Financial Crashes

Topological Data Analysis and Wavelet Analysis

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Framework

- daily adjusted closing prices of four US stock market indices
- from January 1, 1992 to December 31, 2021

Stock Market Index	Stock symbol
S&P 500	GSPC
Dow Jones	DJI
NASDAQ Composite	IXIC
Russell 2000	RUT

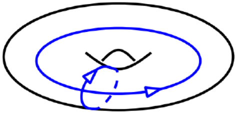
- aim: **detecting market crashes** and finding **early warning signals**
  - **March 10, 2000**: Dotcom Bubble burst
  - **September 15, 2008**: bankruptcy of Lehman Brothers
  - **March 13, 2020**: US national emergency due to COVID-19
- **Topological Data Analysis**
  - persistence homology
  - L^1-norm process
- **Wavelet Analysis**
  - Haar wavelet
  - Walsh-type wavelet packet

Topological Data Analysis of Financial Time Series

Homology

- powerful tool from **algebraic topology**
- algebraic **invariants** that **classify topological spaces**
- For any dimension k, the k-**dimensional "holes"** are represented by the **homology vector space** H_k with coefficients in $\mathbb{Z}_2$ whose **dimension** is the number of such independent features

  - $H_0 \to$ connected components
  - $H_1 \to$ loops
  - $H_2 \to$ cavities



$$\begin{cases} \mathbf{H}_0 & \cong \mathbb{Z}_2 \\ \mathbf{H}_1 & \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ \mathbf{H}_2 & \cong \mathbb{Z}_2 \end{cases}$$

Embedding of a data set[1]

- d time series $\{x^k = (x^k_t)_{t\in T} : k = 1, \dots, d\}$
- $\forall t_i \in T$, $\mathbf{x}(t_i) = (x^1_{t_i}, x^2_{t_i}, \dots, x^d_{t_i}) \in \mathbb{R}^d$
- **sequence of point cloud data sets** of w points in $\mathbb{R}^d$:

$$\mathbb{X}_{t_i} = (\mathbf{x}(t_i), \mathbf{x}(t_{i+1}), \dots, \mathbf{x}(t_{i+w-1}))$$

- 4 time series of the **logarithmic returns** $\log(P_{i,j}/P_{i-1,j})$ where $P_{i,j}$ is the adjusted closing value of stock index j of day i
- every **day** is represented by a **point** in $\mathbb{R}^4$ whose coordinates are the logarithmic returns the indices
- point clouds of $w = 50$ points embedded in $\mathbb{R}^4$

[1] Marian Gidea and Yuri Katz. "Topological data analysis of financial time series: Landscapes of crashes". In: *Physica A: Statistical Mechanics and its Applications* 491 (Oct. 2017).

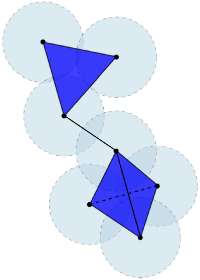
Computation of the persistence homology of point clouds

As our input data we have a point cloud data set consisting of a family of points $X = \{x_1, \dots, x_n\}$ in an Euclidean space $\mathbb{R}^d$.

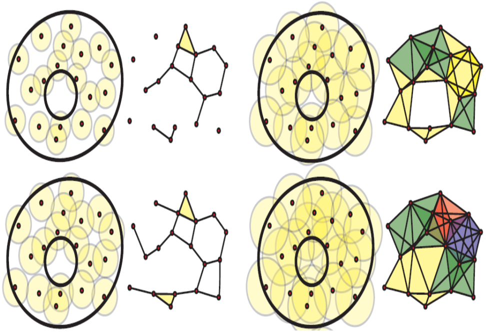
Vietoris-Rips simplicial complex

The **Vietoris-Rips complex** $VR_r(\mathbb{X})$ of $\mathbb{X}$ with radius r is the set of simplexes of the form $[x_0, \dots, x_k]$ such that the mutual distance between any pair of these simplexes is less than r, i.e.,

