Circuit Theory and Electronics Fundamentals

MEAer (Integrated Master In Aerospace Engineering), Técnico, University of Lisbon

Laboratory 2: RC Circuit analysis

Group 3

Diogo Faustino, nº95782
Henry Machado, nº95795
Rúben Novais, nº95843

April 5, 2021

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1 Introduction

The objective of this laboratory assignment is to study the behaviour of an RC circuit powered by a sinusoidal voltage source V_s, as seen in Figure 1. In order to analyse it, we will make use of different techniques such as Thevenin and Phasor analysis in order to establish the natural, forced and total solutions of the circuit. A frequency response analysis then follows, where we look at the magnitude and phase as a function of frequency. Then, we compare the theoretical and simulation data.

In Section 2, we go through the various steps that are needed to obtain the total solution of the circuit (natural plus forced responses) and present the theoretical frequency response analysis. In Section 3, we present the circuit response as simulated in NGSpice. The results in operating point, transient analysis and frequency analysis are compared to the theoretical results obtained in Section 2 and the conclusions of this study are laid out in Section 4.

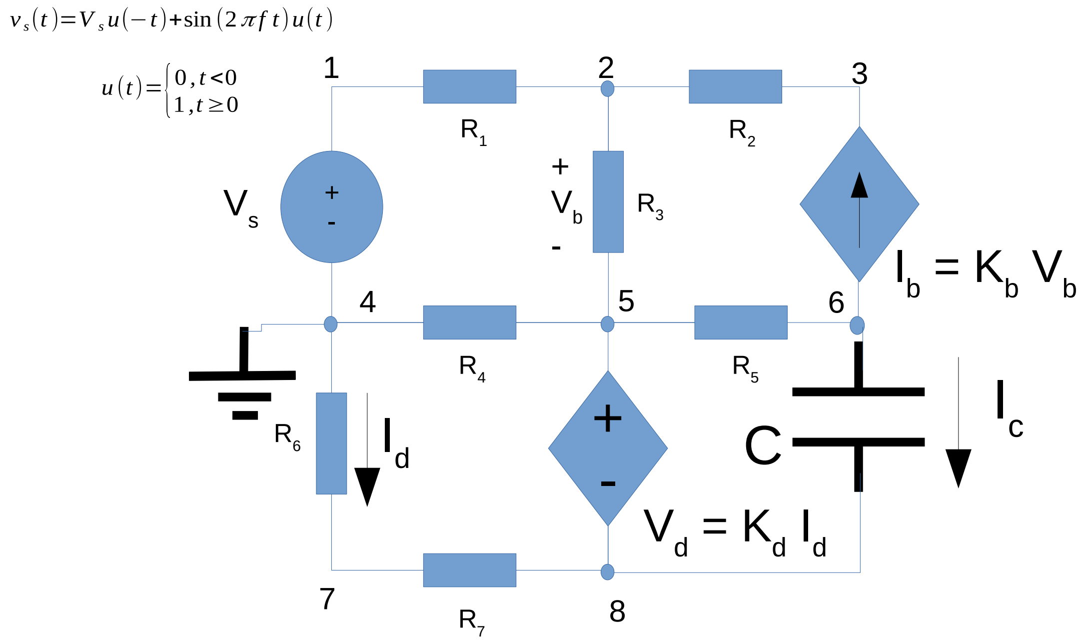

Figure 1: RC circuit.

2 Theoretical Analysis

Here follows the list of the components numeric values that we will use in our theoretical (and simulation) analyses (Table 1) - this values were generated by a python script based on our student ID numbers and can easily be changed in the top Makefile:

2.1 Node Analysis for t<0

We analyse the circuit shown in Figure 1 for t<0 using the nodal method. The nodes are numbered according to what's shown in the picture. In this first instance we are working in a steady state, where no current is flowing through the capacitor: we can replace it with an open circuit. After doing this, it is clear that all the components we are working with are linear and so we will need to solve a system of linear equations to determine the initial values for the subsequent analysis. This way, we've ran the nodal method and solved the linear system on GNU Octave.

Name	Values
R1	1.021831 kOhm
R2	2.005193 kOhm
R3	3.017358 kOhm
R4	4.151360 kOhm
R5	3.044174 kOhm
R6	2.021970 kOhm
R7	1.007843 kOhm
V_s	5.113399 V
C	1.005042 uF
K_b	7.181066 mS
K_d	8.122711 kOhm

Table 1: Components numeric values used in our analysis and simulation.

