The Multisymplectic Framework of Field Theories

Workshop on Geometric Aspects of Material Modelling

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Structure of the talk

The geometry of calculus of variations

What to minimize/maximize? Sections!

Fixed some fibered manifold

$$Y \xrightarrow{\pi_{YX}} X \text{ with coordinates } (x^\mu, y^i) \mapsto x^\mu,$$

we want to find a section

$$\phi : X \to Y,\ (x^\mu) \mapsto (x^\mu, y^i = \phi^i(x^\mu))$$

minimizing/maximizing the functional

$$\mathcal{J}[\phi] = \int_X L\left(x^\mu, \phi^i(x^\mu), \frac{\partial \phi^i}{\partial x^\mu}, \frac{\partial^2 \phi^i}{\partial x^\mu \partial x^\nu}, \dots\right) d^n x.$$

We will focus on first order theories,

$$\mathcal{J}[\phi] = \int_X L\left(x^\mu, \phi^i(x^\mu), \frac{\partial \phi^i}{\partial x^\mu}\right) d^n x.$$

The geometric setting II

We can interpret

$$L\left(x^\mu, \phi^i(x^\mu), \frac{\partial \phi^i}{\partial x^\mu}\right) d^n x$$

as an n-form on the first jet bundle

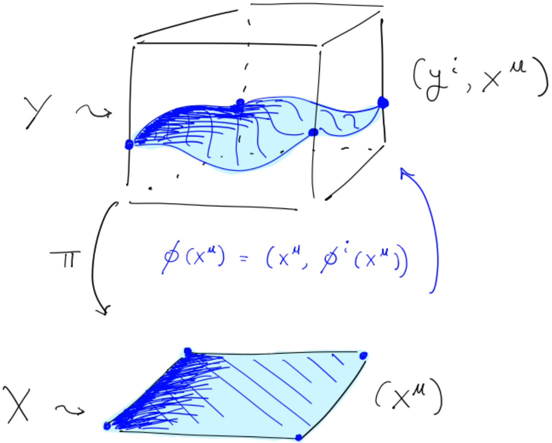
$$J^1\pi_{YX} \text{ with coordinates } (x^\mu, y^i, z^i_\mu).$$

We call it the Lagrangian density

$$\mathcal{L} = L(z^\mu, y^i, z^i_\mu) d^n x.$$

We can rewrite the action as

$$\mathcal{J}[\phi] = \int_X (j^1\phi)^* \mathcal{L}.$$



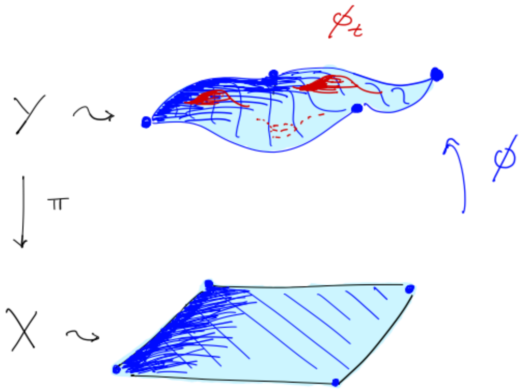
$$\mathcal{J}[\phi] = \int_X (j^1\phi)^*\mathcal{L},$$

where $\mathcal{L} \in \Omega^n(J^1\pi_{YX})$ is the Lagrangian dentisy.

The Euler-Lagrange equations I

If ϕ is a minimizer/maximizer (more generally, stationary section),

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{J}[\phi_t] = 0, \ \forall \text{ variation } \phi_t.$$



The Euler-Lagrange equations II

Equivalently,

$$0 = \int_X \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (j^1 \phi_t)^* \mathcal{L}.$$

Locally, we get

$$\frac{\partial L}{\partial y^i} = \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial L}{\partial z^i_\mu}\right).$$

What about intrinsic Euler-Lagrange equations?

