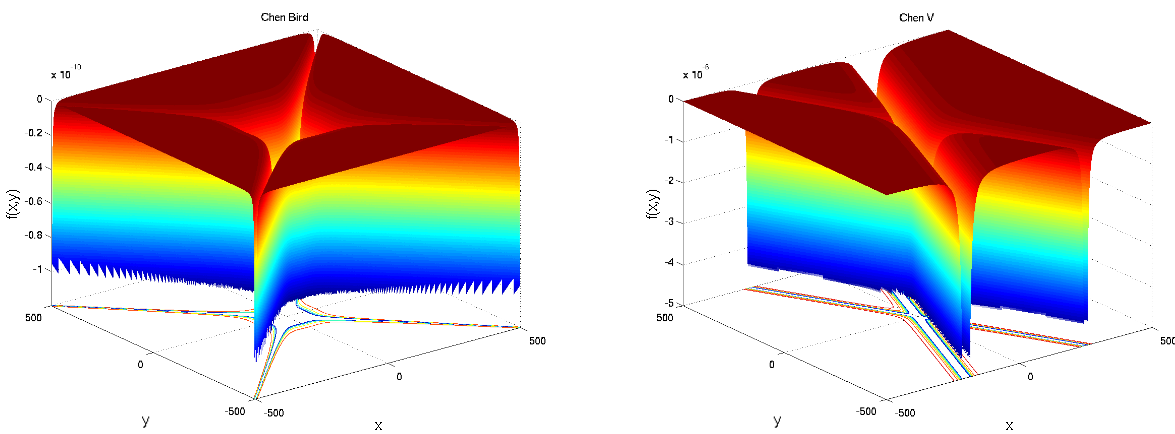
$$f(\mathbf{x}) = -\frac{0.001}{\lfloor 0.001^2 + (x_1^2 + x_2^2 - 1)^2 \rfloor} - \frac{0.001}{\lfloor 0.001^2 + (x_1^2 + x_2^2 - 0.5)^2 \rfloor} - \frac{0.001}{\lfloor 0.001^2 + (x_1^2 - x_2^2)^2 \rfloor} \tag{36}$$

with $x \in [-500, 500]$, for which $f(x^*) = -2000$, with $\mathbf{x}^* = (-0.3888889, 0.7222222)$.

- V Function:

$$f(\mathbf{x}) = -\frac{0.001}{\lfloor 0.001^2 + (x_1 - 0.4x_2)^2 \rfloor} - \frac{0.001}{\lfloor 0.001^2 + (2x_1 + x_2 - 1.5)^2 \rfloor} \tag{37}$$

with $x \in [-500, 500]$, for which $f(x^*) = -2000$, with $\mathbf{x}^* = (-\frac{7}{18}, -\frac{13}{18})$.



(a) Chen Bird

(b) Chen V

Figure 23: Chen Functions for $n = 2$

1.22 Chichinadze

The Chichinadze function is defined in two dimensions as:

$$f(\mathbf{x}) = x_1^2 - 12x_1 + 11 + 10\cos(\pi x_1/2) + 8\sin(5\pi x_1) - (1/5)^{0.5} e^{-0.5(x_2-0.5)^2} \tag{38}$$

with $x_i \in [-30, 30]$ for which $f(\mathbf{x}^*) = -43.3159$ for $\mathbf{x}^* = (5.90133, 0.5)$.

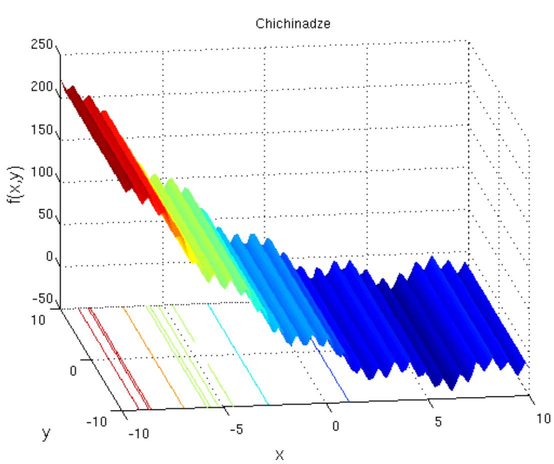


Figure 24: Chichinadze for $n = 2$

1.23 Chung-Reynolds

The Chung-Reynolds function is defined as:

$$f(\mathbf{x}) = \left(\sum_{i=1}^{n} x_i^2\right)^2 \tag{39}$$

with $x_i \in [-100, 100]$ for which $f(\mathbf{x}^*) = 0$ for $\mathbf{x}^* = (0, \ldots, 0)$.

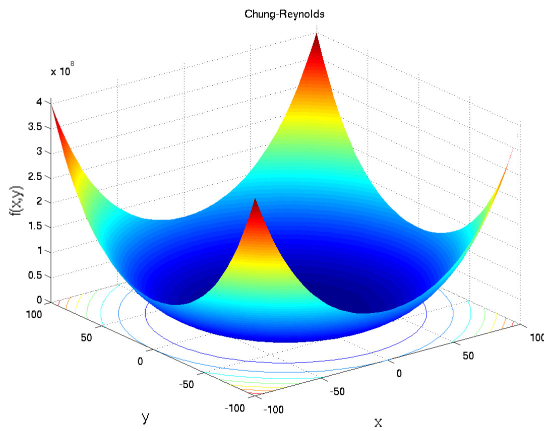


Figure 25: Chung-Reynolds for $n = 2$

1.24 Cigar

The Cigar function is defined as:

$$f(\mathbf{x}) = x_1^2 + 10^6 \sum_{i=2}^{n} x_i^2 \tag{40}$$

with $x_i \in [-100, 100]$ for which $f(\mathbf{x}^*) = 0$ for $\mathbf{x}^* = (0, \ldots, 0)$.

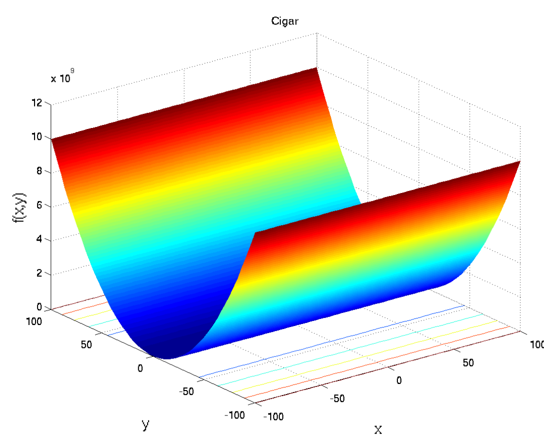


Figure 26: Cigar for $n = 2$

1.25 Colville

The Colville function is defined in four dimensions as:

$$f(\mathbf{x}) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 + 90(x_4 - x_3)^2 + (1 - x_3)^2 + 10.1((x_3 - 1)^2 + (x_4 - 1)^2) + 19.8(x_2 - 1)(x_4 - 1) \quad (41)$$

with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = 0.0$ for $\mathbf{x}^* = (1, \ldots, 1)$.

1.26 Corana

The Corana function is defined in two, four, or ten dimensions as:

$$f(\mathbf{x}) = \sum_{i=1}^{n} y_i \quad (42)$$

where

$$y_j = \begin{cases} c * (z_i - 0.05\mathrm{sgn}(z_i))^2 d_i & \text{if } |v_i| < A \\ d_i x_i^2 & \text{otherwise} \end{cases}$$

with $A = 0.05$, $c = 0.15$, and

$$\begin{aligned} z_i &= 0.2 \left\lfloor \left| \frac{x_i}{0.2} \right| + 0.49999 \right\rfloor \mathrm{sgn}(x_i) \\ v_i &= |x_i - z_i| \end{aligned}$$

For $n = 2$, $\mathbf{d} = (1, 1000)$, for $n = 4$, $\mathbf{d} = (1, 1000, 10, 100)$, and for $n = 10$, $\mathbf{d} = (1, 1000, 10, 100, 1, 10, 100, 1000, 1, 10)$. For the Corona function, $x_i \in [-5, 5]$, for which $f(\mathbf{x}^*) = 0.0$ for $\mathbf{x}^* = (0, \ldots, 0)$.

For the Corana function, the components of the $\mathbf{d}$ vector determine the steepness of the paraboloid along the axes, while c controls the depth of local minima.

1.27 Cosine Mixture

The Cosine Mixture function is defined as:

$$f(\mathbf{x}) = 0.1 \sum_{i=1}^{n} \cos(5\pi x_i) + \sum_{i=1}^{n} x_i^2 \quad (43)$$

with $x_i \in [-1, 1]$, for which $f(\mathbf{x}^*) = -0.1n$ for $\mathbf{x}^* = (0, \ldots, 0)$.

1.28 Cross Functions

Below, three of the cross function variations are generalized:

- The Cross-in-Tray function is defined as:

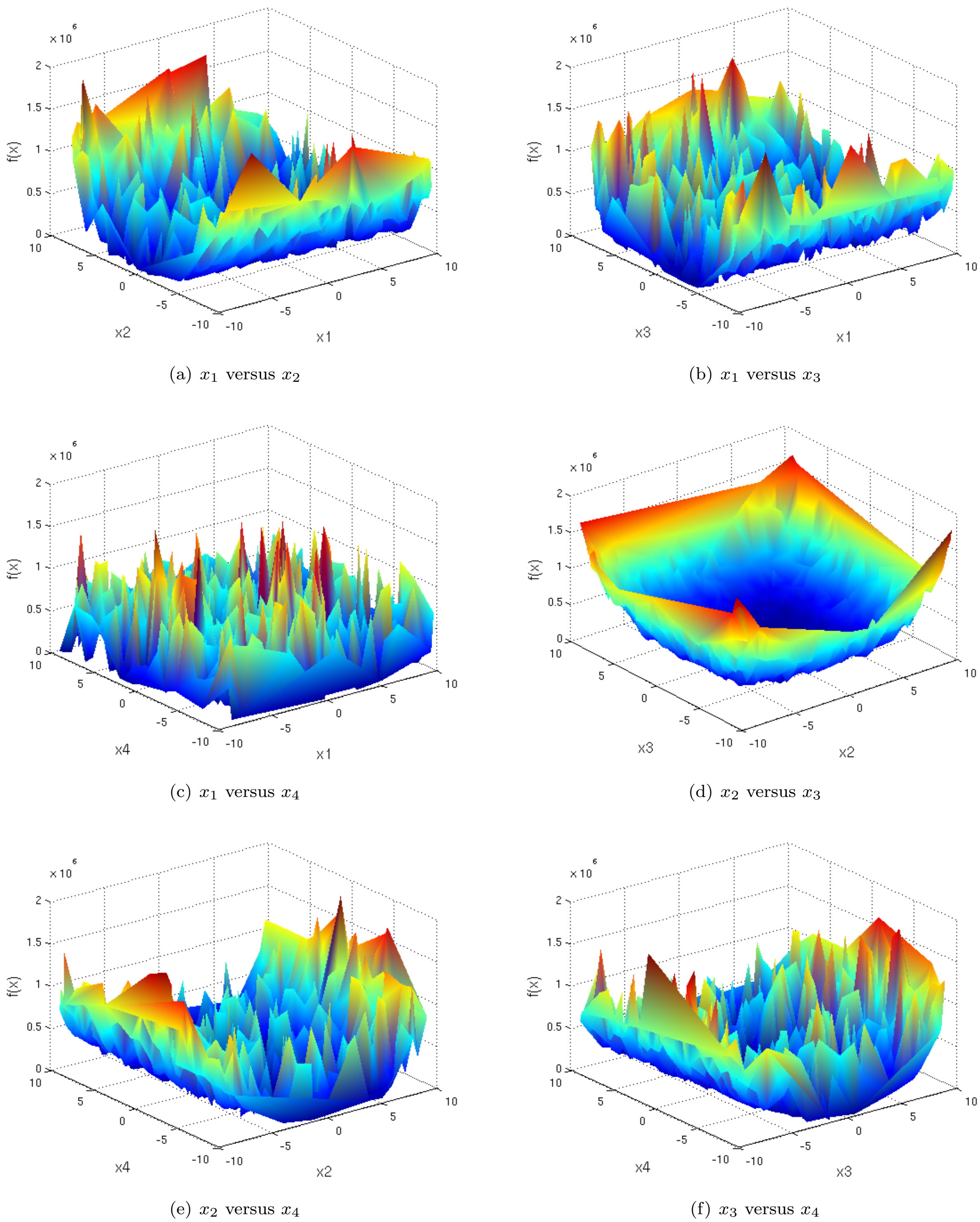
$$f(\mathbf{x}) = -0.0001 \left[\left| \left(\prod_{i=1}^{n} \sin(x_i) \right) e^{|100 - (\sum_{i=1}^{n} x_i^2)^{0.5})/\pi|} \right| + 1 \right]^{0.1} \quad (44)$$

  with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = -2.06261218$ for $\mathbf{x}^* = (\pm{-1.349406685353340}, \pm{-1.349406608602084})$ for $n = 2$.

- The Cross Leg Table function is defined as:

$$f(\mathbf{x}) = -\frac{1}{\left(\left| e^{|100 - \frac{\sqrt{\sum_{i=1}^{n} x_i^2}}{\pi}|} \prod_{i=1}^{n} \sin(x_i) \right| + 1 \right)^{0.1}} \quad (45)$$

  with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = -1$ for $\mathbf{x}^* = (0, \ldots, 0)$.



(a) x_1 versus x_2

(b) x_1 versus x_3

(c) x_1 versus x_4

(d) x_2 versus x_3

(e) x_2 versus x_4

(f) x_3 versus x_4

Figure 27: Colville

- The Crowned Cross function is defined as:

$$f(\mathbf{x}) = 0.0001\left(\left|e^{\left|100-\frac{\sqrt{\sum_{i=1}^{n} x_i^2}}{\pi}\right|}\prod_{i=1}^{n}\sin(x_i)\right|+1\right)^{0.1} \tag{46}$$

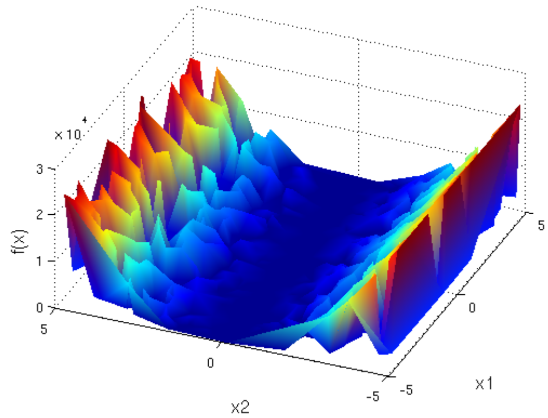


Figure 28: Corana in 2D

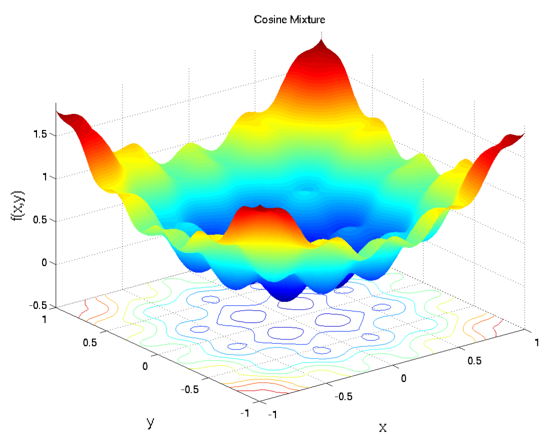


Figure 29: Cosine Mixture 2D

with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = -0.0001$ for $\mathbf{x}^* = (0, \ldots, 0)$.

1.29 Csendes

The Csendes function is defined as:

$$f(\mathbf{x}) = \sum_{i=1}^{n} x_i^6 \left(2 + \sin(\frac{1}{x_i})\right) \tag{47}$$

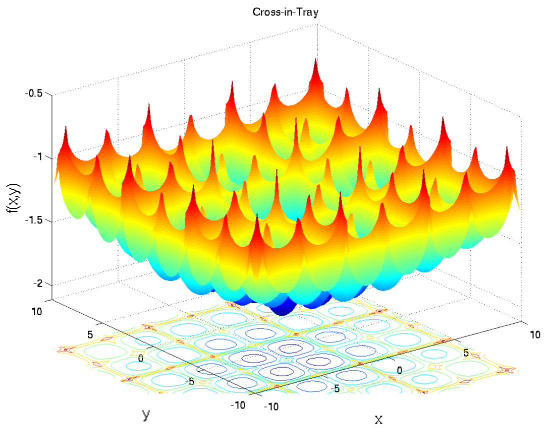
with $x_i \in [-1, 1]$ for which $f(\mathbf{x}^*) = 0$ for $\mathbf{x}^* = (0, \ldots, 0)$.

1.30 Cube

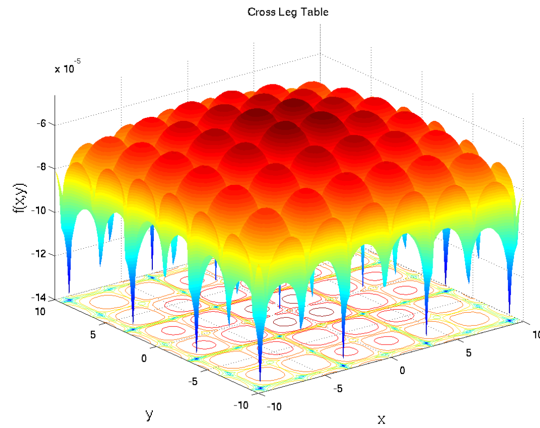
The Cube function is defined in two dimensions as:

$$f(\mathbf{x}) = 100(x_2 - x_1^3)^2 + (1 - x_1)^2 \tag{48}$$

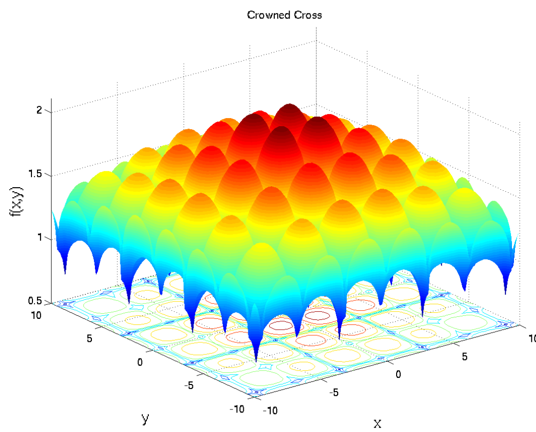
with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = 0$ for $\mathbf{x}^* = (-1, 1)$.



(a) Cross-in-Tray



(b) Cross Leg Table



(c) Cross Leg Table

Figure 30: Cross Functions in 2D

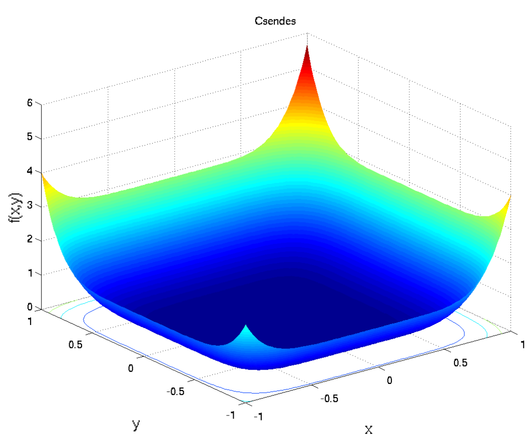


Figure 31: Csendes for $n = 2$

1.31 Damavandi

The Damavandi function is defined in two dimensions as:

$$f(\mathbf{x}) = \left[1 - \left|\frac{\sin(\pi(x_1 - 2))\sin(\pi(x_2 - 2))}{\pi^2(x_1 - 2)(x_2 - 2)}\right|^5\right]\left[2 + (x_1 - 7)^2 + 2(x_2 - 7)^2\right] \tag{49}$$

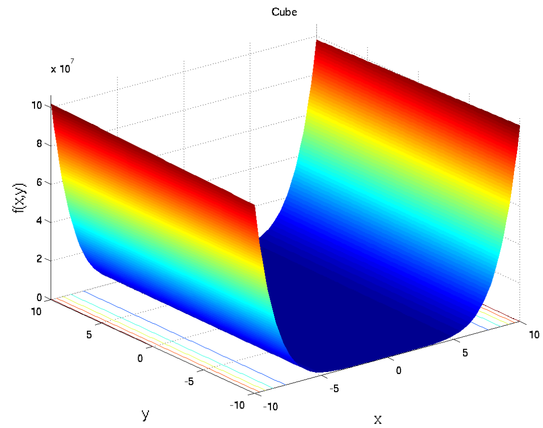


Figure 32: Cube for $n = 2$

with $x_i \in [0, 14]$ for which $f(\mathbf{x}^*) = 0$ for $\mathbf{x}^* = (2, 2)$.

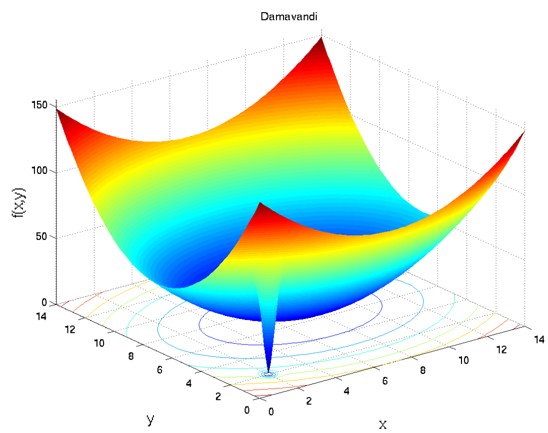


Figure 33: Damavandi for $n = 2$

1.32 Deb Functions

The following two Deb functions are defined:

- Deb 1:

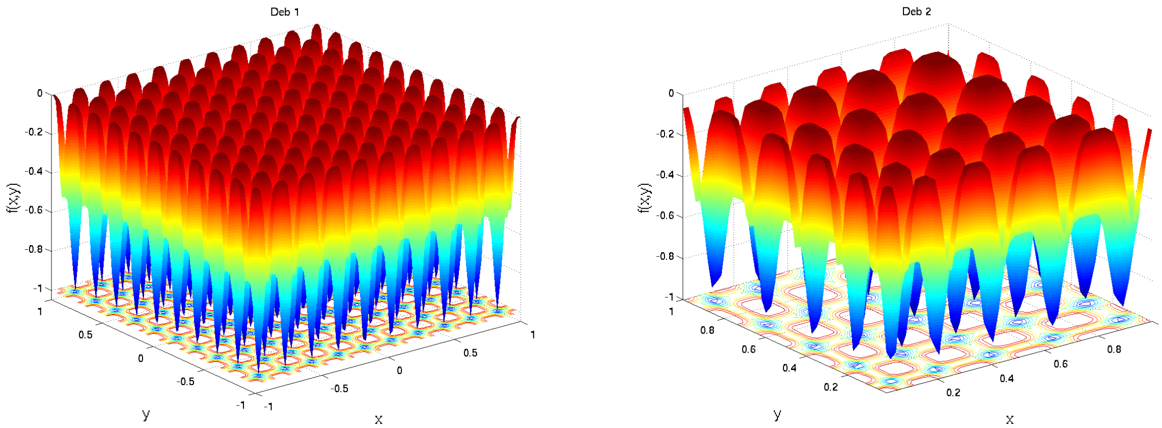
$$f(\mathbf{x}) = -\frac{1}{n}\sum_{i=1}^{n} \sin^6(5\pi x_i) \tag{50}$$

  with $x_i \in [-1, 1]$ for which $f(\mathbf{x}^*) = 0$. The number of global minima is 5^n, and are evenly spaced in the function landscape.

- Deb 2:

$$f(\mathbf{x}) = -\frac{1}{n}\sum_{i=1}^{n} \sin^6(5\pi(x_i^{3/4} - 0.05)) \tag{51}$$

  with $x_i \in [-1, 1]$ for which $f(\mathbf{x}^*) = 0$. The number of global minima is 5^n, and are unevenly spaced in the function landscape.



(a) Deb 1 (b) Deb 2

Figure 34: Deb Functions for $n = 2$

1.33 Decanomial

The Decanomial function is defined in two dimensions as:

$$\begin{aligned} f(\mathbf{x}) &= 0.001 * (|x_2^4 + 12x_2^3 + 54x_2^2 + 108x_2 + 81| \\ &+ |x_1^{10} - 20x_1^9 + 180x_1^8 - 960x_1^7 + 3360x_1^6 - 8064x_1^5 + 13340x_1^4 - 15360x_1^3 + 11520x_1^2 - 5120x_1 + 26240|)^2 \end{aligned} \tag{52}$$

with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = 0$ for $\mathbf{x}^* = (2, -3)$.

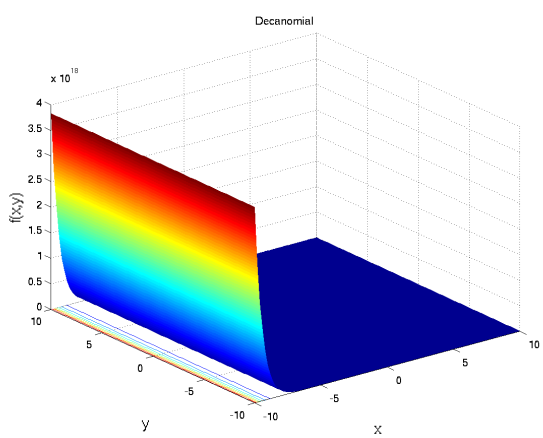


Figure 35: Decanomial for $n = 2$

1.34 Deceptive

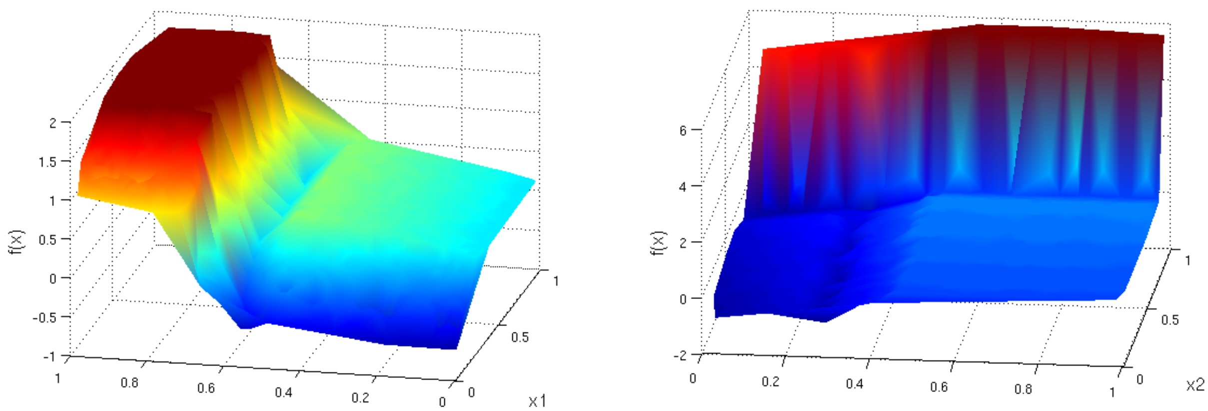
The Deceptive function is defined as:

$$f(\mathbf{x}) = -\left[\frac{1}{n}\sum_{i=1}^{n} g_i(x_i)\right]^\beta \tag{53}$$

where

$$g_i(x_i) = \begin{cases} -\frac{x_i}{\alpha_i} + \frac{4}{5} & 0 \le x_i \le \frac{4}{5}\alpha_i \\ \frac{5x_i}{\alpha_i} - 4 & \frac{4}{5}\alpha_i < x_i \le \alpha_i \\ \frac{5(x_i - \alpha_i)}{\alpha_i - 1} & \alpha_i < x_i \le \frac{1+4\alpha_i}{5} \\ \frac{x_i - 1}{1 - \alpha_i} & \frac{1+4\alpha_i}{5} < x_i \le 1 \end{cases}$$

and $\beta = 2$ is the non-linearity factor. Bounds are $x_i \in [0, 1]$ for which $f(\mathbf{x}^*) = -1$ for $\mathbf{x}^* = (\alpha_1, \dots, \alpha_n)$.



(a) $\alpha_1 = 0.1, \alpha_2 = 0.6$

(b) $\alpha_1 = 0.3, \alpha_2 = 0.9$

Figure 36: Deceptive for $n = 2$

1.35 Deckkers-Aarts

The Deckkers-Aarts function is defined in two dimensions as:

$$f(\mathbf{x}) = 10^5 x_1^2 + x_2^2 - (x_1^2 + x_2^2)^2 + 10^{-5}(x_1^2 + x_2^2)^4 \tag{54}$$

with $x_i \in [-20, 20]$ for which $f(\mathbf{x}^*) = -24777$ for $\mathbf{x}^* = (0, \pm 15)$.

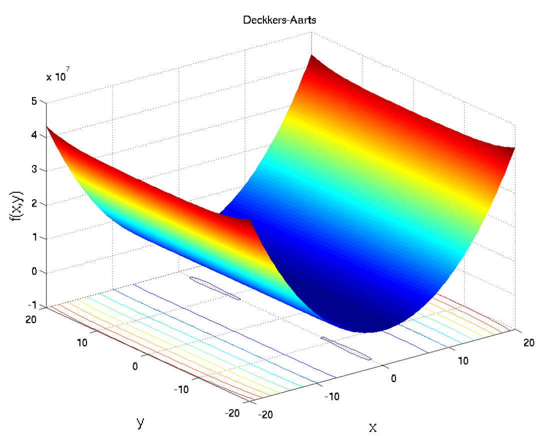


Figure 37: Deckkers-Aarts for $n = 2$

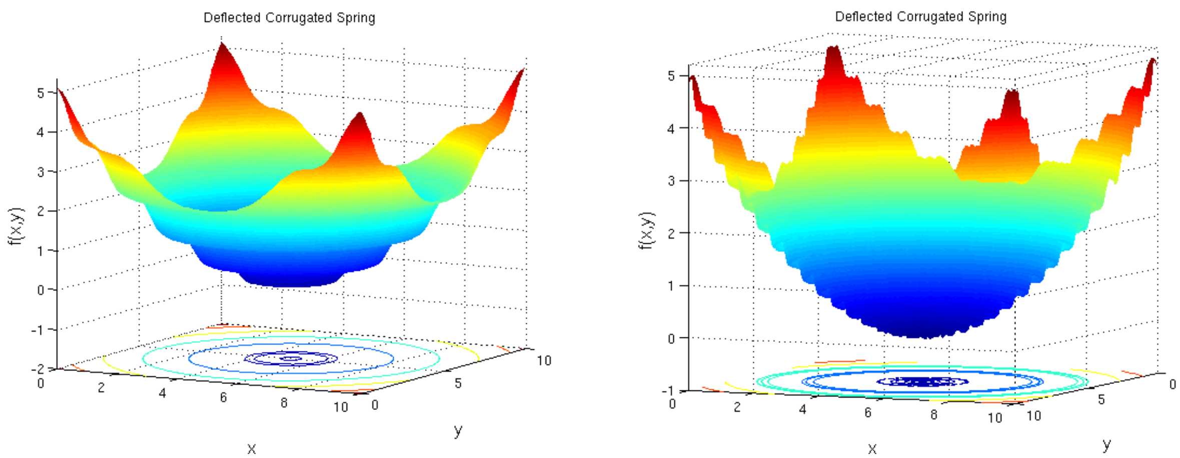
1.36 Deflected Corrugated Spring

The Deflected Corrugated Spring function is defined in two dimensions as:

$$f(\mathbf{x}) = 0.1 \sum_{i=1}^{n} \left[(x_i - \alpha)^2 - \cos\left(K \sqrt{\sum_{i=1}^{n} (x_i - \alpha)^2} \right) \right] \tag{55}$$

where $K = 5$ and $\alpha = 5$, with $x_i \in [0, 2\alpha]$ for which $f(\mathbf{x}^*) = 0$ for $\mathbf{x}^* = (\alpha, \dots, \alpha)$.

The larger the value of K, the more the local minima, due to more levels of the spring function.



(a) $K = 5$ (b) $K = 15$

Figure 38: Deflected Corrugated Spring for $n = 2$

1.37 De Jong F4

The De Jong F4 function is a noisy version of the quartic function, defined as:

$$f(\mathbf{x}) = \sum_{j=1}^{n} (j x_j^4 + N(0,1)) \tag{56}$$

with $x_i \in [-1.28, 1.28]$ for which $f(\mathbf{x}^*) = 0$ for $\mathbf{x}^* = (0, \dots, 0)$.

1.38 DeVilliers-Glasser Functions

The following two DeVilliers-Glasser functions are defined:

- DeVilliers-Glasser 1 in four dimensions:

$$f(\mathbf{x}) = \sum_{i=1}^{24} \left[x_1 x_2^{t_i} \sin(x_3 t_i + x_4) - y_i \right]^2 \tag{57}$$

  where $t_i = 0.1(i-1)$ and $y_i = 60.137 \times 1.371^{t_i} \sin(3.112 t_i + 1.761)$, with $x_i \in [1, 100]$ for which $f(\mathbf{x}^*) = 0$.

- DeVilliers-Glasser 2 in five dimensions:

$$f(\mathbf{x}) = \sum_{i=1}^{16} \left[x_1 x_2^{t_i} \tanh(x_3 t_i + \sin(x_4 t_i)) \cos(t_i e^{x_5}) - y_i \right]^2 \tag{58}$$

  where $t_i = 0.1(i-1)$ and $y_i = 53.81 \times 1.27^{t_i} \tanh(3.012 t_i + \sin(2.13 t_i)) \cos(e^{0.507} t_i)$, with $x_i \in [1, 60]$ for which $f(\mathbf{x}^*) = 0$.