$$d(x_i, x_j) \leq r \quad \forall\, 0 \leq i, j \leq k.$$

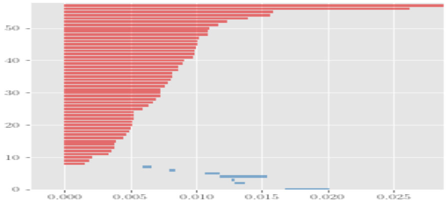


- **filtration** $(K_r)_{r \in T}$ of a simplicial complex

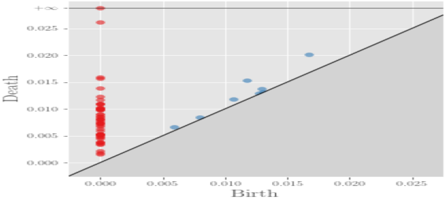


- changes tracked by **persistent homology**

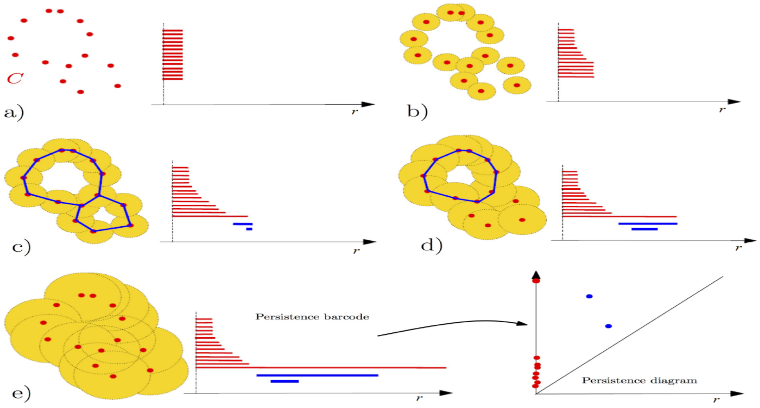
- **persistence barcode**



- **persistence diagram**



Example of filtration associated to a point cloud and the resulting persistent barcode and persistent diagram[2]



[2]Frédéric Chazal and Bertrand Michel. "An Introduction to Topological Data Analysis: Fundamental and Practical Aspects for Data Scientists". In: *Frontiers in Artificial Intelligence* 4 (Oct. 2017).

L^1-norm process

Given a persistent diagram $\mathcal{D} = \{(b_i, d_i)\}_{i \in I}$, where (b_i, d_i) are the birth-death couples, we define a piecewise linear continuous function $\lambda_{(b_i,d_i)} : \mathbb{R} \to \mathbb{R}$

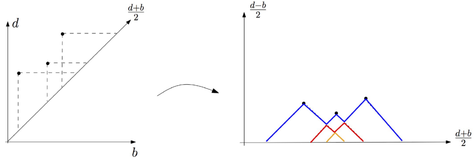
$$\lambda_{(b_i,d_i)}(t) = \begin{cases} t - b_i & \text{if} \quad t \in [b_i, \frac{b_i+d_i}{2}] \\ d_i - t & \text{if} \quad t \in (\frac{b_i+d_i}{2}, d_i] \\ 0 & \text{if} \quad t \notin (b_i, d_i) \end{cases}$$

The **persistence landscape** $\lambda : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ is then defined as

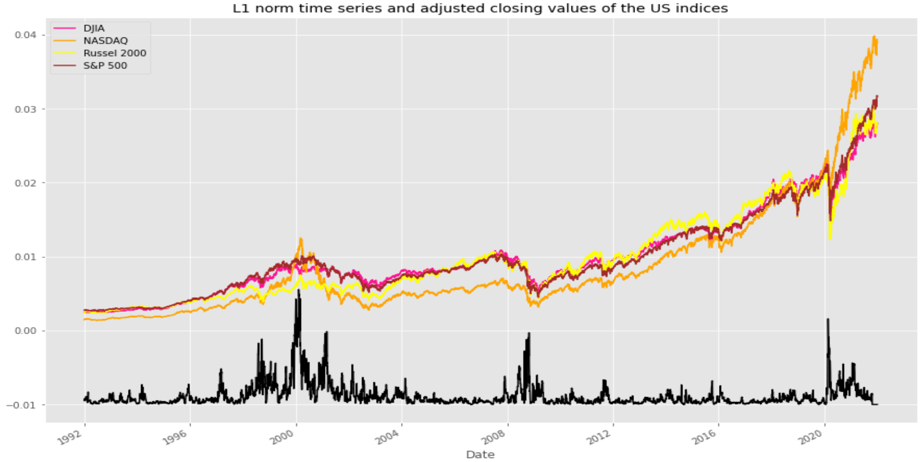
$$\lambda(k, t) = \mathrm{kmax}_{i \in I}\{\lambda_{(b_i,d_i)}(t)\}$$

