Informe: Optimizacion Convexa Taller No. 2

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1. Introducción

Development of exercises on notation, Gradient and demonstrations.

2. Desarrollo

1. Basic notation. In front of each symbol write its meaning
   - R : denotes the set of real numbers.
   - R_+ : denotes the set of positive real numbers.
   - R_{++} : denotes the set of positive real numbers except zero.
   - R_+^{mxn}: denotes the set of matrices of positive real numbers of size $m * n$.
   - N_+^k: denotes the set of vectors of positive natural numbers of size k.
   - S_+^k : denotes the set of symmetric positive semi-defined matrices.
   - $f : R^p \rightarrow R^q$: denotes a function with domain R^p and range in R^q.
2. Calculate the $\nabla f(x)$ respect to x of the following function
   - $||y - Hx||_2^2$: (Demonstration in annex; 1)
   - $||y - Hx||_2$: (Demonstration in annex; 1)
   - $||y - Hx||_1$: (Demonstration in annex; 2)

3. For the special case where A is a symmetric matrix and

$$\alpha = x^T A x$$

where $x \in R^n$ and $A \in R^{nxn}$. Then, calculate $\frac{\partial \alpha}{\partial x}$. Differs the result if A is not symmetric?.

Answ: If A is not symmetric the answer change. (Demonstration in annex;3)

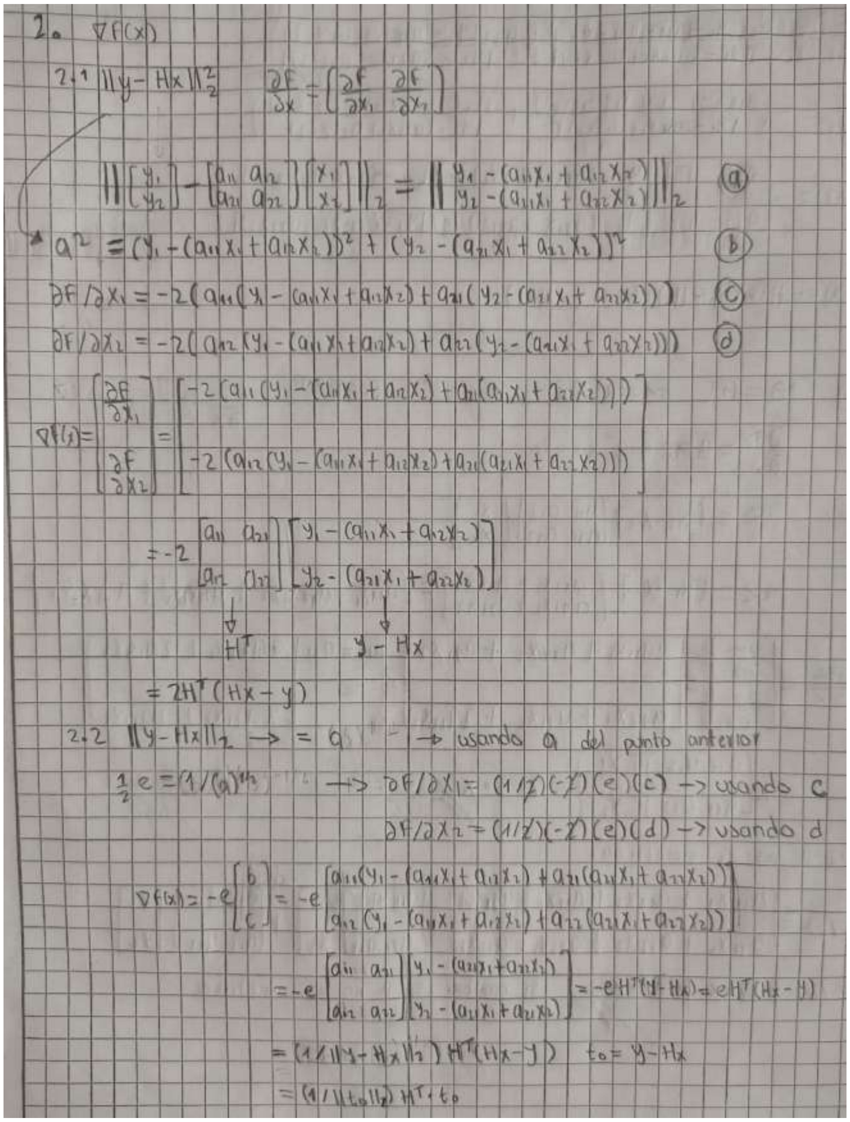
4. Let $A(\alpha)$ be a $m * n$ matrix whose elements are functions of the scalar parameter α. Then the derivative of the matrix $A(\alpha)$ with respect to the scalar parameter α is the $m * n$ matrix of element-by-element derivatives:

$$\frac{\partial \mathbf{A}(\alpha)}{\partial \alpha} = \begin{bmatrix} \frac{\partial a_{11}}{\partial \alpha} & \frac{\partial a_{12}}{\partial \alpha} & \dots & \frac{\partial a_{1n}}{\partial \alpha} \\ \frac{\partial a_{21}}{\partial \alpha} & \frac{\partial a_{22}}{\partial \alpha} & \dots & \frac{\partial a_{2n}}{\partial \alpha} \\ \vdots & \vdots & & \vdots \\ \frac{\partial a_{ma}}{\partial \alpha} & \frac{\partial a_{m2}}{\partial \alpha} & \dots & \frac{\partial a_{man}}{\partial \alpha} \end{bmatrix}$$

Then let $A(\alpha)$ be a nonsingular, $m * m$ matrix whose elements are functions of the scalar parameter α. Then show what:

$\frac{\partial A(\alpha)^{-1}}{\partial \alpha} = -A(\alpha)^{-1} \frac{\partial A(\alpha)}{\partial \alpha} A(\alpha)^{-1}$ (Demonstration in annex; 4)

3. Anexos



2. $\nabla f(x)$

2.1 $\|y - Hx\|_2^2$ $\quad \frac{\partial f}{\partial x} = \left[\frac{\partial f}{\partial x_1} \; \frac{\partial f}{\partial x_2}\right]$

$$\left\| \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_2 = \left\| \begin{matrix} y_1 - (a_{11}x_1 + a_{12}x_2) \\ y_2 - (a_{21}x_1 + a_{22}x_2) \end{matrix} \right\|_2 \quad (a)$$

$$a^2 = (y_1 - (a_{11}x_1 + a_{12}x_2))^2 + (y_2 - (a_{21}x_1 + a_{22}x_2))^2 \quad (b)$$

$$\partial f / \partial x_1 = -2(a_{11}(y_1 - (a_{11}x_1 + a_{12}x_2) + a_{21}(y_2 - (a_{21}x_1 + a_{22}x_2))) \quad (c)$$

$$\partial f / \partial x_2 = -2(a_{12}(y_1 - (a_{11}x_1 + a_{12}x_2) + a_{22}(y_2 - (a_{21}x_1 + a_{22}x_2))) \quad (d)$$

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -2(a_{11}(y_1 - (a_{11}x_1 + a_{12}x_2) + a_{21}(a_{21}x_1 + a_{22}x_2))) \\ -2(a_{12}(y_1 - (a_{11}x_1 + a_{12}x_2) + a_{21}(a_{21}x_1 + a_{22}x_2))) \end{bmatrix}$$

$$= -2 \underbrace{\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}}_{H^T} \underbrace{\begin{bmatrix} y_1 - (a_{11}x_1 + a_{12}x_2) \\ y_2 - (a_{21}x_1 + a_{22}x_2) \end{bmatrix}}_{y - Hx}$$

$$= 2H^T(Hx - y)$$

2.2 $\|y - Hx\|_2 \rightarrow = a$ $\rightarrow$ usando a del punto anterior

$\frac{1}{2} e = 1/(a)^{1/2}$ $\longrightarrow \partial f/\partial x_1 = (1/\cancel{2})(-\cancel{2})(e)(c) \rightarrow$ usando c

$\partial f/\partial x_2 = (1/\cancel{2})(-\cancel{2})(e)(d) \rightarrow$ usando d

