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Description and Solution of an Unreported Intrinsic Bias in Photon Mapping Density Estimation with Constant Kernel

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Abstract
This paper presents an analysis of the irradiance estimator often used in photon mapping algorithms and concludes that the classical approach with a constant kernel overestimates the correct value. We propose a new estimator that solves this problem and provide both theoretical and empirical studies to verify it.

ACM CCS: I.3.7 [Computer Graphics]: Three-Dimensional Graphics and Realism; G.3 [Probability and Statistics]; F.2.2 [Analysis of Algorithms and Problem Complexity]: Non-numerical Algorithms and Problems

1. Introduction

One approach to creating realistic images is the simulation of the light transport in the scene using random walk techniques. In particular, we will consider here the photon mapping algorithm [Jen96]. Photon mapping is the base of a wide variety of Monte Carlo global illumination algorithms, which modify the algorithm to reduce error or increase efficiency in different situations.

Many previous methods have inadvertently introduced an extra bias which becomes increasingly significant as less photons are used in the radiance estimate. In this paper, we identify this bias, and show the correct way to perform photon density estimation when using the k-nearest neighbour method with a constant kernel.

Section 1.1 contains a brief outline of the photon mapping algorithm and a (very incomplete) selection of improvements. Then Section 1.2 introduces Order Statistics, which provides the mathematical basis for our study of photon mapping. Section 2 contains a study of this previously unreported bias, which overestimates irradiance, while Section 3 contains the empirical study. Finally, Section 4 provides conclusions and possible directions of future work. Mathematical proofs are provided in appendices.

1.1. Photon mapping

The photon mapping algorithm includes several computation steps:

Photon tracing: Photons are traced from the light sources and reflected or absorbed upon interaction with the scene, using the bidirectional reflectance distribution function (BRDF) of the corresponding materials. Each impact is recorded in a kd-tree data structure.

Ray tracing (from the eye): Rays are traced from the eye into the scene. On the intersection point with a surface of the scene, either a photon map query is performed, yielding the estimated radiance at the point, or a number of secondary rays are raytraced according to the BRDF (this last step is called *final gather*), and the photon mapping queries from the secondary impacts are integrated to yield an improved radiance estimate. In both cases, density estimation (see later) must be done.

Density estimation (photon map query): For a given point on a surface, the flux of the k nearest impacts recorded in the kd-tree is added, and divided by the area of a disc with the radius equal to the distance between the point and the k^{th} impact.

There have been some studies describing the biases of photon mapping [Rol03][HBS05]. The sources of bias are the following:

Proximity bias: The algorithm converges to the mean irradiance in a neighbourhood of the point (weighted by a kernel function), which may be different from irradiance at the point of interest.

Boundary bias: Because impacts are only recorded on surfaces, if the area where photons are sought crosses the border of the object, no impacts will be found in that zone, and the estimate will be too dark.

Hey and Purgathofer [HP02] use geometry feelers to see if the disc mentioned earlier has an unreachable zone, and update the area accordingly (because no photons can impact this zone, this produces boundary bias). Their method reduces this source of bias significantly.

Another interesting article, by Herzog *et al.* [HHK*07], uses a truncated cone to establish the zone affected by the photon trajectories. Density estimation is calculated by splatting, and they use a kernel with adaptive bandwidth. Czuczor *et al.* [CSKSN05] store photon hits in texture space and provide another method to avoid this bias.

Geometric (or Topological) bias: Authors of other methods, such as density estimation on the tangent plane [LURM02b, LURM02a] and ray maps [HBHS05], notice that the original photon mapping method makes the assumption that surfaces are locally almost flat at the scale of the distance between impact points. This assumption does not hold for highly detailed surfaces with high curvature even at small scale, or with small isolated surfaces; therefore, these methods use the ray trajectory instead of the ray impact to reduce this bias.

This article describes an additional source of bias, which we denote **overestimation bias**, caused by adding the flux of the k nearest photons, instead of only the $k-1$ nearest ones.