Knowing that V4=0 since it is connected to the ground:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -G1 & G1+G2+G3 & -G2 & -G3 & 0 & 0 & 0 \\ 0 & Kb+G2 & -G2 & -Kb & 0 & 0 & 0 \\ -G1 & G1 & 0 & G4 & 0 & G6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -G6-G7 & G7 \\ 0 & 0 & 0 & 1 & 0 & G6*Kd & -1 \\ 0 & -G3 & 0 & G3+G4+G5 & -G5 & G6 & 0 \end{pmatrix} \begin{pmatrix} V1 \\ V2 \\ V3 \\ V5 \\ V6 \\ V7 \\ V8 \end{pmatrix} = \begin{pmatrix} Vs \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{1}$$

We can also easily obtain the current values in the various branches through Ohm's law. This yields the following results:

Name	Value [mA and V]
$@I_{R1}$	0.219167
$@I_{R2}$	0.229771
$@I_{R3}$	0.010604
$@I_{R4}$	1.185502
$@I_{R5}$	0.229771
$@I_{R6}$	0.966335
$@I_{R7}$	0.966335
$@I_b$	-0.229771
$@I_C$	0.000000
$@I_{Vs}$	0.219167
$@I_{Kd}$	0.966335
V_1	5.113399
V_2	4.889447
V_3	4.428712
V_4	0.000000
V_5	4.921444
V_6	5.620907
V_7	-1.953900
V_8	-2.927814

Table 2: Theoretical analysis results for time, t, inferior to 0 . (A variable preceded by @ is of type *current*)

2.2 Equivalent Resistor

In order to compute the equivalent resistor seen through the capacitor, we ran a nodal analysis, making Vs=0 and replacing the capacitor with a voltage source Vx=V6-V8 as calculated in the previous step. This is made to ensure that the voltage in the capacitor is continuous since it does so in reality: this is a capacitor discharging through a resistance, any discontinuity in voltage would require an infinite amount of current; the energy stored in the capacitor must be conserved (E_C = $\frac{1}{2}$CV^2).

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -G1 & G1+G2+G3 & -G2 & -G3 & 0 & 0 & 0 \\ 0 & Kb+G2 & -G2 & -Kb & 0 & 0 & 0 \\ -G1 & G1 & 0 & G4 & 0 & G6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -G6-G7 & G7 \\ 0 & 0 & 0 & 1 & 0 & G6*Kd & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} V1 \\ V2 \\ V3 \\ V5 \\ V6 \\ V7 \\ V8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ Vx \end{pmatrix} \tag{2}$$

With the following definitions:

$$V_x = V_6 - V_8; \tag{3}$$

$$I_x = \frac{V_6 - V_5}{R_5} + \frac{V_3 - V_2}{R_2}; \tag{4}$$

$$Req = \frac{V_x}{I_x} \tag{5}$$

$$\tau = Req * C \tag{6}$$

We were able to obtain the following values, laying the foundation for the natural solution (the characteristic time, τ, will mark the decay of voltage in the capacitor, as we will see in the following subsection):

Name	Value
V_x	8.548721 V
V_6	8.548721 V
V_8	0.000000 V
$@I_x$	2.808224 mA
R_{eq}	3.044174 kOhm
tau	3.059522 ms

Table 3: Theoretical analysis calculations for R_{eq} and tau.

2.3 Natural solution

The natural solution for a node voltage dosen't take into account the independent sources of the circuit. The general solution for the natural regime in RC circuits, making use of the computations in the previous section, yields:

$$V_{6n}(t) = V_{6n}(\infty) + (V_{6n}(0) - V_{6n}(\infty))e^{(-\frac{t}{\tau})}; \tag{7}$$

The initial voltage in node 6 was obtained in the previous subsection, when we determined the equivalent resistor and the characteristic time. Calculating the V_6 node voltage in natural regime for t=∞ is very simple - we need only to consider the same linear system of equations we used for our t<0 analysis, but this time V_s = 0. Without any independent source powering the circuit, it is obvious that after an infinite amount of time the nodal voltages and branch currents will all be null. Nevertheless, we solved the system and our predictions checked out, as we can see in Table 4.