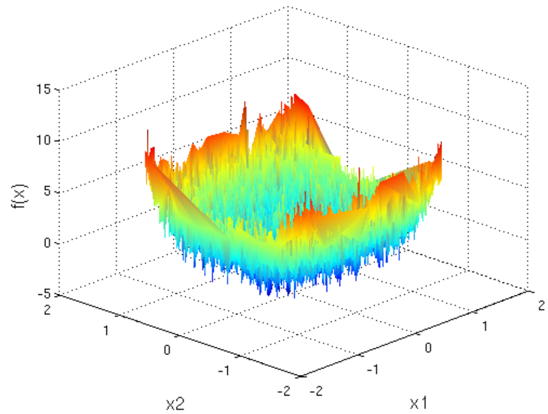
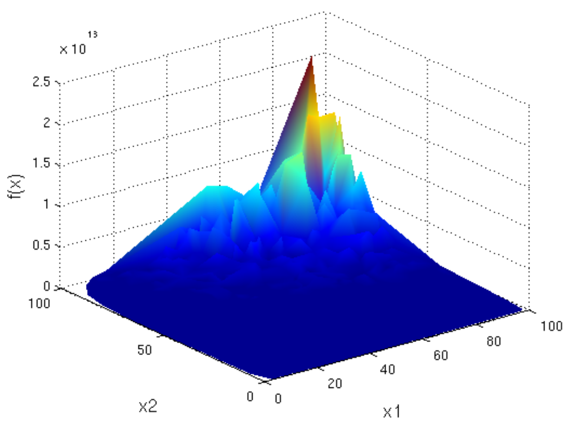
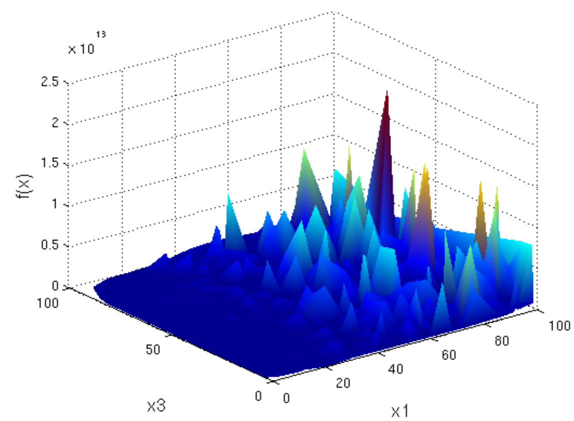


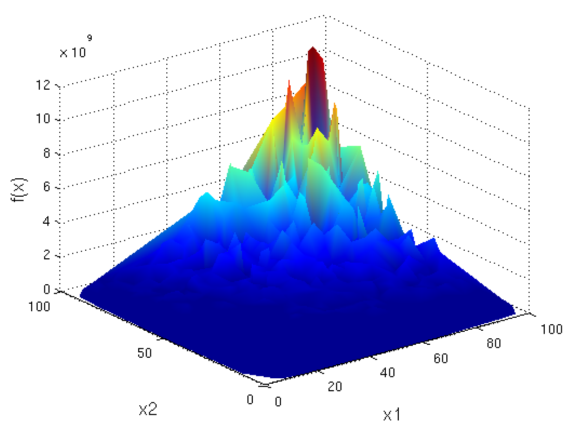
Figure 39: De Jong F4 for $n = 2$



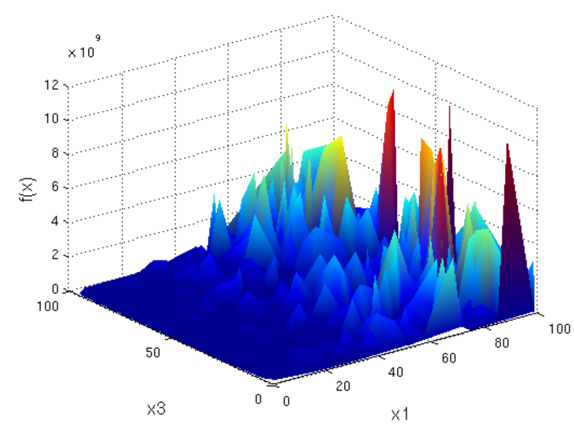
(a) DeVilliers-Glasser 1, $x1$ versus x_2



(b) DeVilliers-Glasser 1, $x1$ versus x_3



(c) DeVilliers-Glasser 2, $x1$ versus x_2



(d) DeVilliers-Glasser 2, $x1$ versus x_3

Figure 40: DeVilliers-Glasser Functions

1.39 Different Powers

The Different Powers function is defined as

$$f(\mathbf{x}) = \sqrt{\sum_{i=1}^{n} |x_i|^{2+4\left(\frac{i-1}{n-1}\right)}} \tag{59}$$

with $x_i \in [-100, 100]$ for which $f(\mathbf{x}^*) = 0$.

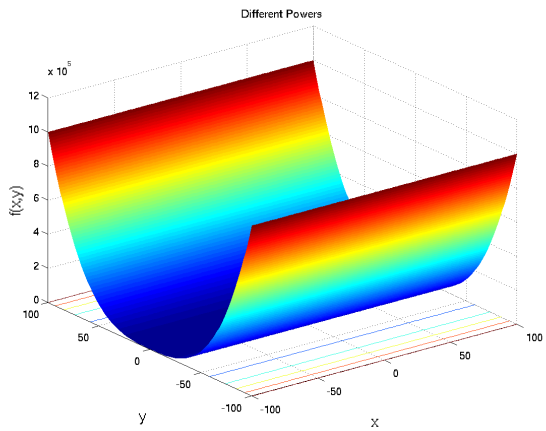


Figure 41: Different Powers for $n = 2$

1.40 Discus

The Discus function is defined as

$$f(\mathbf{x}) = 10^6 x_1^2 + \sum_{i=2}^{n} x_i^2 \tag{60}$$

with $x_i \in [-100, 100]$ for which $f(\mathbf{x}^*) = 0$.

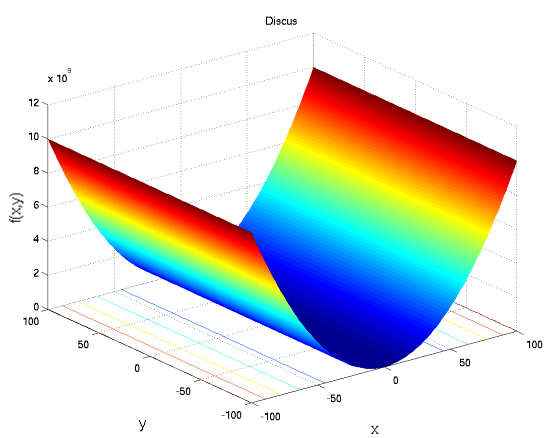


Figure 42: Discus for $n = 2$

1.41 Dixon-Price

The Dixon-Price function is defined as

$$f(\mathbf{x}) = (x_1 - 1)^2 + \sum_{i=1}^{n} i(2x_i^2 - x_{i-1})^2 \tag{61}$$

with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (2^{-(2^i-2)/2^i})$.

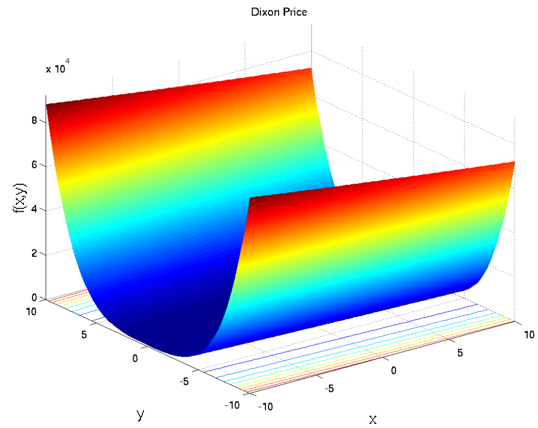


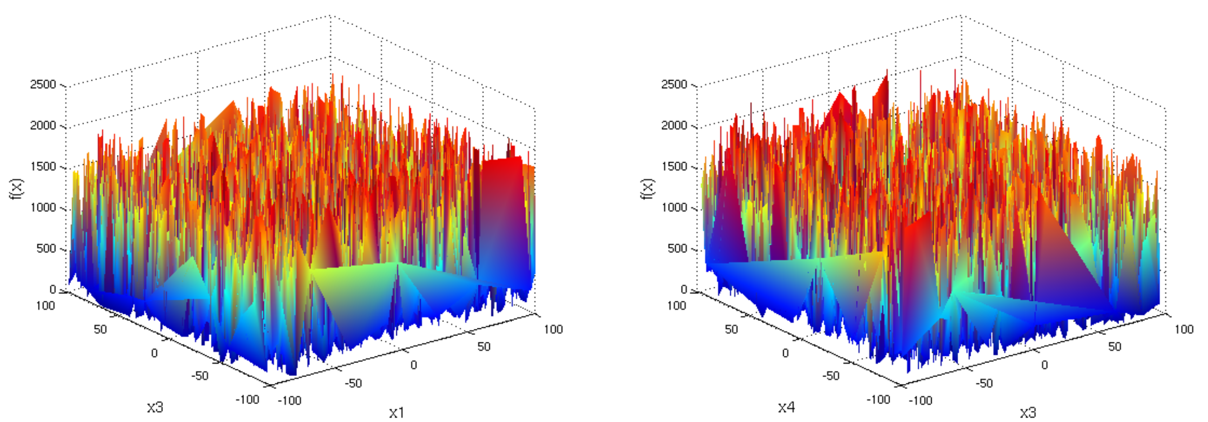
Figure 43: Dixon-Price for $n = 2$

1.42 Dolan

The Dolan function is defined in five dimensions as

$$f(\mathbf{x}) = |(x_1 + 1.7x_2)\sin(x_1) - 1.5x_3 - 0.1x_4\cos(x_4 + x_5 - x_1) + 0.2x_5^2 - x_2 - 1| \tag{62}$$

with $x_i \in [-100, 100]$ for which $f(\mathbf{x}^*) = 10^{-5}$, $\mathbf{x}^* = (8.39045925, 4.81424707, 7.34574133, 68.88246895, 3.85470806)$.



(a) x_1 versus x_3

(b) x_3 versus x_4

Figure 44: Dolan

1.43 Drop Wave

The Drop Wave function is defined as

$$f(\mathbf{x}) = -\frac{1 + \cos(12\sqrt{\sum_{i=1}^{n} x_i^2})}{2 + 0.5\sum_{i=1}^{n} x_i^2} \tag{63}$$

with $x_i \in [-5.12, 5.12]$ for which $f(\mathbf{x}^*) = -1$, $\mathbf{x}^* = (0, \ldots, 0)$.

1.44 Easom

The Easom function, defined in the following two forms:

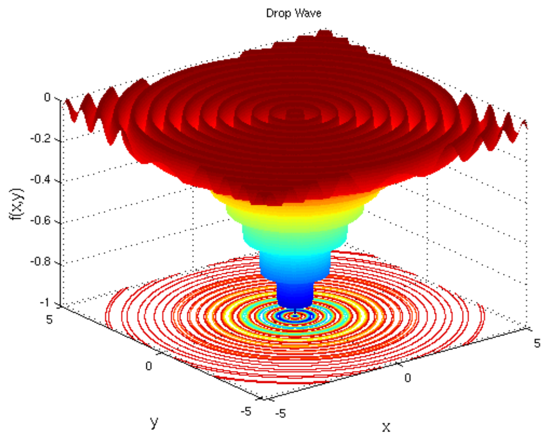


Figure 45: Drop Wave for $n = 2$

- Defined in two dimensions

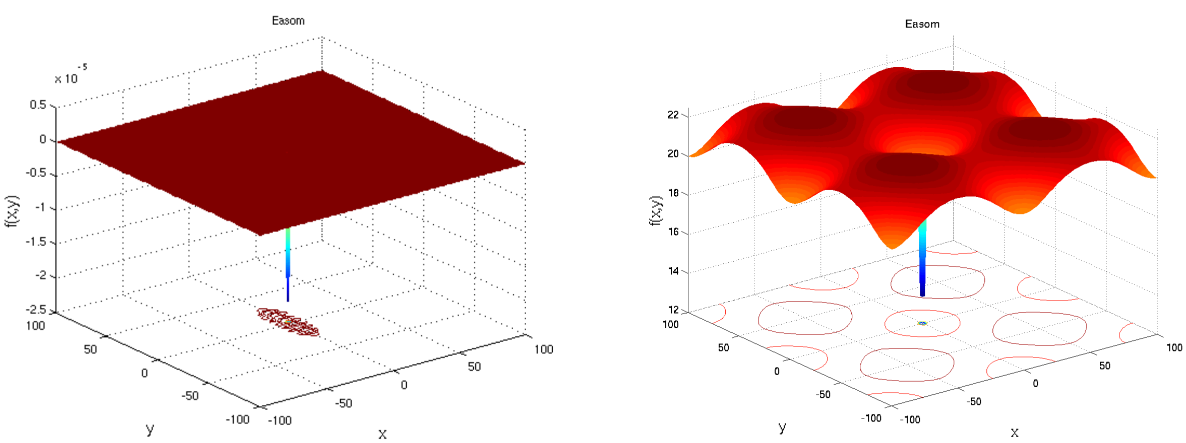
$$f(x_1, x_2) = -\cos(x_1)\cos(x_2)e^{-(x_1-\pi)^2-(x_2-\pi)^2} \tag{64}$$

$x_i \in [-100, 100]$ for which $f(\mathbf{x}^*) = -1$, $\mathbf{x}^* = (\pi, \pi)$. The global minimum has a small area relative to the search space.

- Generalized to any dimension:

$$f(\mathbf{x}) = a - \frac{a}{e^{b\sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n}}}} + e - e^{\frac{\sum_{i=1}^{n}\cos(cx_i)}{n}} \tag{65}$$

with $a = 20, b = 0.2, c = 2\pi$, and $x_i \in [-100, 100]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \dots, 0)$.



(a) 2D Form

(b) Generalized Form

Figure 46: Easom for $n = 2$

1.45 Egg Functions

The following Egg functions

- The Egg Crate function, generalized from its original 2D definition:

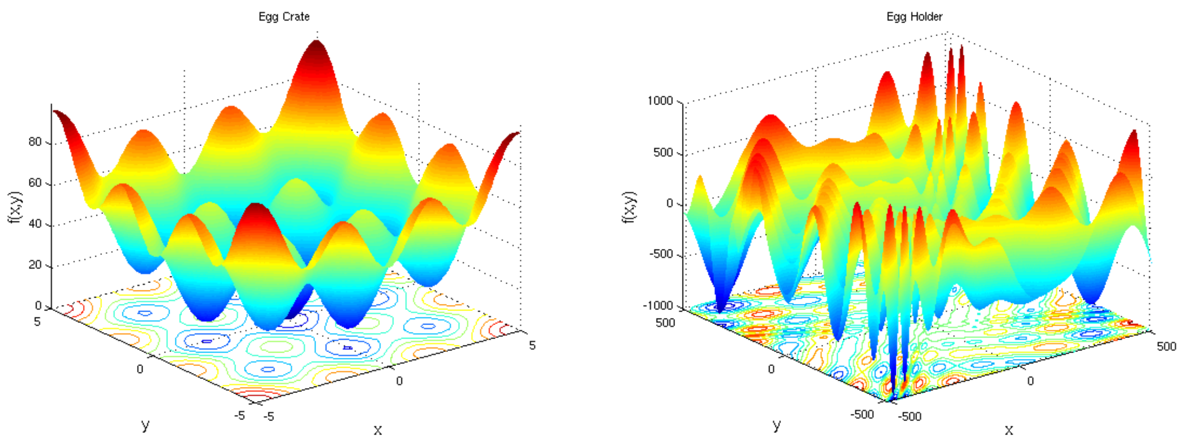
$$f(\mathbf{x}) = \sum_{i=1}^{n} x_i^2 + 24 \sum_{i=1}^{n} \sin(x_i)^2 \tag{66}$$

  with $x_i \in [-5, 5]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.

- Egg Holder function, defined as

$$f(\mathbf{x}) = \sum_{i=1}^{n-1} \left(-(x_{i+1} + 47) \sin(\sqrt{|x_{i+1} + x_i/2 + 47|}) - x_i \sin(\sqrt{|x_i - (x_{i+1} + 47)|}) \right) \tag{67}$$

  with $x_i \in [-512, 512]$ for which $f(\mathbf{x}^*) \approx 959.64$, $\mathbf{x}^* = (512, 404.2319)$ for $n = 2$.



(a) Egg Crate

(b) Egg Holder

Figure 47: Egg Functions for $n = 2$

1.46 El-Attar-Vidyasagar-Dutta

The El-Attar-Vidyasagar-Dutta function is defined in two dimensions as

$$f(\mathbf{x}) = (x_1^2 + x_2 - 10)^2 + (x_1 + x_2^2 - 7)^2 + (x_1^2 + x_2^3)^2 \tag{68}$$

with $x_i \in [-100, 100]$ for which $f(\mathbf{x}^*) = 1.712780354$, $\mathbf{x}^* = (3.40918683, -2.17143304)$.

1.47 Elliptic

The Elliptic (ellipsoidal) function is defined as

$$f(\mathbf{x}) = \sum_{i=1}^{n} (10^6)^{\frac{i-1}{n-1}} x_i^2 \tag{69}$$

with $x_i \in [-100, 100]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$. The function is globally quadratic, and ill-conditioned with smooth local irregularities. The condition number is 10^6.

Here are rotated Ellipse functions, originally defined in two dimensions:

- Rotated Ellipse 1:

$$f(\mathbf{x}) = \sum_{i=1}^{n-1} \left(7x_i^2 - 6\sqrt{3} x_i x_{i+1} + 13x_2^2 \right) \tag{70}$$

  with $x_i \in [-500, 500]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.

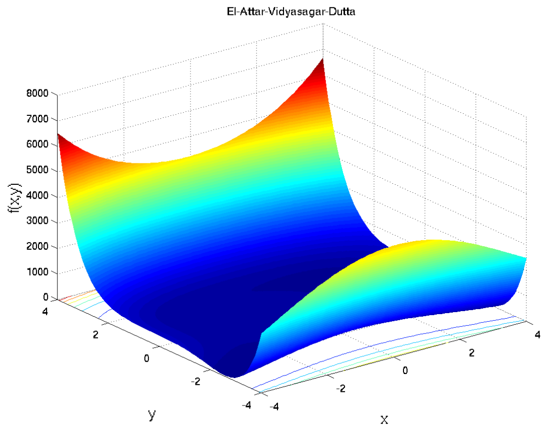


Figure 48: El-Attar-Vidyasagar-Dutta for $n = 2$

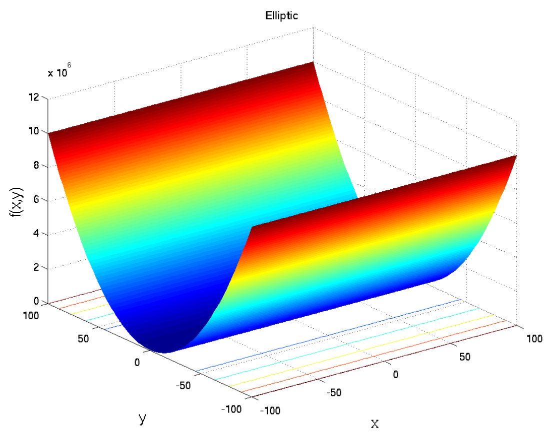


Figure 49: Elliptic for $n = 2$

- Rotated Ellipse 2:

$$f(\mathbf{x}) = \sum_{i=1}^{n-1} \left(x_i^2 - x_i x_{i+1} + x_2^2\right) \tag{71}$$

with $x_i \in [-500, 500]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.

1.48 Exponential

The following exponential functions:

- Exp 1:

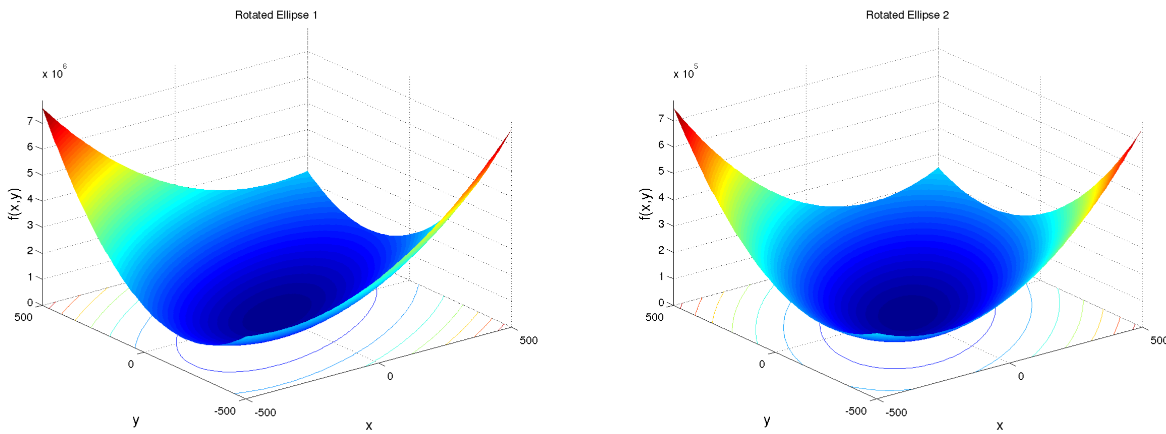
$$f(\mathbf{x}) = -e^{-0.5 \sum_{i=1}^{n} x_i^2} \tag{72}$$

with $x_i \in [-1, 1]$ for which $f(\mathbf{x}^*) = -1$, $\mathbf{x}^* = (0, \ldots, 0)$.

- Exp 2 in two dimensions:

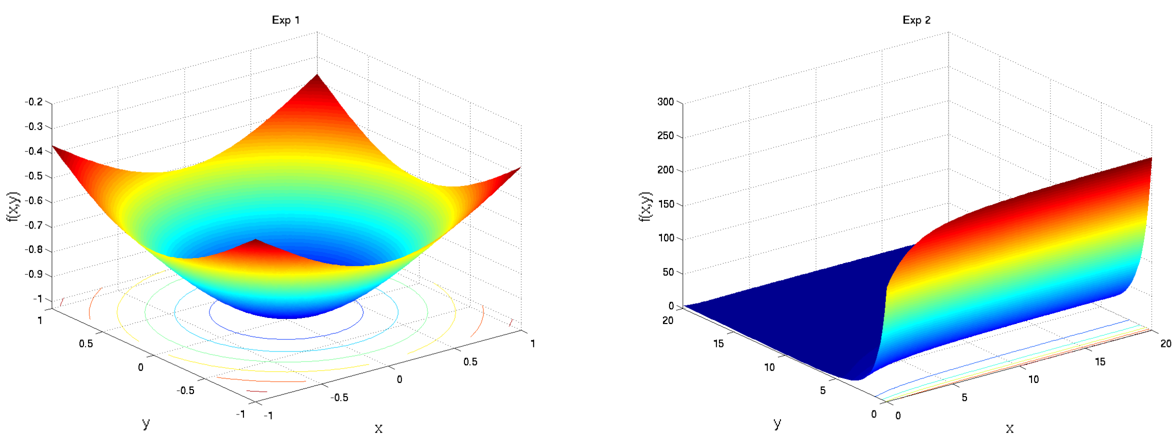
$$f(\mathbf{x}) = \sum_{i=0}^{9} (e^{-ix_1/10} - 5e^{-ix_2/10} - e^{-i/10} + 5e^{-i})^2 \tag{73}$$

with $x_i \in [0, 20]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (1, 0.1)$.



(a) Rotated Ellipse 1 (b) Rotated Ellipse 2

Figure 50: Rotated Ellipse Functions for $n = 2$



(a) Exp 1 (b) Exp 2

Figure 51: Exponential Functions for $n = 2$

1.49 Freudenstein-Roth

The Freudenstein-Roth function is defined in two dimensions as

$$f(\mathbf{x}) = (x_1 - 13 + ((5 - x_2)x_2 - 2)x_2)^2 + (x_1 - 29 + ((x_2 + 1)x_2 - 14)x_2)^2 \tag{74}$$

with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (5, 4)$.

1.50 Gear

The Gear function is defined in four dimensions as

$$f(\mathbf{x}) = \left(\frac{1}{6.931} - \frac{\lfloor x_1 \rfloor \lfloor x_2 \rfloor}{\lfloor x_3 \rfloor \lfloor x_4 \rfloor} \right)^2 \tag{75}$$

with $x_i \in [12, 60]$ for which $f(\mathbf{x}^*) = 2.7 \times 10^{-12}$, $\mathbf{x}^* = (16, 19, 43, 49)$.

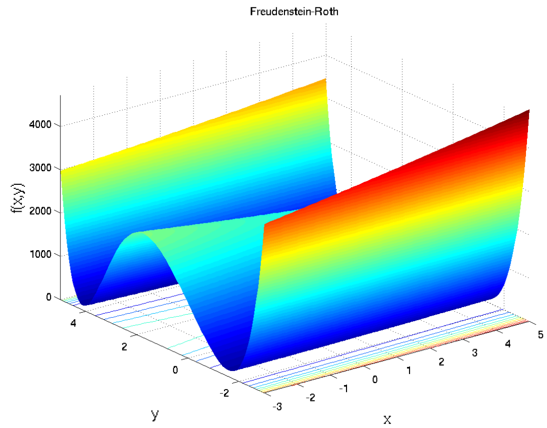
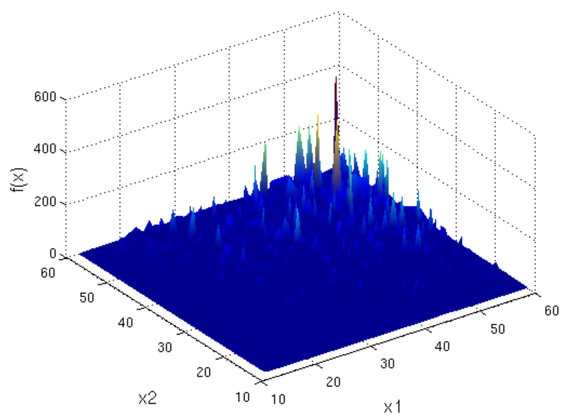
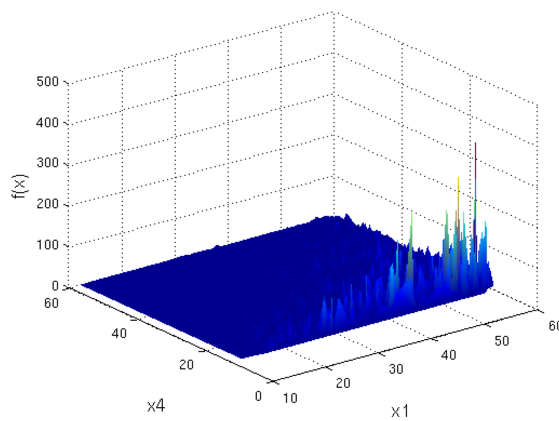


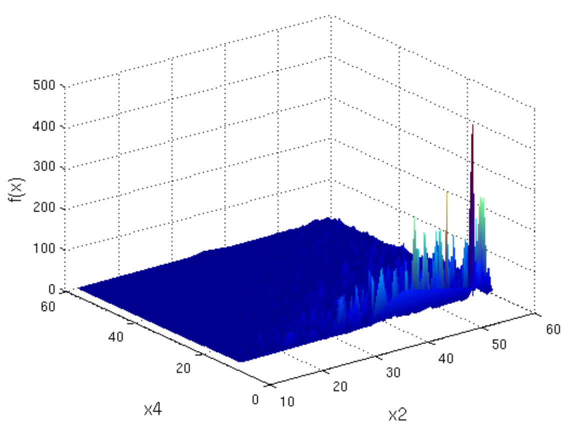
Figure 52: Freudenstein-Roth for $n = 2$



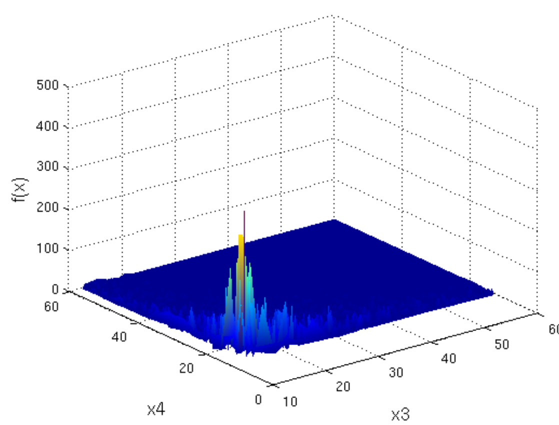
(a) x_1 vs x_2



(b) x_1 vs x_4



(c) x_2 vs x_4



(d) x_3 vs x_4

Figure 53: Gear

1.51 Giunta

The Giunta function is defined as

$$f(\mathbf{x}) = 0.6 + \sum_{i=1}^{2} \left(\sin(\frac{16}{15}x_i - 1) + \sin(\frac{16}{15}x_i - 1)^2 + \frac{1}{50}\sin(4(\frac{16}{15}x_i - 1)) \right) \tag{76}$$

with $x_i \in [-1, 1]$ for which $f(\mathbf{x}^*) = 0.060447$, $\mathbf{x}^* = (0.45834282, 0.45834282)$.

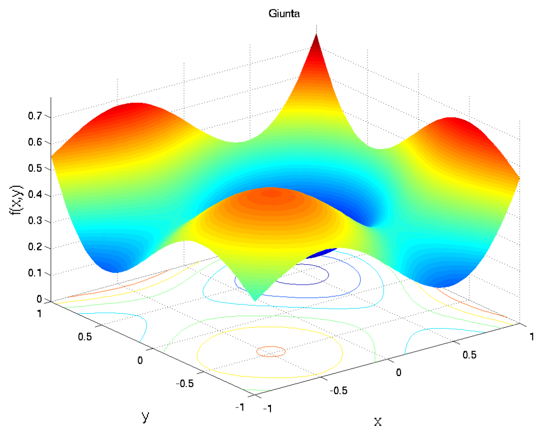


Figure 54: Giunta for $n = 2$

1.52 Goldstein-Price Functions

The Goldstein-Price functions defined in two dimensions

- Goldstein-Price 1:

$$\begin{aligned} f(\mathbf{x}) &= [1 + (x_1 + x_2 + 1)^2(19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2)] \\ &\quad \times[30 + (2x_1 - 3x_2)^2(18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)] \end{aligned} \tag{77}$$

  with $x_i \in [-2, 2]$ for which $f(\mathbf{x}^*) = 3$, $\mathbf{x}^* = (0, -1)$.

  This function has local minima at $(1.2, 0.8)$ of 840, $(1.8, 0.2)$ of 84, and $(-0.6, -0.4)$ of 30.

- Goldstein-Price 2:

$$f(\mathbf{x}) = e^{-0.5(x_1^2+x_2^2-25)^2} + \sin(4x_1 - 4x_2)^4 + 0.5(2x_1 + x_2 - 10)^2 \tag{78}$$

  with $x_i \in [-5, 5]$ for which $f(\mathbf{x}^*) = 1$, $\mathbf{x}^* = (3, 4)$.

1.53 Griewank

The Griewank function is defined as

$$f(\mathbf{x}) = 1 + \frac{1}{4000}\sum_{j=1}^{n} x_j^2 - \prod_{j=1}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right) \tag{79}$$

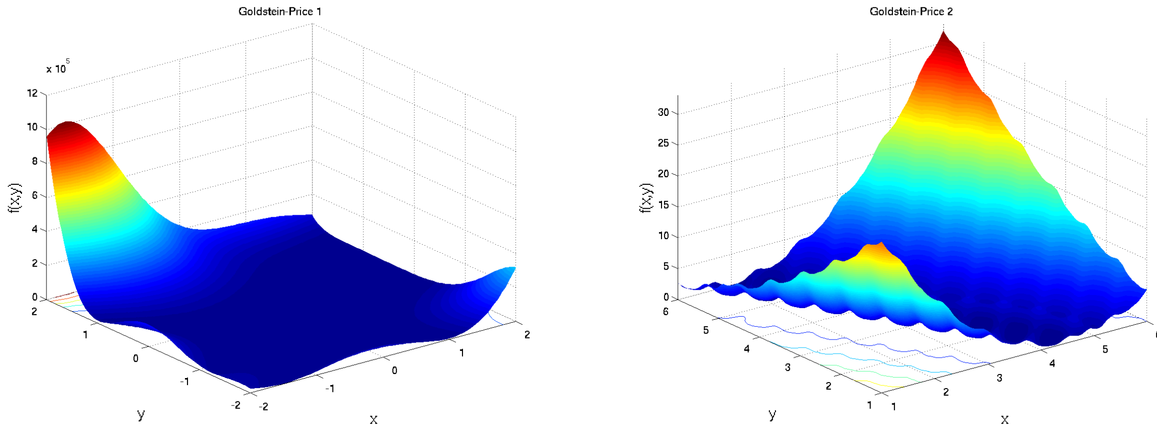
with $x_i \in [-600, 600]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$. Local minima are regularly distributed.