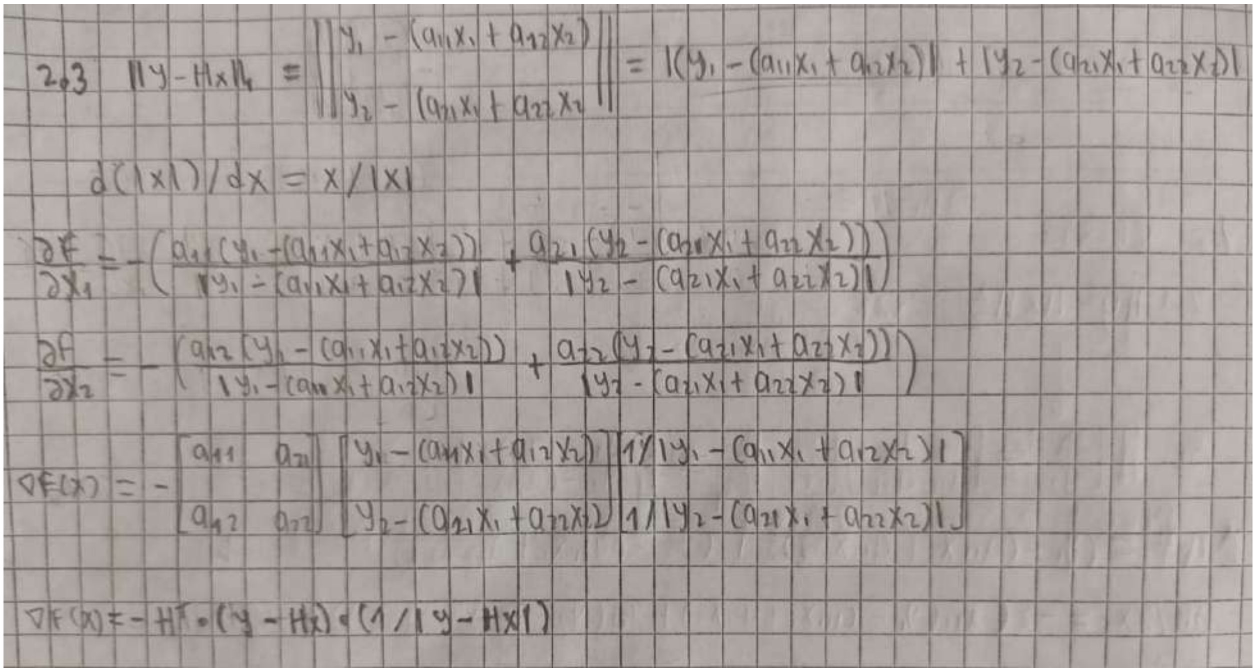
$$\nabla f(x) = -e \begin{bmatrix} b \\ c \end{bmatrix} = -e \begin{bmatrix} a_{11}(y_1 - (a_{11}x_1 + a_{12}x_2) + a_{21}(a_{21}x_1 + a_{22}x_2)) \\ a_{12}(y_1 - (a_{11}x_1 + a_{12}x_2) + a_{22}(a_{21}x_1 + a_{22}x_2)) \end{bmatrix}$$

$$= -e \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 - (a_{11}x_1 + a_{12}x_2) \\ y_2 - (a_{21}x_1 + a_{22}x_2) \end{bmatrix} = -eH^T(y - Hx) = eH^T(Hx - y)$$

$$= (1/\|y - Hx\|_2) H^T(Hx - y) \qquad t_0 = y - Hx$$

$$= (1/\|t_0\|_2) H^T \cdot t_0$$

Figura 1: Demostracion punto 2.1 y 2.2



2.3 $\|y - Hx\|_1 = \left\| \begin{matrix} y_1 - (a_{11}x_1 + a_{12}x_2) \\ y_2 - (a_{21}x_1 + a_{22}x_2) \end{matrix} \right\| = |(y_1 - (a_{11}x_1 + a_{12}x_2)| + |y_2 - (a_{21}x_1 + a_{22}x_2)|$