In some algorithms, results are normalized by dividing by the maximum value obtained, so this overestimation is not an issue. However, because the final gathering phase of photon mapping combines different estimates for direct and indirect illumination [Jen01], this bias should be taken into account, either by discarding the contribution of the k^{th} nearest photon, by using filtering kernels, which assign weight 0 to this photon [WHSG97, Wal98, Jen01], or by using other density estimation techniques (e.g. density estimation on the tangent plane uses a fixed radius to search for photons, whereas photon ray splatting [HHK*07] and beam radiance estimate [JZJ08] use variable kernel density estimation). Care should be taken when creating new algorithms based on photon mapping to avoid this bias, although since the bias is proportional to $\frac{1}{k}$, high-quality algorithms with very large k are almost unaffected.

Interestingly, the statistics literature seems to have struggled with overestimation bias as well. B.W. Silvermann, in the first edition of his book [Sil86], clearly indicates $k-1$ should be used (Equation 2.3 on page 19). However, in reprints and following editions [Sil90], [Sil98], the equation has been changed to use k. Our analysis indicates the correct formula is the original one using $k-1$.

1.2. Order statistics

Order statistics is the part of probability theory, which deals with the probability distribution of elements in ordered lists. In other words, for a given random phenomenon, with a probability distribution f (with codomain $\subseteq \mathbb{R}$), the result of a number n of experiments is a list of real values $[x_1, \ldots, x_n]$. After ordering this list, we obtain $[x_{(1)}, \ldots, x_{(n)}]$, where $x_{(1)}$ is the minimum value, and $x_{(n)}$ the maximum. Even though all x_k have the same probability density function f, the probability distribution of the $x_{(k)}$ depends on f, n and k, by the following formula [DN03]:

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} F^{k-1}(x)(1-F(x))^{n-k} f(x), \quad (1)$$

where F is the cumulative distribution function corresponding to f.

Order statistics will provide a theoretical framework to study photon map queries in Section 2; we will use their applicability in density estimation [Sil86, Sil98], and derive the expected value of the photon mapping irradiance estimate when the number of rays shot tend to infinity in the following section.

2. Theoretical Study of Convergence of Photon Mapping

Convergence will be demonstrated for a constant incoming flux function in a diffuse disc with radius 1. Volumetric effects are studied next. All the symbols used repeatedly in this article are described in Table 1.

2.1. Uniform density of impacts

Let $I(P)$ be the constant value of the incoming flux per unit area in the disc (this can be created, e.g. by a directional light source). The particle tracing step will throw n rays in the disc with uniform area distribution, because the incoming flux is constant. The area of the unit disc is π, and the location of impacts has a constant distribution function

$$g(P) = \frac{1}{\pi}. \quad (2)$$

Let us consider the distribution of the distance from an impact point to the centre, and let us call f its corresponding pdf. The probability (density) for this distance to be exactly x is

Table 1: *Symbols used in this article.*

Photon mapping parameters	
P	Location of photon map query.
n	Number of photon impacts on surfaces.
k	Number of photons sought in a photon map query.
Photon mapping internals	
Φ	Incoming flux on the surface.
r_k	Distance of the k^{th} nearest photon in a query. r_k follows the distribution $f_{X_{(k)}}$.
$\phi = \frac{\Phi}{n}$	Flux of a photon when all photons carry the same flux.
Distribution functions	
g	Area density of photon impacts (redefined to volume density in the volumetric case).
f	1D pdf for the distribution of distances from impact points to disc centre (redefined to interaction points in the volumetric case).
F	Accumulated distribution of f.
$f_{X_{(k)}}$	k^{th} order statistics of f.
Irradiance	
I	Irradiance.
$\widehat{I}$	Photon mapping irradiance estimate at P.
$\widehat{I_{k-1}}$	Corrected photon mapping irradiance estimate at P.
Other symbols	
E[]	Expected value.

the length of the circumference of radius x (divided by π, because f must be normalized), so it holds

$$f(x) = 2x. \tag{3}$$

Thus, the cumulative distribution function F is

$$F(x) = \int_0^x f(x')\,\mathrm{d}x' = x^2. \tag{4}$$

The distribution of the k^{th} order statistic (the distance of the impact sought by photon mapping) is

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} F^{k-1}(x)(1-F(x))^{n-k} f(x). \tag{5}$$

Substituting, we get

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} 2x^{2k-1}\left(1-x^2\right)^{n-k}. \tag{6}$$

The estimate of incoming flux is calculated in photon mapping by adding the flux of the k photons nearest to the point of interest (we chose the centre of the disc for simplicity), and dividing by the area covered. In this case, all photons carry the same flux ϕ:

$$\phi = \frac{\pi I(P)}{n} \tag{7}$$

(remember that a unit disc has area π).