1.54 Hansen

The Hansen is defined in two dimensions as

$$f(\mathbf{x}) = \left[\sum_{i=0}^{4}(i+1)\cos(ix_1 + i + 1)\right]\left[\sum_{j=0}^{4}(j+1)\cos((j+2)x_2 + j + 1)\right] \tag{80}$$

with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = -176.54$, $\mathbf{x}^* = \{(-7.58993, -7.708314), (-7.589893, -1.425128),$ $(-7.589893, 4.858057), (-1.306708, -7.708314), (-1.306708, 4.858057), (4.976478, 4.858057), (4.976478, -1.425128),$ $(4.976478, -7.708314)\}$.



(a) Goldstein-Price 1

(b) Goldstein-Price 2

Figure 55: Goldstein-Price Functions for $n = 2$

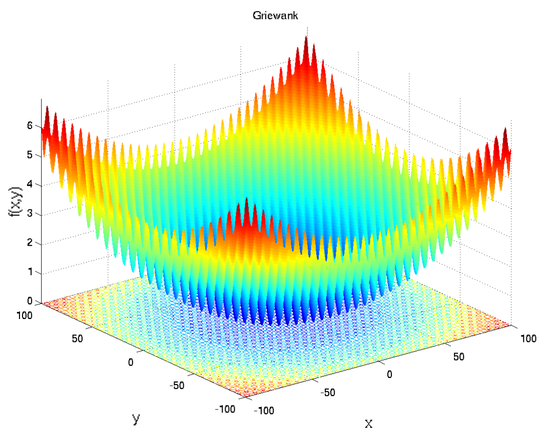


Figure 56: Griewank for $n = 2$

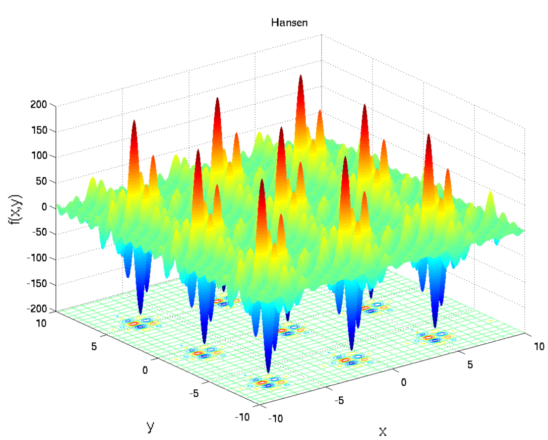


Figure 57: Gulf Research Problem for $n = 2$

1.55 Hartman Functions

The Hartman functions are defined as

- Hartman 3 in three dimensions:

$$f(\mathbf{x}) = -\sum_{i=1}^{4} c_i e^{-\sum_{j=1}^{3} a_{ij}(x_j - p_{ij})^2} \tag{81}$$

where

i	a_{ij}			c_i	p_{ij}		
1	3.0	10.0	30.0	1.0	0.689	0.1170	0.2673
2	0.1	10.0	35.0	1.2	0.4699	0.4387	0.7470
3	3.0	10.0	30.0	3.0	0.1091	0.8732	0.5547
4	0.1	10.0	35.0	3.2	0.0381	0.5743	0.8828

with $x_i \in [0, 1]$ for which $f(\mathbf{x}^*) \approx -3.862782$, $\mathbf{x}^* = (0.1140, 0.556, 0.852)$.

- Hartman 6 in six dimensions:

$$f(\mathbf{x}) = -\sum_{i=1}^{4} c_i e^{-\sum_{j=1}^{6} a_{ij}(x_j - p_{ij})^2} \tag{82}$$

where

i	a_{ij}						c_i
1	10.0	3.0	17.0	3.50	1.70	8.00	1.0
2	0.05	10.0	17.0	0.10	8.00	14.00	1.2
3	3.00	3.50	1.70	10.0	17.00	8.00	3.0
4	17.00	8.00	0.05	10.00	0.10	14.00	3.2

i	p_{ij}					
1	0.1312	0.1696	0.5569	0.0124	0.8283	0.5886
2	0.2329	0.4135	0.8307	0.3736	0.1004	0.9991
3	0.2348	0.1451	0.3522	0.2883	0.3047	0.6650
4	0.4047	0.8828	0.8732	0.5743	0.1091	0.0381

with $x_i \in [-5, 5]$ for which $f(\mathbf{x}^*) \approx -3.32236$, $\mathbf{x}^* = (0.201690, 0.150011, 0.476874, 0.275332, 0.311652, 0.657301)$.

1.56 Helical Valley

The Helical Value is defined in three dimensions:

$$f(\mathbf{x}) = 100[z - 10\theta(x_1, x_2)]^2 + \left(\sqrt{x_1^2 + x_2^2} - 1\right)^2 + x_3^2 \tag{83}$$

where

$$2\pi\theta(x, y) = \begin{cases} \arctan(y/x) & \text{for } x \geq 0 \\ \pi + \arctan(y/x) & \text{for } x < 0 \end{cases}$$

with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (1, 0, 0)$.

1.57 Himmelblau

The Himmelblau function is defined in two dimensions:

$$f(\mathbf{x}) = (x_1^2 + x_2 - 11)^2 + (x_1 + x_2^2 - 7)^2 \tag{84}$$

with $x_i \in [-6, 6]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (3, 2)$.

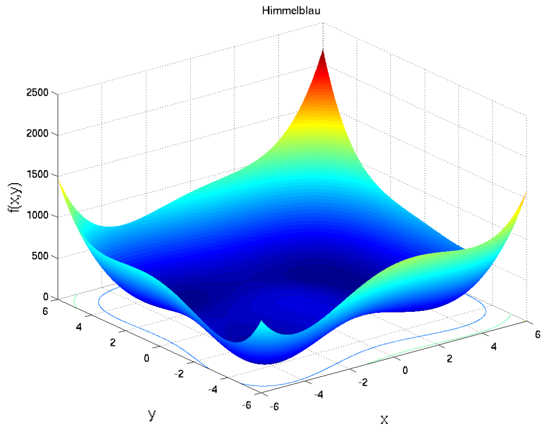


Figure 58: Himmelblau for $n = 2$

1.58 Holzman

The Holzman function is defined in three dimensions as

$$f(\mathbf{x}) = \sum_{i=0}^{99} \left[e^{\frac{(u_i - x_2)^{x_3}}{x_1}} - 0.1(i+1) \right]^2 \tag{85}$$

where

$$u_i = 25 + (-50 \log(0.01(i+1)))^{2/3}$$

with $x_1 \in [0.1, 100], x_2 \in [0, 25.6]$ and $x_3 \in [0, 5]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (50, 25, 1.5)$.

1.59 Hosaki

The Hosaki function is defined in two dimensions as

$$f(\mathbf{x}) = \left(1 - 8x_1 + 7x_1^2 - \frac{7}{3}x_1^3 + \frac{1}{4}x_1^4\right) x_2^2 e^{-x_2} \tag{86}$$

where with $x_i \in [0, 10]$ for which $f(\mathbf{x}^*) \approx -2.3458$, $\mathbf{x}^* = (4, 2)$.

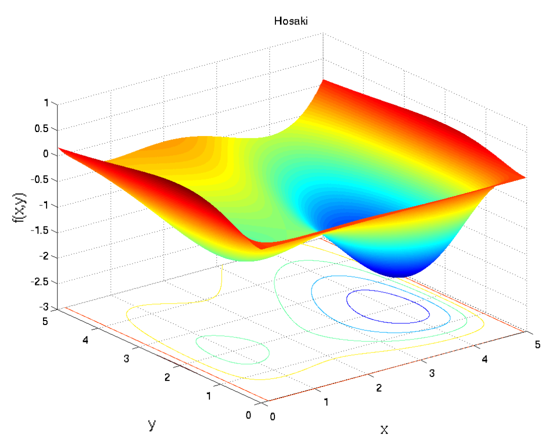


Figure 59: Hosaki for $n = 2$

1.60 Hyperellipsoid

The Hyperellipsoid (weighted Sphere) function is defined as

$$f(\mathbf{x}) = \sum_{i=1}^{n} i x_i^2 \tag{87}$$

with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.

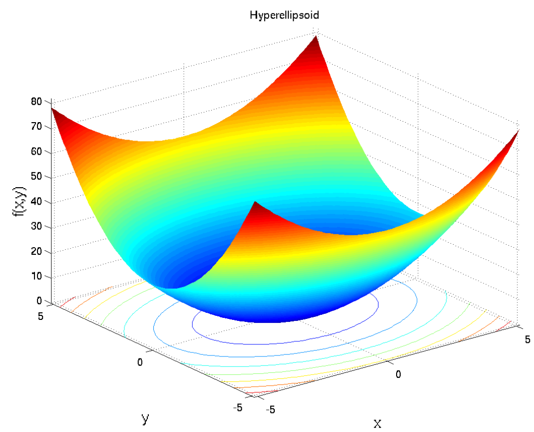


Figure 60: Hyperellipsoid for $n = 2$

The rotated hyperellipsoid is defined as

$$f(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{i} x_j^2 \tag{88}$$

with $x_i \in [-65.536, 65.536]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$. The function is convex.

1.61 Infinity

The Infinity function is defined as

$$f(\mathbf{x}) = \sum_{i=1}^{n} x_i^6 (\sin(\frac{1}{x_i}) + 2) \tag{89}$$

where with $x_i \in [-1, 1]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.

1.62 Jennrich-Sampson

The Jennrich-Sampson function is defined in two dimensions as

$$f(\mathbf{x}) = \sum_{i=1}^{10} (2 + 2i - (e^{ix_1} e^{ix_2}))^2 \tag{90}$$

with $x_i \in [-1, 1]$ for which $f(\mathbf{x}^*) = 124.3612$, $\mathbf{x}^* = (0.257825, 0.257825)$.

1.63 Judge

The Judge function is defined in two dimensions as

$$f(\mathbf{x}) = \sum_{i=1}^{20} \left[(x_1 + A_i x_2 + B_i x_2^2) - C_i \right] \tag{91}$$

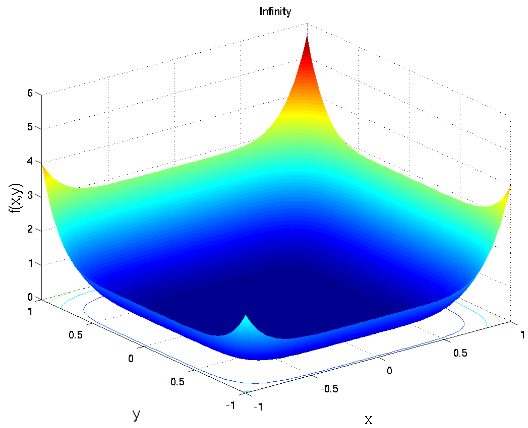
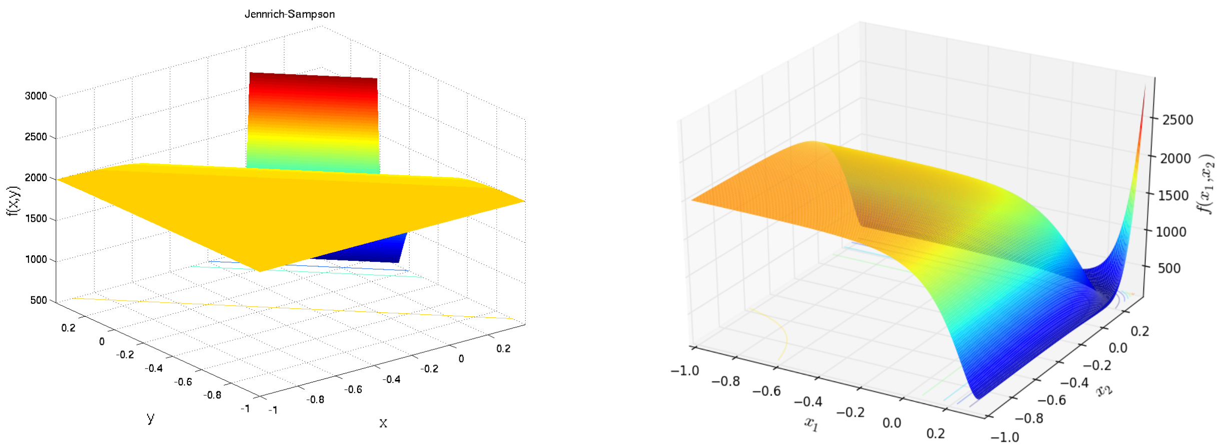


Figure 61: Infinity for $n = 2$



(a) According to my Matlab code

(b) According to infinity77.net

Figure 62: Jennrich-Sampson for $n = 2$

where

$A = [4.284, 4.149, 3.877, 0.533, 2.211, 2.389, 2.145, 3.231, 1.998, 1.379, 2.106, 1.428, 1.011, 2.179, 2.858, 1.388, 1.651, 1.593, 1.046, 2.152]$

$B = [0.286, 0.973, 0.384, 0.276, 0.973, 0.543, 0.957, 0.948, 0.543, 0.797, 0.936, 0.889, 0.006, 0.828, 0.399, 0.617, 0.939, 0.784, 0.072, 0.889]$

$C = [0.645, 0.585, 0.310, 0.058, 0.455, 0.779, 0.259, 0.202, 0.028, 0.099, 0.142, 0.296, 0.175, 0.180, 0.842, 0.039, 0.103, 0.620, 0.158, 0.704]$

with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = 16.0817307$, $\mathbf{x}^* = (0.86479, 1.2357)$.

1.64 Katsuura

The Katsuura function is defined as

$$f(\mathbf{x}) = \prod_{i=0}^{n-1} \left[1 + (i+1) \sum_{k=1}^{d} \lfloor (2^k x_i) \rfloor 2^{-k} \right] \tag{92}$$

where $d = 32$, and with $x_i \in [0, 100]$ for which $f(\mathbf{x}^*) = 1$, $\mathbf{x}^* = (0, \ldots, 0)$.

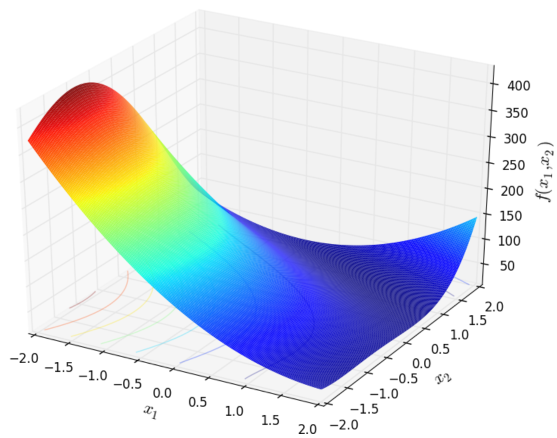


Figure 63: Judge for $n = 2$

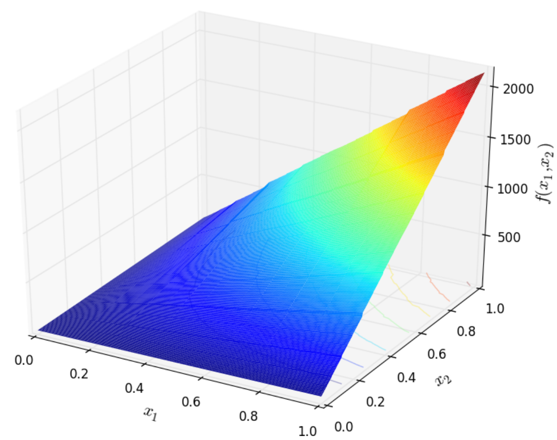


Figure 64: Katsuura for $n = 2$

1.65 Keane

The Keane function is defined in two dimensions as

$$f(\mathbf{x}) = \frac{\sin^(x_1 - x_2)\sin^(x_1 + x_2)^2}{\sqrt{x_1^2 + x_2^2}} \tag{93}$$

with $x_i \in [0, 10]$ for which $f(\mathbf{x}^*) = 0.673668$, $\mathbf{x}^* = (0.192833, 0.190836, 0.123117, 0.135766)$.

1.66 Kowalik

The Kowalik function is defined in four dimensions as

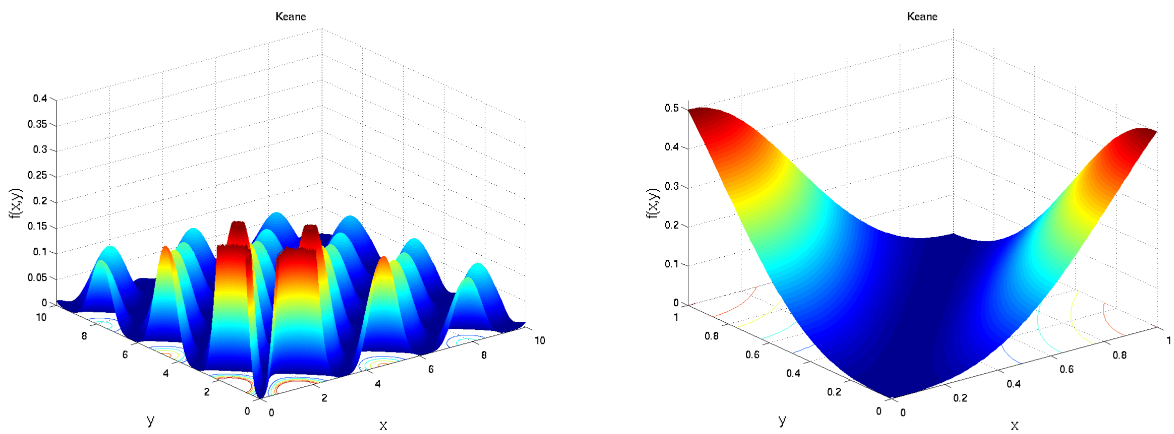
$$f(\mathbf{x}) = \sum_{i=0}^{10} \left[a_i - \frac{x_1(b_i^2 + b_i x_2)}{b_i^2 + b_i x_3 + x_4} \right]^2 \tag{94}$$

where

$$\mathbf{a} = [4, 2, 1, 1/2, 1/41/8, 1/10, 1/12, 1/14, 1/16]$$

$$\mathbf{b} = [0.1957, 0.1947, 0.1735, 0.1600, 0.0844, 0.0627, 0.0456, 0.0342, 0.0323, 0.0235, 0.0246]$$

with $x_i \in [-5, 5]$ for which $f(\mathbf{x}^*) = 0.0003074861$, $\mathbf{x}^* = (0.192833, 0.190836, 0.123117, 0.135766)$.



(a) Full domain

(b) Smaller domain

Figure 65: Keane for $n = 2$

1.67 Langermann 5

The Langermann 5 function is defined as

$$f(\mathbf{x}) = -\sum_{i=1}^{m} c_i e^{-\frac{1}{\pi}\sum_{j=1}^{n}(x_j - a_{ij})^2} \cos\left(\pi \sum_{j=1}^{n}(x_j - a_{ij})^2\right) \tag{95}$$

where

$$A = [a_{ij}] = \begin{bmatrix} 9.681 & 0.667 & 4.783 & 9.095 & 3.517 & 9.325 & 6.544 & 0.211 & 5.122 & 2.020 \\ 9.400 & 2.041 & 3.788 & 7.931 & 2.882 & 2.672 & 3.568 & 1.284 & 7.033 & 7.374 \\ 8.025 & 9.152 & 5.114 & 7.621 & 4.564 & 4.711 & 2.996 & 6.126 & 0.734 & 4.982 \\ 2.196 & 0.415 & 5.649 & 6.979 & 9.510 & 9.166 & 6.304 & 6.054 & 9.377 & 1.426 \\ 8.074 & 8.777 & 3.467 & 1.863 & 6.708 & 6.349 & 4.534 & 0.276 & 7.633 & 1.567 \end{bmatrix}$$

and

$$\mathbf{c} = [c_i]^T = \begin{bmatrix} 0.806 \\ 0.517 \\ 1.500 \\ 0.908 \\ 0.965 \end{bmatrix}$$

with $m = 5$, $x_i \in [0, 10]$ for which $f(\mathbf{x}^*) = -5.1621259$, $\mathbf{x}^* = (2.00299219, 1.006096)$ for $n = 2$. Local minima are unevenly distributed.

1.68 Lanczos Functions

The Lanczos functions, derived from Lanczos interpolation, are defined as

- Lanczos 1:

$$f(\mathbf{x}) = \sum_{i=1}^{n} \mathrm{sinc}(x_i)\mathrm{sinc}(x_i/k) \tag{96}$$

where $\mathrm{sinc}(x) = \frac{sin(x)}{x}$. For larger k, more local minima are produced. For $x_i \in [-20, 20]$, $f(\mathbf{x}^*) = 1(?)$, and $\mathbf{x}^* = (0, \ldots, 0)$.

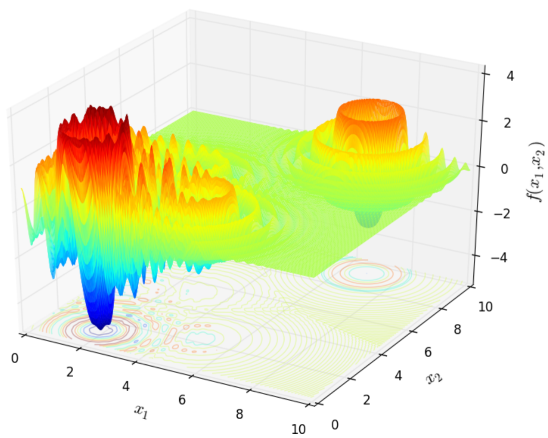


Figure 66: Langermann 5 for $n = 2$

- Lanczos 2:
$$f(\mathbf{x}) = \sum_{i=1}^{n-1} \text{sinc}(x_i)\text{sinc}(x_{i+1}/k) + \text{sinc}(x_{i+1})\text{sinc}(x_i/k) \tag{97}$$
where $\text{sinc}(x) = \frac{sin(x)}{x}$. For $x_i \in [-20, 20]$, $f(\mathbf{x}^*) =?$, and $\mathbf{x}^* = (0, \dots, 0)$.

- Lanczos 3:
$$f(\mathbf{x}) = \prod_{i=1}^{n} \text{sinc}(x_i)\text{sinc}(x_i/k) \tag{98}$$
where $\text{sinc}(x) = \frac{sin(x)}{x}$. For $x_i \in [-20, 20]$, $f(\mathbf{x}^*) = 1(?)$, and $\mathbf{x}^* = (?, \dots, ?)$.

1.69 Lennard-Jones

The Lennard-Jones function is defined as

$$f(\mathbf{x}) = \sum_{i=0}^{n-2} \sum_{j=2}^{n-1} \left(\frac{1}{r_{ij}^{12}} - \frac{1}{r_{ij}^{6}} \right) \tag{99}$$

where

$$r_{ij} = \sqrt{(x_{3i} - x_{rj})^2 + (x_{3i+1} + x_{3j+1})^2 + (x_{3i+2} - x_{3j+2})^2}$$

which is valid for dimensions $n = 3k, k = 2, 3, 4, \dots, 20$. with $x_i \in [-4, 4]$ for which $f(\mathbf{x}^*) = minima[k-2] + 0.0001$, and

$$minima = [-1., -3., -6., -9.103852, -12.712062, -16.505384, -19.821489, -24.113360, -28.422532, -32.765970, -37.967600,$$
$$-44.326801, -47.845157, -52.322627, -56.815742, -61.317995, -66.530949, -72.659782, -77.1777043]$$

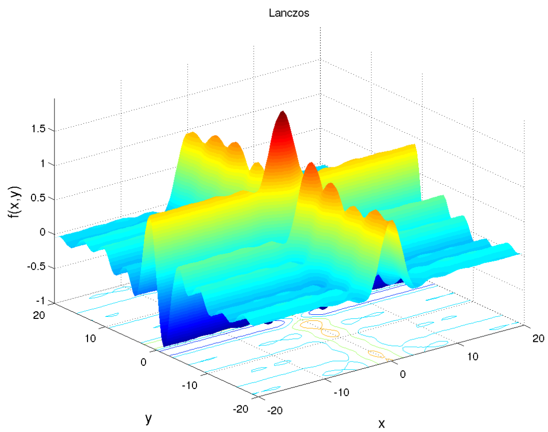
The function has one global minimum.

1.70 Leon

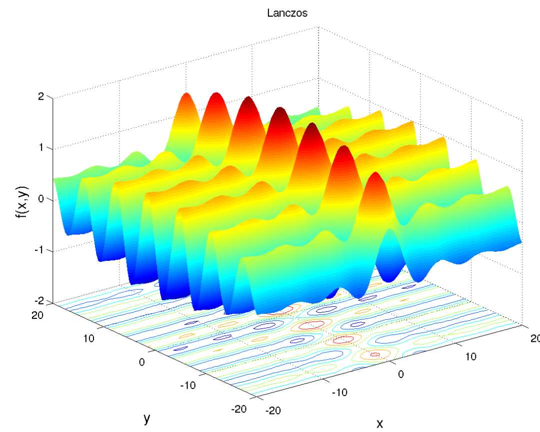
The Leon (2D Rosenbrock) function is defined in two dimensions as

$$f(\mathbf{x}) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 \tag{100}$$

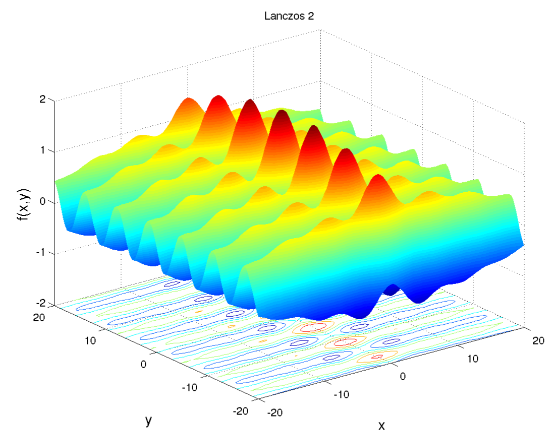
with $x_i \in [-1.2, 1.2]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (1, 1)$.



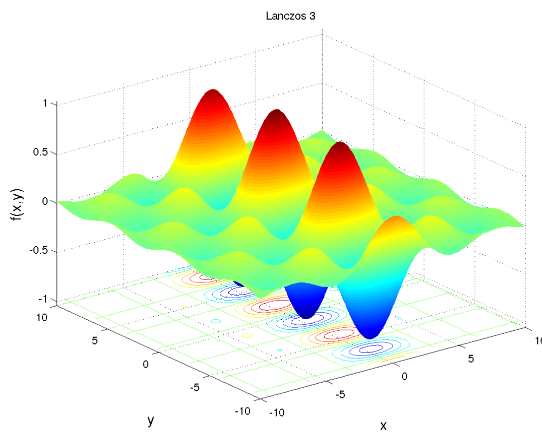
(a) Lanczos 1, $k = 2$



(b) Lanczos 1, $k = 10$



(c) Lanczos 2, $k = 10$



(d) Lanczos C, $k = 10$

Figure 67: Lanczos Functions for $n = 2$

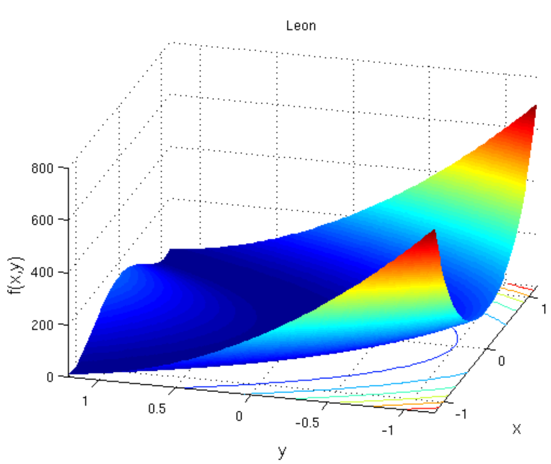


Figure 68: Leon for $n = 2$

1.71 Levy Functions

The following Levy functions:

- Levy 3:

$$f(\mathbf{x}) = \sin^2(\pi y_1) + \sum_{i=1}^{n-1}(y_i - 1)^2[1 + 10\sin^2(\pi y_{i+1})] + (y_n - 1)^2 \tag{101}$$

where

$$y_i = 1 + \frac{x_i - 1}{4}$$

with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (1, \dots, 1)$.

- Levy 5 defined in two dimensions:

$$f(\mathbf{x}) = \sum_{i=1}^{5} i\cos\left[(i-1)x_1 + i\right] \times \sum_{j=1}^{5} j\cos\left[(j+1)x_2 + j\right] + (x_1 + 1.42513)^2 + (x_2 + 0.80032)^2 \tag{102}$$

with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = -176.1375$, $\mathbf{x}^* = (-1.3068, -1.4248)$.

- Levy 13 defined in two dimensions:

$$f(\mathbf{x}) = (x_1 - 1)^2\left[\sin^2(3\pi x_2) + 1\right] + (x_2 - 1)^2\left[\sin^2(2\pi x_2) + 1\right] + \sin^2(3\pi x_1) \tag{103}$$

with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (1, \dots, 1)$.

- Levy-Montalvo 2 (generalization of Levy 13):

$$f(\mathbf{x}) = 0.1\left(\sin(3\pi x_1)^2 + \sum_{i=1}^{n-1}(x_i - 1)^2\left[\sin^2(3\pi x_{i+1}) + 1\right] + (x_n - 1)^2\left[\sin^2(2\pi x_n) + 1\right]\right) \tag{104}$$

with $x_i \in [-5, 5]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (1, \dots, 1)$.

1.72 Matyas

The Matyas function is generalized from its two dimensional definition:

$$f(\mathbf{x}) = 0.26(x_1^2 + x_2^2) - 0.48x_1x_2 \tag{105}$$

with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \dots, 0)$.

1.73 McCormick

The McCormick function is defined in two dimensions as:

$$f(\mathbf{x}) = \sin(x_1 + x_2) + (x_1 - x_2)^2 - 1.5x_1 + 2.5x_2 + 1 \tag{106}$$

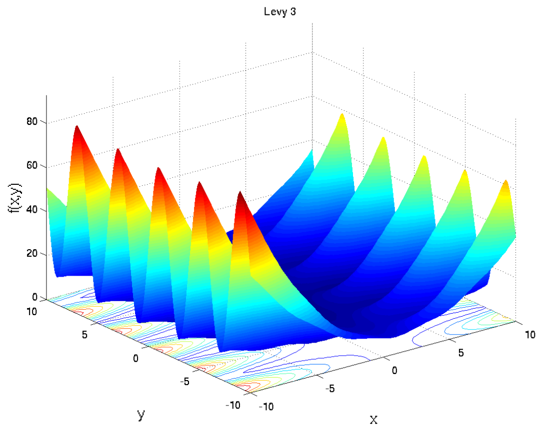
with $x_1 \in [-1.5, 1.5]$ and $x_2 \in [-3, 4]$ for which $f(\mathbf{x}^*) \approx -1.9133$, $\mathbf{x}^* = (-0.547, -1.547)$.

1.74 Michalewicz

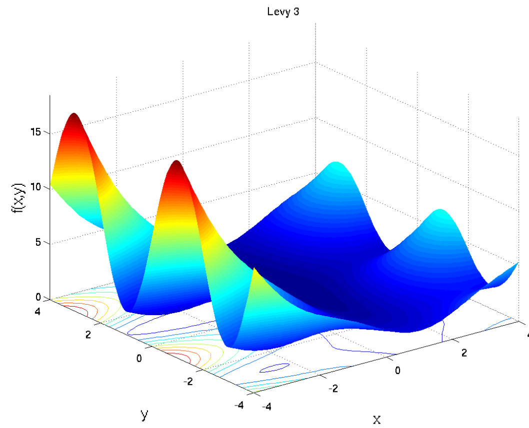
The Michalewicz function is defined as:

$$f(\mathbf{x}) = -\sum_{i=1}^{n}\sin(x_i)\left(\sin\left(\frac{ix_i^2}{\pi}\right)\right)^{2m} \tag{107}$$

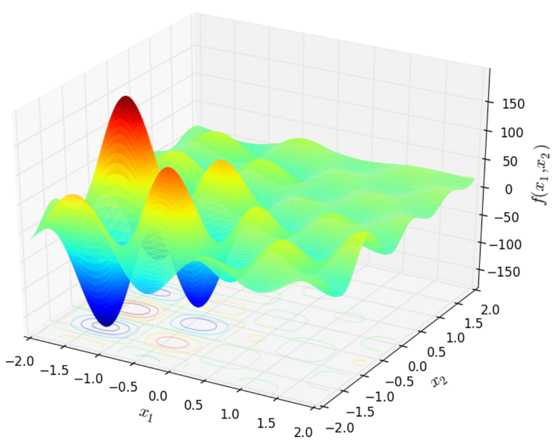
where $m = 10$ and with $x_i \in [0, \pi]$ for which $f(\mathbf{x}^*) = -0.966n$, $\mathbf{x}^* = (0, \dots, 0)$. The function has $n!$ local minima. The parameter m defines the steepness of the valleys or edges. Larger m leads to more difficult search. For very large m, the function behaves like a needle in the haystack, with the functions values in the space outside the narrow peaks giving little information on the location of the global minimum.