A persistence landscape $\Lambda : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ is endowed with the L^1-**norm**

$$||\lambda||_1 = \sum_{k=1}^{\infty} ||\lambda_k||_1$$



L^1-norm time series



Early warning system

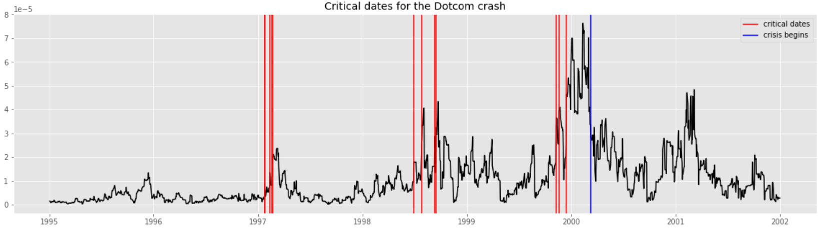
- For every trading day j:
  - N_j: the L^1-norm of the persistence landscape of the point cloud of the w trading days following day j
  - B_j: the average L^1-norm of the point clouds referring to t consecutive trading days before j, with $t >> w$

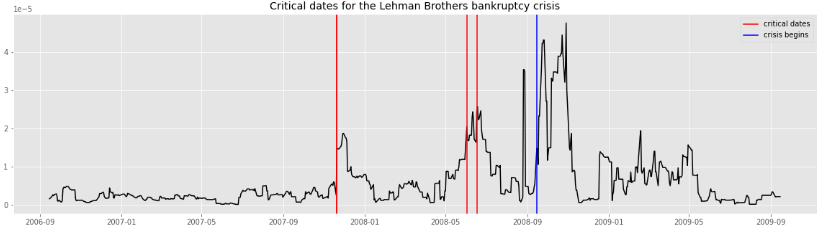
  Trading day j is an **early warning signal** if:

  1. $\frac{N_j}{N_{j-1}} \geq \alpha$ for a fixed $\alpha > 1$;
  2. $N_j / B_j \geq \beta$ for a fixed $\beta \geq 2$.
- We set $\alpha = 1.4$, $\beta = 3$ and $t = 250$[3].

[3] Hongfeng Guo et al. "Empirical study of financial crises based on topological data analysis". In: 558 (July 2020), p. 124956.

Early warning signals for the TDA approach



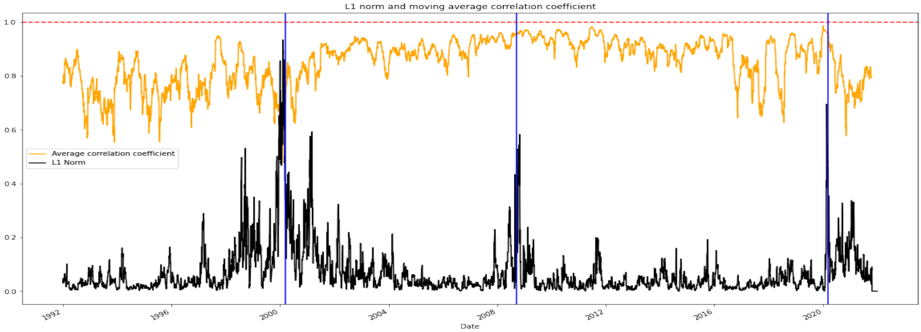


Crisis	Critical Dates
2000	Mar. 1997 (4)
	Aug. 1998 (2)
	Oct. 1998 (2)
	Dic. 1999 (2)
	Jan, 2000 (1)
2008	Dic. 2007 (1)
	Jul. 2008 (2)
2020	-

L^1-norm and average moving correlation coefficient time series

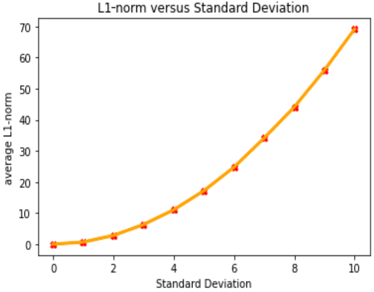
- strong synchronization of the stock indices
- nullified the persistence of the 1-dimensional components

	Dow Jones	NASDAQ C.	Russell 2000	S&P 500
Dow Jones	1	0.986	0.973	0.996
NASDAQ C.	0.986	1	0.964	0.995
Russell 2000	0.973	0.964	1	0.974
S&P 500	0.996	0.995	0.974	1

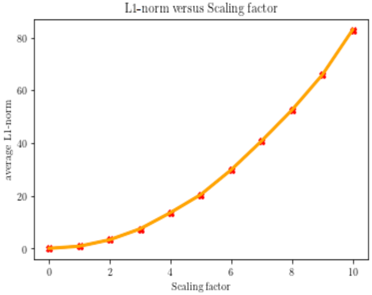


Testing with independent random variables[4]

