Application of the H-DES Quantum Algorithm to the Black-Scholes Equation

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Introduction

I have extended your quantum algorithm method to the field of quantitative finance by solving the Black-Scholes Partial Differential Equation (PDE). This equation is fundamental in pricing European call options, a key problem in financial mathematics.

The Black-Scholes PDE is given by:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - rf = 0,$$

where:

- S: Asset price,
- $f(S,t)$: Option price,
- σ: Volatility,
- r: Risk-free interest rate,
- t: Time, with $t = 0$ as the current time and $t = T$ as the maturity.

The terminal condition at maturity is the payoff function:

$$f(S,T) = \max(S-K, 0),$$

where K is the strike price. Additional boundary conditions include:

- $f(0,t) = 0$: No value if the asset price is zero.
- $f(S,t) \sim S - Ke^{-r(T-t)}$: Large-$S$ behavior.

Analytical Solution

For the European call option, the Black-Scholes formula provides the closed-form solution:

$$f(S,t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$

where:

$$d_1 = \frac{\ln(S/K) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Here, $\Phi(x)$ is the cumulative distribution function of the standard normal distribution.

Quantum Algorithm Adaptation

To apply your quantum algorithm:

1. The S- and t-domains were encoded into the quantum circuit via Chebyshev polynomials.

2. Boundary and terminal conditions were embedded into the solution using a suitable scaling function.

3. The residual loss for the PDE and boundary conditions was minimized using a hybrid quantum-classical approach with a BFGS optimizer.

The quantum circuit was built with 5 qubits and a depth of 2. The variational parameters were optimized to approximate the solution $f(S, t)$.

Results

The final result compares the VQA approximation to the analytical Black-Scholes solution at $t = 0.5$. The plot below illustrates the option price $f(S, t = 0.5)$ as a function of S.

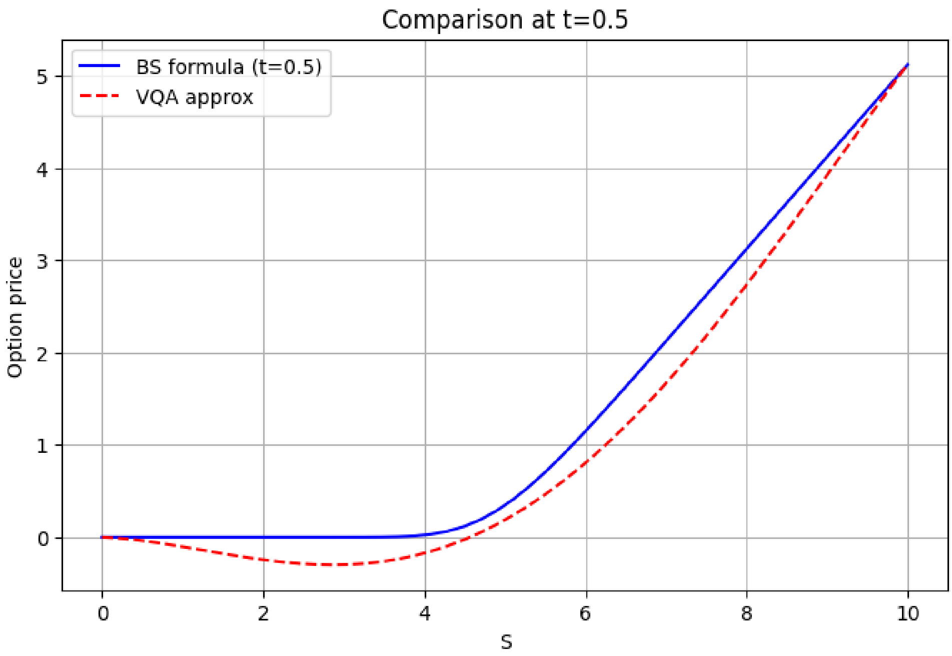

Figure 1: Comparison of the Black-Scholes analytical solution (blue) and VQA approximation (red) at $t = 0.5$.

The results highlight several important aspects:

- **Low-Circuit Depth and Simplicity:** The solution was achieved with a shallow circuit depth and a limited number of parameters, demonstrating the potential for further improvement with deeper circuits or more expressive ansätze.

- **Boundary Conditions:** The boundary conditions were well enforced, ensuring that $f(0, t) = 0$ and that the large-S behavior $f(S, t) \sim S - Ke^{-r(T-t)}$ was accurately captured.

- **Scalability and Feasibility:** This proof of concept highlights the computational feasibility of using quantum algorithms for financial PDEs, even with near-term quantum hardware constraints.

Conclusion

This work demonstrates the potential of quantum algorithms to solve financial PDEs such as the Black-Scholes equation. While the original paper focused on univariate differential equations for results and examples, this study extends the results to a multivariate PDE, demonstrating its applicability to more complex equations like those in finance. The results highlight the viability of quantum techniques for option pricing and open avenues for further exploration in quantitative finance.