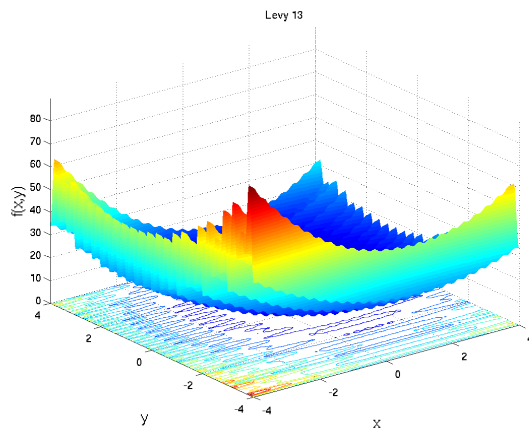
(a) Levy 3 large domain



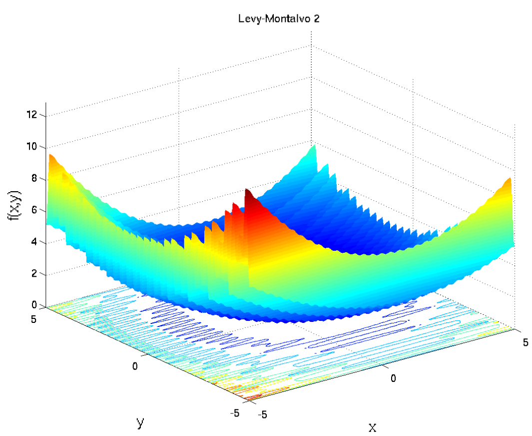
(b) Levy 3 smaller domain



(c) Levy 5



(d) Levy 13



(e) Levy-Montalvo 2

Figure 69: Levy for $n = 2$

1.75 Miele-Cantrell

The Miele-Cantrell function is defined in four dimensions as:

$$f(\mathbf{x}) = (e^{-x_1} - x_2)^4 + 100(x_2 - x_3)^6 + (\tan(x_3 - x_4))^4 + x_1^8 \tag{108}$$

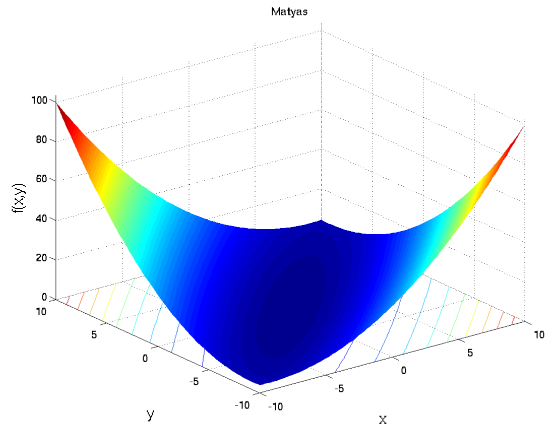


Figure 70: Matyas for $n = 2$

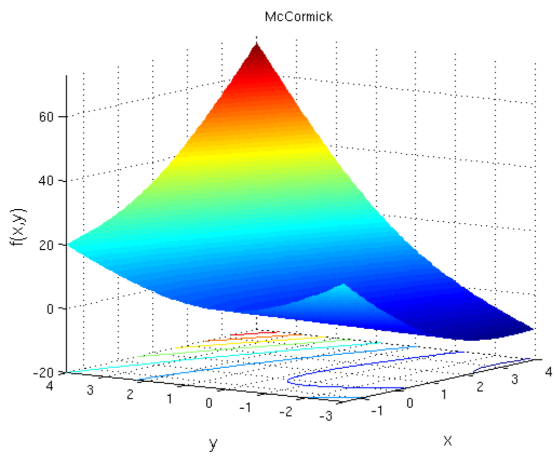


Figure 71: McCormick for $n = 2$

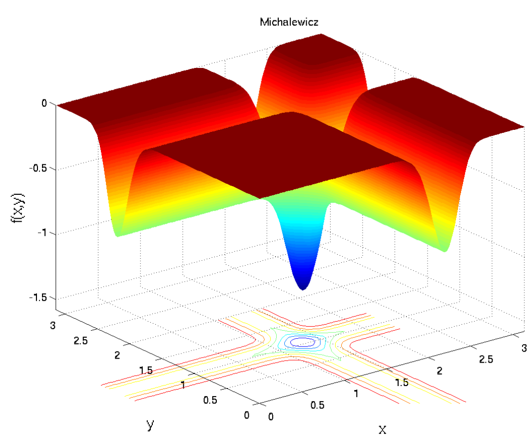


Figure 72: Michalewicz for $n = 2$

with $x_i \in [-1, 1]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, 1, 1, 1)$.

1.76 Mishra Functions

The following Mishra functions:

- Mishra 1:

$$f(\mathbf{x}) = \left(1 + n - \sum_{i=1}^{n-1} x_i\right)^{(n - \sum_{i=1}^{n-1} x_i)} \tag{109}$$

  with $x_i \in [0, 1]$ for which $f(\mathbf{x}^*) = 2$, $\mathbf{x}^* = (1, \ldots, 1)$.

- Mishra 2:

$$f(\mathbf{x}) = \left(1 + n - \sum_{i=1}^{n-1} 0.5(x_i + x_{i+1})\right)^{(n - \sum_{i=1}^{n-1} 0.5*(x_i - x_{i+1})} \tag{110}$$

  with $x_i \in [0, 1]$ for which $f(\mathbf{x}^*) = 2$, $\mathbf{x}^* = (1, \ldots, 1)$.

- Mishra 3, generalized from its original two-dimensional version:

$$f(\mathbf{x}) = \sqrt{\left|\cos\sqrt{|\sum_{i=1}^{n} x_i^2|}\right|} + 0.01 \sum_{i=1}^{n} x_i \tag{111}$$

  with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = -0.18467$, $\mathbf{x}^* = (-8.466, -10)$ for $n = 2$

- Mishra 4, generalized from its original two-dimensional version:

$$f(\mathbf{x}) = \sqrt{\left|\sin\sqrt{|\sum_{i=1}^{n} x_i^2|}\right|} + 0.01 \sum_{i=1}^{n} x_i \tag{112}$$

  with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = -0.18467$, $\mathbf{x}^* = (-9.94112, -10)$ for $n = 2$

- Mishra 5, in two dimensions:

$$f(\mathbf{x}) = \left[\sin^2((\cos(x_1) + \cos(x_2))^2) + \cos^2((\sin(x_1) + \sin(x_2))^2) + x_1\right]^2 + 0.01(x_1 + x_2) \tag{113}$$

  with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = -1.01983$, $\mathbf{x}^* = (-1.98682, -10)$ for $n = 2$

- Mishra 5 modified, in two dimensions:

$$f(\mathbf{x}) = \left[\sin^2((\cos(x_1) + \cos(x_2))) + \cos^2((\sin(x_1) + \sin(x_2))^2) + x_1\right]^2 + 0.01(x_1 + x_2) \tag{114}$$

  with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = -1.01983$, $\mathbf{x}^* = (-1.98682, -10)$.

- Mishra 6, in two dimensions:

$$f(\mathbf{x}) = -\log\left[\sin^2((\cos(x_1) + \cos(x_2))^2) - \cos^2((\sin(x_1) + \sin(x_2))^2) + x_1\right]^2 + 0.01\left[(x_1 - 1)^2 + (x_2 - 1)^2\right] \tag{115}$$

  with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = -2.28395$, $\mathbf{x}^* = (2.88631, 1.82326)$.

- Mishra 7:

$$f(\mathbf{x}) = \left[\prod_{i=1}^{n} x_i - n!\right]^2 \tag{116}$$

  with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (\sqrt{n}, \ldots, \sqrt{n})$.

- Mishra 8 in two dimensions:

$$\begin{aligned} f(\mathbf{x}) &= 0.001\left[|x_1^{10} - 20x_1^9 + 180x_1^8 - 960x_1^7 + 3360x_1^6 - 8064x_1^5 + 13340x_1^4 - 15360x_1^3 + 11520x_1^2 - 5120x_1 + 2624|\right. \\ &\times \left. |x_2^4 + 12x_2^3 + 54x_2^2 + 108x_2 + 81|\right]^2 \end{aligned} \tag{117}$$

  with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (2, -3)$.

- Mishra 9 in three dimensions:

$$f(\mathbf{x}) = \left[ab^2c + abc^2 + b^2 + (x_1 + x_2 - x_3)^2\right]^2 \tag{118}$$

  where

$$a = 2x_1^3 + 5x_1x_2^2 + 4x_3 - 2x_1^2x_3 - 18$$
$$b = x_1 + x_2^3 + x_1x_3^2 - 22$$
$$c = 8x_1^2 + 2x_2x_3 + 2x_2^2 + 3x_2^3 - 52$$

  with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (1, 2, 3)\}$.

- Mishra 10 in two dimensions:

$$f(\mathbf{x}) = [\lfloor x_1 \perp x_2 \rfloor - \lfloor x_1 \rfloor - \lfloor x_2 \rfloor]^2 \tag{119}$$

  with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = \{(0, 0), (2, 2)\}$.

- Mishra 11:

$$f(\mathbf{x}) = \left[\frac{1}{n}\sum_{i=1}^{n}|x_i| - \left(\prod_{i=1}^{n}|x_i|\right)^{\frac{1}{n}}\right]^2 \tag{120}$$

  with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.

1.77 Multi Gaussian

The Multi Gaussian function is defined in two dimensions as:

$$f(\mathbf{x}) = \sum_{i=1}^{5} a_i e^{(-((x_1-b_i)^2+(x_2-c_i)^2))/d_i^2} \tag{121}$$

where

$$\begin{aligned} \mathbf{a} &= (0.5, 1.2, 1.0, 1.0, 1.2) \\ \mathbf{b} &= (0.0, 1.0, 0.0, -0.5, 0.0) \\ \mathbf{c} &= (0.0, 0.0, -0.5, 0.0, 1.0) \\ \mathbf{d} &= (0.1, 0.5, 0.5, 0.5, 0.5) \end{aligned}$$

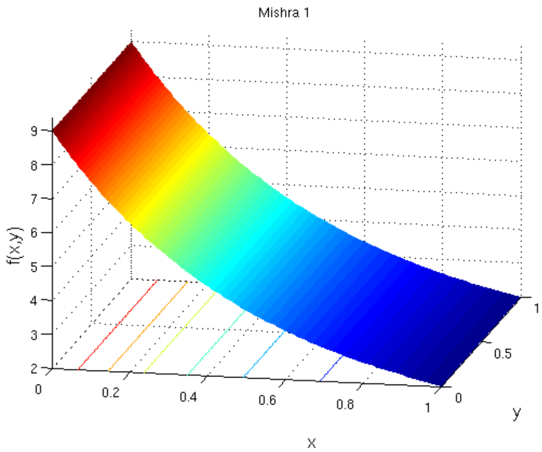
with $x_i \in [-2, 2]$ for which $f(\mathbf{x}^*) \approx 1.29695$, $\mathbf{x}^* = (-0.01356, -0.01356)$.

1.78 Multi Modal

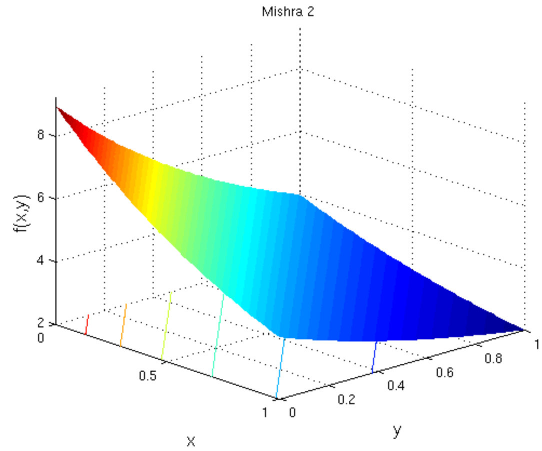
The Multi Modal function is defined as:

$$f(\mathbf{x}) = \left(\sum_{i=1}^{n}|x_i|\right)\left(\prod_{i=1}^{n}|x_i|\right) \tag{122}$$

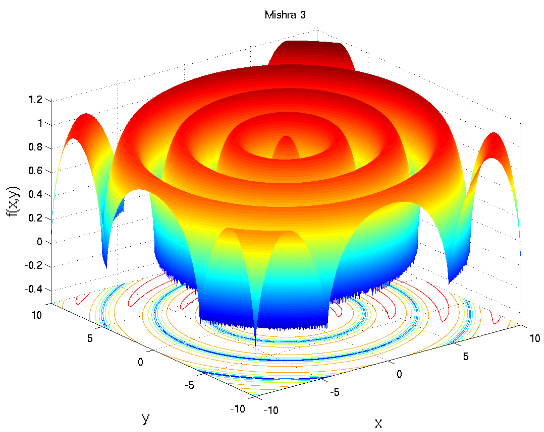
with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.



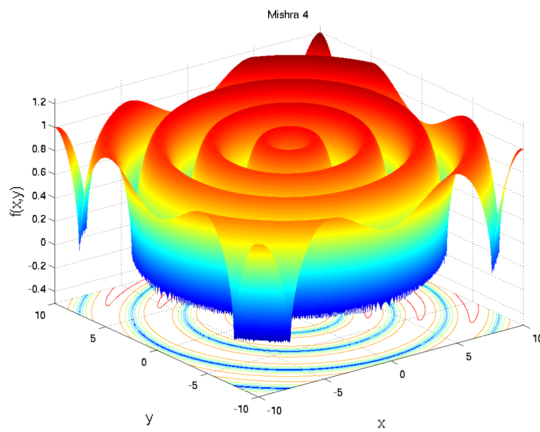
(a) Mishra 1



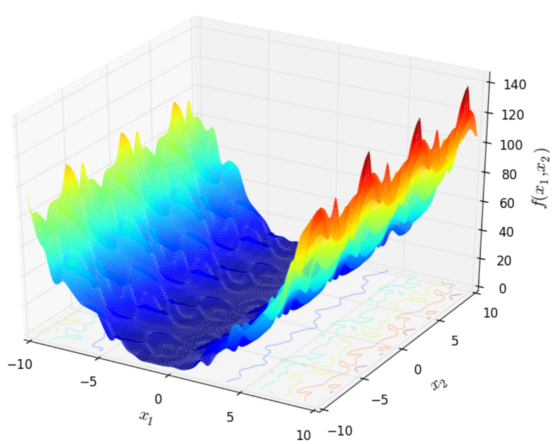
(b) Mishra 2



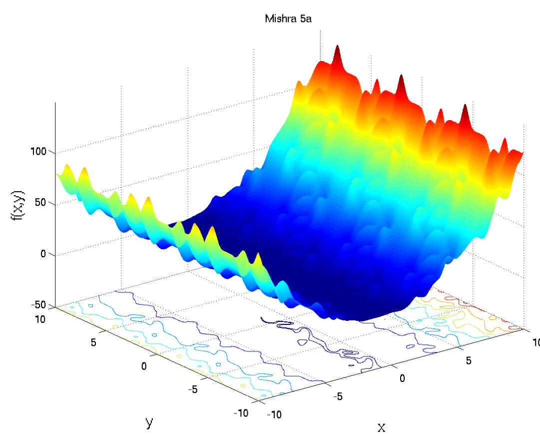
(c) Mishra 3



(d) Mishra 4



(e) Mishra 5



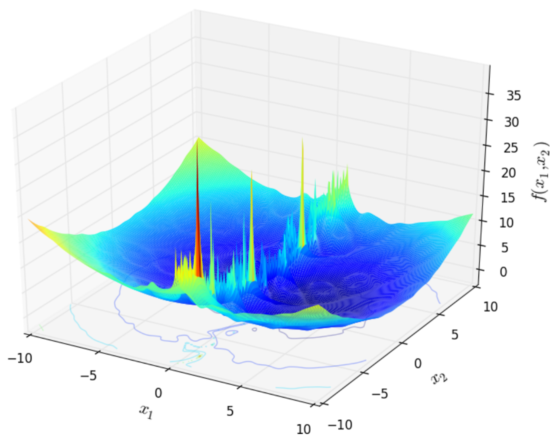
(f) Mishra 5 Modified

Figure 73: Mishra Functions for $n = 2$

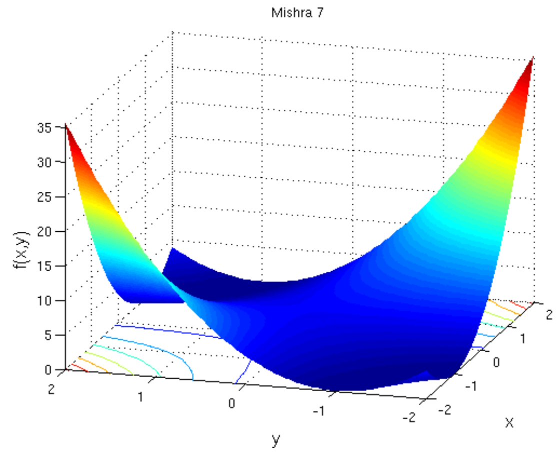
1.79 Needle Eye

The Needle Eye function is defined as:

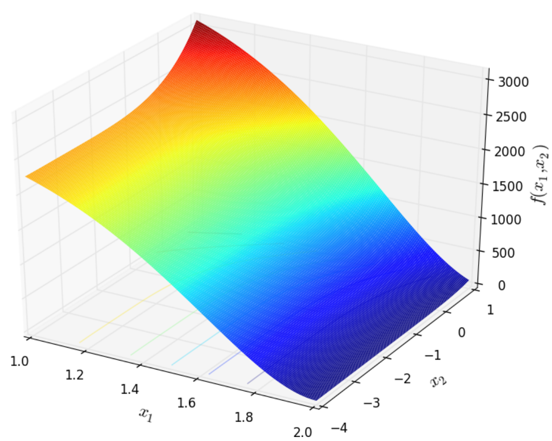
$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } |x_i| < eye \ \ \forall i \\ \sum_{i=1}^{n}(100 + |x_i|) & \text{if } |x_i| > eye \\ 0 & \text{otherwise} \end{cases} \tag{123}$$



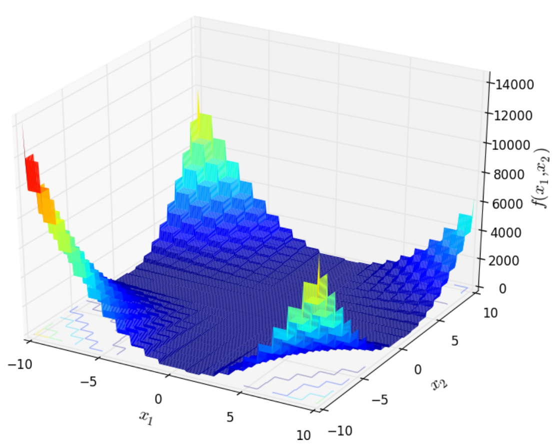
(a) Mishra 6



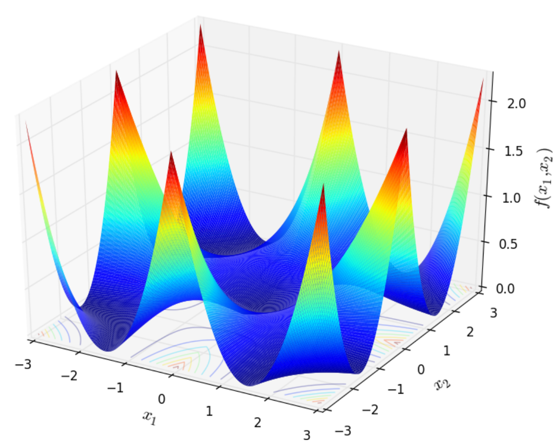
(b) Mishra 7



(c) Mishra 8



(d) Mishra 10



(e) Mishra 11

Figure 74: Mishra Functions for $n = 2$

where $eye = 0.0001$, and with $x_i \in [-10, 10]$ for which $f(\mathbf{x}^*) = 1$, $\mathbf{x}^* = (-1, \ldots, -1)$.

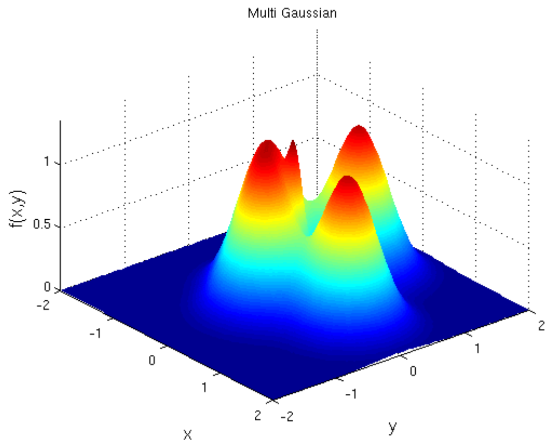


Figure 75: Multi Gaussian for $n = 2$

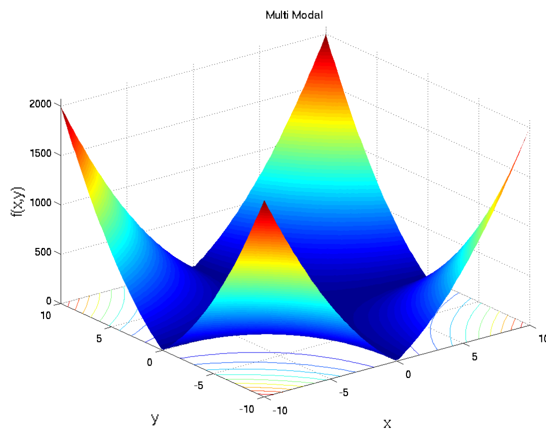


Figure 76: Multi Modal for $n = 2$

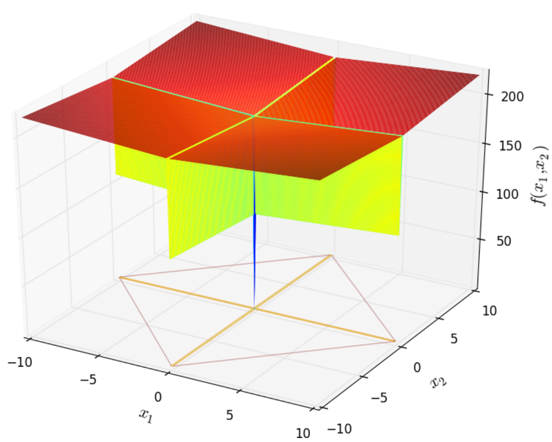


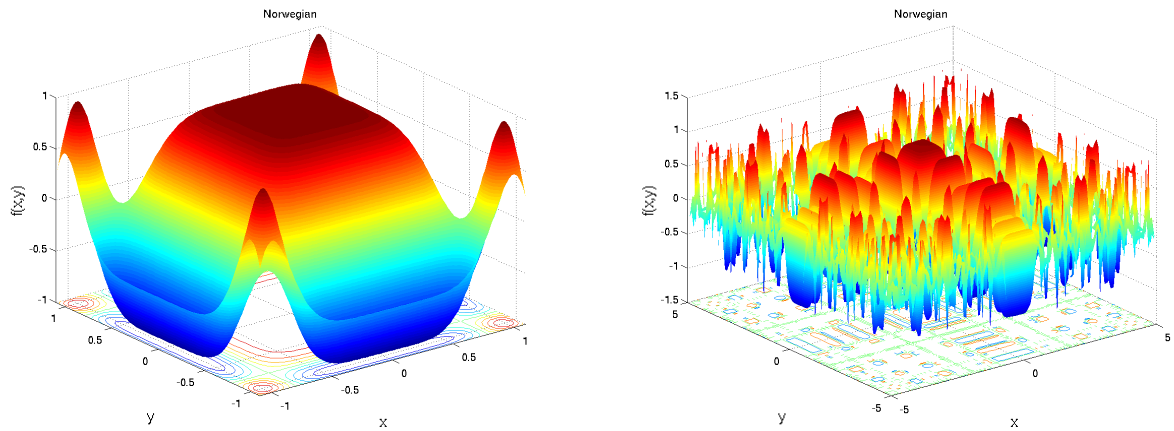
Figure 77: Needle Eye for $n = 2$

1.80 Norwegian

The Norwegian is defined as:

$$f(\mathbf{x}) = \prod_{j=1}^{n_x} \left(\cos(\pi x_j^3) \left(\frac{99 + x_j}{100} \right) \right) \tag{124}$$

with $x_i \in [-1.1, 1.1]$ for which $f(\mathbf{x}^*) = ?$, $\mathbf{x}^* = (??)$.



(a) $x_i \in -1.1, 1.1$

(b) $x_i \in -5, 5$

Figure 78: Norwegian for $n = 2$

1.81 Odd Square

The Odd Square function is defined as:

$$f(\mathbf{x}) = -e^{-\frac{d}{2\pi}} \cos(\pi d) \left(1 + \frac{0.02h}{d + 0.01}\right) \tag{125}$$

where

$$d = n \cdot \max_{1 \le i \le n} \left[(x_i - b_i)^2\right]$$

$$h = \sum_{i=1}^{n} (x_i - b_i)^2$$

$$\mathbf{b} = [1, 1.3, 0.8, -0.4, -1.3, 1.6, -0.2, -0.6, 0.5, 1.4, 1, 1.3, 0.8, -0.4, -1.3, 1.6, -0.2, -0.6, 0.5, 1.4]$$

with $x_i \in [-5\pi, 5\pi]$, $n \le 20$, for which $f(\mathbf{x}^*) = -1.0084$, $\mathbf{x}^* = (\approx b)$.

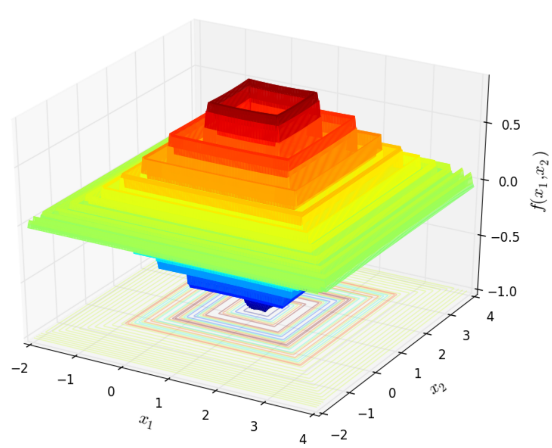


Figure 79: Odd Square for $n = 2$

1.82 Parsopoulos

The Parsopoulos function is defined in two dimensions as:

$$f(\mathbf{x}) = \cos(x_1)^2 + \sin(x_2)^2 \tag{126}$$

with $x_i \in [-5, 5]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (\kappa\frac{\pi}{2}, \lambda\pi)$, for $\kappa = \pm1, \pm3, \ldots$ and $\lambda = 0, \pm1, \pm2, \ldots$. In this domain, there are 12 global optima.

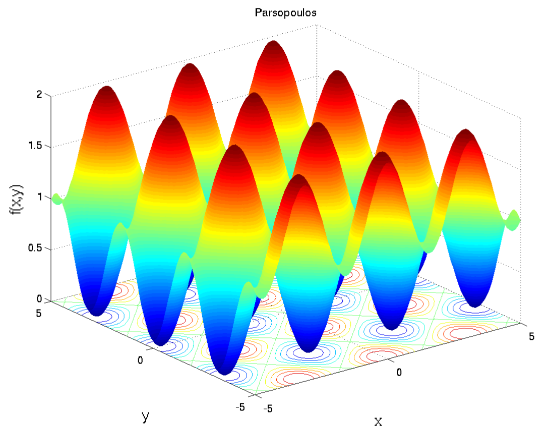


Figure 80: Parsopoulos for $n = 2$

1.83 Pathological

The Pathological function is defined as:

$$f(\mathbf{x}) = \sum_{i=1}^{n-1} \left(\frac{\sin^2 \sqrt{100x_i^2 + x_{i+1}^2} - 0.5}{0.5 + 0.001(x_i - x_{i+1})^4} \right) \tag{127}$$

with $x_i \in [-100, 100]$, $n \geq 2$, for which $f(\mathbf{x}^*) = 0$, and $\mathbf{x}^* = (0, \ldots, 0)$.

1.84 Paviani

The Paviani function is defined as:

$$f(\mathbf{x}) = \sum_{i=1}^{n} \left[\log^2(10 - x_i) + \log^2(x_i - 2)\right] - \left(\prod_{i=1}^{n} x_i^{10}\right)^{0.2} \tag{128}$$

with $x_i \in [2.001, 9.999]$, for which $f(\mathbf{x}^*) = -45.7784684040686$, $\mathbf{x}^* = (9.350266)$.

1.85 Penalty Functions

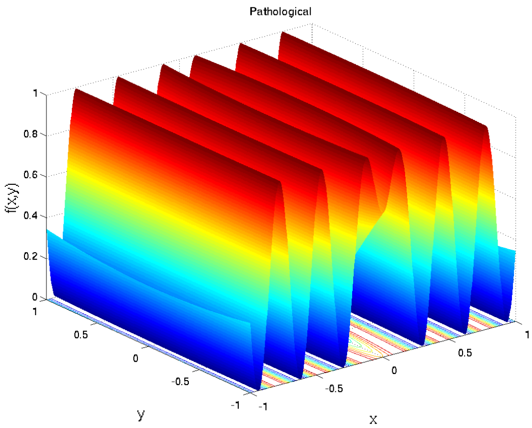
The following Penalty functions:

- Penalty 1:

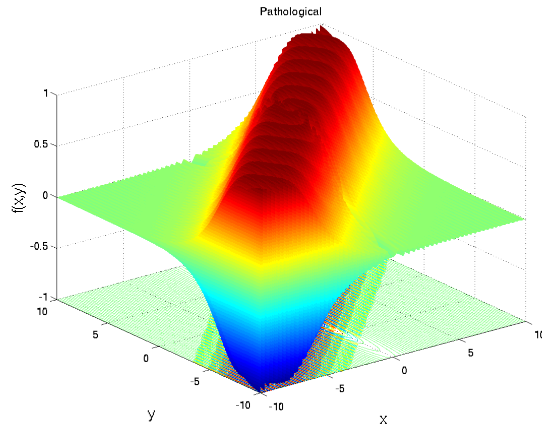
$$f(\mathbf{x}) = \frac{\pi}{30}\left[10\sin^2(\pi y_1) + \sum_{i=1}^{n-1}(y_i - 1)^2\left[1 + 10\sin^2(\pi y_{i+1})\right] + (y_n - 1)^2\right] + \sum_{i=1}^{n} u(x_i, 10, 100, 4) \tag{129}$$

  where

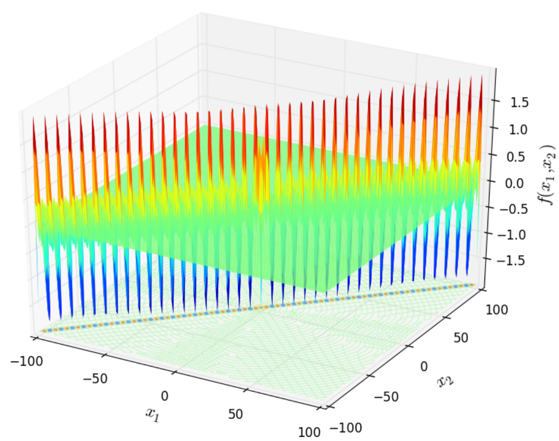
$$y_i = 1 + \frac{1}{4}(x_i + 1)$$



(a) $x_i \in -1, 1$



(b) $x_i \in -10, 10$



(c) $x_i \in -100, 100$

Figure 81: Pathologocal for $n = 2$

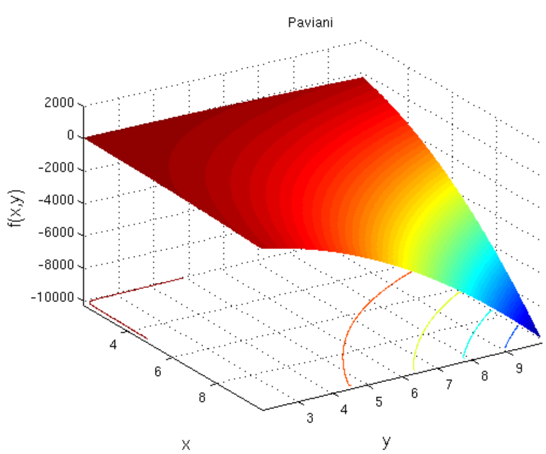


Figure 82: Paviani for $n = 2$

and

$$u(x_i, a, k, m) = \begin{cases} k(x_i - a)^m & \text{if } x_i > a \\ 0 & \text{if } -a \le x_i \le a \\ k(-x_i - a)^m & \text{if } x_i < -a \end{cases}$$

with $x_i \in [-50, 50]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (-1, \dots, -1)$.

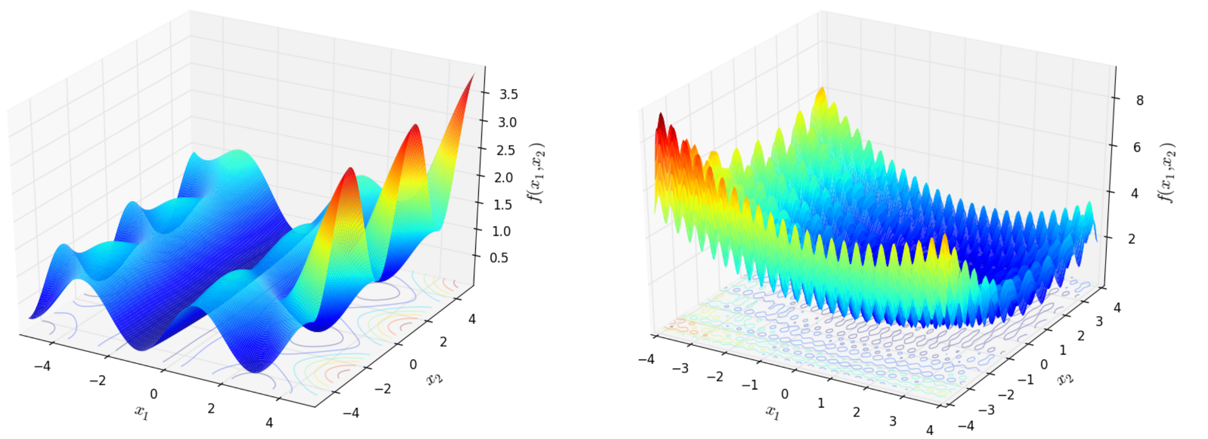
- Penalty 2:

$$f(\mathbf{x}) = 0.1 \left[\sin^2(3\pi x_1) + \sum_{i=1}^{n-1} (x_i - 1)^2 \left[1 + \sin^2(3\pi x_{i+1})\right] + (x_n - 1)^2 \left[1 + \sin^2(2\pi x_n)\right] \right] + \sum_{i=1}^{n} u(x_i, 5, 100, 4) \tag{130}$$

where

$$u(x_i, a, k, m) = \begin{cases} k(x_i - a)^m & \text{if } \ x_i > a \\ 0 & \text{if } \ -a \le x_i \le a \\ k(-x_i - a)^m & \text{if } \ x_i < -a \end{cases}$$

with $x_i \in [-50, 50]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (1, \dots, 1)$.