The estimate of irradiance depends on the distance of the k^{th} nearest photon, which we will denote r_k. We denote irradiance with the symbol $I()$ to avoid confusion with the symbol for expected value, which we denote $E[]$

$$\widehat{I}(r_k) = \frac{k\phi}{\pi r_k^2} = \frac{kI(P)}{nr_k^2}. \tag{8}$$

The expectation of the estimate of irradiance in the photon mapping method is

$$E[\widehat{I}(r_k)] = \int_0^1 \widehat{I}(r_k) f_{X_{(k)}}(r_k)\mathrm{d}r_k = \frac{k}{k-1} I(P), \tag{9}$$

if $k > 1$. $E[\widehat{I}(r_k)]$ is divergent for $k = 1$ (see Appendix B for the details).

This is a very surprising result, because it means that the estimate used does not converge to the correct value. It also shows a way to fix the estimate: adding the contribution of the $k-1$ nearest photons while still dividing by the disc with radius r_k. Then, the corrected irradiance estimate becomes

$$\widehat{I_{k-1}}(r_k) = \frac{(k-1)\phi}{\pi r_k^2} = \frac{(k-1)I(P)}{nr_k^2} \tag{10}$$

and the expected value is

$$E[\widehat{I_{k-1}}(r_k)] = \int_0^1 \widehat{I_{k-1}}(r_k) f_{X_{(k)}}(r_k)\mathrm{d}r_k = \frac{k-1}{k-1} I(P) = I(P). \tag{11}$$

If $k = 1$, our estimate always yields the wrong value 0 (see Appendix C for the proof of Equation 11). We can see that for $k = 1$, neither photon mapping nor our correction give useful estimates, so k should always be greater than 1.

2.2. Volumetric effects in photon mapping

Volumetric effects arise when light is not travelling in a vacuum. Instead of travelling on straight lines until they reach a surface, a photon may interact with the medium at any point; the probability of interaction depends on scattering, absorption and extinction coefficients, which may also vary in the medium.

One example of using photon mapping to calculate the volumetric effects was presented by Jensen [JC98].

We will study here the case of an homogeneous, isotropic sphere of unit radius. In this case, the probability density of the photon interactions is constant in the sphere (we reuse and redefine the notation used in the 2D case)

$$g(v) = \frac{1}{\frac{4}{3}\pi} = \frac{3}{4\pi} \tag{12}$$

for all v inside the sphere, and $g(v) = 0$ otherwise.

The cumulative distribution of impacts located at a distance smaller than x from the centre of the sphere is

$$F(x) = \frac{4}{3}\pi x^3 \frac{3}{4\pi} = x^3 \tag{13}$$

and the density of impacts located at distance x is

$$f(x) = 3x^2. \tag{14}$$

The density of the k^{th} nearest impact $f_{X_{(k)}}(x)$ is

$$\begin{aligned}&\frac{n!}{(k-1)!(n-k)!}F^{k-1}(x)(1-F(x))^{n-k}f(x)\\ &= \frac{n!}{(k-1)!(n-k)!}\left(x^3\right)^{k-1}\left(1-x^3\right)^{n-k}3x^2.\end{aligned} \tag{15}$$

For this example, we will assume a uniform power density in the sphere, which although artificial, will allow us to study easily the expected value of the estimates.

If we denote the power contained in the sphere by W and the volume of the sphere by V, the power density is

$$PD = \frac{W}{V} = \frac{3W}{4\pi}. \tag{16}$$

If all photons carry the same power ϕ,

$$\phi = \frac{W}{n}. \tag{17}$$

The volumetric photon mapping algorithm estimates the power density by taking the k nearest photons, adding their flux and dividing by the volume of the sphere containing the k photons. The power density estimate is, therefore,

$$\widehat{PD}(r_k) = \frac{k\phi}{\frac{4}{3}\pi r_k^3}. \tag{18}$$

The expected value of the estimate can be calculated similarly to previous sections (details in Appendix D)

$$E[\widehat{PD}(r_k)] = \int_0^1 \widehat{PD}(r_k) f_{X_{(k)}}(r_k)\,\mathrm{d}r_k = \frac{k}{k-1}PD. \tag{19}$$

We can see clearly that the power density is being overestimated by the factor $\frac{k}{k-1}$. To calculate irradiance $\widehat{I}$, we need this power density, the scattering coefficient and the phase function. After integration along the line of sight, we can obtain an estimate of irradiance at the surface. However, because the power density is being overestimated and integrals are linear operators, the irradiance estimate is overestimated by $\frac{k}{k-1}$.