If we define

$$\xi := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \phi_t = \xi^i \frac{\partial}{\partial y^i} \in \mathfrak{X}(Y),$$

$$\xi^{(1)} := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} j^1 \phi_t = \xi^i \frac{\partial}{\partial y^i} + \left(\frac{\partial \xi^i}{\partial x^\mu} + \frac{\partial x^i}{\partial y^j} z^j_\mu\right) \frac{\partial}{\partial z^j_\mu} \in \mathfrak{X}(J^1\pi_{YX}).$$

The Euler-Lagrange equations III

If we define

$$\xi := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi_t = \xi^i \frac{\partial}{\partial y^i} \in \mathfrak{X}(Y),$$

$$\xi^{(1)} := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} j^1\phi_t = \xi^i \frac{\partial}{\partial y^i} + \left(\frac{\partial \xi^i}{\partial x^\mu} + \frac{\partial x^i}{\partial y^j} z^j_\mu\right) \frac{\partial}{\partial z^j_\mu} \in \mathfrak{X}(J^1\pi_{YX}),$$

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{J}[\phi_t] = \int_X (j^1\phi)^* \pounds_{\xi^{(1)}} \mathcal{L}, \text{ for every vertical } \xi \in \mathfrak{X}(Y)$$

Applying Stokes' Theorem

$$0 = \int_X (j^1\phi)^* \iota_{\xi^{(1)}} d\mathcal{L} + \int_X d\iota_{\xi^{(1)}} \mathcal{L} = \int_X (j^1\phi)^* \iota_{\xi^{(1)}} d\mathcal{L}.$$

The Euler-Lagrange equations IV

$$0 = \int_X (j^1\phi)^* \iota_{\xi^{(1)}} d\mathcal{L} \text{ for every vertical } \xi \in \mathfrak{X}(Y).$$

Does not yield equations.

Idea: modify $\mathcal{L}$

We want to find an n-form $\Theta_{\mathcal{L}}$ satisfying

$$(j^1\phi)^*\mathcal{L} = (j^1\phi)^*\Theta_{\mathcal{L}}$$

such that ϕ is an stationary field of the action if and only if

$$0 = \int_X (j^1\phi)^* \iota_{\eta} d\Theta_{\mathcal{L}} \text{ for every } \eta \in \mathfrak{X}(J^1\pi_{YX}).$$

The Euler-Lagrange equations V

Proposition
There is such $\Theta_{\mathcal{L}}$, and can be intrinsically defined (using the geometry of $J^1\pi_{YX}$).

Locally,

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial z^i_\mu} dy^i \wedge d^{n-1}x_\mu - \left(\frac{\partial L}{\partial z^i_\mu} z^i_\mu - L \right) d^n x$$

and it is called the Poincaré-Cartan form.

Corollary (Intrinsic Euler-Lagrange equations)
A field $\phi : X \to Y$ is stationary if and only if it satifies

$$(j^1\phi)^* \iota_\eta d\Theta_{\mathcal{L}} = 0, \text{ for every } \eta \in \mathfrak{X}(J^1\pi_{YX}).$$

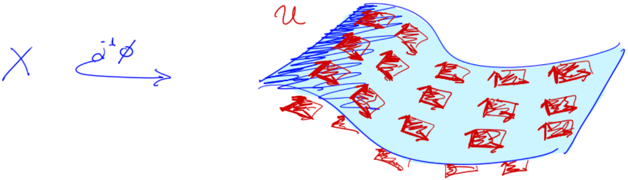
Looking for solutions

To find solutions, we can look for distributions on $J^1\pi_{YX} \to$ such that an integral section of such this distribution $\sigma : X \to J^1\pi_{YX}$ satisfies

$$\sigma^* \iota_\eta \Omega_{\mathcal{L}} = 0, \forall \eta \in \mathfrak{X}(J^1\pi_{YX}).$$

We can define such distributions via decomposable n-multivector fields

$$U = X_1 \wedge \cdots \wedge X_n.$$



Then, being stationary is characterized by $\iota_U \Omega_{\mathcal{L}} = 0$.

Disclaimer

Giving such a multivector field U does not immediately give a solution:

- We need to make sure that the corresponding distribution is integrable.
- Even if it is integrable, it may not be holonomic. That is, that the corresponding integral section $\sigma : X \to J^1\pi_{YX}$ could fail to be the jet lift of some section

  $$\phi : X \to Y.$$

  When $\mathcal{L}$ is regular, this is not an issue.
- Even if it satisfies the previous conditions, there may not exist global sections of $Y \xrightarrow{\pi_{YX}} X$.