From this equation, we computed the natural response in node 6, in the [0,20] milisseconds interval of time (Figure 2).

Name	Value [V]
V_1	0.000000
V_2	0.000000
V_3	-0.000000
V_4	0.000000
V_5	0.000000
V_6	-0.000000
V_7	-0.000000
V_8	0.000000

Table 4: Node voltages for t=∞.

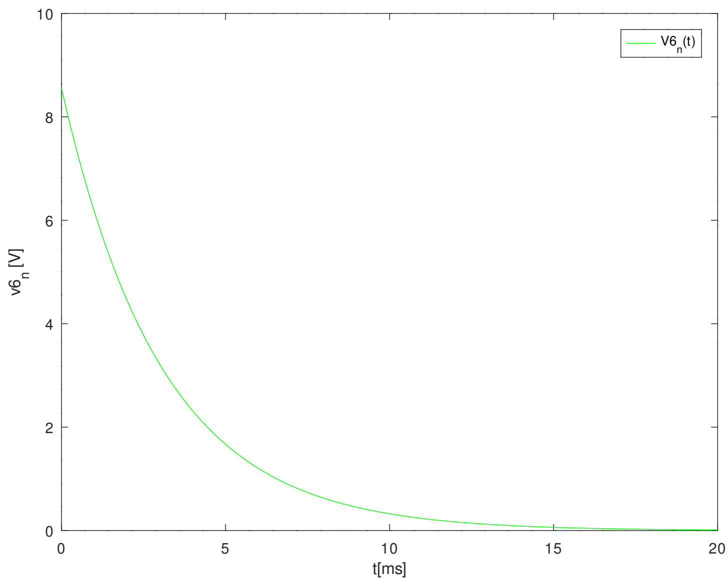

Figure 2: Natural solution to V_6 node voltage.

2.4 Forced solution

In order to study the forced response of the system, we perform a nodal analysis to obtain the phasor voltage in every node. The voltage in every node will now be a function of the following type: $V = |V|e^{j(wt+\phi)}$, with the phasor expression given by $\tilde{V} = |V|e^{j\phi}$. Since there is only a single independent source acting in the circuit, every node voltage will oscillate with the same frequency - the source frequency - so we don't need to take frequency into acount in our nodal analysis and we can work solely on phasors.

In this forced regime, the independent voltage source has a sinusoidal signal, with the following expression, $sin(wt)$, with $w = 2\pi * f$ (in this study, we consider $f = 1kHz$). To perform the nodal analysis and determine the phasors' magnitudes and phases we must now work with impedances, instead of resistances and capacitance - the impedance of a resistor is R and the impedance of a capacitor is given by $(\frac{1}{jwC})$.

With all these definitions, we can now solve our linear system of equations:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -G1 & G1+G2+G3 & -G2 & -G3 & 0 & 0 & 0 \\ 0 & Kb+G2 & -G2 & -Kb & 0 & 0 & 0 \\ -G1 & G1 & 0 & G4 & 0 & G6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -G6-G7 & G7 \\ 0 & 0 & 0 & 1 & 0 & G6*Kd & -1 \\ 0 & -G3 & 0 & G3+G4+G5 & -G5-jwC & G6 & jwC \end{pmatrix} \begin{pmatrix} \tilde{V1} \\ \tilde{V2} \\ \tilde{V3} \\ \tilde{V5} \\ \tilde{V6} \\ \tilde{V7} \\ \tilde{V8} \end{pmatrix} = \begin{pmatrix} -j \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{8}$$

We then get these values:

Name	Amplitude [V]	Phase [Degrees]
V_1	1.000000	-90.000000
V_2	0.956203	-90.000000
V_3	0.866099	-90.000000
V_4	0.000000	0.000000
V_5	0.962460	-90.000000
V_6	0.574648	98.680964
V_7	0.382114	90.000000
V_8	0.572577	90.000000

Table 5: Phasor voltage for forced regime in every node.