We can also see that for $k = 1$ the estimate won't converge, so apart from adding the contribution of only the $k-1$ nearest photons, the minimum value of k should be 2.

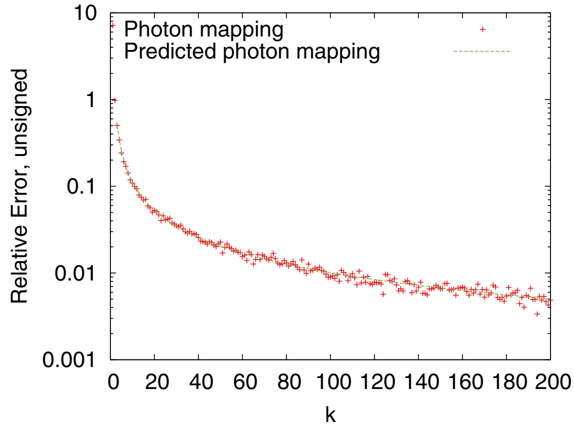

Figure 1: *Relative error of photon mapping for a uniform distribution of photons, as a function of k; theoretical prediction of the error (logarithmic plot).*

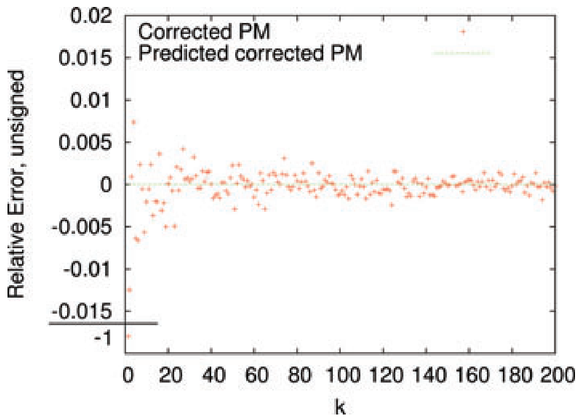

Figure 2: *Relative error of our corrected photon mapping for a uniform distribution of photons, as a function of k; theoretical prediction of the error.*

3. Empirical Study

We have implemented in C++ the simple scene described in the theoretical study, and we have calculated the photon mapping irradiance estimate for different values of n and k. We have repeated the simulations ten thousand times to calculate an estimate of the convergence of the algorithm.

Figures 1, 2 and 3 contain the averaged results for executions of the algorithm with k between 1 and 200, for $n = 100000$. Corrected PM stands for corrected photon mapping (i.e. using $k-1$ photons while dividing by the distance to the k^{th}).

The predicted line for photon mapping is calculated by the formula $\frac{k}{k-1} - 1$, while our corrected photon mapping

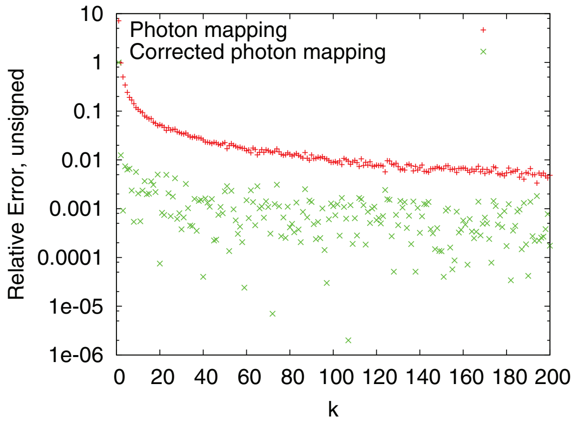

Figure 3: *Relative error of standard and corrected photon mapping as a function of k (logarithmic plot).*