$d(|x|)/dx = x/|x|$

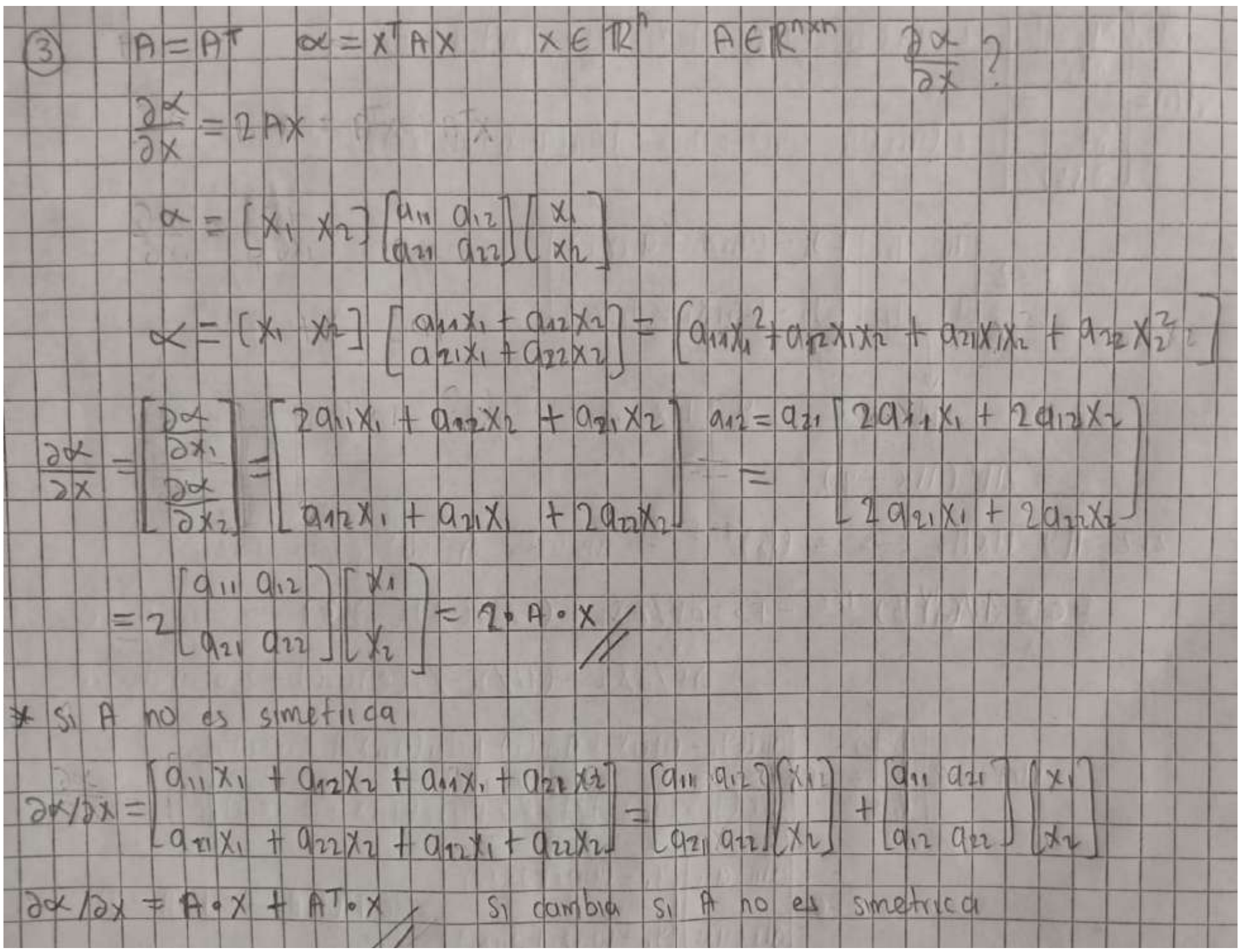
$$\frac{\partial f}{\partial x_1} = -\left(\frac{a_{11}(y_1 - (a_{11}x_1 + a_{12}x_2))}{|y_1 - (a_{11}x_1 + a_{12}x_2)|} + \frac{a_{21}(y_2 - (a_{21}x_1 + a_{22}x_2))}{|y_2 - (a_{21}x_1 + a_{22}x_2)|} \right)$$

$$\frac{\partial f}{\partial x_2} = -\left(\frac{a_{12}(y_1 - (a_{11}x_1 + a_{12}x_2))}{|y_1 - (a_{11}x_1 + a_{12}x_2)|} + \frac{a_{22}(y_2 - (a_{21}x_1 + a_{22}x_2))}{|y_2 - (a_{21}x_1 + a_{22}x_2)|} \right)$$

$$\nabla f(x) = -\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 - (a_{11}x_1 + a_{12}x_2) \\ y_2 - (a_{21}x_1 + a_{22}x_2) \end{bmatrix} \begin{bmatrix} 1/|y_1 - (a_{11}x_1 + a_{12}x_2)| \\ 1/|y_2 - (a_{21}x_1 + a_{22}x_2)| \end{bmatrix}$$

$$\nabla f(x) = -H^T \cdot (y - Hx) \cdot (1/|y - Hx|)$$

Figura 2: Demostracion punto 2.3



3) $A=A^T$ $\quad \alpha = x^T A x$ $\quad x \in \mathbb{R}^n$ $\quad A \in \mathbb{R}^{n\times n}$ $\quad \frac{\partial \alpha}{\partial x}$?