(a) Penalty 1

(b) Penalty 2

Figure 83: Penalty Functions for $n = 2$

1.86 Pen Holder

The Pen Holder function is defined as:

$$f(\mathbf{x}) = -e^{\left| e^{\left| -\frac{\sqrt{x_1^2 + x_2^2}}{\pi} + 1 \right|} \cos(x_1)\cos(x_2) \right|^{-1}} \tag{131}$$

with $x_i \in [-11, 11]$, for which $f(\mathbf{x}^*) = -0.96354$, $\mathbf{x}^* = (\pm 9.646168)$.

1.87 Periodic

The Periodic function is defined as:

$$f(\mathbf{x}) = 1 + \sum_{i=1}^{n} \sin^2(x_i) - 0.1 e^{-\sum_{i=1}^{n} x_i^2} \tag{132}$$

with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = 0.9$, $\mathbf{x}^* = (0, \dots, 0)$.

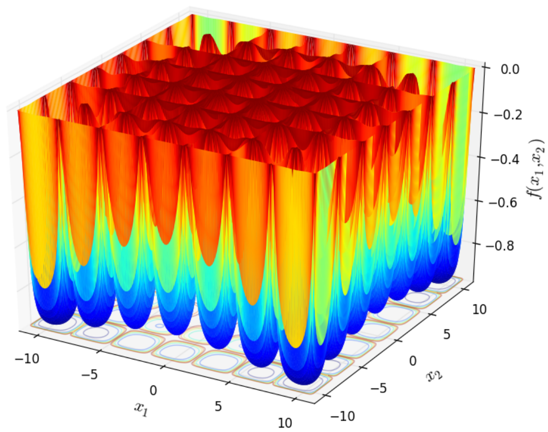


Figure 84: Pen Holder for $n = 2$

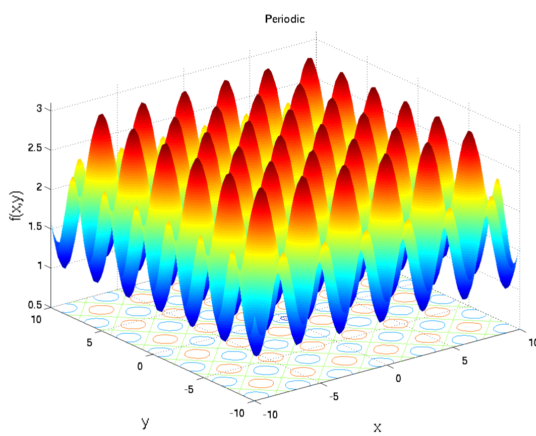


Figure 85: Periodic for $n = 2$

1.88 Perm Functions

The following Perm functions:

- Perm 1:

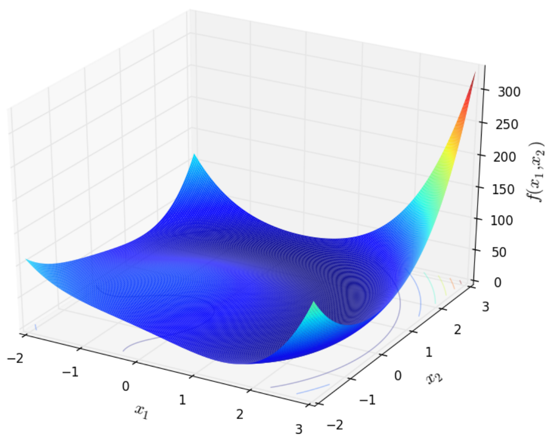
$$f(\mathbf{x}) = \sum_{k=1}^{n} \left\{ \sum_{j=1}^{n} (j^k + \beta) \left[\left(\frac{x_j}{j} \right)^k - 1 \right] \right\}^2 \tag{133}$$

  with $x_i \in [-n, n+1]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (1, 2, 3, \ldots, n)$.

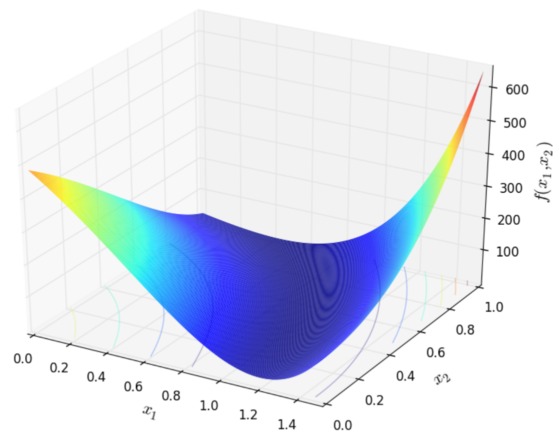
- Perm 2:

$$f(\mathbf{x}) = f(\mathbf{x}) = \sum_{k=1}^{n} \left\{ \sum_{j=1}^{n} (j + \beta) \left[x_j^k - \frac{1}{j} \right] \right\}^2 \tag{134}$$

  with $x_i \in [-n, n+1]$ for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (1/1, 1/2, 1/3, \ldots, 1/n)$.



(a) Perm 1



(b) Perm 2

Figure 86: Perm Functions for $n = 2$

1.89 Pintér

The Pintér function is defined as:

$$f(\mathbf{x}) = \sum_{i=1}^{n} i x_i^2 + \sum_{i=1}^{n} 20 i \sin^2 A + \sum_{i=1}^{n} i \log_{10}(1 + iB^2) \tag{135}$$

where

$$A = x_{i-1} \sin x_i + \sin x_{i+1}$$
$$B = x_{i-1}^2 - 2x_i + 3x_{i+1} - \cos x_i + 1$$

with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.

1.90 Powell Functions

The following Powell functions are defined:

- Powell, in four dimensions:

$$f(\mathbf{x}) = (x_3 + 10x_1)^2 + 5(x_2 - x_4)^2 + (x_1 - 2x_2)^4 + 10(x_3 - x_4)^4 \tag{136}$$

  with $x_i \in [-4, 5]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, 0, 0, 0)$.

- Powell Singular (generalization of Powell):

$$f(\mathbf{x}) = \sum_{i=1}^{n/4} (x_{4i-3} + 10x_{4i-2})^2 + 5(x_{4i-1} - x_{4i})^2 + (x_{4i-2} - x_{4i-1})^4 + 10(x_{4i-3} - x_{4i})^4 \tag{137}$$

  with $x_i \in [-4, 5]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (3, -1, 0, 1, \ldots, 3, -1, 0, 1)$.

- Powell Singular 2:

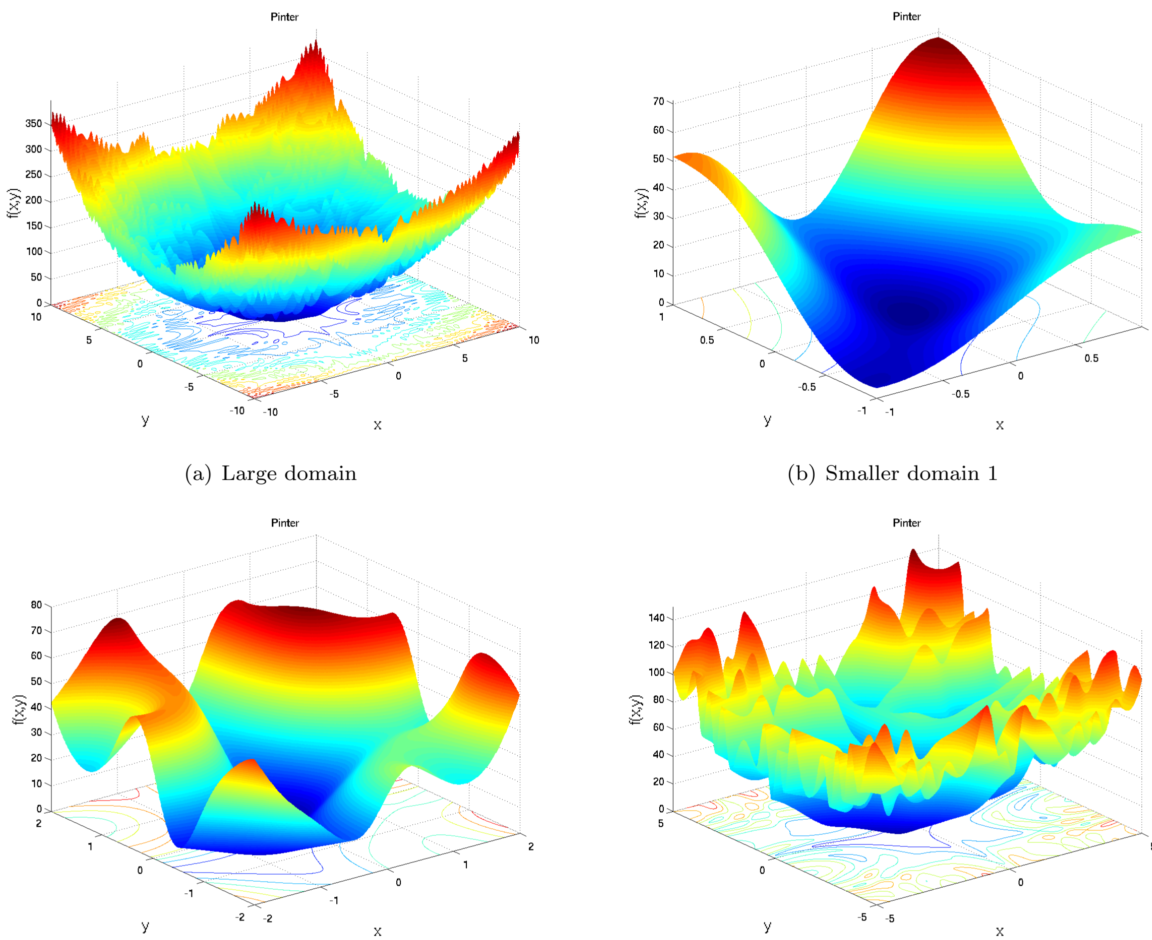
$$f(\mathbf{x}) = \sum_{i=1}^{n-2} (x_{i-1} + 10x_i)^2 + 5(x_{i+1} - x_{i+2})^2 + (x_i - 2x_{i+1})^4 + 10(x_{i-1} - x_{i+2})^4 \tag{138}$$

  with $x_i \in [-4, 5]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (?)$.

- Powell Sum:

$$f(\mathbf{x}) = \sum_{i=1}^{n} |x_i|^{i+1} \tag{139}$$

  with $x_i \in [-1, 1]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.



(a) Large domain

(b) Smaller domain 1

(c) Smaller domain 2

(d) Smaller domain 3

Figure 87: Pinter for $n = 2$

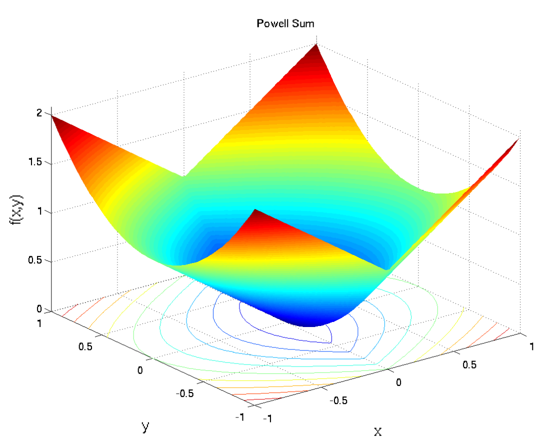


Figure 88: Powell Functions for $n = 2$

1.91 Power Sum / Neumaier 2

The Power Sum (Neumaier 2) function is defined in four dimensions as:

$$f(\mathbf{x}) = \sum_{k=1}^{n} \left[\left(\sum_{i=1}^{n} x_i^k \right) - b_k \right]^2 \tag{140}$$

where

$$\mathbf{b} = [8, 18, 44, 114]$$

with $x_i \in [0, 4]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (1, 2, 2, 3)$.

1.92 Price Functions

The following Price functions, which are generalized from their original two dimensional forms where possible:

- Price 1:
$$f(\mathbf{x}) = \sum_{i=1}^{n} (|x_i| - 5)^2 \tag{141}$$
with $x_i \in [-500, 500]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (\pm 5)$, for n=2.

- Price 2:
$$f(\mathbf{x}) = 1 + \sum_{i=1}^{n} \sin^2(x_i) - 0.1e^{-\sum_{i=1}^{n} x_i^2} \tag{142}$$
with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = 0.9$, $\mathbf{x}^* = (0, \ldots, 0)$.

- Price 3:
$$f(\mathbf{x}) = 100(x_2 - x_1^2)^2 + 6[6.4(x_2 - 0.5)^2 - x_1 - 0.6]^2 \tag{143}$$
with $x_i \in [-500, 500]$, for which $f(\mathbf{x}^*) = 0.9$, $\mathbf{x}^* = (\pm 5)$.

- Price 4:
$$f(\mathbf{x}) = \sum_{i=1}^{n-1} (2x_i^3 x_{i+1} - x_{i+1}^3)^2 + (6x_i - x_{i+1}^2 + x_{i+1})^2 \tag{144}$$
with $x_i \in [-500, 500]$, for which $f(\mathbf{x}^*) = 0.9$. For $n = 2$, there are three global minima at $(0, 0), (2, 4)$ and $(1.464, -2.506)$.

1.93 Qing

The Qing function is defined as:

$$f(\mathbf{x}) = \sum_{i=1}^{n} (x_i^2 - i)^2 \tag{145}$$

with $x_i \in [-500, 500]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (\pm\sqrt{i})$.

1.94 Quadratic

The Quadratic function is defined in two dimensions as:

$$f(\mathbf{x}) = -3803.84 - 138.08x_1 - 232.92x_2 + 128.08x_1^2 + 203.64x_2^2 + 182.25x_1x_2 \tag{146}$$

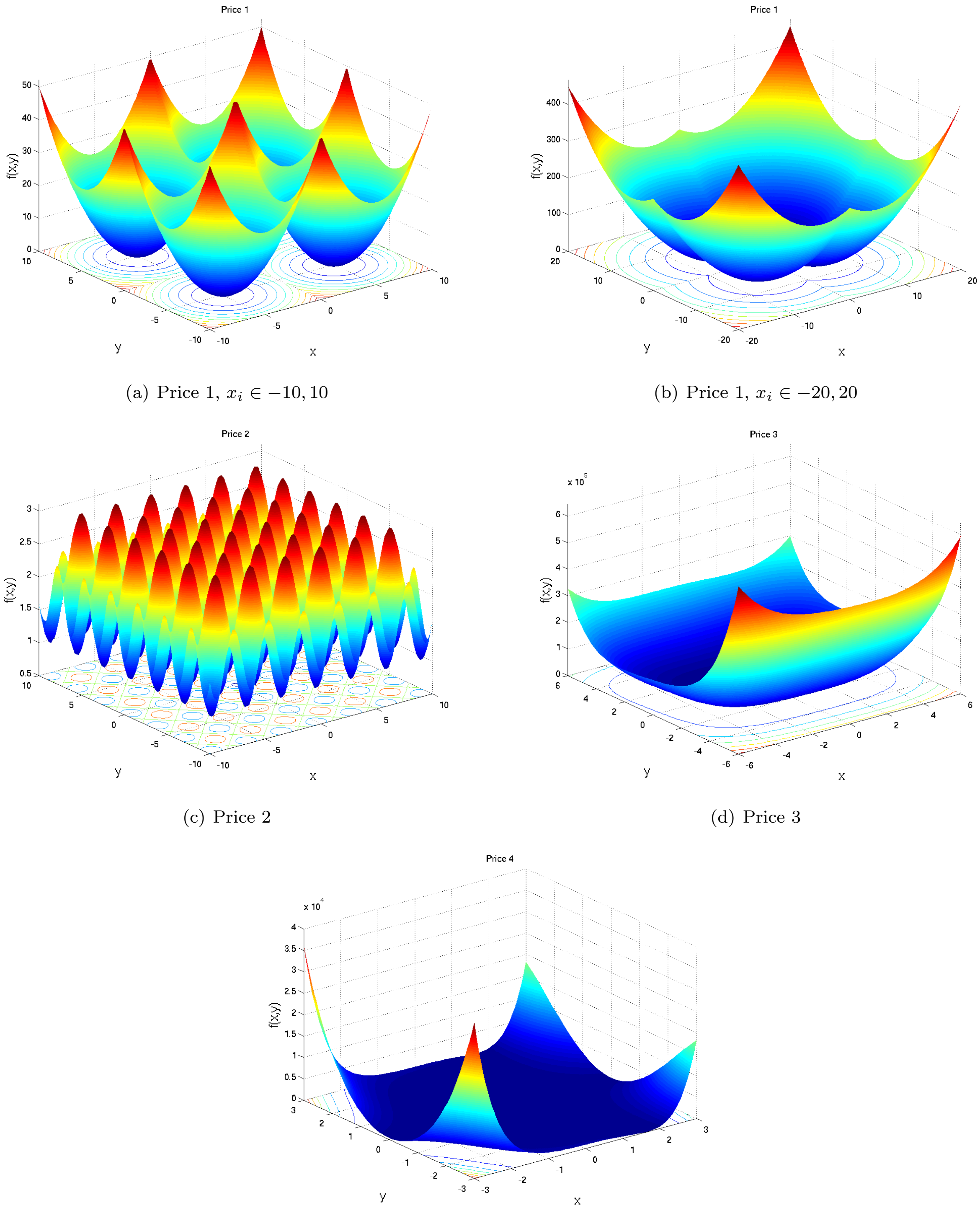
with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = -3873.7243$, $\mathbf{x}^* = (0.19388, 0.48513)$.

1.95 Quadric

The Quadric (Schwefel 1.2) function is defined as:

$$f(\mathbf{x}) = \sum_{i=1}^{n} \left(\sum_{j=1}^{i} x_j \right)^2 \tag{147}$$

with $x_i \in [-100, 100]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.



(a) Price 1, $x_i \in -10, 10$

(b) Price 1, $x_i \in -20, 20$

(c) Price 2

(d) Price 3

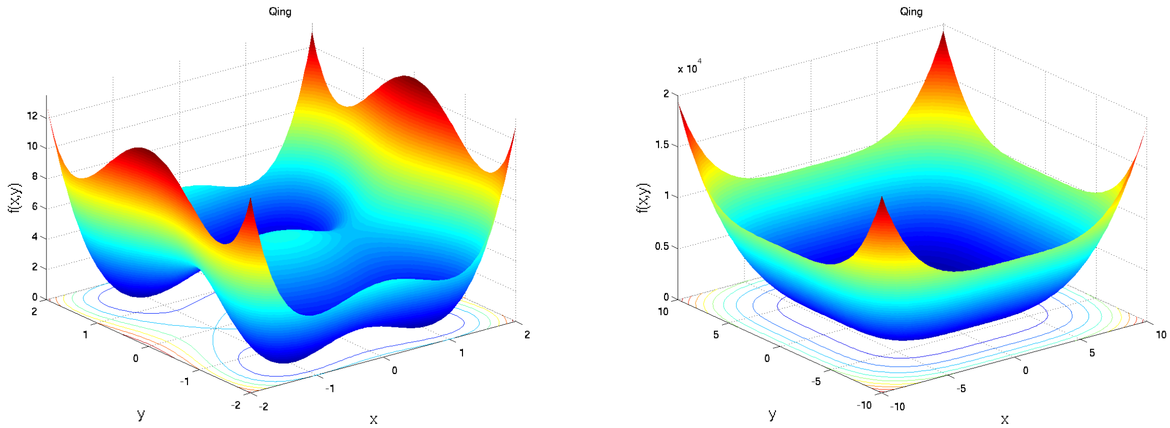
(e) Price 4

Figure 89: Price Functions for $n = 2$

1.96 Quartic

The Quartic, or De Jong F4 function is defined as:

$$f(\mathbf{x}) = \sum_{i=1}^{n} i x_i^4 + U(0,1) \tag{148}$$



(a) $x_i \in -2, 2$

(b) $x_i \in -10, 10$

Figure 90: Qing for $n = 2$

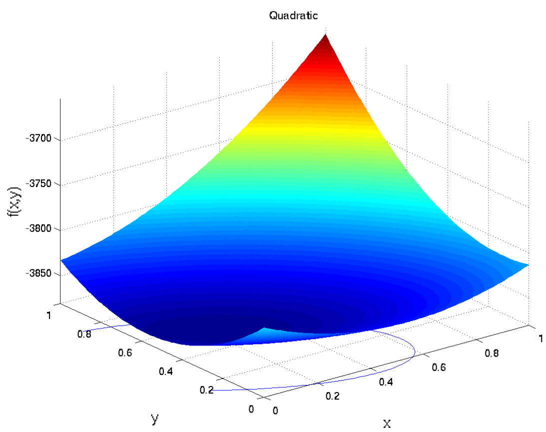


Figure 91: Quadratic for $n = 2$

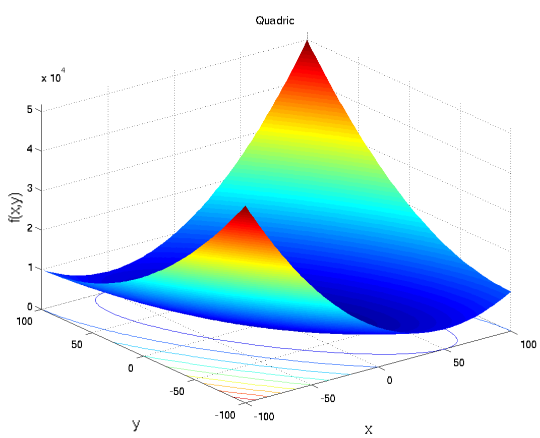


Figure 92: Quadric for $n = 2$

with $x_i \in [-1.28, 1.28]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.

Without noise, the function is referred to as the Holzman 2 function.

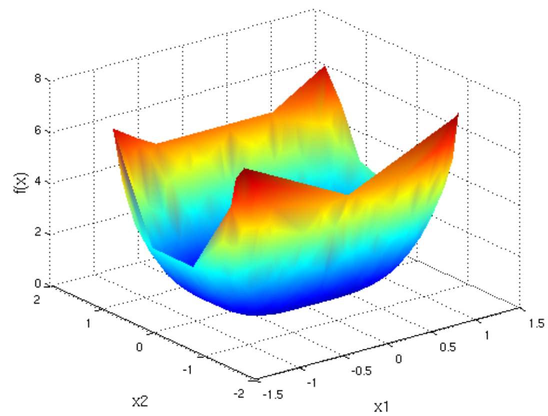


Figure 93: Quartic for $n = 2$

1.97 Quintic

The Quintic function is defined as:

$$f(\mathbf{x}) = \sum_{i=1}^{n} \left| x_i^5 - 3x_i^4 + 4x_i^3 + 2x_i^2 - 10x_i - 4 \right| \tag{149}$$

with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (-1 \text{ or } 2, \ldots, -1 \text{ or } 2)$.

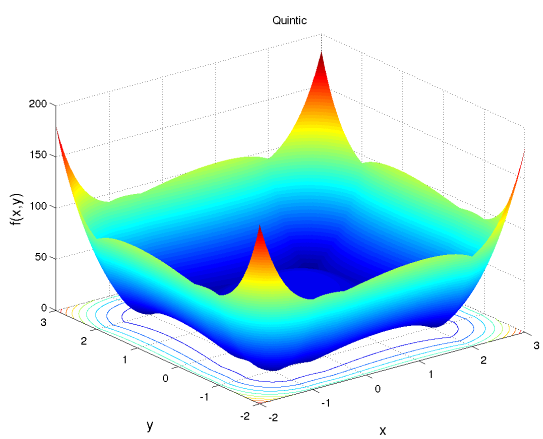


Figure 94: Quintic for $n = 2$

1.98 Rana

The Rana function is defined as:

$$f(\mathbf{x}) = \sum_{i=1}^{n-1} (x_{i+1} + 1)\cos(t_2)\sin(t_1) + x_i \cos(t_1)\sin(t_2) \tag{150}$$

where

$$t_1 = \sqrt{|x_{i+1} + x_i + 1|}$$
$$t_2 = \sqrt{|x_{i+1} - x_i + 1|}$$

and with $x_i \in [-500, 500]$, for which $f(\mathbf{x}^*) = -928.5478$ (?), $\mathbf{x}^* = (-500, \dots, -500)$.

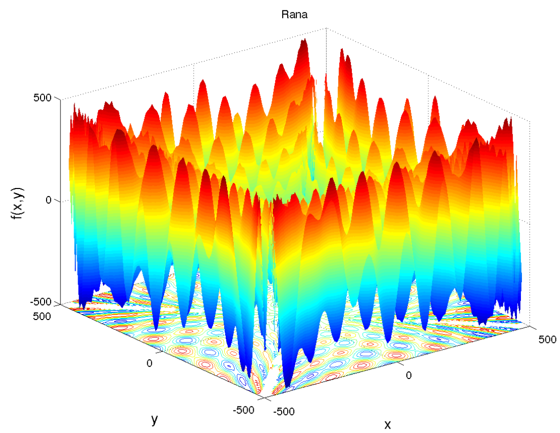


Figure 95: Rana for $n = 2$

1.99 Rastrigin

The Rastrigin function is defined as:

$$f(\mathbf{x}) = 10n + \sum_{i=1}^{n} \left(x_i^2 - 10\cos(2\pi x_i)\right) \tag{151}$$

with $x_i \in [-5.12, 5.12]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \dots, 0)$. Local minima are regularly distributed.

The function has roughly 10^n optima, and has a condition number of roughly 10.

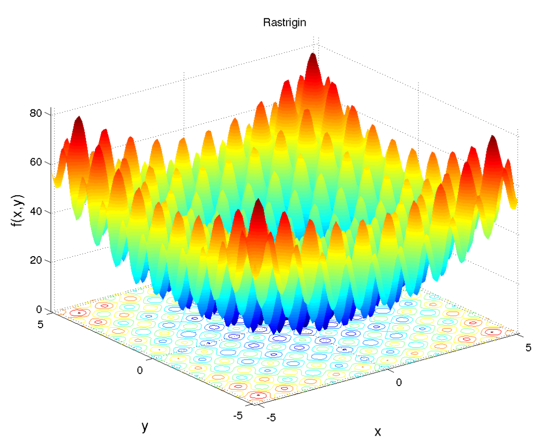


Figure 96: Rastrigin for $n = 2$

1.100 Ripple Functions

The following Ripple functions, originally defined in two dimensions:

- Ripple 1:

$$f(\mathbf{x}) = \sum_{i=1}^{n} -e^{-2\log_2(\frac{x_i-0.1}{0.8})^2\left(\sin^6(5\pi x_i)+0.1\cos^2(500\pi x_i)\right)} \tag{152}$$

  with $x_i \in [0, 1]$, for which $f(\mathbf{x}^*) = -2.2$, $\mathbf{x}^* = (0.1, \ldots, 0.1)$.

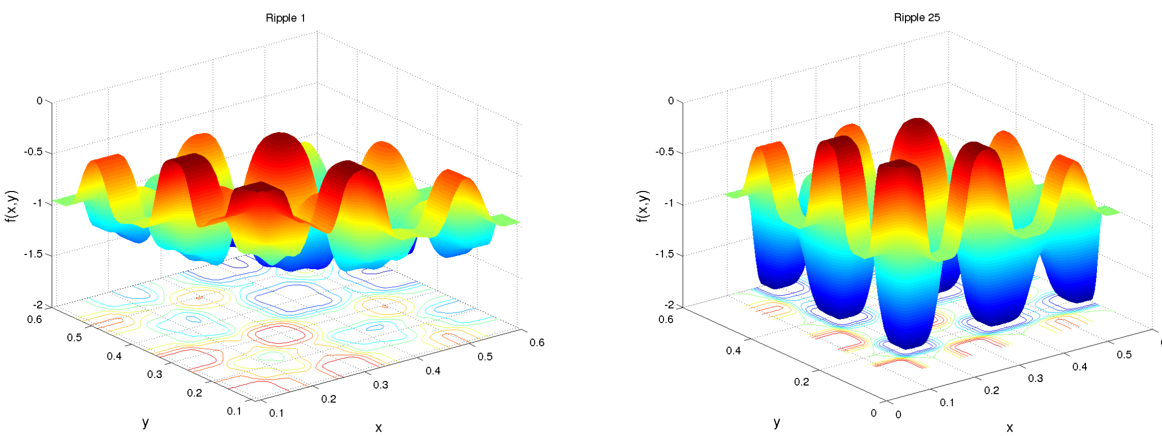
  Ripple 1 has one global minimum and 252004 local minima for $n = 2$, for which the global form of the function consists of 25 holes, forming a 5×5 regular grid. The enitre function landscape is full of small ripples caused by the high frequency cosine function which creates the large number of local minima.

- Ripple 25:

$$f(\mathbf{x}) = \sum_{i=1}^{n} -e^{-2\log_2(\frac{x_i-0.1}{0.8})^2\left(\sin^6(5\pi x_i)\right)} \tag{153}$$

  with $x_i \in [0, 1]$, for which $f(\mathbf{x}^*) = -2$, $\mathbf{x}^* = (0.1, \ldots, 0.1)$.

  The function has one global form of the Ripple 1 function, but no ripples.



(a) Ripple 1

(b) Ripple 25

Figure 97: Ripple Functions for $n = 2$

1.101 Rosenbrock Functions

The following Rosenbrock functions:

- Rosenbrock (De Jong Function 2):

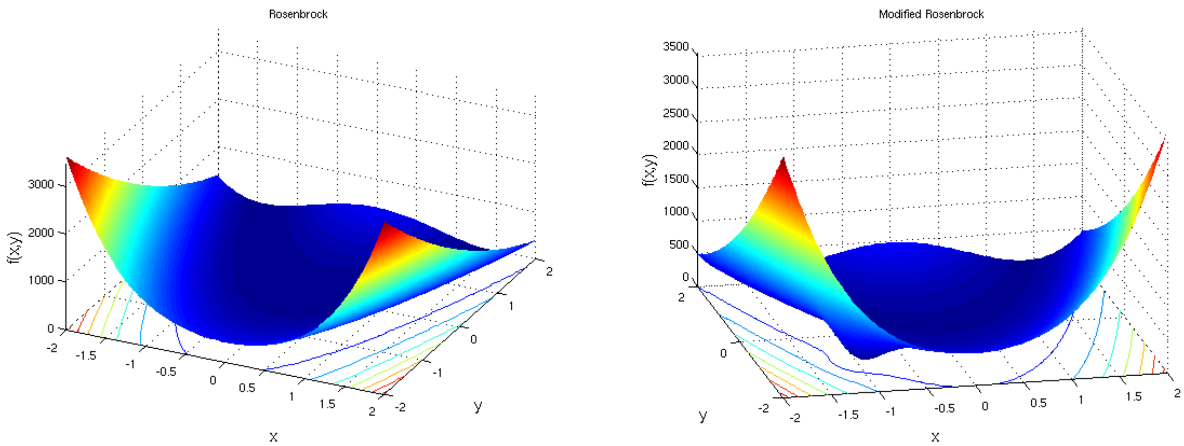
$$f(\mathbf{x}) = \sum_{i=1}^{n-1} \left(100(x_{i+1} - x_i^2)^2 + (x_i - 1)^2\right) \tag{154}$$

  with $x_i \in [-30, 30]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (1, \ldots, 1)$. Note that Rosenbrock becomes multi-modal for dimensions larger than 4. Also referred to as the Banana function, the global optimum is inside a long, narrow, parabolic shaped flat valley.

- Modified Rosenbrock (originally defined in two dimensions):

$$f(\mathbf{x}) = 74 + \sum_{i=1}^{n-1} \left(100(x_{i+1} - x_i^2)^2 + (x_i - 1)^2\right) - 400e^{-\frac{(x_i+1)^2+(x_{i+1}+1)^2}{0.1}} \tag{155}$$

  with $x_i \in [-2, 2]$, for which $f(\mathbf{x}^*) = 34.37$, $\mathbf{x}^* = (-0.9, \ldots, -0.95)$.