Normally distributed variables with a growing variance



Scaled Student's t-distributed variables with growing scaling parameter

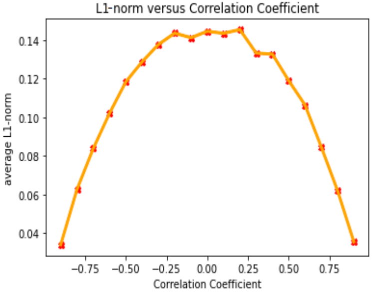


- There is quadratic dependency of the L^1-norm with respect to the standard deviation or to the scaling factor.

[4] L. Aromi Leaverton. *Analysis of Financial Time Series using TDA: Theoretical and Empirical Results*. Universitat de Barcelona, 2020.

Testing with correlated random variables[5]

Normally distributed variables with constant variance



It is clear that as $|\rho| \to 1$, the L^1-norm approaches zero, meaning that high correlation between variables can completely nullify the L^1-norm of the persistence landscape.

[5] Lloyd Aromi, Yuri Katz, and Josep Vives. "Topological features of multivariate distributions: Dependency on the covariance matrix". In: *Communications in Nonlinear Science and Numerical Simulation* 103 (Aug. 2021), p. 105996.

Analysis of financial time series through Haar wavelets

Haar Wavelets[6]

- signal $f = (f_1, f_2, \dots, f_N) \in \mathbb{R}^N$

1-level Haar wavelets	1-level Haar scalings
$w_1^1 := \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \dots, 0, 0, 0\right)$	$v_1^1 := \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0, \dots, 0\right)$
$w_2^1 := \left(0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \dots, 0\right)$	$v_2^1 := \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0\right)$
$\vdots$	$\vdots$
$w_{N/2}^1 := \left(0, \dots, 0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$v_{N/2}^1 := \left(0, \dots, 0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

- set $\left\{v_1^1, \dots, v_{N/2}^1, w_1^1, \dots, w_{N/2}^1\right\}$ is an orthonormal basis of $\mathbb{R}^N$ and we call $V^1 := \mathrm{span}\{\mathrm{v}_1^1, \dots, \mathrm{v}_{N/2}^1\}$ and $W^1 = \mathrm{span}\{\mathrm{w}_1^1, \dots, \mathrm{w}_{N/2}^1\}$.

[6] James Walker. *A Primer on Wavelets and Their Scientific Applications*. Nov. 2020.

first trend	first fluctuation
$a^1 = (f \cdot v_1^1, \ldots, f \cdot v_{N/2}^1)$	$d^1 = (f \cdot w_1^1, \ldots, f \cdot w_{N/2}^1)$

first averaged signal	first detail signal
$A^1 = \sum_{j=1}^{N/2} (f \cdot v_j^1) v_j^1$	$D^1 = \sum_{j=1}^{N/2} (f \cdot w_j^1) w_j^1$

- A^1 and D^1 are respectively the orthogonal projections of f on the spaces V^1 and W^1
- a signal is expressed as $f = A^1 + D^1$
- The **first level Haar transform** is $H^1 : \mathbb{R}^N \to \mathbb{R}^N$ and it is defined as

$$H^1(\mathrm{f}) = \left(\mathrm{a}^1 | \mathrm{d}^1\right)$$

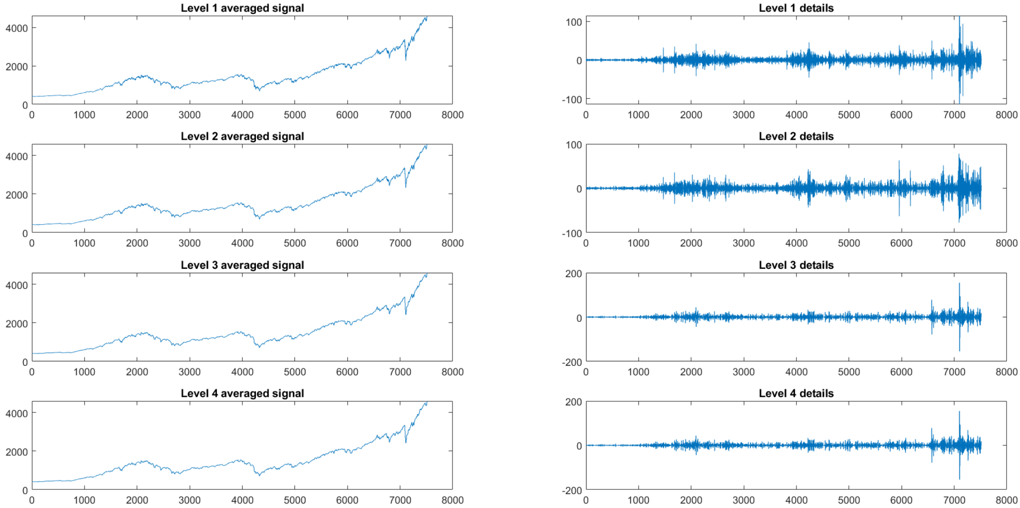
- Analogously, the m-level MRA, where m is such that N is divisible m times by 2, is