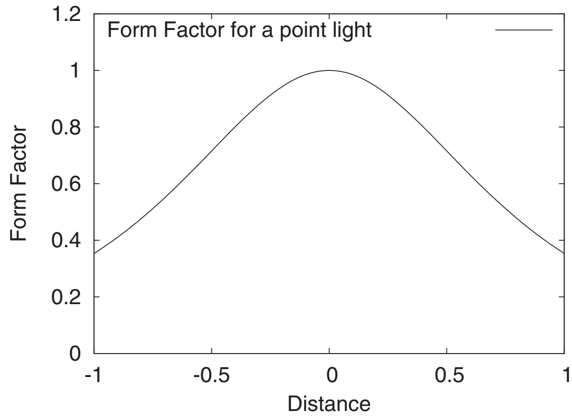

Figure 5: *Form factor for a omnidirectional light source located above the centre of a disc, at one unit of distance.*

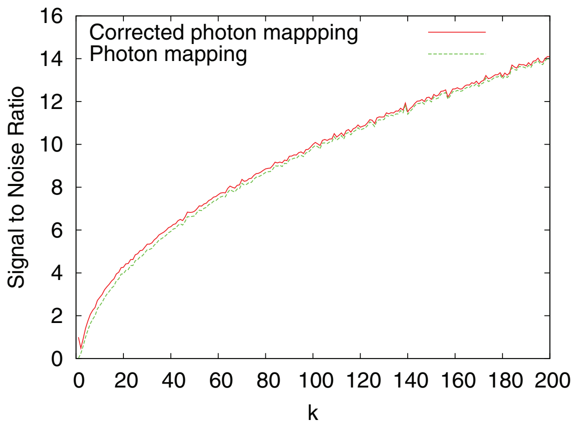

Figure 4: *Signal-to-noise ratio of photon mapping for a uniform distribution of photons, as a function of k.*

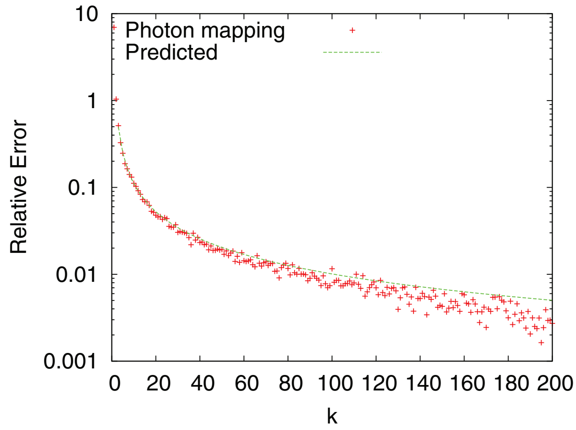

Figure 6: *Relative error of photon mapping for a distribution of photons arising from a point light source, as a function of k; theoretical prediction of the error (logarithmic plot).*

algorithm has a predicted bias of -1 for $k = 1$ and 0 for $k > 1$.

The figures show clearly an important reduction of error when using the corrected estimate for photon mapping, averaged over all the repetitions of the simulation. The theoretical prediction matches the value obtained closely.

If we look at the signal-to-noise ratio (Figure 4), it is also apparent why this bias has not been noticed before: when running the photon mapping algorithm only once in typical scenes, the variance of the algorithm completely masks the effect.

Looking at the first point at $k = 1$, we can see confirmation of the theoretical results mentioned in the previous section, i.e. neither basic photon mapping nor the corrected version are usable.

We can see that for $k = 2$, while the basic photon mapping algorithm obtains the double of the correct results, our corrected algorithm has errors in the range $1\% - 4\%$. However, the value of the signal-to-noise ratio is so low that, if only one execution of the algorithm is done, the results will not be accurate. Nevertheless, a high increase of the value of the signal-to-noise ratio is seen with respect to basic photon mapping. Therefore, even though $k = 2$ is an extreme setting, we can see that the value of k can be modified to change the computation time, at the cost of increased variance, but without the increased bias of basic photon mapping.