(a) Rosenbrock (b) Modified Rosenbrock

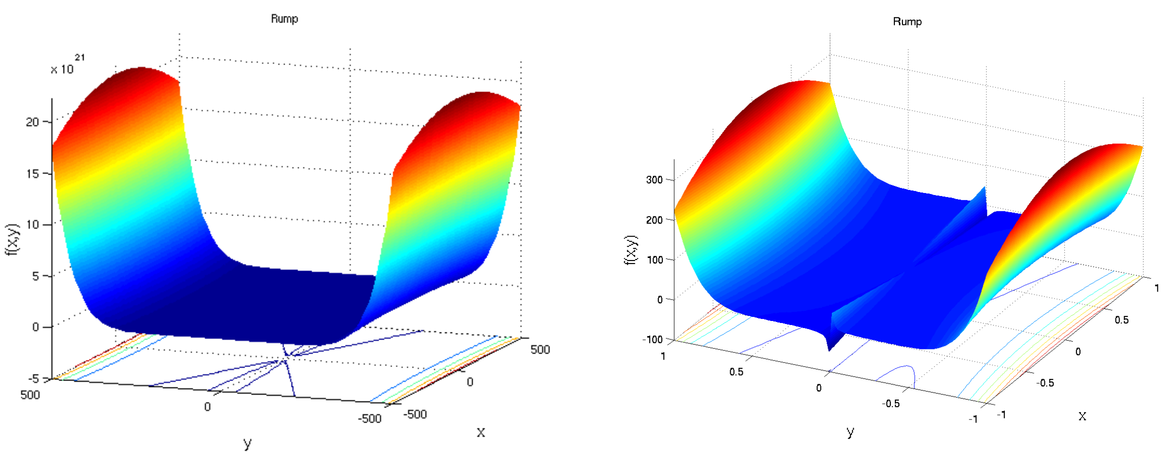
Figure 98: Rosenbrock Functions for $n = 2$

1.102 Rump

The Rump function is defined as (originally defined in two dimensions):

$$f(\mathbf{x}) = \sum_{i=1}^{n-1} (333.75 - x_i^2)x_{i+1}^6 + x_i^2(11x_i^2 x_{i+1}^2 - 121x_{i+1}^4 - 2) + 5.5x_{i+1}^8 + \frac{x_i}{2x_{i+1}} \tag{156}$$

with $x_i \in [-500, 500]$, for which $f(\mathbf{x}^*) = 0)$, $\mathbf{x}^* = (0, \ldots, 0)$.



(a) Large Domain (b) Smaller Domain

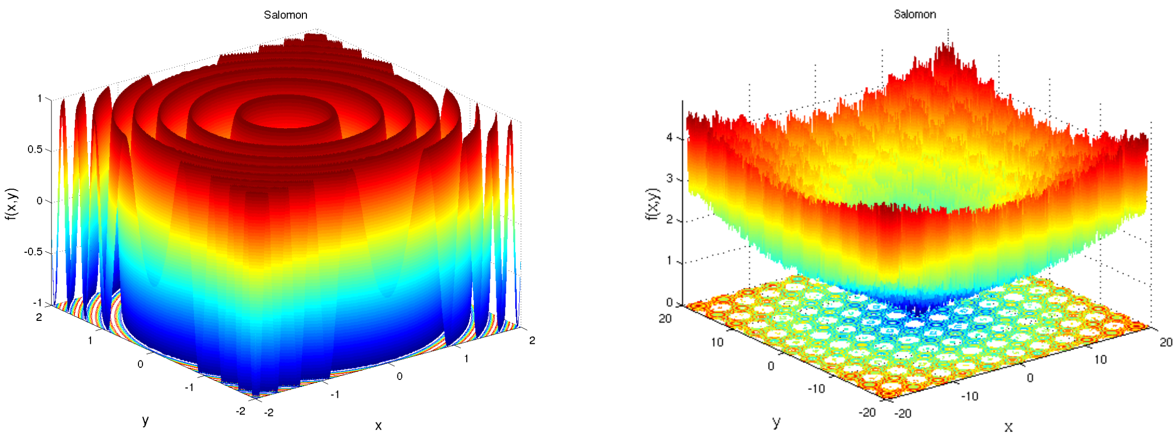
Figure 99: Rump Function for $n = 2$

1.103 Salomon

The Salomon function is defined as:

$$f(\mathbf{x}) = -\cos(2\pi \sum_{i=1}^{n} x_i^2) + 0.1\sqrt{\sum_{i=1}^{n} x_i^2} + 1 \tag{157}$$

with $x_i \in [-100, 100]$, for which $f(\mathbf{x}^*) = 0)$, $\mathbf{x}^* = (0, \ldots, 0)$.



(a) Salomon small domain

(b) Salomon large domain

Figure 100: Salomon Function for $n = 2$

1.104 Sargan

The Sargan function is defined as:

$$f(\mathbf{x}) = \sum_{i=1}^{n} n(x_i^2 + 0.4 \sum_{j \neq i}^{n} x_i x_j) \tag{158}$$

with $x_i \in [-100, 100]$, for which $f(\mathbf{x}^*) = 0)$, $\mathbf{x}^* = (0, \ldots, 0)$.

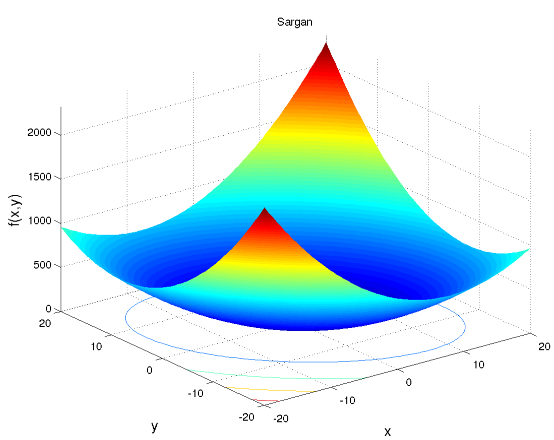


Figure 101: Sargan Function for $n = 2$

1.105 Schaffer Functions

The following Schaffer functions, originally defined in two dimensions:

- Schaffer 1:

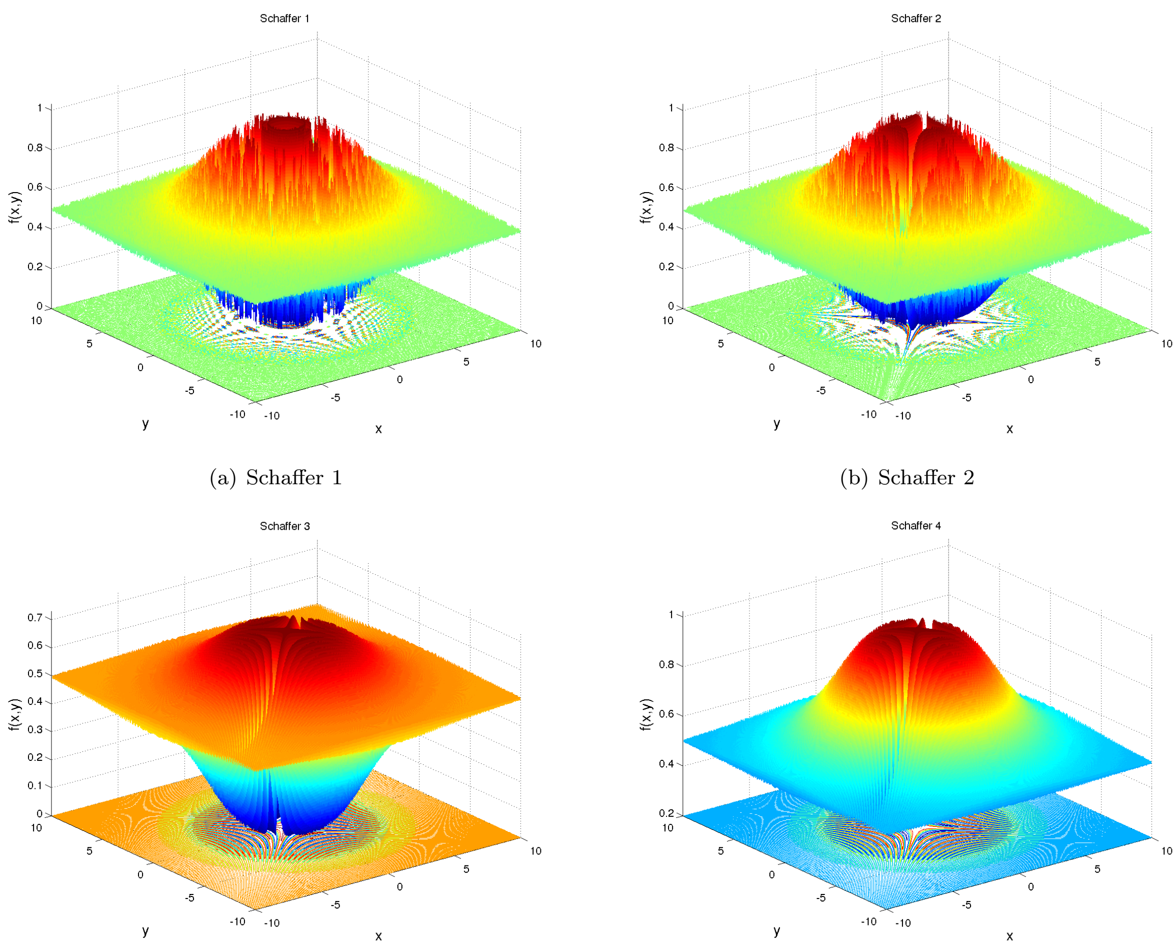
$$f(\mathbf{x}) = \sum_{i=1}^{n-1} \left(0.5 + \frac{\sin^2(x_i^2 + x_{i+1}^2)^2 - 0.5}{(1 + 0.001(x_i^2 + x_{i+1}^2))^2} \right) \tag{159}$$

  with $x_i \in [-100, 100]$, for which $f(\mathbf{x}^*) = 0)$, $\mathbf{x}^* = (0, \ldots, 0)$.

- Schaffer 2:
$$f(\mathbf{x}) = \sum_{i=1}^{n-1} \left(0.5 + \frac{\sin^2(x_i^2 - x_{i+1}^2)^2 - 0.5}{(1 + 0.001(x_i^2 + x_{i+1}^2))^2} \right) \tag{160}$$
with $x_i \in [-100, 100]$, for which $f(\mathbf{x}^*) = 0)$, $\mathbf{x}^* = (0, \dots, 0)$.

- Schaffer 3:
$$f(\mathbf{x}) = \sum_{i=1}^{n-1} \left(0.5 + \frac{\sin^2(\cos(|x_i^2 - x_{i+1}^2|))^2 - 0.5}{(1 + 0.001(x_i^2 + x_{i+1}^2))^2} \right) \tag{161}$$
with $x_i \in [-100, 100]$, for which $f(\mathbf{x}^*) = 0)$, $\mathbf{x}^* = (0, \dots, 0)$.

- Schaffer 4:
$$f(\mathbf{x}) = \sum_{i=1}^{n-1} \left(0.5 + \frac{\cos^2(\sin(x_i^2 - x_{i+1}^2))^2 - 0.5}{(1 + 0.001(x_i^2 + x_{i+1}^2))^2} \right) \tag{162}$$
with $x_i \in [-100, 100]$, for which $f(\mathbf{x}^*) = 0)$, $\mathbf{x}^* = (0, \dots, 0)$.



(a) Schaffer 1

(b) Schaffer 2

(c) Schaffer 3

(d) Schaffer 4

Figure 102: Schaffer Functions for $n = 2$

1.106 Schmidt-Vetters

The Schmidt-Vetters function is defined in three dimensions:

$$f(\mathbf{x}) = \frac{1}{1 + (x_1 - x_2)^2} + \sin(\frac{\pi x_2 + x_3}{2}) + e^{(\frac{x_1+x_2}{x_2} - 2)^2} \tag{163}$$

with $x_i \in [0, 10]$, for which $f(\mathbf{x}^*) = 3$, $\mathbf{x}^* = (0.78547, 0.78547, 0.78547)$.

1.107 Schumer-Steiglitz

The Schumer-Steiglitz function is defined as: three dimensions:

$$f(\mathbf{x}) = \sum_{i=1}^{n} x_i^4 \tag{164}$$

with $x_i \in [-100, 100]$, for which $f(\mathbf{x}^*) = 0)$, $\mathbf{x}^* = (0, \ldots, 0)$.

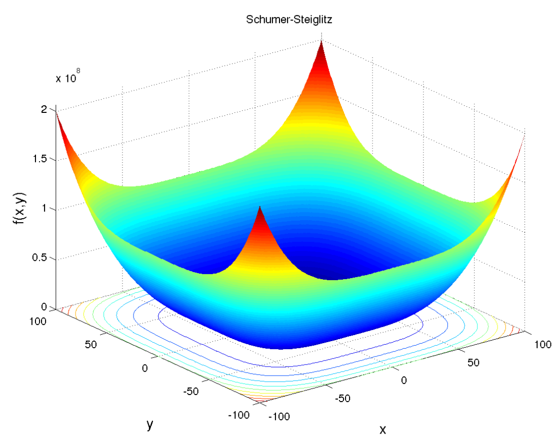


Figure 103: Schumer-Steiglitz Function for $n = 2$

1.108 Schwefel Functions

The following Schwefel functions:

- Schwefel 1:

$$f(\mathbf{x}) = \left(\sum_{i=1}^{n} x_i^2\right)^{\alpha} \tag{165}$$

  where $\alpha \geq 0$, with $x_i \in [-100, 100]$, for which $f(\mathbf{x}^*) = 0)$, $\mathbf{x}^* = (0, \ldots, 0)$. For this document, $\alpha = \sqrt{\pi}$.

- Schwefel 2.4:

$$f(\mathbf{x}) = \sum_{i=1}^{n} ((x_i - 1)^2 + (x_1 - x_i^2)^2) \tag{166}$$

  with $x_i \in [0, 10]$, for which $f(\mathbf{x}^*) = 0)$, $\mathbf{x}^* = (1, \ldots, 1)$.

- Schwefel 2.6:

$$f(\mathbf{x}) = \max\{|x_1 + 2x_2 - 7|, |2x_1 + x2 - 5|\} \tag{167}$$

  with $x_i \in [-100, 100]$, for which $f(\mathbf{x}^*) = 0)$, $\mathbf{x}^* = (1, 3)$.

- Schwefel 2.13:
$$f(\mathbf{x}) = \sum_{i=1}^{n} (\mathbf{A}_i - \mathbf{B}_i(\mathbf{x}))^2 \tag{168}$$
where
$$\mathbf{A}_i = \sum_{j=1}^{n_x} (a_{ij} \sin \alpha_j + b_{ij} \cos \alpha_j)$$
$$\mathbf{B}_i(\mathbf{x}) = \sum_{j=1}^{n} (a_{ij} \sin x_j + b_{ij} \cos x_j)$$
where $a_{ij} \in \mathbf{A}$ and $b_{ij} \in \mathbf{B}$ with $a_{ij}, b_{ij} \sim U(-100, 100)$, and $\alpha_j \sim U(-\pi, \pi)$, and with $x_i \in [-\pi, \pi]$, for which $f(\mathbf{x}^*) = 0)$, $\mathbf{x}^* = (0, \ldots, 0)$.

- Schwefel 2.21:
$$f(\mathbf{x}) = \max_{i=1,\ldots,n} |x_i| \tag{169}$$
with $x_i \in [-100, 100]$, for which $f(\mathbf{x}^*) = 0)$, $\mathbf{x}^* = (0, \ldots, 0)$.

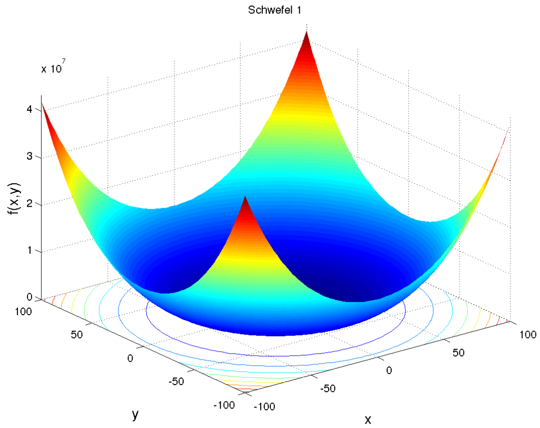
- Schwefel 2.22:
$$f(\mathbf{x}) = \sum_{i=1}^{n} |x_i| + \prod_{i=1}^{n} |x_i| \tag{170}$$
with $x_i \in [-100, 100]$, for which $f(\mathbf{x}^*) = 0)$, $\mathbf{x}^* = (0, \ldots, 0)$.

- Schwefel 2.23:
$$f(\mathbf{x}) = \sum_{i=1}^{n} x_i^{10} \tag{171}$$
with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = 0)$, $\mathbf{x}^* = (0, \ldots, 0)$.

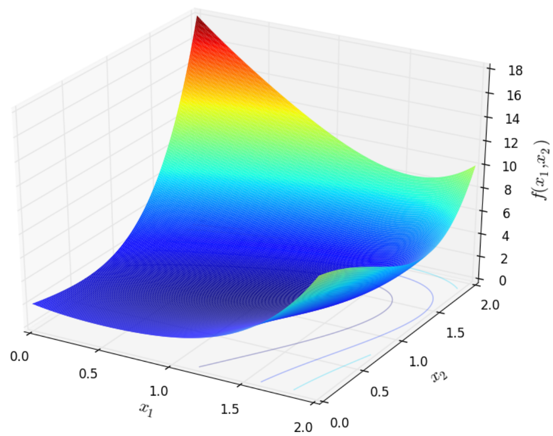
- Schwefel 2.26 (modified version):
$$f(\mathbf{x}) = 418.9829n - \sum_{i=1}^{n} x_i \sin(\sqrt{|x_i|}) \tag{172}$$
with $x_i \in [-500, 500]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (420.968746, \ldots, 420.968746)$. The function is deceptive in that the global minimum is geometrically distant from the next best local minimum.

  The original formulation is
$$f(\mathbf{x}) = -\sum_{i=1}^{n} x_i \sin(\sqrt{|x_i|}) \tag{173}$$
in which case $f(\mathbf{x}^*) = -12569.5$

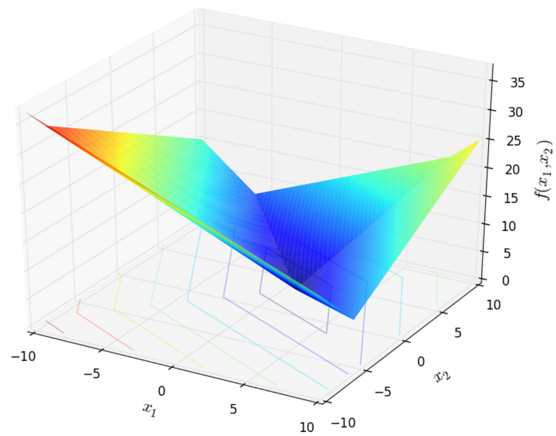
- Schwefel 2.36 (originally defined for two dimensions):
$$f(\mathbf{x}) = -\prod_{i=1}^{n} x_i (72 - 2\sum_{i=1}^{n} x_i) \tag{174}$$
with $x_i \in [0, 500]$, for which $f(\mathbf{x}^*) = -3456$ (for $n = 2$), $\mathbf{x}^* = (12, \ldots, 12)$.



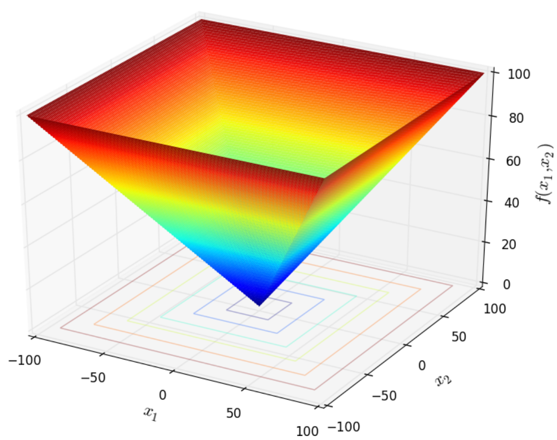
(a) Schwefel 1



(b) Schwefel 2.4



(c) Schwefel 2.6

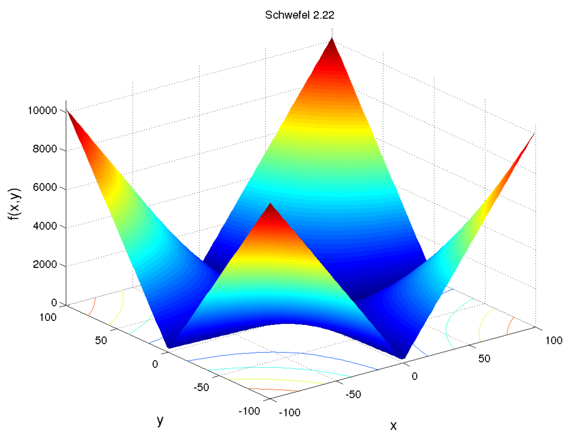


(d) Schwefel 2.21

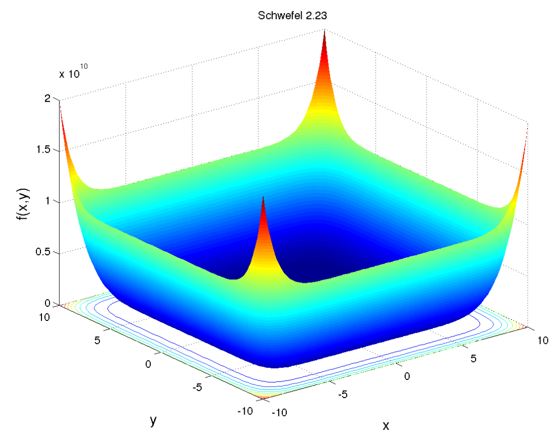
Figure 104: Schwefel Functions for $n = 2$

1.109 Shekel Functions

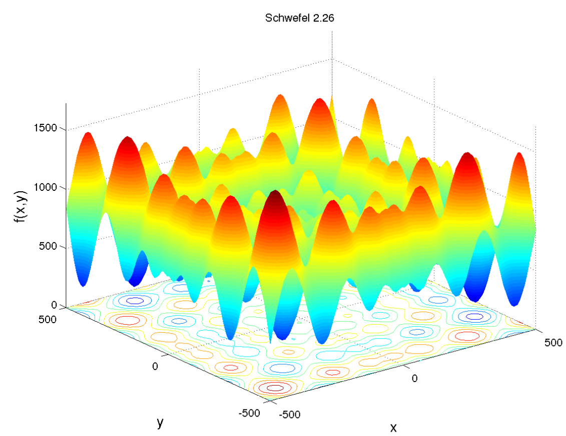
The following Shekel functions:



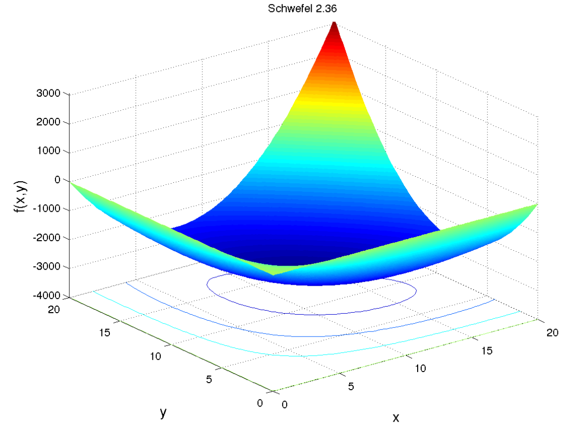
(a) Schwefel 2.22



(b) Schwefel 2.23



(c) Schwefel 2.26



(d) Schwefel 2.36

Figure 105: Schwefel Functions for $n = 2$ (cont)

- Shekel 5:

$$f(\mathbf{x}) = -\sum_{i=1}^{5} \frac{1}{\sum_{j=1}^{4}(x_j - a_{ij})^2 + c_i} \tag{175}$$

where $\mathbf{A} = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 \\ 8 & 8 & 8 & 8 \\ 6 & 6 & 6 & 6 \\ 3 & 7 & 3 & 7 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.2 \\ 0.4 \\ 0.4 \end{bmatrix}$
with $x_i \in [0, 10]$, for which $f(\mathbf{x}^*) \approx -10.1499$, $\mathbf{x}^* = (4, \dots, 4)$

- Shekel 7:

$$f(\mathbf{x}) = -\sum_{i=1}^{7} \frac{1}{\sum_{j=1}^{4}(x_j - a_{ij})^2 + c_i} \tag{176}$$

where $\mathbf{A} = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 \\ 8 & 8 & 8 & 8 \\ 6 & 6 & 6 & 6 \\ 3 & 7 & 3 & 7 \\ 2 & 9 & 2 & 9 \\ 5 & 5 & 3 & 3 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.2 \\ 0.4 \\ 0.4 \\ 0.6 \\ 0.3 \end{bmatrix}$
with $x_i \in [0, 10]$, for which $f(\mathbf{x}^*) \approx -10.3999$, $\mathbf{x}^* = (4, \ldots, 4)$

- Shekel 10:

$$f(\mathbf{x}) = -\sum_{i=1}^{10} \frac{1}{\sum_{j=1}^{4}(x_j - a_{ij})^2 + c_i} \tag{177}$$

where $\mathbf{A} = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 \\ 8 & 8 & 8 & 8 \\ 6 & 6 & 6 & 6 \\ 3 & 7 & 3 & 7 \\ 2 & 9 & 2 & 9 \\ 5 & 5 & 3 & 3 \\ 8 & 1 & 8 & 1 \\ 6 & 2 & 6 & 2 \\ 7 & 3.6 & 7 & 3.6 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.2 \\ 0.4 \\ 0.4 \\ 0.6 \\ 0.3 \\ 0.7 \\ 0.5 \\ 0.5 \end{bmatrix}$
with $x_i \in [0, 10]$, for which $f(\mathbf{x}^*) \approx -10.5319$, $\mathbf{x}^* = (4, \ldots, 4)$

1.110 Shubert Functions

The following Shubert functions:

- Shubert:

$$f(\mathbf{x}) = \prod_{i=1}^{n} \left(\sum_{j=1}^{5} j \cos((j+1)x_i + j) \right) \tag{178}$$

  with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) \approx -186.7309$. There are 18 global optima for $n = 2$.

- Shubert 3:

$$f(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{5} j \sin((j+1)x_i + j) \tag{179}$$

  with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) \approx -29.6733337$. There are multiple global optima.

- Shubert 4:

$$f(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{5} j \cos((j+1)x_i + j) \tag{180}$$

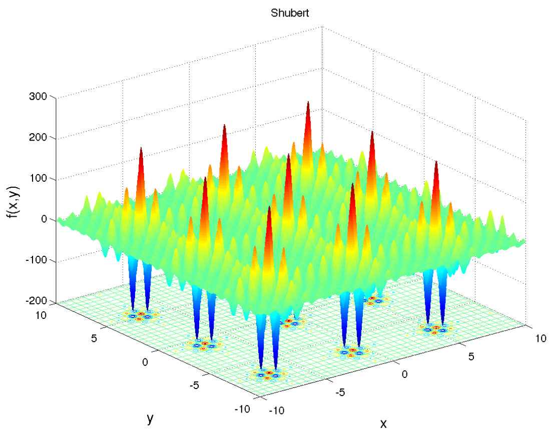
  with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) \approx -25.740858$. There are multiple global optima.

1.111 Sine Envelope

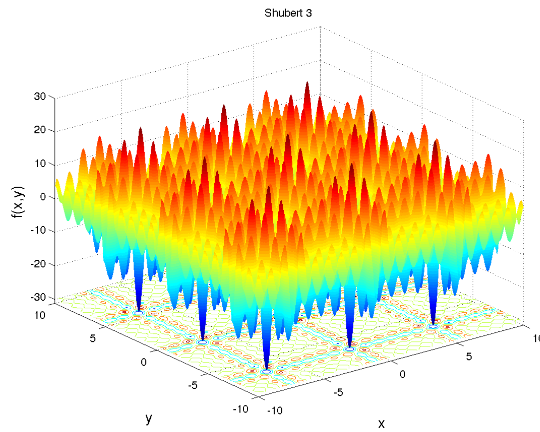
The Sine Envelope function (related to Schaffer 1) is defined as

$$f(\mathbf{x}) = \sum_{i=1}^{n-1} \left(0.5 + \frac{\sin^2 \sqrt{x_i^2 + x_{i+1}^2} - 0.5}{(1 + 0.001(x_i^2 + x_{i+1}^2))^2} \right) \tag{181}$$

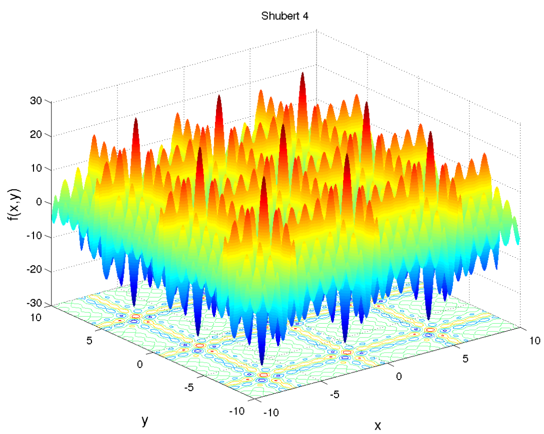
with $x_i \in [-100, 100]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.



(a) Shubert



(b) Shubert 3



(c) Shubert 4

Figure 106: Shubert Functions for $n = 2$

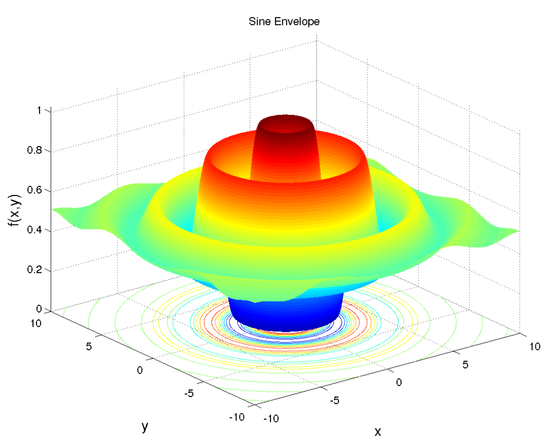


Figure 107: Sine Envelope for $n = 2$

1.112 Sinusoidal

The Sinusoidal function is defined as

$$f(\mathbf{x}) = -\left[A \prod_{i=1}^{n} \sin(x_i - z) + \prod_{i=1}^{n} \sin(B(x_i - z))\right] \tag{182}$$

where $A = 2.5, B = 5$ and $z = 30$, with $x_i \in [0, 180]$, for which $f(\mathbf{x}^*) = 90 + z$, $\mathbf{x}^* = (-(A+1), \ldots, -(A+1))$. When $B = 5$, the number of local minima are

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \left(\frac{n!}{(n-2i)!(2i)!} \times 3^{n-2i} 2^{2i} \right)$$

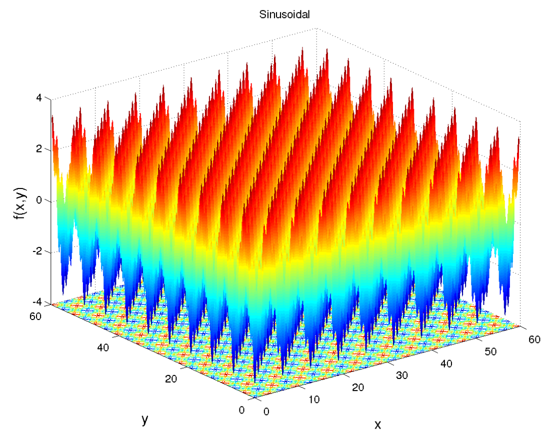


Figure 108: Sinusoidal for $n = 2$

1.113 Spherical

The Spherical function is defined as

$$f(\mathbf{x}) = \sum_{i=1}^{n} x_i^2 \tag{183}$$

with $x_i \in [-100, 100]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.

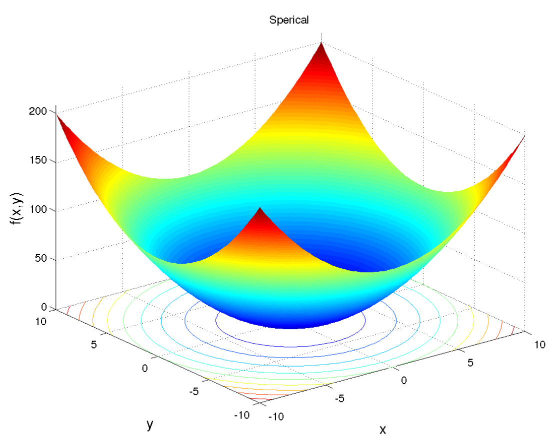


Figure 109: Spherical for $n = 2$

1.114 Step Functions

The following Step functions:

- Step 1:

$$f(\mathbf{x}) = \sum_{i=1}^{n} \lfloor |x_i| \rfloor \tag{184}$$

with $x_i \in [-100, 100]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.

- Step 2:

$$f(\mathbf{x}) = \sum_{i=1}^{n} \lfloor x_i + 0.5 \rfloor^2 \tag{185}$$

with $x_i \in [-100, 100]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.

- Step 3:

$$f(\mathbf{x}) = \sum_{i=1}^{n} \lfloor x_i^2 \rfloor \tag{186}$$

with $x_i \in [-5.12, 5.12]$, for which $f(\mathbf{x}^*) = c$, $\mathbf{x}^* = (0, \ldots, 0)$.

- Plateau:

$$f(\mathbf{x}) = c + \sum_{i=1}^{n} \lfloor x_i \rfloor \tag{187}$$

with $x_i \in [-5.12, 5.12]$, for which $f(\mathbf{x}^*) = c$, $\mathbf{x}^* = (0, \ldots, 0)$. Here, $c = 30$.