$$
\begin{aligned}
f &= A^m + D^m + \cdots + D^2 + D^1 \\
A^m &= \sum_{j=1}^{N/2^m} (f \cdot v_j^m) v_j^m \\
D^m &= \sum_{j=1}^{N/2^m} (f \cdot w_j^m) w_j^m
\end{aligned}
$$

- the m-**level Haar transform** is $\mathrm{H}^m : \mathbb{R}^N \to \mathbb{R}^N$ defined as

$$\mathrm{H}^m(\mathrm{f}) = \left(\mathrm{a}^m | \mathrm{d}^m | \mathrm{d}^{m-1} | \dots | \mathrm{d}^1\right).$$

Averaged signals and details for S&P 500

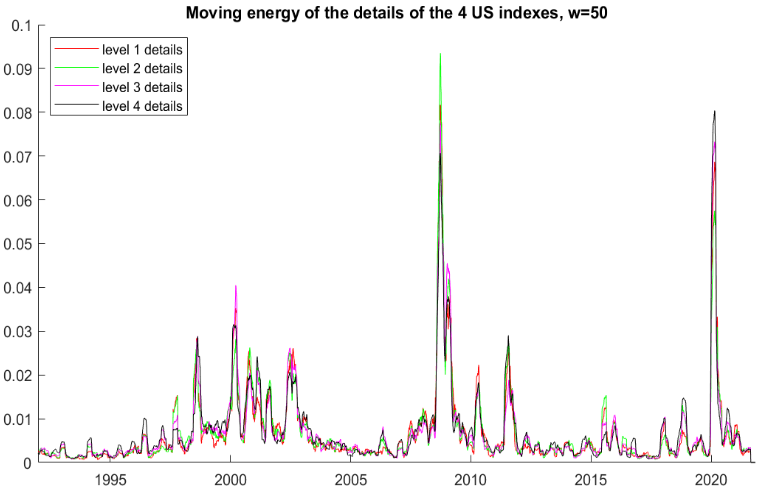


Methodology for the Haar wavelet analysis

- multiresolution analysis using the Haar transform up to level 4.
- define four time series H_i for $i = 1, \ldots, 4$ using details and averaged signals of each index up to i-th level

$$H_i(t) = \sum_{z \in \text{indices}} \frac{\sum_{j=1}^{i} |\mathrm{D}_z^j(t)|}{\mathrm{A}_z^i(t)} \quad i = 1, \ldots, 4$$

Moving energy of the time series H_i with $i = 1, \ldots, 4$



Noise reduction

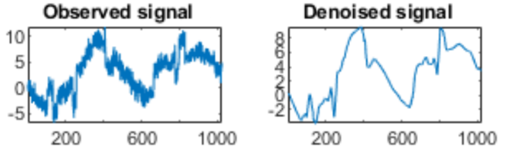
- **energy** of a signal $E(f) = f_1^2 + f_2^2 + \cdots + f_N^2$
- arrange the absolute values of the Haar transform in decreasing order

$$L_1 \geq L_2 \geq L_3 \geq \cdots \geq L_N$$

- Letting p be the percentage of energy we want to retain, we identify j as the first index such that

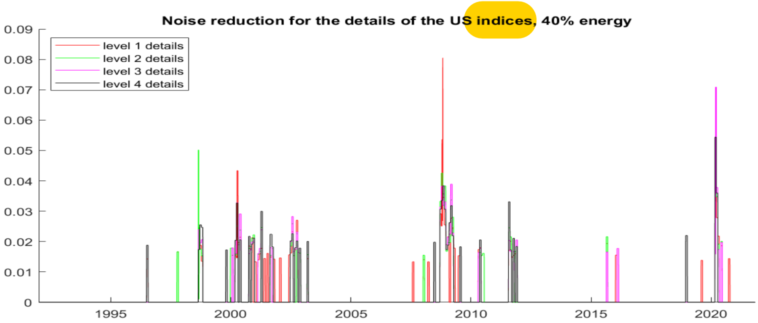
$$\frac{L_1^2 + L_2^2 + \cdots + L_j^2}{E(\mathrm{f})} \geq p$$