3.1. Simulation of non-constant irradiance

To simulate more realistic light conditions, while at the same time being able to calculate analytically the solution, we

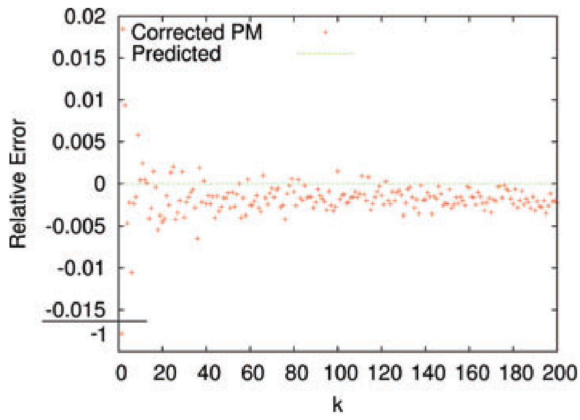

Figure 7: *Relative error of our corrected photon mapping for a distribution arising from a point light source of photons, as a function of* k*; theoretical prediction of the error.*

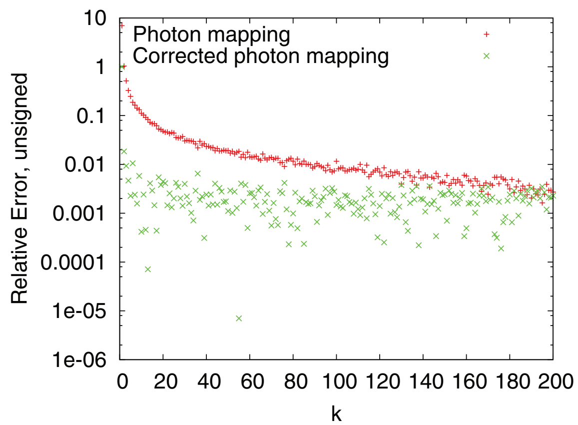

Figure 8: *Relative error of standard and corrected photon mapping for a distribution arising from a point light source of photons, as a function of* k*.*

simulated a scene composed of a disc illuminated by a point light source located above the disc centre.

The form factor is $\frac{\cos(\theta)}{||P-L||^2}$, with θ denoting the angle between the light vector and the normal, L the light position and P the point of interest. Figure 5 has a graph of the form factor as a function of the distance to the centre of the disc (the form factor is radially symmetric in this scene).

Using this function, we simulated a photon map query on the point $(0, 0)$. The theoretical solution is $g(0, 0) = \frac{4+\sqrt{2}}{7\pi}$.

Figures 6, 7 and 8 show the relative error of the photon mapping algorithm and our correction. Figure 9 shows the signal-to-noise ratio. We can see that the behaviour of the algorithm is very similar to the one shown in previous figures;

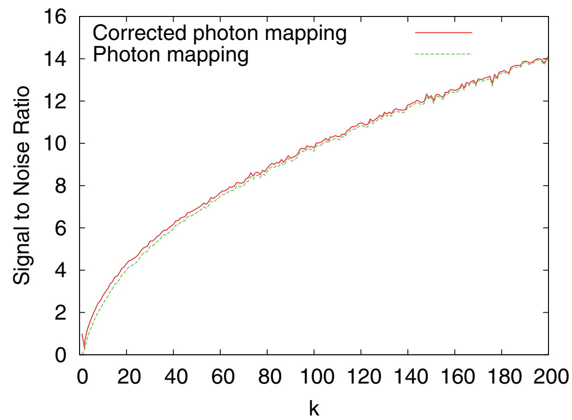

Figure 9: *Signal-to-noise ratio of photon mapping for a distribution of photons arising from a point light source, as a function of* k*.*

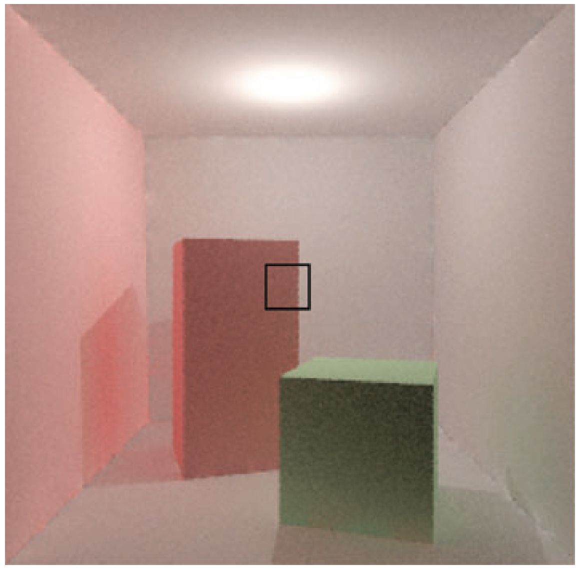

Figure 10: *Cornell box scene, with* $n = 100000$ *and* $k = 50$*. Center pixel highlighted.*

although because we are estimating irradiance at a maximum, the smoothing kernels produce a slight decrease in value. This is the reason why the new data points are not centred on the theoretical predictions. The agreement between empirical and theoretical results is, however, quite good, with less than 1% difference.