We're now able to make sense of the equation that describes the forced solution to the voltage at node 6.

$$V_{6f}(t) = V_{6r}cos(wt + V_{6\phi}) = 0.574648cos(2000\pi * t + 1.7223); \tag{9}$$

2.5 Final total solution

Now we convert the phasors to real time functions and consider an angular frequency of 2000π. By superimposing both natural and forced responses we get the total solution:

$$V_6(t) = V_{6f}(t) + V_{6n}(t); \tag{10}$$

By plotting both $V_s(t)$ and $V_6(t)$ from -5ms to 20ms we can see that both plots are constant before t=0 (Figure 3). The evolution of V6 is as expected, as we can clearly see the negative exponential behaviour as well as the induced frequency.

2.6 Frequency response

As we can see in the following plots, for low frequencies ($<$1Hz) every voltage is in phase. The capacitor is given enough time to charge up (asymptotically) to the same voltage as the voltage source, whose phase and magnitude remain constant throughout the analysed period. This means that at 0.1 Hz (period of 10 seconds), the capacitor charges and discharges in a fraction of a second and behaves like an open circuit for the rest of the cycle. We start seeing divergences in the graphs around the "cut-off frequency", defined as $\frac{1}{2\pi\tau}$, as we will see, which in our case is around 52Hz, explaining the changes happening between 10 and 10^2Hz. When we consider a time interval in the same order of magnitude of τ, the time it takes for a capacitor to deplete 36.8% of its charge through a resistance, we are no longer giving the capacitor

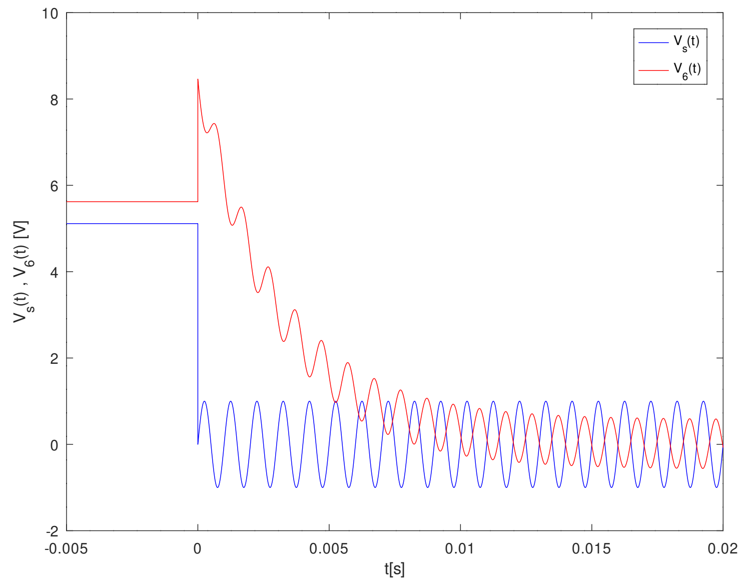

Figure 3: Total solution to V_6 node voltage, compared to V_s voltage source.

enough time to charge or discharge (almost) all the way. For much higher frequencies than the cut off, the capacitor starts behaving like a short circuit since it can no longer oppose the change of current (its reactance drops). The apparent phase discontinuity is only due to the use of the arctan function, whose output is bounded by [-180,180] until it circles back around to the top. So the phase is in fact continuous. The following equations help describe the behaviour shown in the plots. They come from reducing the circuit to a sinusoidal voltage source, equivalent resistance and a capacitor.

$$\tilde{V_c} = \frac{\tilde{V_i}}{1 + j(R_{eq}C)2\pi * f} \tag{11}$$

$$|\tilde{V_c}| = \frac{|\tilde{V_i}|}{\sqrt{1 + ((R_{eq}C)2\pi * f)^2}} \tag{12}$$

$$\phi V_c = -\frac{\pi}{2} - arctan((R_{eq})C2\pi * f) \tag{13}$$

Converting the magnitude equation to dB, we encounter a term from which the "cut-off" angular frequency appears, equal to $\frac{1}{R_{eq}C}$: $-10\log_{10}(1 + w^2(R_{eq}C)^2)$. In an other form, the "cut-off" frequency can be defined as the point of intersection of two asymptotes: the horizontal asymptote for frequencies close to zero and the oblique asymptote, for very high frequencies. Once again, after this point the voltage stored in the capacitor starts dropping considerably.