- set the threshold $T := L_j$ and transmit only the significant, non-zero values of the transform through the **hard thresholding function** and a **significance map**
- inverse wavelet transform



Noise reduction of the time series H_i

- Energy retention 40%
- A trading day i is called a **critical date** if the corresponding denoised signal is different from zero



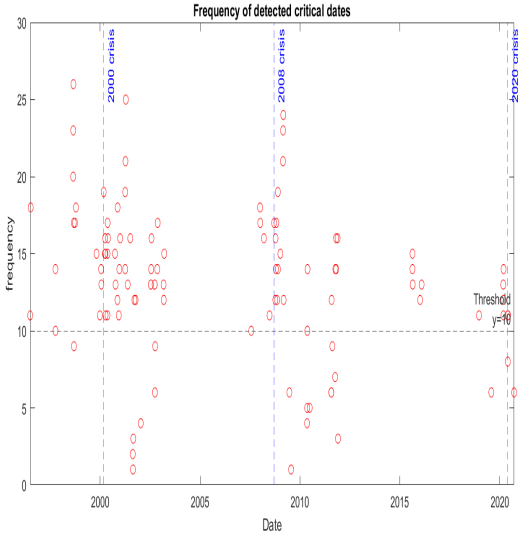
Early warning system

- We associate a counter to each critical date
- For each day $x \in (\textit{critical date} - 15, \textit{critical date} + 15]$, we consider the price time series from January 1, 1992 up to day x
- We concatenate the truncated series with its flipped version and repeat this pattern until we obtain a time series of the same length as the original data set

$$p(1), p(2), \ldots, p(x-2), p(x-1), p(x), p(x-1), p(x-2), \ldots, p(2), p(1), p(2), \ldots$$

- We apply the wavelet method to the newly created price series
- If the obtained denoised signal is different from zero in correspondence of the critical date, we increase the counter by one If a critical date has associated counter ≥ 10, then it is an early warning signal

Early warning signals

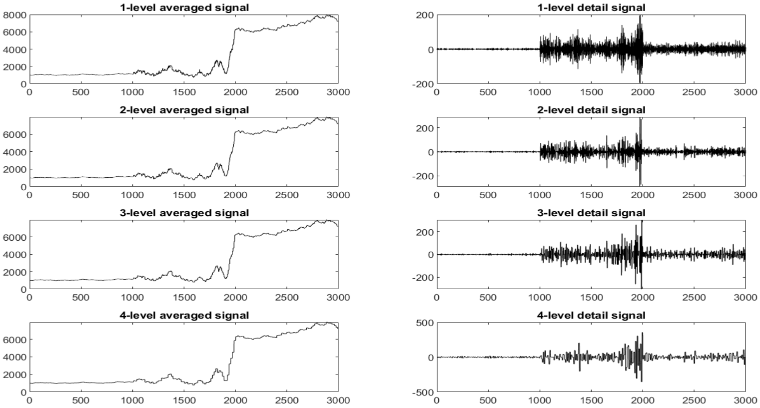


Crisis	Critical Dates
2000	July 1996 (2)
	October 1997 (2)
	August 1998 (2)
	September 1998 (3)
	October 1998 (2)
	December 1999 (1)
	January 2000 (2)
	March 2000 (1)
2008	July 2007 (1)
	January 2008 (2)
	March 2008 (1)
	June 2008 (1)
2020	March 2020 (4)

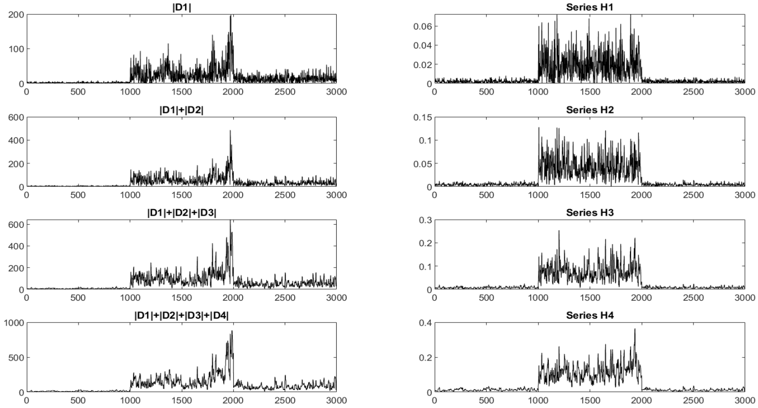
Testing for the Haar wavelet analysis

- assumption of normally distributed log-returns
- the price series can be generated using $s_i = \exp\left(r_i + \log(s_{i-1})\right)$