We also simulated the Cornell box scene with different parameters one hundred times to calculate the signal-to-noise ratio and its inverse, the coefficient of variation, on the central pixel of the scene (Figure 10 and Table 2). The overestimation is four times larger than the coefficient of variation when using only 50 photons in the search, but

Table 2: *Signal-to-noise ratio, coefficient of variation and overestimation bias for different parameters of the photon mapping algorithm with the Cornell box scene.*

n	k	*SNR*	$C_v = \frac{1}{SNR}$	O. Bias
100000	50	215.10	4.649	20.400
1000000	500	293.77	3.404	2.004

when using 500, it is small enough to hide in the noise. In addition, using high quality photon mapping implementations with filtering kernels (which assign weight zero to the k^{th} nearest photon) would remove the overestimation bias completely.

4. Conclusions and Future Work

We have presented here an overestimation bias present in the nearest neighbour density estimation technique commonly used in the photon mapping algorithm. This source of bias has not been presented previously in the photon mapping literature. We have shown, both theoretically and empirically, that this bias is proportional to the inverse of the number of photons sought in a query, and presented a way to eliminate it.

Our near-term future work will deal with formally studying the case of non-uniform distribution of impacts, photons with different flux values, general BRDFs and final gathering. As later future work, we would like to continue using Order Statistics (which has never been applied to the study of global illumination algorithms) to derive other properties of the algorithms. We would like to study the commonly used kernels [Rol03] (cone filter, 2D Epanechnikov [Wal98], Silverman's [WHSG97]) in the context of photon mapping. In addition, we are interested in results concerning the variance of photon mapping and its asymptotic behaviour.

Acknowledgements

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Appendix A: Preliminaries

We will start by solving the following integral:

$$\int_0^1 F^{k-2}(r_k)(1 - F(r_k))^{n-k} f(r_k)\mathrm{d}r_k, \tag{A1}$$

where $F(r_k) = r_k^2$ and $f(r_k) = 2r_k$ (Equations 4 and 3), and $1 < k < n + 1$.

We do a change of variable:

$$t = F(r_k), \quad \mathrm{d}t = f(r_k)\mathrm{d}r_k, \tag{A2}$$

so Equation (A1) becomes

$$\begin{aligned}
&\int_0^1 F^{k-2}(r_k)(1 - F(r_k))^{n-k} f(r_k)\mathrm{d}r_k \\
&= \int_{F(0)}^{F(1)} t^{k-2}(1-t)^{n-k}\mathrm{d}t \\
&= \int_0^1 t^{k-2}(1-t)^{n-k}\mathrm{d}t = \frac{(k-2)!(n-k)!}{(n-1)!}.
\end{aligned} \tag{A3}$$

The last equality only holds for integer values n and k such that $1 < k < n + 1$. In photon mapping, obviously n and k are integers, and inequality $k < n + 1$ always holds, as there are only n photons to choose from. In the case $k = 1$, the expansion of $t^{-1}(1 - t)^{n-1}$ includes the summand $1/t$, which cannot be integrated (diverges) as it has a singularity in $t = 0$ (the lower integration limit).

Appendix B: Proof of the Expected Value of the Photon Mapping Irradiance Estimate in the Case of Constant Incoming Flux

Now, we deal with the expected value of the photon mapping estimate of irradiance (Equation 9).