1.115 Storn's Tchebychev Problem

Storn's Tchebychev problem is defined as

$$f(\mathbf{x}) = p_1 + p_2 + p_3 \tag{188}$$

where

$$p_1 = \begin{cases} (u-d)^2 & \text{if } u < d \\ 0 & \text{if } u \geq d \end{cases}$$

$$p_2 = \begin{cases} (v-d)^2 & \text{if } v < d \\ 0 & \text{if } v \geq d \end{cases}$$

$$p_3 = \sum_{j=0}^{m} p_i^{'}$$

with

$$p_j^{'} = \begin{cases} (w_j - 1)^2 & \text{if } w_j > 1 \\ (w_j + 1)^2 & \text{if } w_j < -1 \\ 0 & \text{otherwise} \end{cases}$$

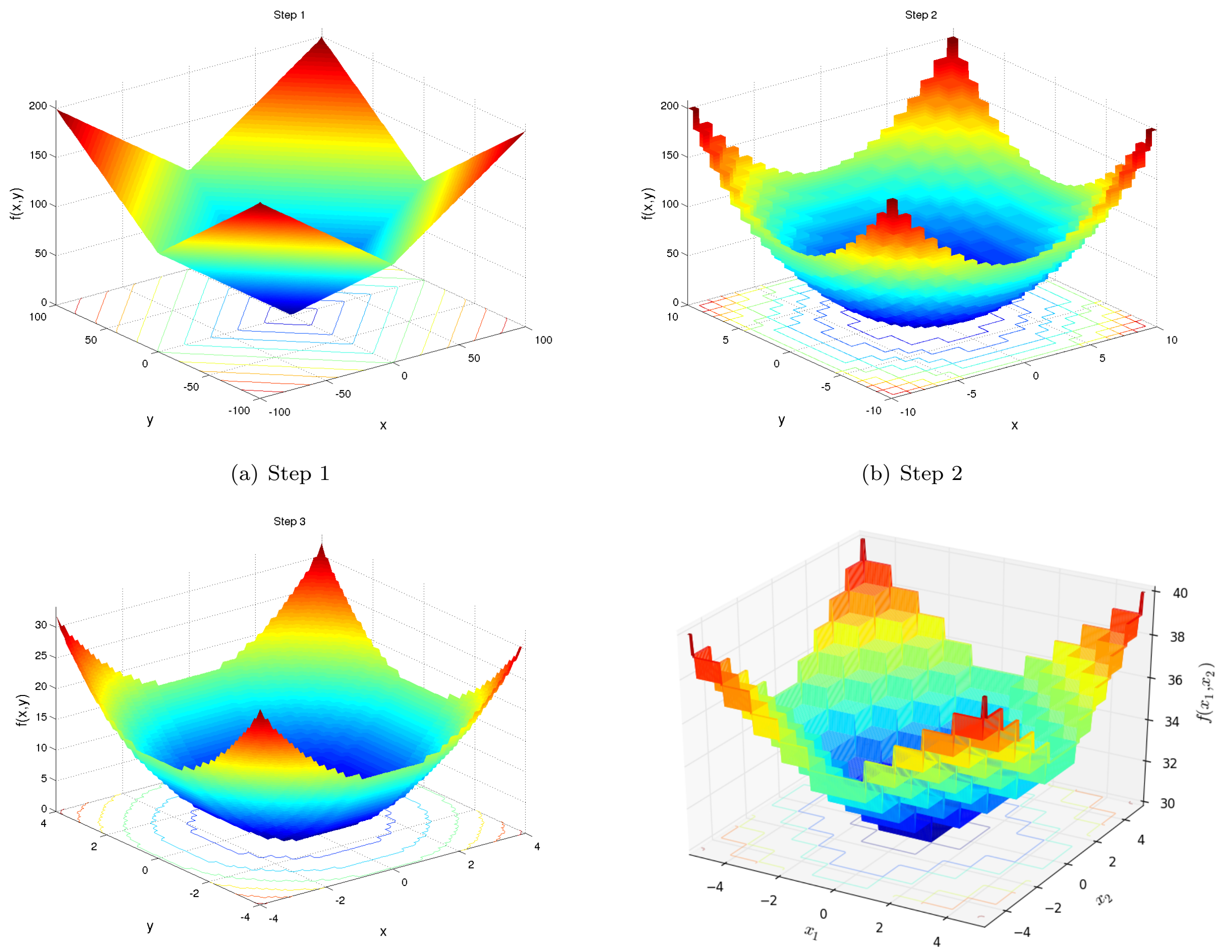
and

$$u = \sum_{i=1}^{n} 1.2^{n-i} x_i$$

$$v = \sum_{i=1}^{n} (-1.2)^{n-i} x_i$$

$$w_j = \sum_{i=1}^{n} \left(\frac{2j}{m} - 1 \right)^{n-i} x_i$$

For the above, $d = 72.661$ and $m = 60$. For $n = 9$, $x_i \in [-128, 128]$ and for $n = 17$, $x_i \in [-32768, 32768]$.



(a) Step 1

(b) Step 2

(c) Step 3

(d) Plateau

Figure 110: Step Functions for $n = 2$

1.116 Stretched V Sine Wave

The Stretched V Sine Wave function is defined as

$$f(\mathbf{x}) = \sum_{i=1}^{n-1} (x_i^2 + x_{i+1}^2)^{0.25} [\sin^2(50(x_i^2 + x_{i+1}^2)^{0.1}) + 0.1] \tag{189}$$

with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = 0$. When $n = 2$, $\mathbf{x}^* = (-9.38723188, 9.34026753)$.

1.117 Sum Squares

The Sum Squares Function is defined as

$$f(\mathbf{x}) = \sum_{i=1}^{n} i x_i^2 \tag{190}$$

with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.

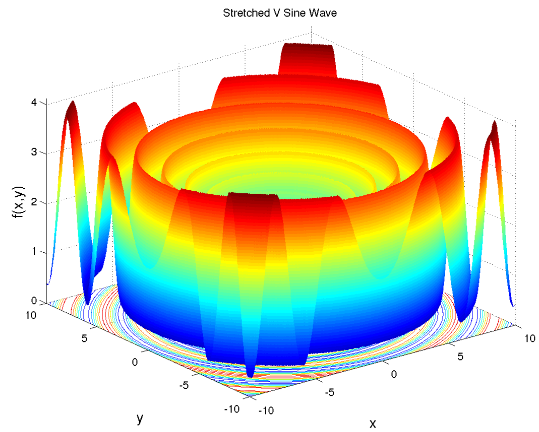


Figure 111: Stretched V Sine Wave Function for $n = 2$

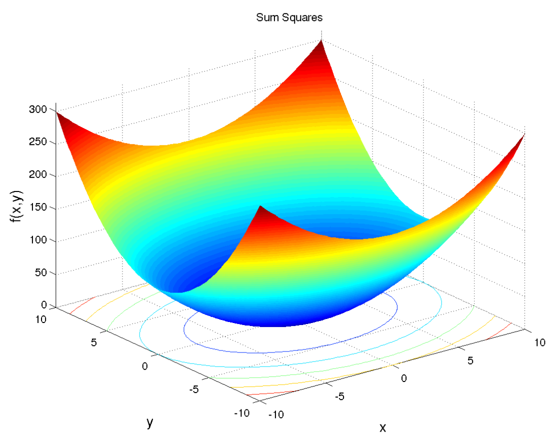


Figure 112: Sum Squares Function for $n = 2$

1.118 Sum of Different Powers

The Sum of Different Powers Function is defined as

$$f(\mathbf{x}) = \sum_{i=1}^{n} |x_i|^{i+1} \tag{191}$$

with $x_i \in [-1, 1]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.

1.119 Styblinski-Tang

The Styblinski-Tang is defined as

$$f(\mathbf{x}) = \frac{1}{2} \sum_{i=1}^{n} (x_i^4 - 16x_i^2 + 5x_i) \tag{192}$$

with $x_i \in [-5, 5]$, for which $f(\mathbf{x}^*) = -39.16616570377142 \times n$, $\mathbf{x}^* = (-2.90353401818596, \ldots, -2.90353401818596)$.

1.120 Table Functions

The following Table functions are defined as (originally in two dimensions, bur generalized below where possible):

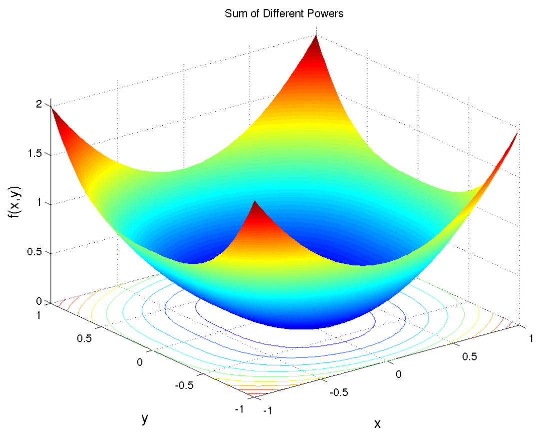


Figure 113: Sum of Different Powers Function for $n = 2$

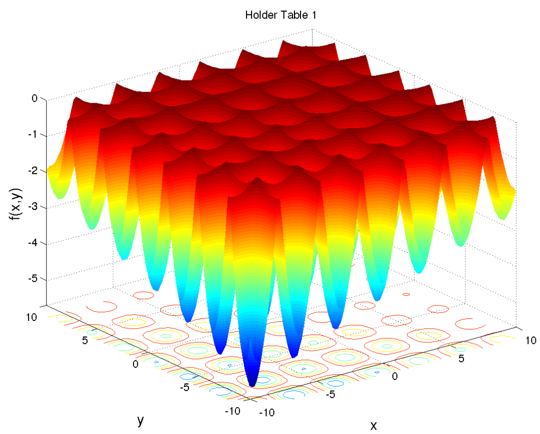
- Holder Table 1:
$$f(\mathbf{x}) = -\left| \left(\prod_{i=1}^{n} \cos(x_i) \right) e^{\left|1-\sqrt{\sum_{i=1}^{n} x_i}/\pi\right|} \right| \tag{193}$$
with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = -26.920336$, $\mathbf{x}^* = (\pm 9.646168, \ldots, \pm 9.646168)$.
- Holder Table 2:
$$f(\mathbf{x}) = -\left| \sin(x_1)\cos(x_2) e^{\left|(1-\sqrt{x_1+x_2})/\pi\right|} \right| \tag{194}$$
with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = -19.2085$, $\mathbf{x}^* = (\pm 8.055023472141116, \pm 9.664590028909654$.
- Holder Table 3:
$$f(\mathbf{x}) = \frac{-\left(\left(\prod_{i=1}^{n} \cos(x_i)\right) e^{\left|1-\sqrt{\sum_{i=1}^{n} x_i}/\pi\right|} \right)^2}{30} \tag{195}$$
with $x_i \in [-10, 10]$, for which, for $n = 2$, $f(\mathbf{x}^*) = -24.1568155$, $\mathbf{x}^* = (\pm 9.646157266348881, \pm 9.646134286497169)$.
- Test Tube Holder:
$$f(\mathbf{x}) = -4 \left| \sin(x_1)\cos(x_2) e^{\left|\cos((x_1^2+x_2^2)/200)\right|} \right| \tag{196}$$
with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = -10.8723$, and two global minima, $\mathbf{x}^* = (\pm\pi/2, 0)$.

1.121 Trecanni

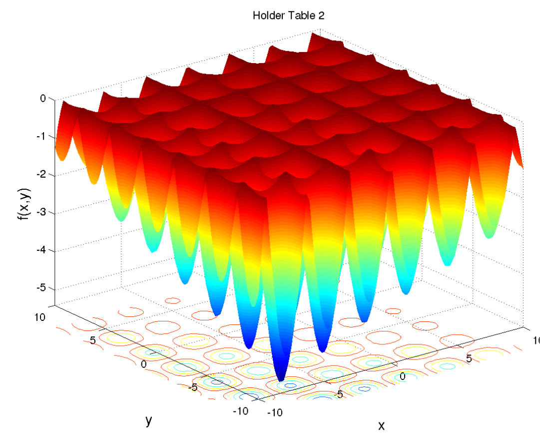
The Trecanni function is defined as
$$f(\mathbf{x}) = x_1^4 - 4x_1^3 + 4x_1 + x_2^2 \tag{197}$$
with $x_i \in [-5, 5]$, for which $f(\mathbf{x}^*) = 0$, and two global optima, $\mathbf{x}^* = \{(0, 0), (-2, 0)\}$.

1.122 Trefethen

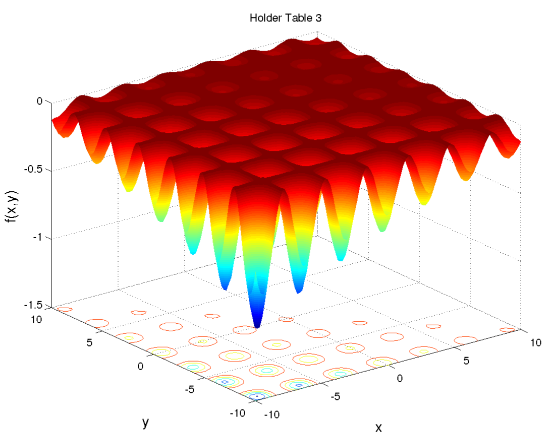
The Trefethen 4 function is defined as
$$f(\mathbf{x}) = e^{\sin(50x_1)} + \sin(60e^{x_2}) + \sin(70\sin(x_1)) + \sin(\sin(80x_2)) - \sin(10(x + y)) + \frac{1}{4}(x_1^2 + x_2^2) \tag{198}$$
with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = -3.30686865$, $\mathbf{x}^* = (-0.024403, 0.210612)$.



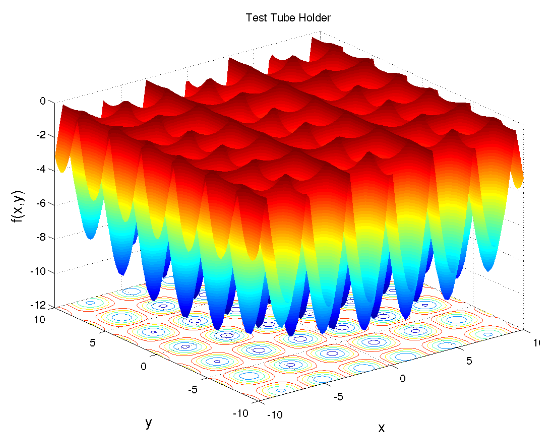
(a) Holder Table 1



(b) Holder Table 2



(c) Holder Table 3



(d) Test Tube Holder

Figure 114: Holder Table 1 Function for $n = 2$

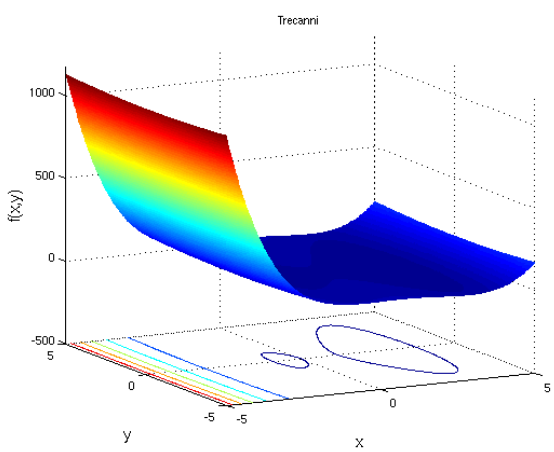


Figure 115: Trecanni for $n = 2$

1.123 Trid

The Trid (Neumaier 3) function is defined as:

$$f(\mathbf{x}) = \sum_{i=1}^{n}(x_i - 1)^2 - \sum_{i=2}^{n} x_i x_{i-1} \tag{199}$$

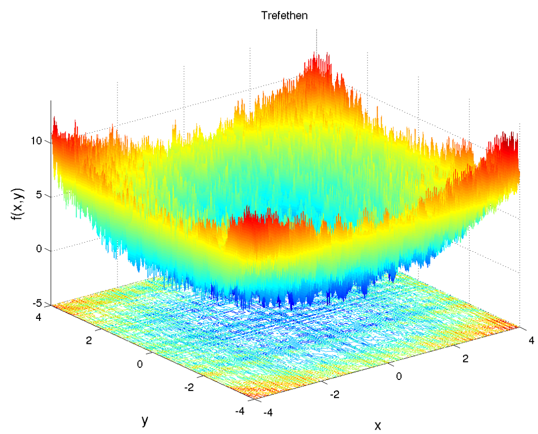


Figure 116: Trefethen for $n = 2$

with $x_i \in [-20, 20]$. For $n = 6$, $f(\mathbf{x}^*) = -50$, $\mathbf{x}^* = (6, 10, 12, 12, 20, 6)$. For $n = 10$, $f(\mathbf{x}^*) = -200$.

1.124 Trigonometric

The following Trigonometric functions:

- Trigonometric 1:

$$f(\mathbf{x}) = \sum_{i=1}^{n} \left[n - \sum_{j=1}^{n} \cos(x_j) + i(1 - \cos(x_i) - sin(x_i)) \right]^2 \tag{200}$$

  with $x_i \in [0, \pi]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.

- Trigonometric 2:

$$f(\mathbf{x}) = 1 + \sum_{i=1}^{n} 8\sin^2(7(x_i - x_i^*)^2) + 6\sin^2(14(x_1 - x_1^*)^2) + (x_i - x_i^*)^2 \tag{201}$$

  with $x_i \in [-500, 500]$, for which $f(\mathbf{x}^*) = 1$, and $\mathbf{x}^*$ any vector as substituted in the equation, e.g. $x_i^* = 0.9$.

1.125 Tripod

The Tripod function is defined as

$$f(\mathbf{x}) = p(x_2)(1 + p(x_1)) + |x_1 + 50p(x_2)(1 - 2p(x_1))| + |x_2 + 50(1 - 2p(x_2))| \tag{202}$$

where $p(x) = 1$ for $x \geq 0$ and $p(x) = 0$ otherwise; $x_i \in [-100, 100]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, -50)$.

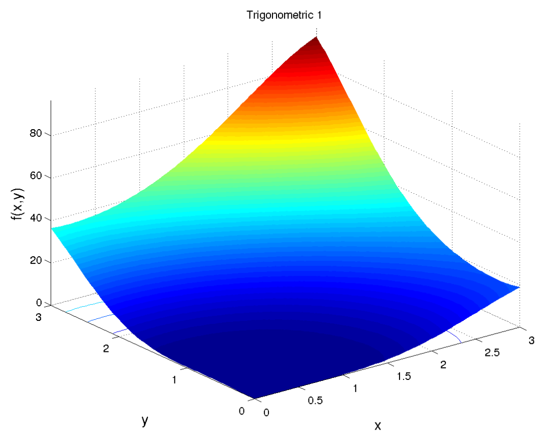
1.126 Ursem Functions

The following Ursem functions are defined as

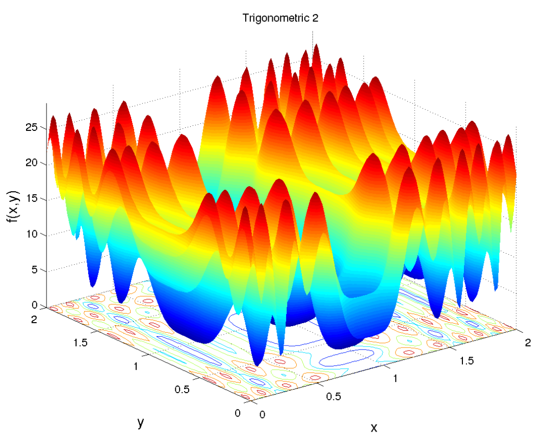
- Ursem 1:

$$f(\mathbf{x}) = -\sin(2x_1 - 0.5\pi) - 3\cos(x_2) - 0.5x_1 \tag{203}$$

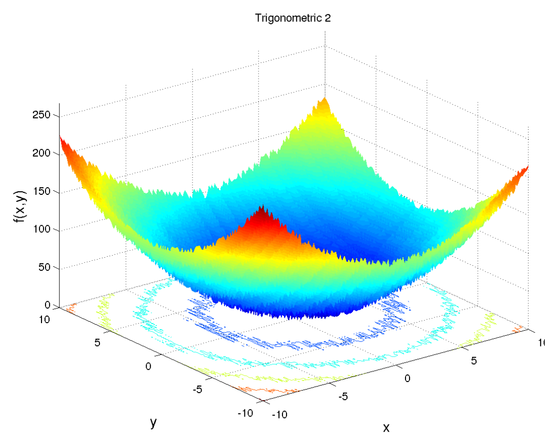
  with $x_1 \in [-2.5, 2.5], x_2 \in [-2, 2]$, for which $f(\mathbf{x}^*) = -4.8168$, $\mathbf{x}^* = (1.69714, 0)$.



(a) Trigonometric 1



(b) Trigonometric 2 small domain



(c) Trigonometric 2 large domain

Figure 117: Trigonometric Functions for $n = 2$

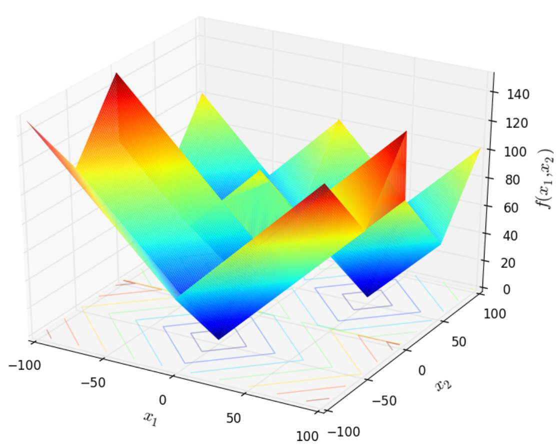


Figure 118: Tripod for $n = 2$

- Ursem 3:

$$f(\mathbf{x}) = -\sin(2.2\pi x_1 + 0.5\pi)\frac{2 - |x_1|}{2}\frac{(3 - |x_1|)}{2} - \sin(0.5\pi x_2^2 + 0.5\pi)\frac{2 - |x_2|}{2}\frac{(3 - |x_2|)}{2} \qquad (204)$$

with $x_1 \in [-2, 2], x_2 \in [-1.5, 1.5]$, for which $f(\mathbf{x}^*) = -3$, $\mathbf{x}^* = (0, 0)$. The function has four regularly spaced local minima positioned in a direct line, with the global minimum in the middle.

- Ursem 4:

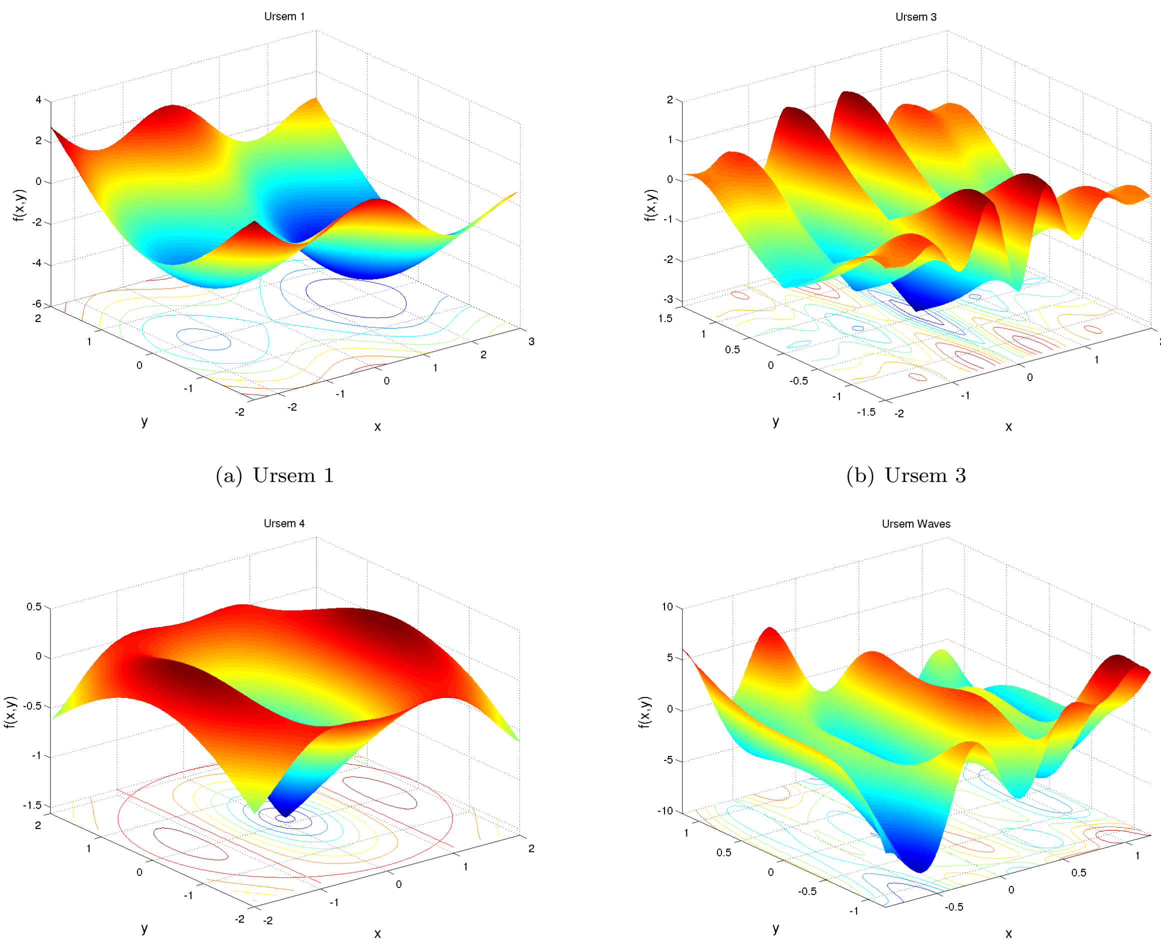
$$f(\mathbf{x}) = -3\sin(0.5\pi x_1 + 0.5\pi)\frac{(2 - \sqrt{x_1^2 + x_2^2})}{4} \tag{205}$$

with $x_i \in [-2, 2]$, for which $f(\mathbf{x}^*) = -1.5$, $\mathbf{x}^* = (0, 0)$. The global minimum is in the middle, and four local minima are at the corners of the search space.

- Ursem Waves:

$$f(\mathbf{x}) = -0.9x_1^2 + (x_2^2 - 4.5x_2^2)x_1x_2 + 4.7\cos(3x_1 - x_2^2(2 + x_1))\sin(2.5\pi x_1) \tag{206}$$

with $x_1 \in [-0.9, 1.2], x_2 \in [-1.2, 1.2]$, for which $f(\mathbf{x}^*) = -8.5536$, $\mathbf{x}^* = (1.2, 1.2)$. The function has nine irregularly spaced local minima.



(a) Ursem 1 (b) Ursem 3

(c) Ursem 4 (d) Ursem Waves

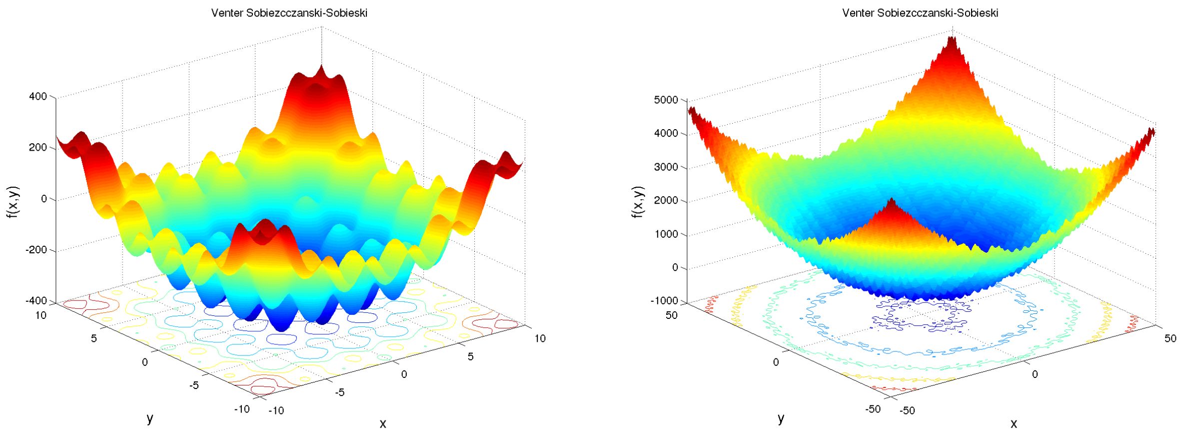
Figure 119: Ursem 1 for $n = 2$

1.127 Venter Sobiezcczanski-Sobieski

The Venter Sobiezcczanski-Sobieski is defined as:

$$f(\mathbf{x}) = x_1^2 - 100\cos(x_1)^2 - 100\cos(x_1^2/30) + x_2^2 - 100\cos(x_2)^2 - 100\cos(x_2^2/30) \tag{207}$$

with $x_i \in [-50, 50]$, for which $f(\mathbf{x}^*) = -400$, $\mathbf{x}^* = (0, 0)$.



(a) Venter Sobiezcczanski-Sobieski small domain

(b) Venter Sobiezcczanski-Sobieski large domain

Figure 120: Venter Sobiezcczanski-Sobieski for $n = 2$

1.128 Vincent

The Vincent function is defined as:

$$f(\mathbf{x}) = -\sum_{i=1}^{n} \sin(10 \log(x)) \tag{208}$$

with $x_i \in [0.25, 10]$, for which $f(\mathbf{x}^*) = -n$, $\mathbf{x}^* = (7.70628098, \ldots, 7.70628098)$.

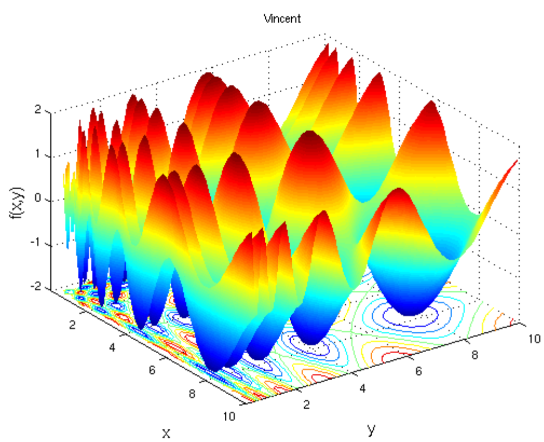


Figure 121: Vincent Function for $n = 2$

1.129 Watson

The Watson function is defined as:

$$f(\mathbf{x}) = \sum_{i=0}^{29} \left[\sum_{j=0}^{4} ((j-1) a_i^j x_{j+1}) - \left(\sum_{j=0}^{5} a_i^j x_{j+1} \right)^2 - 1 \right]^2 + x_1^2 \tag{209}$$

with $|x_i| \leq 10$, $a_i = i/29$, for which $f(\mathbf{x}^*) = 0.002288$, $\mathbf{x}^* = (-0.0158, 1.012, -0.2329, 1.260, -1.513, 0.9928)$.

1.130 Wavy

The Wavy function is defined as:

$$f(\mathbf{x}) = 1 - \frac{1}{n}\sum_{i=1}^{n}\cos(kx_i)e^{\frac{-x_i^2}{2}} \tag{210}$$

with $x_i \in [-\pi, \pi]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$. There are kn and $(k+1)n$ local minima for odd and even k respectively. For $n = 2$ and $k = 10$, there are 121 local minima.

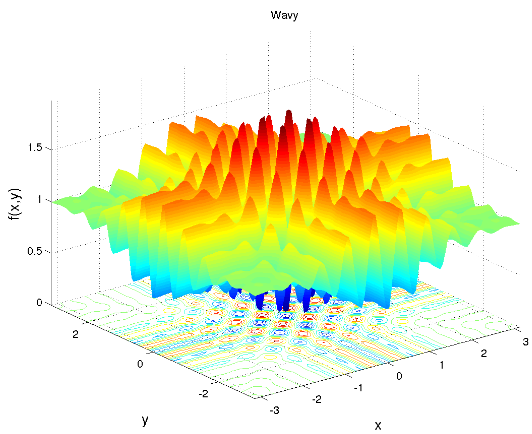


Figure 122: Wavy Function for $n = 2$

1.131 Wayburn-Seader

The following Wayburn-Seader functions are defined as:

- Wayburn-Seader 1:
$$f(\mathbf{x}) = (x_1^6 + x_2^4 - 17)^2 + (2x_1 + x_2 - 4)^2 \tag{211}$$
with $x_i \in [-5, 5]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (1, 2)$.

- Wayburn-Seader 2:
$$f(\mathbf{x}) = \left[1.613 - 4(x_1 - 0.3125)^2 - 4(x_2 - 1.625)^2\right]^2 + (x_2 - 1)^2 \tag{212}$$
with $x_i \in [-500, 500]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0.2, 1)$.

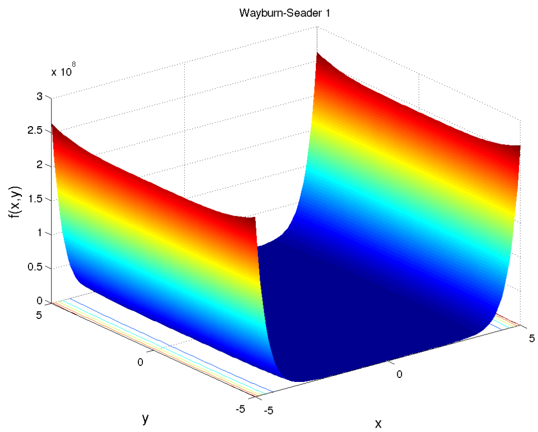
- Wayburn-Seader 3:
$$f(\mathbf{x}) = \frac{2x_1^3}{3} - 8x_1^2 + 33x_1 - x1x_2 + 5 + \left[(x_1 - 4)^2 + (x_2 - 5)^2 - 4\right]^2 \tag{213}$$
with $x_i \in [-500, 500]$, for which $f(\mathbf{x}^*) = 21.35$, $\mathbf{x}^* = (5.611, 6.187)$.