Thanks to the MRA we are able to capture the **short-term price changes** of an index in the details, while the **price trend** is captured by the averaged signals.



The time series H_i for $i = 1, \ldots, 4$ are able to capture the **relative price change** rather than their absolute price changes.



Discussion and Conclusions[7]

Achievements:

- development of **two methods** through TDA and wavelet analysis
- **detection** of crashes
- implementation of an **early warning system** triggered by high market volatility

Limitations:

- **cannot** give early warning signals for financial meltdowns caused by sudden **exogenous shocks**
- both methods **cannot forecast** crashes
- strongly depend on **hyperparameters**

[7]Yuri Katz and Alain Biem. "Time-resolved topological data analysis of market instabilities". In: *Physica A: Statistical Mechanics and its Applications* 571 (Feb. 2021), p. 125816.

Thanks for your attention

Analysis of financial time series through Walsh-type wavelet packet

Walsh-type wavelet packet

- generalization of the Haar transforms
- when we perform the wavelet packet transform of a signal, we compute the wavelet **transform** not only of the trend subsignals but also of the **fluctuation** subsignals
- 1-level Walsh transform

$$Wh_1(\mathbf{f}) = (\mathbf{a}^1|\mathbf{d}^1) = H_1(\mathbf{f}).$$

- 2-level Walsh transform

$$Wh_2(\mathbf{f}) = (H_1(\mathbf{a}^1)|H_1(\mathbf{d}^1)) = (\mathbf{a}^2|\mathbf{d}^2|H_1(\mathbf{d}^1)) = (\mathbf{a}^2|\mathbf{d}^2|\mathbf{d}^{1,a}|\mathbf{d}^{1,d})$$

Methodology for the Walsh-type wavelet packet analysis

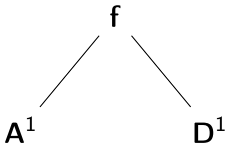
$$W_1(t) = \sum_{z \in \text{indices}} \frac{|\mathrm{D}_z^1(t)|}{\mathrm{A}_z^1(t)}$$

$$W_2(t) = \sum_{z \in \text{indices}} \frac{|\mathrm{D}_z^2(t)| + |\mathrm{D}_z^{1,d}(t)|}{\mathrm{A}_z^2(t)}$$

$$W_3(t) = \sum_{z \in \text{indices}} \frac{|\mathrm{D}_z^3(t)| + |\mathrm{D}_z^{2,d}(t)| + \sum_{j=\{a,d\}} |\mathrm{D}_z^{1,j,d}(t)|}{\mathrm{A}_z^3(t)}$$

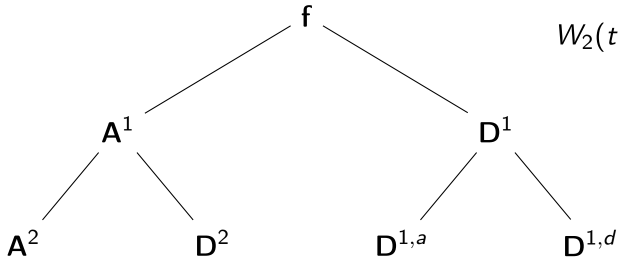
$$W_4(t) = \sum_{z \in \text{indices}} \frac{|\mathrm{D}_z^4(t)| + |\mathrm{D}_z^{3,d}(t)| + \sum_{j=\{a,d\}} |\mathrm{D}_z^{2,j,d}(t)| + \sum_{j,k=\{a,d\}} |\mathrm{D}_z^{1,j,k,d}(t)|}{\mathrm{A}_z^4(t)}$$

Time series W_1:



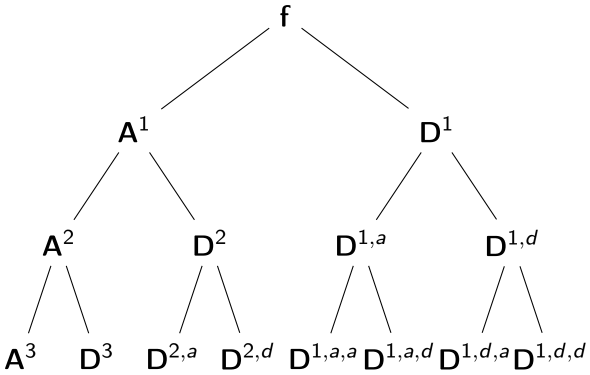
$$W_1(t) = \sum_{z \in \text{indices}} \frac{|\mathrm{D}^1_z(t)|}{\mathrm{A}^1_z(t)}$$

Time series W_2:



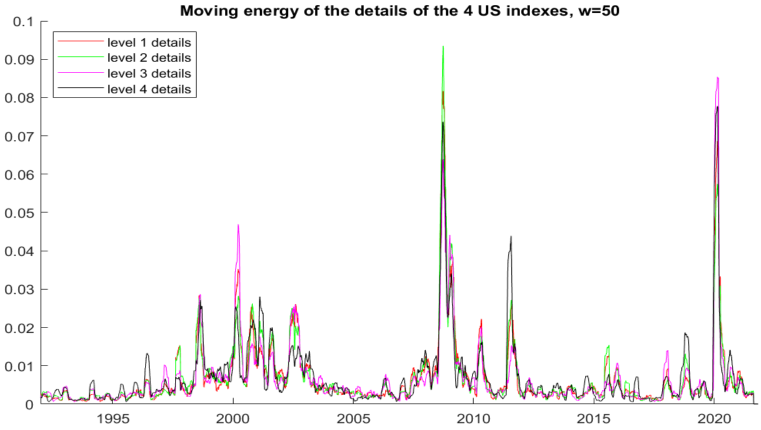
$$W_2(t) = \sum_{z \in \text{indices}} \frac{|\mathrm{D}^2_z(t)| + |\mathrm{D}^{1,d}_z(t)|}{\mathrm{A}^2_z(t)}$$

Time series W_3:



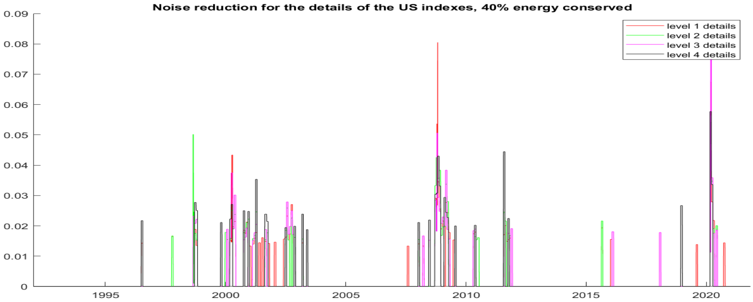
$$W_3(t) = \sum_{z \in \text{indices}} \frac{|\mathrm{D}_z^3(t)| + |\mathrm{D}_z^{2,d}(t)| + \sum_{j=\{a,d\}} |\mathrm{D}_z^{1,j,d}(t)|}{\mathrm{A}_z^3(t)}$$

Moving energy of the time series W_i

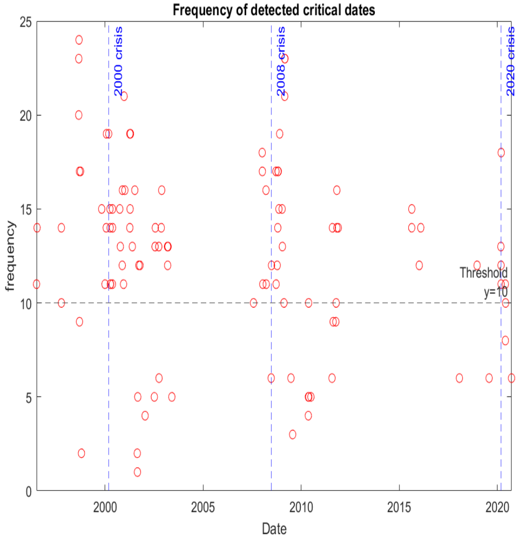


Noise reduction of the time series W_i

- Energy retention 40%



Early warning signals



Crisis	Critical Dates
2000	July 1996 (1)
	October 1997 (2)
	August 1998 (2)
	September 1998 (3)
	October 1998 (1)
	December 1999 (1)
	January 2000 (2)
	March 2000 (1)
2008	January 2008 (2)
	March 2008 (2)
2020	March 2020 (4)