Summary

- Fields, denoted by ϕ, are sections of a fibered manifold $Y \xrightarrow{\pi_{YX}} X$.
- A first order variational problem is defined through a Lagrangian density $\mathcal{L}$ on $J^1\pi_{YX}$ (which defines an n-form on X at each point), and the action can be expressed as

$$\mathcal{J}[\phi] = \int_X (j^1\phi)^* \mathcal{L}.$$

- If we define the multisymplectic form as

$$\Omega_{\mathcal{L}} := -d\Theta_{\mathcal{L}},$$

stationary fields are characterized by

$$(j^1\phi)^* \iota_\eta \Omega_{\mathcal{L}} = 0, \text{ for every } \eta \in \mathfrak{X}(J^1\pi_{YX}).$$

In particular, we can look for decomposable horizontal n-multivector fields U satisfying

$$\iota_U \Omega_{\mathcal{L}} = 0.$$

Multisymplectic geometry

Basic definitions I

Definition

A multisymplectic manifold of order n is a pair (M, ω), where M is a smooth manifold, and ω is a closed $(n+1)-$form.

An immediate example is the bundle of n-forms on a manifold Q.

$$M := \bigwedge^{n} T^*Q \xrightarrow{\tau} Q$$

has a canonical n-form,

$$\Theta|_{\alpha}(v_1, \ldots, v_n) := \alpha(\tau_* v_1, \ldots, \tau_* v_n)$$

and

$$\Omega := -d\Theta$$

defines a multisymplectic structure on M.

Basic definitions II

Definition
Let (M, ω) be a multisymplectic manifold of order n. A q-multivector field U on M ($q \leq n$) is called Hamiltonian if

$$\iota_U \omega = d\alpha,$$

for certain $(n-q)$-form α, which will also be called Hamiltonian.

- Top degree Hamiltonian multivector fields (n-multivector fields) represent solutions to the variational problem,

$$\iota_U \omega = dH, H \in C^\infty(M).$$

- Hamiltonian vector fields $X \in \mathfrak{X}(M)$ are symmetries, $\pounds_X \omega = 0$ and the corresponding $(n-1)-$form can be thought of as the Noether current of the symmetry.

Noether's Theorem I

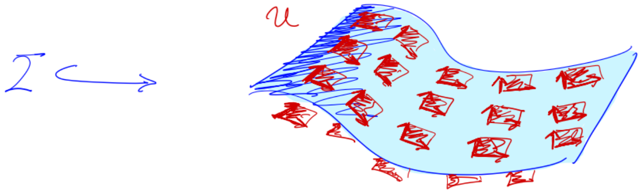
Given a Hamiltonian function H, and a top degree decomposable multivector field U such that

$$\iota_U \omega = dH,$$

let

$$j : \Sigma \hookrightarrow M$$

be an n-dimensional integral submanifold of U (which can be thought of as a distribution).



Noether's Theorem II

Theorem
Let X be a Hamiltonian vector field and α the corresponding $(n-1)$-form (the current),

$$\iota_X \omega = d\alpha.$$

Then, if X is a symmetry of H, that is,

$$X(H) = 0,$$

α is a conserved current on Σ, this means $d(j^*\alpha) = 0$.

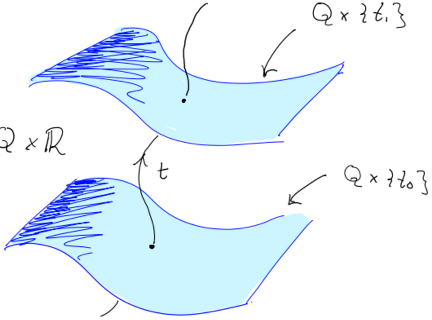
Proof.
Indeed,

$$(d\alpha)(U) = \iota U \iota_X \Omega = (-1)^n \iota_X \iota_U \Omega = (-1)^n X(H) = 0.$$

$\square$

What does conserved means here?

$\Sigma = Q \times \mathbb{R}$



$$\int_{Q\times\{t_0\}} \alpha = \int_{Q\times\{t_1\}} \alpha$$

How can we obtain the corresponding current from the symmetry defined by X?

Theorem

Let (M, ω) be an exact multisymplectic manifold, that is, there exists a multisymplectic potential

$$\omega = -d\theta.$$

Then, if X is a symmetry of θ, $\pounds_X \theta = 0$ (and hence of ω),

$$\alpha := -\iota_X \theta$$

is a current for X.