In the magnitude plot of the V_6 node voltage, we have two horizontal asymptotes, one for frequencies very close to zero, other for very high frequencies, which means the node voltage remains constant for low frequencies, transforming then when the "cut-off" frequency is achieved, stabilizing afterwards at a lower value for very high frequencies. The same process happens regarding the V_6 phase: at an early stage the phase is of course the source's -90 degrees, but then transforms stabilizing at an opposite phase of 90 degrees. The V_c phase variation is well described in Equation 13, starting at the before mentioned -90 degrees, and settling at a -180 degrees phase - for very high frequencies, the V_c and V_6 voltages are not in phase anymore - with such a high frequency, V_8 is not given enough time to phase-in in relation to V_6.

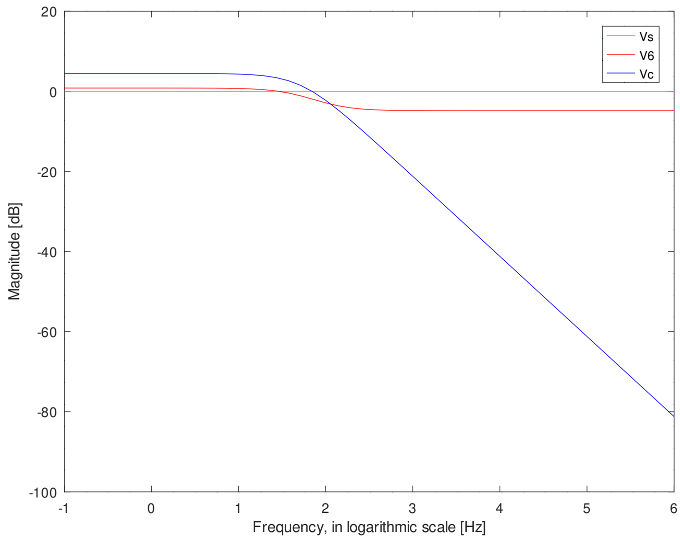

Figure 4: Variation with frequency of V_s, V_6 and V_c voltage magnitudes (in dB).

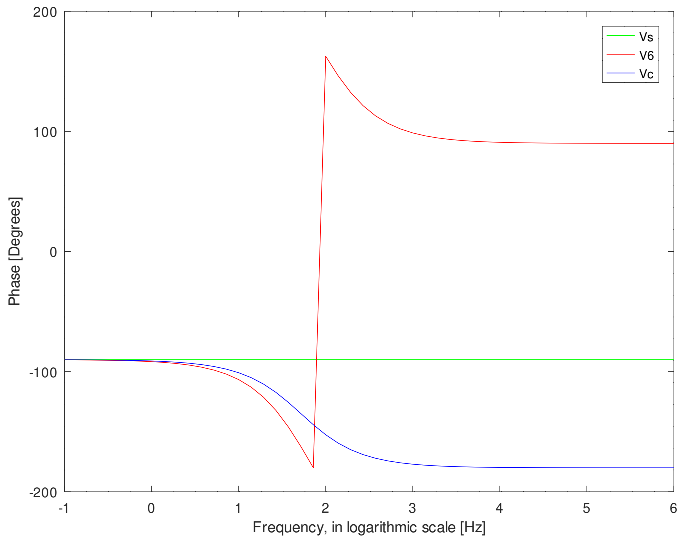

Figure 5: Variation with frequency of V_s, V_6 and V_c voltage phases.

3 Simulation Analysis

Since this circuit has a sinusoidal voltage source, the voltage and current values of the various components vary in time. Therefore, we must perform a transient analysis to simulate the circuit's total response. We also ran operating point analysis for both $t < 0$ and $t = 0$ to determine what the initial conditions were and establish the boundary conditions. Finally, we ran a frequency response analysis of the capacitor voltage and node 6 voltage.

3.1 Operating Point Analysis

The tables below show the simulated operating point results for both $t < 0$ (where we assume no current is flowing through the capacitor) and t=0 where we shut down the voltage source

V_s and we replace the capacitor with a voltage source V_x, in order to determine the equivalent resistor seen through the capacitor.

Name	Value [A or V]
@c[i]	0.000000e+00
@g[i]	-2.29771e-04
@r1[i]	2.191669e-04
@r2[i]	2.297712e-04
@r3[i]	1.060424e-05
@r4[i]	1.185502e-03
@r5[i]	2.297712e-04
@r6[i]	9.663348e-04
@r7[i]	9.663348e-04
v(1)	5.113399e+00
v(2)	4.889448e+00
v(3)	4.428712e+00
v(4)	-1.95390e+00
v(5)	4.921444e+00
v(6)	5.620908e+00
v(7)	-1.95390e+00
v(8)	-2.92781e+00

Table 6: Operating point analysis for $t < 0$. A variable preceded by @ is of type *current* other variables are of type *voltage* and expressed in Volt.