1.132 Weierstrass

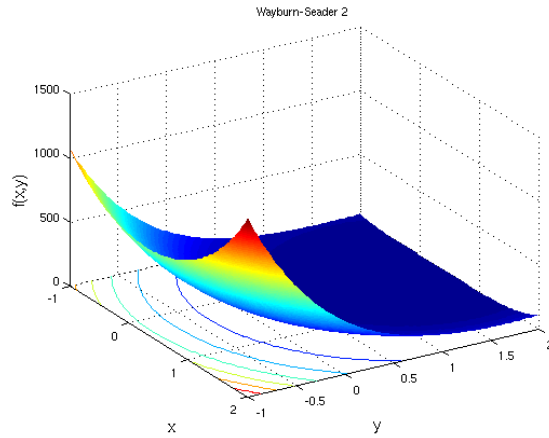
The Weierstrass function is defined as:

$$f(\mathbf{x}) = \sum_{i=1}^{n}\left(\sum_{j=0}^{j_{max}} a^j\cos(2\pi b^j(x_i + 0.5)) - n\sum_{j=1}^{j_{max}} a^j\cos(\pi b^j)\right) \tag{214}$$

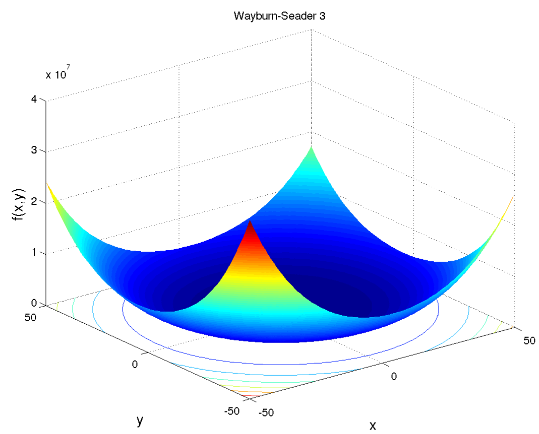
with $j_{max} = 20, a = 0.5, b = 3$, $x_i \in [-0.5, 0.5]$, for which $f(\mathbf{x}^*) = 4$, $\mathbf{x}^* = (0, \ldots, 0)$.



(a) Wayburn-Seader 1



(b) Wayburn-Seader 2



(c) Wayburn-Seader 3

Figure 123: Wayburn-Seader Functions for $n = 2$

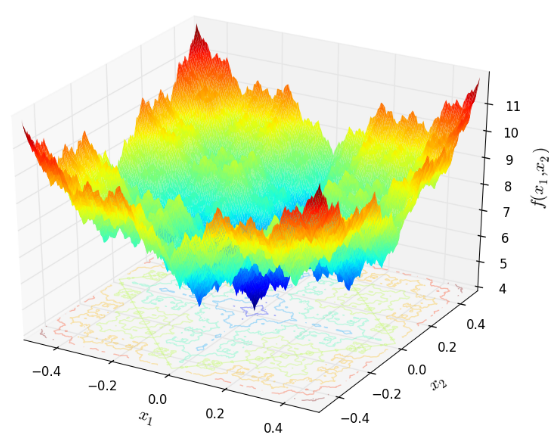


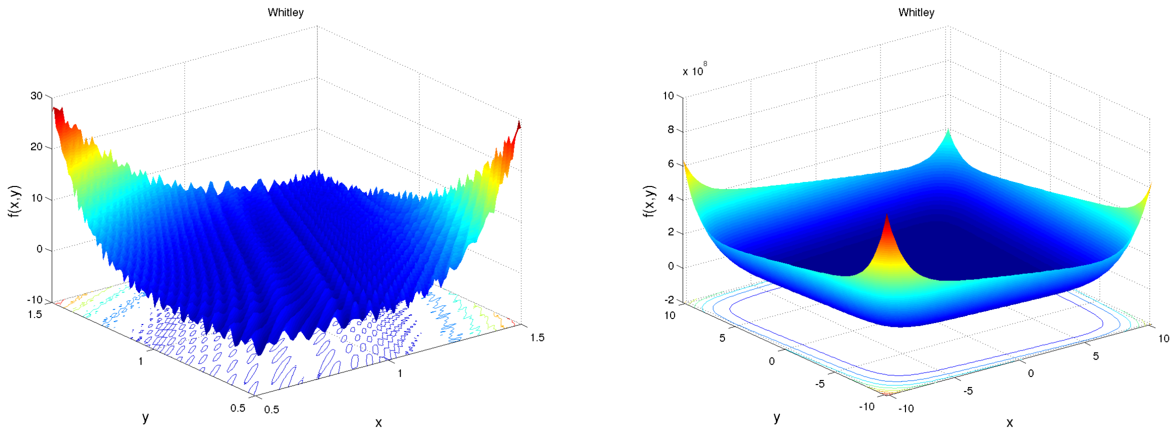
Figure 124: Weierstrass Function for $n = 2$

1.133 Whitley

The Whitley function is defined as:

$$f(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\frac{(100(x_i^2 - x_j)^2 + (1 - x_j)^2)^2}{4000} - \cos(100(x_i^2 - x_j)^2 + (1 - x_j)^2 + 1) \right] \tag{215}$$

with $x_i \in [-10.24, 10.24]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (1, \ldots, 1)$.



(a) Whitley small domain

(b) Whitley large domain

Figure 125: Whitley Function for $n = 2$

1.134 Wolfe

The Wolfe function is defined as:

$$f(\mathbf{x}) = \frac{4}{3}(x_1^2 + x_2^2)^{0.75} + x_3 \tag{216}$$

with $x_i \in [0, 2]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.

1.135 Xin-She Yang Functions

The following Xin-She Yang functions are defined as

- Xin-She Yang 1:

$$f(\mathbf{x}) = \sum_{i=1}^{n} \epsilon_i |x_i|^i \tag{217}$$

  with $\epsilon_i \sim U(0, 1)$, $x_i \in [-5, 5]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.

- Modified Xin-She Yang 1:

$$f(\mathbf{x}) = \sum_{i=1}^{n} \epsilon_i |x_i - \frac{1}{i}| \tag{218}$$

  with $\epsilon_i \sim U(0, 1)$, $x_i \in [-5, 5]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (1/n, \ldots, 1/n)$.

- Xin-She Yang 2:

$$f(\mathbf{x}) = \left(\sum_{i=1}^{n} |x_i| \right) e^{-\sum_{i=1}^{n} \sin(x_i^2)} \tag{219}$$

  with $x_i \in [-2\pi, 2\pi]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.

- Xin-She Yang 3:

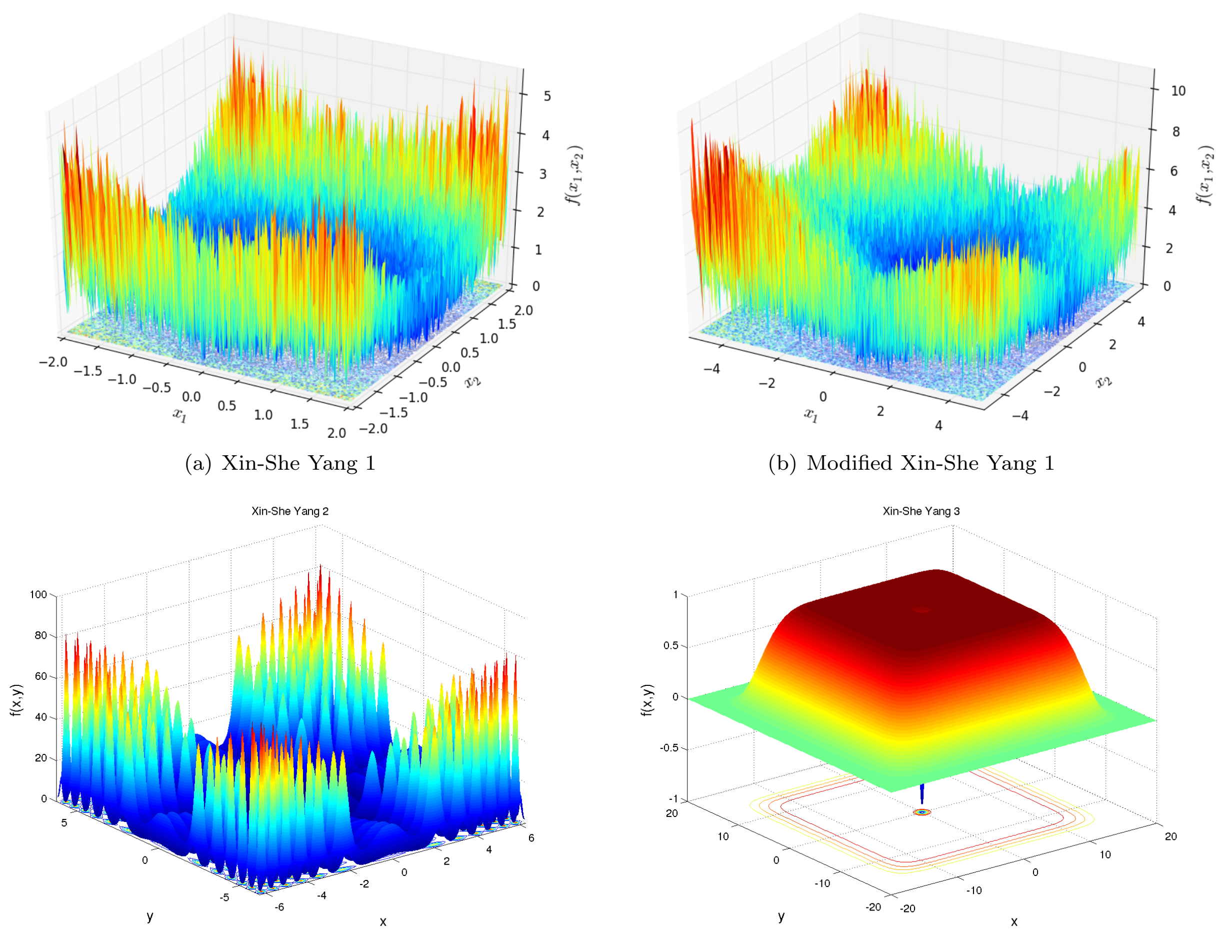
$$f(\mathbf{x}) = \left[e^{-\sum_{i=1}^{n} (x_i/\beta)^{2m}} - 2e^{-\sum_{i=1}^{n} x_i^2} \prod_{i=1}^{n} \cos^2(x_i) \right] \tag{220}$$

  with $x_i \in [-20, 20]$, for which $f(\mathbf{x}^*) = -1$, $\mathbf{x}^* = (0, \ldots, 0)$ when $m = 5$ and $\beta = 15$.

- Xin-She Yang 4:

$$f(\mathbf{x}) = \left[\sum_{i=1}^{n} \sin^2(x_i) - e^{-\sum_{i=1}^{n} x_i^2}\right] e^{-\sum_{i=1}^{n} \sin^2 \sqrt{|x_i|}} \tag{221}$$

with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = -1$, $\mathbf{x}^* = (0, \ldots, 0)$.



(a) Xin-She Yang 1

(b) Modified Xin-She Yang 1

(c) Xin-She Yang 2

(d) Modified Xin-She Yang 3

Figure 126: Xin-She Yang Functions for $n = 2$

1.136 Yao-Liu

The Yao-Liu 4 function is defined as:

$$f(\mathbf{x}) = \max_i\{|x_i|\} \tag{222}$$

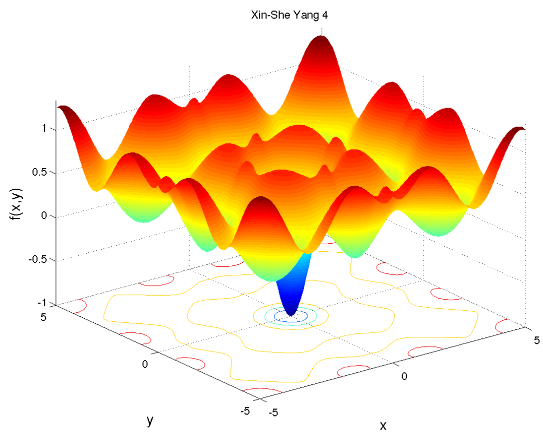
with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (0, \ldots, 0)$.

1.137 Zakharov

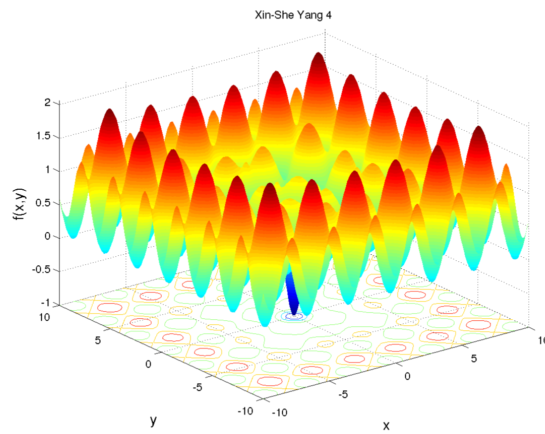
The Zakharov function is defined as:

$$f(\mathbf{x}) = \sum_{i=1}^{n} x_i^2 + \left(\frac{1}{2}\sum_{i=1}^{n} i x_i\right)^2 \left(\frac{1}{2}\sum_{i=1}^{n} i x_i\right)^4 \tag{223}$$

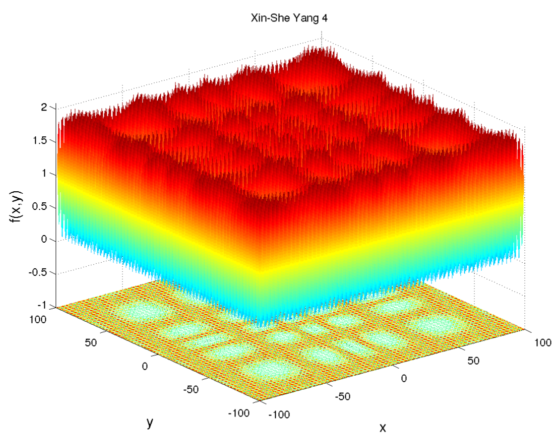
with $x_i \in [-5, 10]$, for which $f(\mathbf{x}^*) = -1$, $\mathbf{x}^* = (0, \ldots, 0)$.



(a) Xin-She Yang 4 small domain



(b) Xin-She Yang 4 large domain



(c) Xin-She Yang 4 larger domain

Figure 127: Xin-She Yang Functions for $n = 2$ (cont)

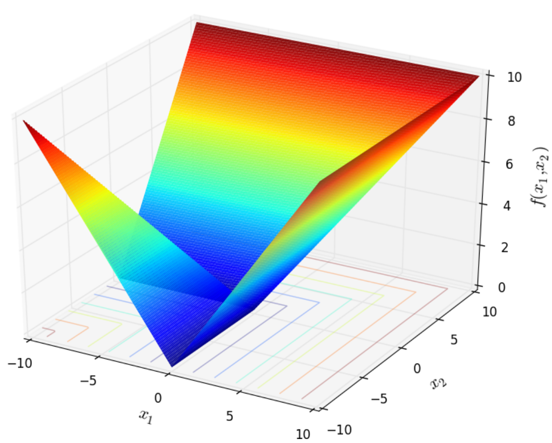


Figure 128: Yao-Liu Function for $n = 2$

1.138 Zero Sum

The Zero Sum function is defined as:

$$f(\mathbf{x}) = \begin{cases} 0 & \text{if } \sum_{i=1}^{n} = 0 \\ 1 + (10000|\sum_{i=1}^{n} x_i|)^{0.5} & \text{otherwise} \end{cases} \tag{224}$$

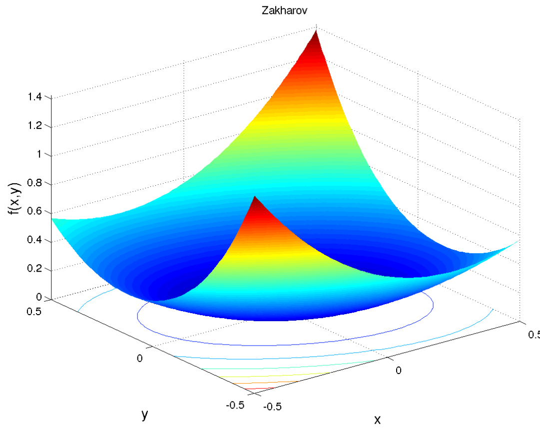


Figure 129: Zakharov Function for $n = 2$

with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) = 0$ where $\sum_{i=1}^{n} x_i = 0$.

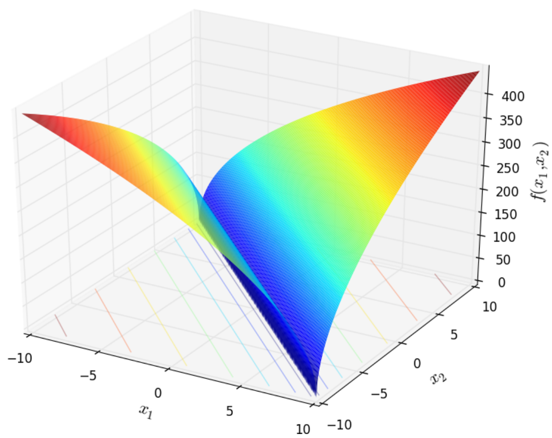


Figure 130: Zero Sum Function for $n = 2$

1.139 Zettle

The Zettle function is defined as:

$$f(\mathbf{x}) = (x_1^2 + x_2^2 - 2x_1)^2 + 0.25x_1 \tag{225}$$

with $x_i \in [-1, 5]$, for which $f(\mathbf{x}^*) = -0.003791$, $\mathbf{x}^* = (-0.0299, 0)$.

1.140 Zimmerman

The Zimmerman function is defined as:

$$f(\mathbf{x}) = \max\{Zh_1(\mathbf{x}), Zp(Zh_2(\mathbf{x}))\text{sgn}(Zh_2(\mathbf{x})), Zp(Zh_3(\mathbf{x}))\text{sgn}(Zh_3(\mathbf{x})), Zp(-x_1)\text{sgn}(x_1), Zp(-x_2)\text{sgn}(x_2)\} \tag{226}$$

where

$$Zh_1(\mathbf{x}) = 9 - x_1 - x_2$$
$$Zh_2(\mathbf{x}) = (x_1 - 3)^2 + (x_2 - 2)^2$$
$$Zh_3(\mathbf{x}) = x_1 x_2 - 14$$

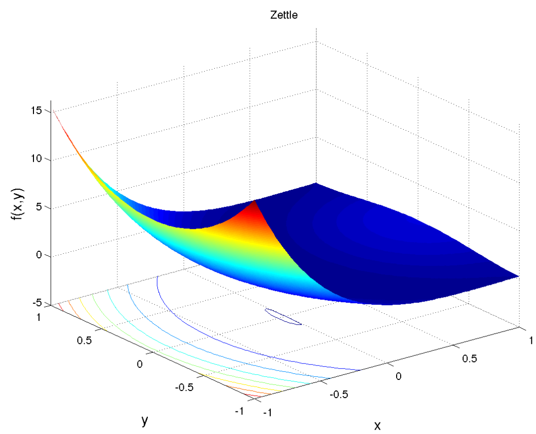


Figure 131: Zettle Function for $n = 2$

$$Zp(t) = 100(1 + t)$$

with $x_i \in [0, 100]$, for which $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* = (7, 2)$.

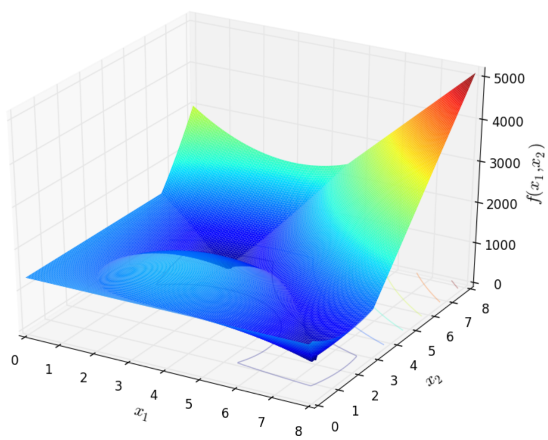


Figure 132: Zimmerman Function for $n = 2$

1.141 Zirilli

The Zirilli function is defined as:

$$f(\mathbf{x}) = 0.25x_1^4 - 0.5x_1^2 + 0.1x_1 + 0.5x_2^2 \tag{227}$$

with $x_i \in [-10, 10]$, for which $f(\mathbf{x}^*) \approx -0.3523$, $\mathbf{x}^* = (-1.0465, 0)$.

Table 1: Main Function Characterisitics (C=continuous, NC=non-continuous, D=differentiable, ND=non-differentiable, S=separable, NS=non-separable, MM=multi-modal, UM=unimodal)

Function	**Equation**	**Continuous**	**Differentiable**	**Separable**	**Multi-Modal**

Absolute Value	1	C	ND	S	UM
Ackley 1	2	C	D	NS	MM
Ackley 2	3	C	D	NS	UM
Ackley 3	4	C	D	NS	MM
Ackley 4	5	C	D	NS	MM
Adjiman	6	C	D	NS	MM
Alpine 1	7	C	ND	S	MM
Alpine 2	8	C	D	S	MM
Arithmetic Mean	8	C	D	NS	MM
Bartels-Conn	10	C	ND	NS	UM
Beale	11	C	D	NS	UM
Biggs EXP 2	12	C	D	NS	MM
Biggs EXP 3	13	C	D	NS	MM
Biggs EXP 4	14	C	D	NS	MM
Biggs EXP 5	15	C	D	NS	MM
Biggs EXP 6	16	C	D	NS	MM
Bird	17	C	D	NS	MM
Bohachevsky1	18	C	D	S	MM
Bohachevsky2	19	C	D	NS	MM
Bohachevsky3	20	C	D	NS	MM
Bonyadi-Michalewicz	21	C	D	NS	MM
Booth	22	C	D	NS	UM
Box-Betts Quadratic Sum	23	C	D	NS	MM
Brad	24	C	D	NS	MM
Brainin RCOS 1	25	C	D	NS	MM
Brainin RCOS 2	26	C	D	NS	MM
Brent	27	C	D	NS	UM
Brown	28	C	D	NS	UM
Bukin 2	29	C	D	S	??
Bukin 2 Adapted	30	C	D	S	??
Bukin 4	31	C	ND	S	??
Bukin 6	31	C	ND	NS	MM
Camel Three Hump	33	C	ND	NS	MM
Camel Six Hump	34	C	ND	NS	MM
Central Two Peak Trap	35	C	ND	S	MM
Chen Bird	36	C	D	NS	MM
Chen V	37	C	D	NS	MM
Chichinadze	38	C	D	S	MM
Chung-Reynolds	39	C	D	≈S	UM
Cigar	40	C	D	S	UM
Colville	41	C	D	NS	MM
Corana	42	NC	ND	S	MM
Cosine Mixture	43	C	D	S	MM
Cross-in-Tray	44	C	ND	NS	MM
Cross Leg Table	45	?	ND	?	MM
Crowned Cross	46	C	ND	?	MM
Csendes	47	C	D	S	?
Cube	48	C	D	NS	UM
Deb 1	50	C	D	S	MM
Deb 2	51	C	D	S	MM
Decanomial	52	C	ND	S	?
Deceptive	53	C	?	S	MM
Deckkers-Aarts	54	C	D	NS	MM
Deflected Corrugated Spring	55	C	D	??	MM

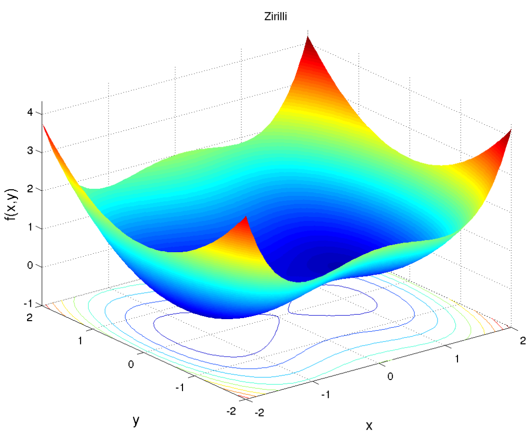
De Jong F4	56	NC	D	S	MM
DeVilliers-Glasser 1	57	C	D	NS	MM
DeVilliers-Glasser 2	58	C	D	NS	MM
Different Powers	59	C	D	S	UM
Discuss	60	C	D	S	UM
Dixon-Price	61	C	D	NS	UM
Dolan	62	C	ND	NS	MM
Drop Wave	63	C	D	??	MM
Easom	65	C	D	S	MM
Egg Crate	66	C	D	S	MM
Egg Holder	67	C	ND	NS	MM
El-Attar-Vidyasagar-Dutta	68	C	D	NS	UM
Elliptic	69	C	D	S	UM
Rotated Ellipse 1	70	C	D	NS	UM
Rotated Ellipse 2	71	C	D	NS	UM
Exp 1	72	C	D	??	UM
Exp 1	72	C	D	NS	UM
Freudenstein-Roth	74	C	D	NS	MM
Gear	75	?	?	NS	MM
Giunta	76	C	D	S	MM
Goldstein-Price 1	77	C	D	NS	MM
Goldstein-Price 2	78	C	D	NS	MM
Griewank	79	C	D	NS	MM
Hansen	80	C	D	NS	MM
Hartman 3	81	C	D	NS	MM
Hartman 6	82	C	D	NS	MM
Himmelblau	84	C	D	NS	MM
Holzman	85	C	D	NS	MM
Hosaki	86	C	D	NS	MM
Infinity	89	C	D	S	??
Hyperellipsoid	87	C	D	S	UM
Jennrich-Sampson	90	C	D	NS	MM
Judge	89	C	D	NS	MM
Katsuura	92	C	D	NS	??
Keane	93	C	D	NS	MM
Kowalik	94	C	D	NS	MM
Langermann 5	95	C	D	NS	MM
Lanczos 1	96	C	D	S	MM
Lanczos 2	97	C	D	NS	MM
Lanczos 3	98	C	D	NS	MM
Lennard-Jones	99	C	D	NS	MM
Leon	100	C	D	NS	UM
Levy 3	101	C	D	NS	MM
Levy 5	102	C	D	NS	MM
Levy 13	103	C	D	NS	MM
Levy-Montalvo 2	104	C	D	NS	MM
Matyas	105	C	D	NS	UM
McCormick	106	C	D	NS	MM
Michalewicz	107	C	D	S	MM
Miele-Cantrell	108	C	D	NS	MM
Mishra 1	109	C	D	NS	MM
Mishra 2	110	C	D	NS	UM
Mishra 3	111	C	ND	NS	MM
Mishra 4	112	C	ND	NS	MM

Mishra 5	113	C	D	NS	MM
Mishra 5 modified	114	C	D	NS	MM
Mishra 6	115	C	D	NS	MM
Mishra 7	116	C	D	NS	MM
Mishra 8	117	C	D	NS	MM
Mishra 9	118	C	D	NS	MM
Mishra 10	119	C	??	NS	MM
Mishra 11	120	C	ND	NS	MM
Multi Gaussian	121	C	D	S	MM
Multi Modal	122	C	ND	NS	MM
Needle Eye	123	C	ND	S	MM
New Function 3	**??**	C	D	NS	MM
Norwegian	124	C	D	NS	MM
Odd Square	125	?	?	NS	MM
Parsopoulos	126	C	D	S	MM
Pathological	127	C	D	NS	MM
Paviani	128	C	D	NS	MM?
Penalty 1	129	C	D	NS	MM
Penalty 2	130	C	D	NS	MM
Pen Holder	131	C	D	NS	MM
Periodic	132	C	D	S	MM
Perm 1	133	C	D	S	MM
Perm 2	134	C	D	S	MM
Pinter	135	C	D	NS	MM
Powell	136	C	D	NS	MM
Powell Singular	137	C	D	NS	UM
Powell Singular 2	138	C	D	NS	UM
Powell Sum	139	C	ND	S	UM
Power Sum	140	C	ND	NS	MM
Price 1	141	C	ND	NS	MM
Price 2	142	C	D	NS	MM
Price 3	143	C	D	NS	MM
Price 4	144	C	D	NS	MM
Qing	145	C	D	S	MM
Quadratic	146	C	D	NS	UM
Quadric (Schwefel 1.2)	147	C	D	NS	UM
Quartic	148	C	D	S	UM
Quintic	149	C	ND	S	MM
Rana	150	C	ND	NS	MM
Rastrigin	151	C	D	S	MM
Ripple 1	152	C	D	S	MM
Ripple 25	153	C	D	S	MM
Rosenbrock	154	C	D	NS	UM/MM[1]
Modified Rosenbrock	155	C	D	NS	MM
Rump	156	C	D	NS	UM
Salomon	157	C	D	NS	MM
Sargan	158	C	D	NS	UM
Schaffer 1	159	C	D	NS	MM
Schaffer 2	160	C	D	NS	MM
Schaffer 3	161	C	ND	NS	MM
Schaffer 4	162	C	D	NS	MM
Schmidt-Vetters	163	C	D	NS	MM
Schumer-Steiglitz	164	C	D	S	UM

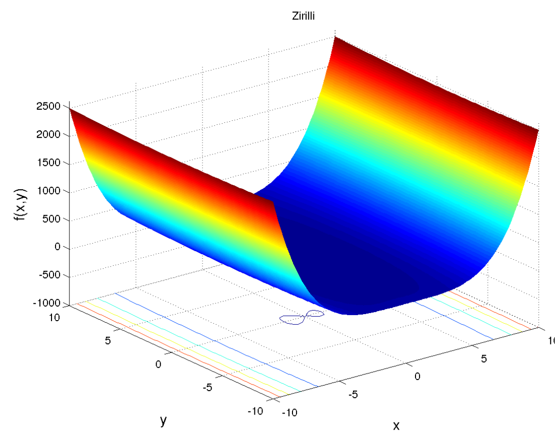
[1]UM for small n

Schwefel 1	165	C	D	S	UM
Schwefel 2.4	167	C	D	NS	MM
Schwefel 2.6	168	C	ND	NS	UM
Schwefel 2.13	169	C	D	NS	MM
Schwefel 2.21	170	C	ND	S	UM
Schwefel 2.22	171	C	ND	NS	UM
Schwefel 2.23	172	C	D	S	UM
Schwefel 2.26	173	C	ND	S	MM
Schwefel 2.36	175	C	D	NS	UM (?)
Shekel 5	176	C	D	NS	MM
Shekel 7	177	C	D	NS	MM
Shekel 10	178	C	D	NS	MM
Shubert	179	C	D	NS	MM
Shubert 3	180	C	D	S	MM
Shubert 4	181	C	D	S	MM
Sine Envelope	182	C	D	NS	MM
Sinusoidal	183	C	D	NS	MM
Spherical	184	C	D	S	UM
Step 1	185	NC	ND	S	UM
Step 2	186	NC	ND	S	MM
Step 3	187	NC	ND	S	MM
Step Plateau	188	NC	ND	S	MM
Storn's Tchebyshev Problem	189	C (?)	D	NS	MM
Stretched V Sine Wave	190	C	D	NS	MM
Sum Squares	191	C	D	S	UM
Sum of Different Powers	192	C	D	S	UM
Styblinski-Tang 193	C	D	S	MM	
Holder Table 1	194	C	ND	NS	MM
Holder Table 2	195	C	ND	NS	MM
Holder Table 3	196	C	D	NS	MM
Test Tube Holder	197	C	ND	NS	MM
Trecanni	198	C	D	S	MM
Trid	200	C	D	NS	MM
Trefethen	199	C	D	NS	MM
Trigonometric 1	201	C	D	NS	MM
Trigonometric 2	202	C	D	NS (?)	MM
Tripod	203	C	ND	NS	MM
Ursem 1	204	C	D	S	MM
Ursem 3	205	C	ND	NS (?)	MM
Ursem 4	206	C	D	NS	MM
Ursem Waves	207	C	D	NS	MM
Venter Sobiezcczanski-Sobieski	208	C	D	S	MM
Vincent	209	C	D	S	MM
Watson	210	C	D	NS	UM
Wavy	211	C	D	S	UM
Wayburnseader 1	212	C	D	NS	UM
Wayburnseader 2	213	C	D	NS	UM
Wayburnseader 3	214	C	D	NS	UM
Weierstrass	215	C	D	S	MM
Whitley	216	C	D	NS	MM
Wolfe	217	C	D	NS	MM
Xin-She Yang 1 218	NC (?)	D (?)	S	MM	
Xin-She Yang 1 Modified 219	NC (?)	D (?)	S	MM	
Xin-She Yang 2 220	C	ND	NS	MM	

Xin-She Yang 3 221	C	D	NS	MM	
Xin-She Yang 4 222	C	ND	NS	MM	
Yao-Liu 4	223	C	ND	S	MM
Zakharov	224	C	D	NS	UM
Zero Sum	225	NC (?)	ND	NS	MM
Zettle	226	C	D	NS	MM (?)
Zimmerman	227	C	D	NS	MM
Zirilli	228	C	D	S	MM



(a) Small Domain



(b) Large Domain

Figure 133: Zirilli Function for $n = 2$