Proof.
Indeed,

$$d\alpha = -d\iota_X \theta = -\iota_X d\theta = \iota_X \omega.$$

$\square$

Brackets I

Proposition
Let (M, ω) be a multisymplectic manifold and α, β be Hamiltonian forms, with Hamiltonian multivector fields, X, Y, respectively. Then

$$\{\alpha, \beta\} := \iota_Y \iota_X \omega$$

is a Hamiltonian form. Its Hamiltonian multivector field is $-[X, Y]$ (the Schouten-Nijenhuis bracket).

Definition
Define the Poisson bracket of two Hamiltonian forms by

$$\{\alpha, \beta\} := -(-1)^{(k-1-\operatorname{ord}\alpha)} \iota_Y \iota_X \omega,$$

which is again Hamiltonian by the previous proposition.

Brackets II

What are the properties that $\{\cdot,\cdot\}$ satisfies?

If we define a new degree:

$$\deg\alpha := k-1-\mathrm{ord}\alpha,$$

- It is *graded-skew-symmetric*, that is,

$$\{\alpha,\beta\} = (-1)^{\deg\alpha\deg\beta}\{\beta,\alpha\}.$$

- Its satisfies *graded-Jacobi identity* (up to an exact form)

$$(-1)^{\deg\alpha\deg\gamma}\{\{\alpha,\beta\},\gamma\} + \mathrm{cycl.} = \text{exact term}$$

Brackets III

Theorem
Let (M, ω) be a multisymplectic manifold. Then, the space of all Hamiltonian forms modulo exact forms is a graded Lie algebra.

Some remarks:

- When restricted to the subspace of forms of $\deg \alpha = 0$, that is, alpha $= k - 1$, we have a Lie algebra, the Lie algebra of currents.
- Some brackets are zero just by degree considerations, more particularly, when

  $$\deg \alpha + \deg \beta > k - 1,$$

  that is, the bracket is trivial when

  $$\operatorname{ord} \alpha + \operatorname{ord} \beta < k - 1.$$

- Dynamics can be characterized by this Poisson bracket. Indeed, fixed a Hamiltonian, an n- multivector field U is a solution ($\iota_U \omega = dH$) if

  $$\{\alpha, H\} = (d\alpha)(U).$$

...and more!

Multisymplectic geometry is a very active area of research, and there has been a lot of interest in generalizing classical results from symplectic geometry to the multisymplectic setting.

- Reduction by symmetries.
- Coisotropic reduction.
- Constraint analysis.
- Darboux-like Theorems.
- Is everything a Lagrangian submanifold? (Weinstein's creed)
- Alogue to Poisson geometry and Dirac geometry (work in progress...).

Summary

- Multisymplectic geometry gives an abstract formulation of field theories (calculus of variations).
- We can talk about the dynamics and conserved quantities with Hamiltonian multivector fields and forms.
- We can prove Noether's Theorem in this formalism.
- Hamiltonian forms are endowed with a graded Lie algebra structure (when quotiented by exact forms) which yields a Lie algebra when restricted to currents, $(n-1)-$forms.

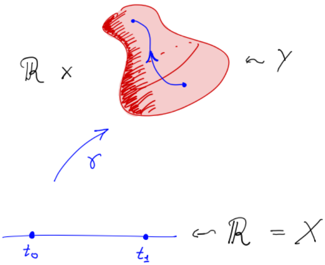
Examples

Classical Mechanics I

We recover Classical Mechanics by taking the bundle

$$\mathbb{R} \times Q \xrightarrow{\pi} \mathbb{R}.$$

Then, a section is just a curve $\gamma : \mathbb{R} \to Q$.



$$\mathcal{S}[\gamma] = \int_{t_0}^{t_1} L(t, \gamma(t), \dot{\gamma}(t))\, dt$$

The jet bundle:

$$J^1\pi = \mathbb{R} \times TQ,$$

and the Poincaré-Cartan form fixed a Lagrangian (which will be identified with a function)

$$L : \mathbb{R} \times TQ \to \mathbb{R}$$

is

$$\theta_L = \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i - \left(\frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L \right) dt.$$

Dyanmics are vector fields X satisfying

- *Stationary condition*, $\iota_X d\theta_L = 0$.
- *Normalization*, $dt(X) = 1$.