Name	Value [A or V]
@g[i]	-4.25263e-18
@r1[i]	4.056361e-18
@r2[i]	4.252625e-18
@r3[i]	1.962643e-19
@r4[i]	-8.55795e-19
@r5[i]	2.808224e-03
@r6[i]	-8.67362e-19
@r7[i]	-1.78493e-18
v(1)	0.000000e+00
v(2)	-4.14491e-15
v(3)	-1.26722e-14
v(4)	1.753779e-15
v(5)	-3.55271e-15
v(6)	8.548722e+00
v(7)	1.753779e-15
v(8)	3.552714e-15
vx	8.548722e+00
@vc[i]	-2.80822e-03

Table 7: Operating point analysis for $t = 0$. A variable preceded by @ is of type *current* and expressed in Ampere; other variables are of type *voltage* and expressed in Volt.

Table 6 shows the simulated node voltages and branch currents for $t < 0$. Table 7 gives us the current flowing through our voltage source, V_x. With the voltage source value calculated in the previous operating poing simulation, we can now calculate the equivalent resistance, R_{eq}, and characteristic time, τ:

$$R_{eq} = \frac{V_x}{@V_c[i]} \Leftrightarrow R_{eq} = \frac{8.548722}{2.80822e-03} \Leftrightarrow R_{eq} = 3044.178163 Ohm$$

$$\tau = R_{eq} \times C \Leftrightarrow \tau = (3044.178163 * C)s$$

3.2 Natural response

In this section we simulate the natural response of the circuit in the [0,20] ms time interval using the transient analysis simulation in NGSpice and the computations from the previous section.

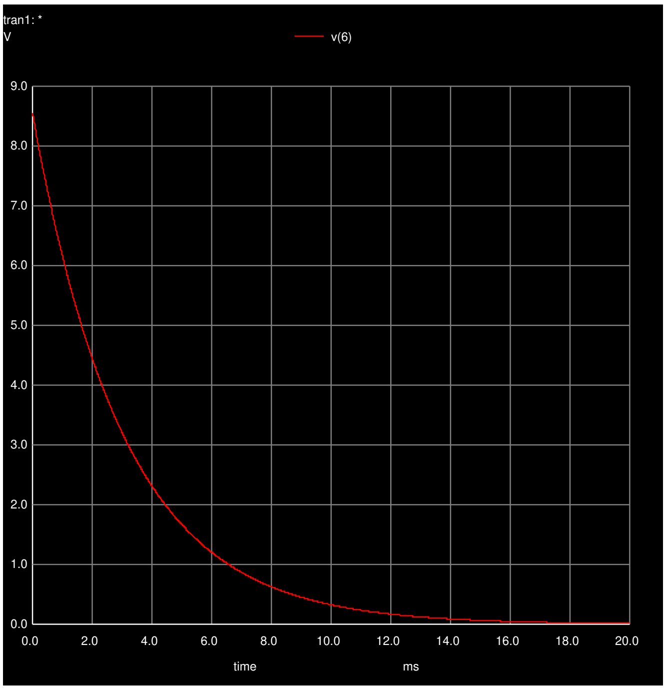

Figure 6: Simulated natural response for V_6 node voltage.

3.3 Natural and forced response

We repeat the previous step now with the sinusoidal voltage source $V_s(t)$ considering a frequency of 1kHz. Below is the plot for both the stimulus and the response:

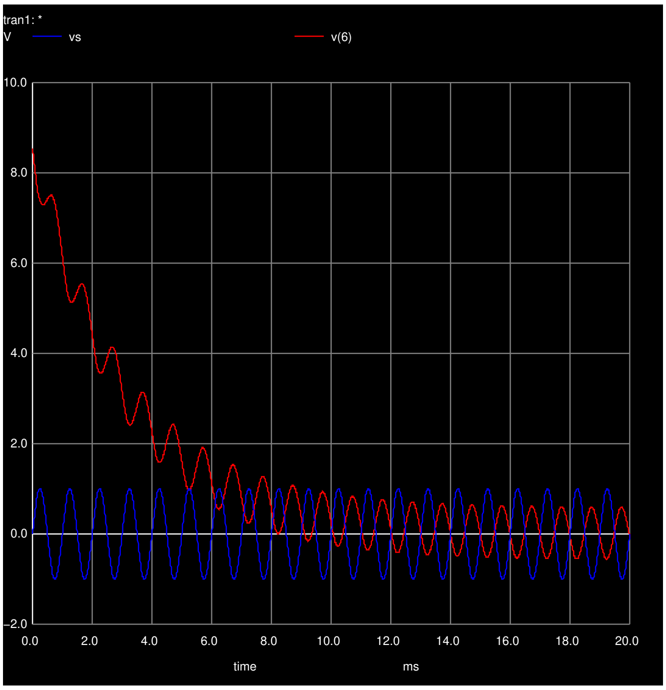

Figure 7: Simulated response for V_6 node voltage and stimulated voltage V_S.

3.4 Frequency response

We simulated the frequency response to the voltage source V_s, both in node 6 and in the capacitor itself, in NGSpice, and we plotted the magnitude, in dB, and phase, in degrees, of the three voltages in relation of course to a variation of frequency. The frequency varies from 0.1 Hz to 1MHz. Analysing figures 8 and 9, they confirm what we foresaw in the theoretical analysis. The various reasons on why certain voltage parameters vary or are equivalent are laid out and explained in the $Frequency\ Response$ subsection of the theoretical analysis.