The irradiance estimate for $k > 1$ is

$$\begin{aligned}
E[\widehat{I}(r_k)] &= \int_0^1 \widehat{I}(r_k) f_{X_{(k)}}(r_k)\mathrm{d}r_k = \int_0^1 \frac{kI(P)}{nr_k^2} f_{X_{(k)}}(r_k)\mathrm{d}r_k \\
&= \int_0^1 \frac{kI(P)}{nr_k^2} \frac{n!}{(k-1)!(n-k)!} \\
&\quad \times F^{k-1}(r_k)\,(1 - F(r_k))^{n-k} f(r_k)\mathrm{d}r_k \\
&= \int_0^1 \frac{kI(P)}{nF(r_k)} \frac{n!}{(k-1)!(n-k)!} \\
&\quad \times F^{k-1}(r_k)\,(1 - F(r_k))^{n-k} f(r_k)\mathrm{d}r_k \\
&= \int_0^1 \frac{kI(P)}{n} \frac{n!}{(k-1)!(n-k)!} \\
&\quad \times F^{k-2}(r_k)\,(1 - F(r_k))^{n-k} f(r_k)\mathrm{d}r_k \\
&= \frac{kI(P)}{n} \frac{n!}{(k-1)!(n-k)!} \\
&\quad \times \int_0^1 F^{k-2}(r_k)\,(1 - F(r_k))^{n-k} f(r_k)\mathrm{d}r_k \\
&= \frac{kI(P)}{n} \frac{n!}{(k-1)!(n-k)!} \frac{(k-2)!(n-k)!}{(n-1)!} \\
&= \frac{k}{k-1} I(P).
\end{aligned} \tag{B1}$$

Note that, as we assume uniform impact density, we can use Equation 4, thus $F(r_k) = r_k^2$. When $k = 1$, the integral is not convergent, and neither is the photon mapping estimator.

Appendix C: Proof of the Expected Value of Our Corrected Irradiance Estimate in the Case of Constant Incoming Flux

Here, we will calculate the expected value of the corrected photon mapping estimate of irradiance. We begin by formulating $\widehat{I_{k-1}}$ in terms of $\widehat{I}$ using their definitions (Equations 8 and 10), again assuming $k > 1$:

$$\widehat{I_{k-1}}(r_k) = \frac{(k-1)I(P)}{nr_k^2} = \frac{(k-1)}{k}\frac{kI(P)}{nr_k^2} = \frac{k-1}{k}\widehat{I}(r_k). \tag{C1}$$

Then, obviously,

$$E[\widehat{I_{k-1}}(r_k)] = \frac{k-1}{k}E[\widehat{I}(r_k)] = \frac{k-1}{k}\frac{k}{k-1}I(P) = I(P). \tag{C2}$$

When $k = 1$, Equation C1 simplifies to $\widehat{I_{k-1}}(r_k) = 0$.

Appendix D: Proof of the Expected Value of the Photon Mapping Power Density Estimate in the Constant Case

We begin with the power density estimate from Equation 18,

$$\widehat{PD}(r_k) = \frac{k\phi}{\frac{4}{3}\pi r_k^3} \tag{D1}$$

and the probability density function of the distance of the k^{th} nearest photon interaction (Equation 15)

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!}\left(x^3\right)^{k-1}\left(1-x^3\right)^{n-k}3x^2. \tag{D2}$$

The expected value of the estimate can be calculated similarly to previous sections,

$$\begin{aligned} &E[\widehat{PD}(r_k)] \\ &= \int_0^1 \widehat{PD}(r_k) f_{X_{(k)}}(r_k)\mathrm{d}r_k \\ &= \int_0^1 \frac{k\phi}{\frac{4}{3}\pi r_k^3}\frac{n!}{(k-1)!(n-k)!}\left(r_k^3\right)^{k-1}\left(1-r_k^3\right)^{n-k}3r_k^2\mathrm{d}r_k \\ &= \frac{n!}{(k-1)!(n-k)!}\frac{k\phi}{\frac{4}{3}\pi}\int_0^1 \left(r_k^3\right)^{k-2}\left(1-r_k^3\right)^{n-k}3r_k^2\mathrm{d}r_k \\ &= \frac{n!}{(k-1)!(n-k)!}\frac{k\phi}{\frac{4}{3}\pi}\frac{(k-2)!(n-k)!}{(n-1)!} = \frac{k}{k-1}\frac{n\phi}{\frac{4}{3}\pi} \\ &= \frac{k}{k-1}\frac{W}{\frac{4}{3}\pi} = \frac{k}{k-1}\frac{3W}{4\pi} = \frac{k}{k-1}PD, \end{aligned} \tag{D3}$$

where we used Appendix A to solve the integral, and Equation 16 (the definition of PD) in the last step. The same caveats about $k = 1$ mentioned previously, apply here as well.

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