We recover cosymplectic geometry!

Classical Mechanics III

What about Noether Theorem? Suppose L t-invariant. Then, time translations $\frac{\partial}{\partial t}$ define a symmetry of the corresponding multisymplectic form. Hence, by previous considerations, $\iota_{\frac{\partial}{\partial t}}\theta_L$ is an conserved current, that is, a conserved quantity. Locally,

$$\iota_{\frac{\partial}{\partial t}}\theta_L = \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L = H,$$

obtaining conservation of energy.

Hyperelastic materials I

We will give an example of hyperelastic dynamics in a fixed background.

- Fix a body $(B, G), \rho$, where B is a smooth manifold, G is a Riemannian metric on B, and ρ is the mass density.
- Fix a background manifold (M, g), with M a smooth manifold and g a Riemannian metric on B.

Dynamics of B on M are time-dependent embeddings

$$\phi_t : B \to M.$$

We model these embeddings as fields

$$Y \xrightarrow{\pi} X.$$

Hyperelastic materials II

- $X := \mathbb{R} \times B$.
- $Y := \mathbb{R} \times B \times M$.
- The projection π is the trivial choice.

There is an issue:

- Arbitrary fields of the previous bundle $\phi : X \to Y$ *do not* necessarily correspond to time-dependent embeddings.

But not for long...

- Nevertheless, we can still apply the theory developed because embeddings are stable under local perturbations (variations).

What is the Lagrangian?

Notation:

- Coordinates on B are denoted by $(x^i) = x^1, \ldots, x^{n-1}$.
- When adding the time coordinate $t = x^0$, we get coordinates (x^μ) on X.
- Coordinates on M are denoted by (y^a).

Then, the Lagrangian is:

$$
\begin{aligned}
\mathcal{L} &= \mathbb{K} - \mathbb{P} \\
&= \frac{1}{2}\sqrt{\det G}\rho g_{ab} z_0^a z_0^b d^{n+1}x - \sqrt{\det G}\rho W(x^\mu, G, g, z_i^a) d^{n+1}x,
\end{aligned}
$$

where W is the stored energy.

Hyperelastic materials IV

The Poincaré-Cartan form is

$$\Theta_{\mathcal{L}} = \rho g_{ab} z_0^b \sqrt{\det G} dy^a \wedge d^n x_0 - \rho \frac{\partial V}{\partial z_i^a} \sqrt{\det G} dy^a \wedge d^n x_i$$
$$- \left(-\frac{\partial W}{\partial z_i^a} z_i^a + \frac{1}{2} g_{ab} z_0^a z_0^b + W \right) \rho \sqrt{\det G} d^{n+1} x$$

How can we apply Noether's Theorem?

- There is a clear symmetry, time-invariance.
- Then, the current obtained through the theory is

$$\alpha = -\rho \frac{\partial W}{\partial z_i^a} \sqrt{\det G} dy^i d^n x_{i0}$$
$$+ \left(\frac{1}{2} g_{ab} z_0^a z_0^b + W - \frac{\partial W}{\partial z_i^a} z_i^a \right) \rho \sqrt{\det G} d^{n+1} x_0.$$

Hyperelastic materials V

on holonomic sections it takes the expression

$$\alpha = \left(\frac{1}{2} g_{ab} z_0^a z_0^b + W\right) \rho \sqrt{\det G} d^{n+1} x_0 + \rho \frac{\partial W}{\partial z_i^a} z_0^a \sqrt{\det G} d^n x_i$$

Since

$$e = \left(\frac{1}{2} g_{ab} z_0^a z_0^b + W\right) \rho \sqrt{\det G} d^{n+1} x_0$$

can be though of as the energy dentisy. This gives us a conservation law, where

$$\rho \frac{\partial W}{\partial z_i^a} z_0^a \sqrt{\det G} d^n x_i$$

is the energy flux.

Final remarks

- Multisymplectic geometry is a tool that allows us to study variational problems (field theory, Classical Mechanics, some problems in material modelling...)
- It is a very active area of research, both from the mathematical and the physical point of view.
- Applications to material modelling seem interesnting, have been practically unexplored.

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Thank you for your attention!