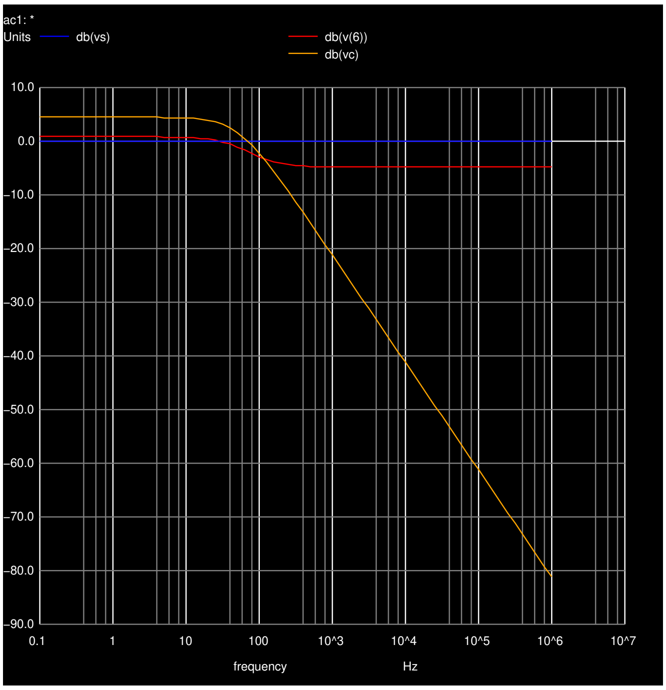

Figure 8: Magnitude of V_s, V_c and V_6 as a function of frequency

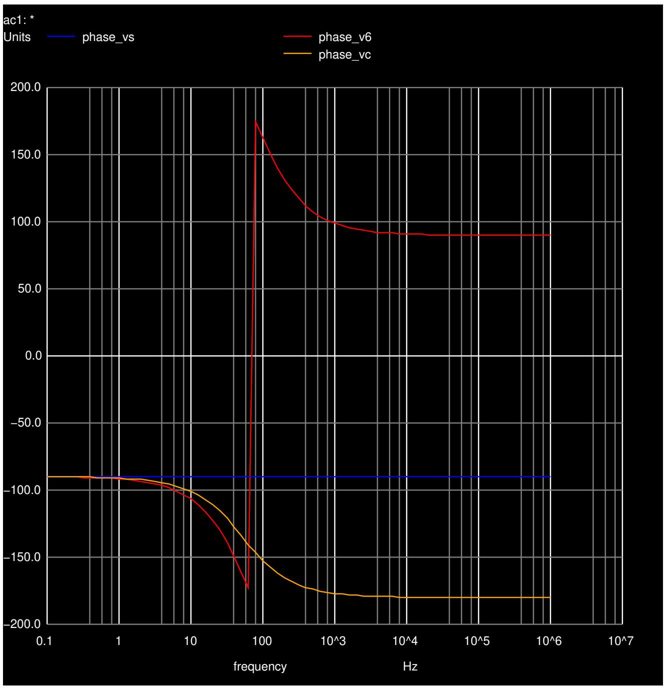

Figure 9: Phase of V_s, V_c and V_6 as a function of frequency

4 Conclusion

Now, we proceed to compare our theoretical and simulation results. Looking at all plots generated either by GNU Octave or NGSpice, the theoretical plots are checked out by NGspice with very high precision (plots relative to the natural solution, the complete solution and the frequency responses). When it comes to the tables generated by the operating point analysis regarding $t < 0$ and $t = 0$, we have assembled a table comparing the theoretical analysis run by Octave and the simulated data from NGSpice (Table 8).

All compared figures are equal, if we discard effectuated roundings, mostly operated by Octave, and if we disregard differences in the number of significant algharisms presented by both softwares. In the final case of the equivalent resistance, R_{eq}, the error is bigger because we calculated it not using either of the softwares (we took the V_x and I_x data from NGspice and operated the division ourselves). All comparisons made, all results predicted in the theoretical section were matched by the simulations executed.

All things taken into account, in this laboratory assignment, we were successful in producing coherent calculations using the Octave Maths tool and the circuit simulation, done using the Ngspice tool. Static and transient analysis were performed both theoretically and using a circuit simulation. Some results in the simulation that were predicted to yield zero current or voltage were actually very slightly off - in the magnitude of 1e-15, a similar magnitude to the floating number precision used in the software we utilised, which shows these errors are most likely

Theoretical	**Value**	**Simulation**	**Value**
$@I_{R1}$	0.219167 mA	@r1[i]	2.191669e-04 A
$@I_{R2}$	0.229771 mA	@r2[i]	2.297712e-04 A
$@I_{R3}$	0.010604 mA	@r3[i]	1.060424e-05 A
$@I_{R4}$	1.185502 mA	@r4[i]	1.185502e-03 A
$@I_{R5}$	0.229771 mA	@r5[i]	2.297712e-04 A
$@I_{R6}$	0.966335 mA	@r6[i]	9.663348e-04 A
$@I_{R7}$	0.966335 mA	@r7[i]	9.663348e-04 A
V_1	5.113399 V	v(1)	5.113399e+00 V
V_2	4.889447 V	v(2)	4.889448e+00 V
V_3	4.428712 V	v(3)	4.428712e+00 V
V_5	4.921444 V	v(5)	4.921444e+00 V
V_6	5.620907 V	v(6)	5.620908e+00 V
V_7	-1.953900 V	v(7)	-1.95390e+00 V
V_8	-2.927814 V	v(8)	-2.92781e+00 V
–	–	–	–
-	-	@g[i]	-4.25263e-18 A
-	-	@r1[i]	4.056361e-18 A
-	-	@r2[i]	4.252625e-18 A
-	-	@r3[i]	1.962643e-19 A
-	-	@r4[i]	-8.55795e-19 A
-	-	@r5[i]	2.808224e-03 A
-	-	@r6[i]	-8.67362e-19 A
-	-	@r7[i]	-1.78493e-18 A
-	-	v(1)	0.000000e+00 V
-	-	v(2)	-4.14491e-15 V
-	-	v(3)	-1.26722e-14 V
-	-	v(5)	-3.55271e-15 V
V_6	8.548721 V	v(6)	8.548722e+00 V
-	-	v(7)	1.753779e-15 V
V_8	0.000000 V	v(8)	3.552714e-15 V
V_x	8.548721 V	vx	8.548722e+00 V
$@I_x$	2.808224 mA	@vc[i]	2.80822e-03 A
R_{eq}	3.044174 kOhm	R_{eq}	3044.178163 Ohm

Table 8: Comparison of the theoretical and simulated data results, regarding $t < 0$, and $t = 0$ (initial conditions).

due to roundings effectuated either by GNU Octave or NGSpice in their calculations, most specifically in linear systems. This could also be do to the writing and reading of the different text files truncating the numbers at specific decimal places. The whole report has been automatised and we're confident that it would yield consistent results with a new set of data.