Coisotropic reduction in different phase spaces

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Rubén Izquierdo, Manuel De León Wednesday 21st February, 2024

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Outline

Coisotropic reduction in non-dissipative mechanics

Coisotropic reduction in dissipative mechanics

non-dissipative mechanics

The canonical phase spaces

• If Q is the configuration space of a mechanical system, the phase space $M:=T^*Q$ inherits a canonical symplectic structure (M,ω) ,

$$\omega:=\omega_Q=-d\lambda_Q=dq^i\wedge dp_i.$$

• The phase space $M:=T^*Q\times\mathbb{R}$ inherits a canonical cosymplectic structure, $(M,\omega,\theta),$

$$\omega = \omega_Q = dq^i \wedge dp_i, \quad \theta = dt.$$

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Symplectic and cosympelctic manifolds are Poisson manifolds with the bracket

$$\{f,g\} = \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}.$$

In each of these cases, the bracket is induced by the Poisson bivector

$$\Lambda = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}, \ \{f,g\} = \Lambda(df,dg).$$

We have an induced map

$$\sharp_{\Lambda}: T^{*}M \to TM, \quad \alpha \mapsto \iota_{\alpha}\Lambda.$$

Denote

$$\mathcal{H} := \operatorname{im} \sharp_{\Lambda} = \langle \frac{\partial}{\partial q^i}, \frac{\partial}{\partial p_i} \rangle$$

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If $\Delta \subseteq T_x M$, we define the orthogonal as

$$\Delta^{\perp_{\Lambda}} := \sharp_{\Lambda}(\Delta^0),$$

where $\Delta^0 \subseteq T_x^*M$ is the annihilator of Δ . We say that Δ is

· Coisotropic, if

$$\Delta^{\perp_{\Lambda}} \subseteq \Delta,$$

· Lagrangian, it

$$\Delta^{\perp_{\Lambda}} = \Delta \cap \mathcal{H}.$$

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These definitions apply to submanifolds $N \hookrightarrow M$ as well.

Let (M,ω) be a symplectic manifold and $i:N\hookrightarrow M$ be a coisotropic submanifold.

Proposition

 $(TN)^{\perp_{\Lambda}} \subseteq TN$ is an involutive distribution.

Define $\mathcal F$ to be the maximal foliation associated to $(TN)^{\perp_\Lambda}$. We will assume that $N/\mathcal F$ admits a manifold structure such that the canonical projection $\pi:N\to N/\mathcal F$ is a summersion.

Theorem ([?]Weinstein)

$$\pi^*\omega_N = i^*\omega.$$

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Let (M, ω, θ) be a cosymplectic manifold and $i: N \hookrightarrow M$ be a coisotropic submanifold.

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Suppose N/\mathcal{F} admits a manifold structure such that $\pi:N\to N/\mathcal{F}$ defines a summersion.

- If $TN \subseteq \mathcal{H}$, N/\mathcal{F} admits an unique symplectic structure compatible with the structure defined on M.
- If $\frac{\partial}{\partial t} \in TN$, N/\mathcal{F} admits an unique cosymplectic sturcture compatible with the one defined on M.

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Coisotropic reduction in dissipative

mechanics

The canonical phase spaces

• The phase space of an autonomous dissipative system is $T^*Q \times \mathbb{R}$, with its canonical contact structure

$$\eta = dz - p_i dq^i.$$

• If we want to study time-dependent dissipative mechanics the phase space is $T^*Q\times \mathbb{R}\times \mathbb{R}$ endowed with its cocontact structure

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The Jacobi bracket

In both of these phase spaces there is a Jacobi bracket which is locally given by

$$\{f,g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} + p_i\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial z}\right) + g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}.$$

This Jacobi bracket is defined through the Jacobi bivector and a vector field

$$\begin{split} \Lambda &= \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial z}, \\ E &= -\frac{\partial}{\partial z}, \end{split}$$

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If $N \hookrightarrow M$ is a coisotropic submanifold, then $(TN)^{\perp_{\Lambda}}$ is involutive and thus arises from a maximal foliation \mathcal{F} .

We assume that $\frac{\partial}{\partial z} \in TN$.

Theorem

- If $\frac{\partial}{\partial t} \in TN$, N/\mathcal{F} inherits an unique cocontact structure from M.
- If $TN \subseteq \operatorname{im} \sharp_{\Lambda} \oplus \langle \frac{\partial}{\partial z} \rangle$, N/\mathcal{F} inherits an unique contact structure from M.

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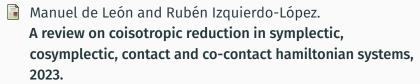
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