$$\frac{\partial \alpha}{\partial x} = 2Ax$$

$$\alpha = [x_1 \ x_2] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\alpha = [x_1 \ x_2] \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = [a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_1x_2 + a_{22}x_2^2]$$

$$\frac{\partial \alpha}{\partial x} = \begin{bmatrix} \frac{\partial \alpha}{\partial x_1} \\ \frac{\partial \alpha}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2a_{11}x_1 + a_{12}x_2 + a_{21}x_2 \\ a_{12}x_1 + a_{21}x_1 + 2a_{22}x_2 \end{bmatrix} \quad a_{12} = a_{21} \quad = \begin{bmatrix} 2a_{11}x_1 + 2a_{12}x_2 \\ 2a_{21}x_1 + 2a_{22}x_2 \end{bmatrix}$$

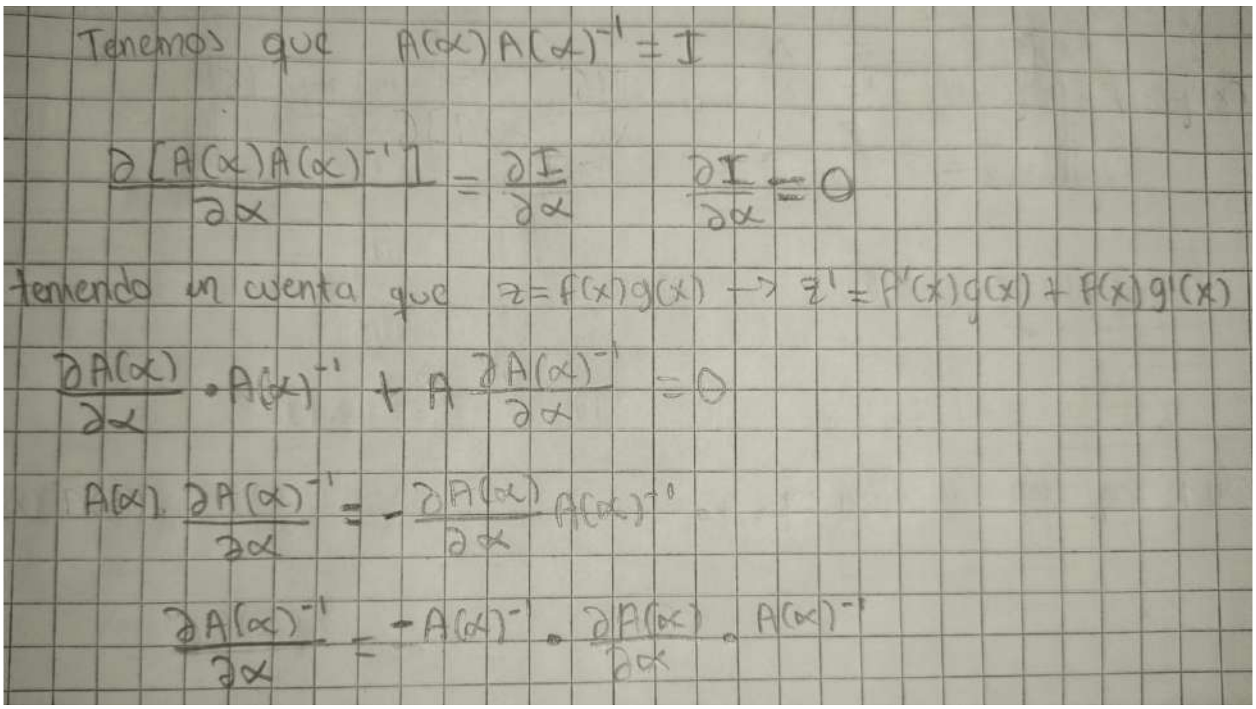
$$= 2 \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2 \cdot A \cdot x$$

* Si A no es simetrica

$$\partial\alpha/\partial x = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{11}x_1 + a_{22}x_2 \\ a_{21}x_1 + a_{22}x_2 + a_{12}x_1 + a_{22}x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\partial\alpha/\partial x = A \cdot x + A^T \cdot x$$ Si cambia si A no es simetrica

Figura 3: Demostracion punto 3



Tenemos que $A(\alpha)A(\alpha)^{-1} = I$

$$\frac{\partial[A(\alpha)A(\alpha)^{-1}]}{\partial\alpha} = \frac{\partial I}{\partial\alpha} \qquad \frac{\partial I}{\partial\alpha} = 0$$

teniendo en cuenta que $z = f(x)g(x) \rightarrow z' = f'(x)g(x) + f(x)g'(x)$

$$\frac{\partial A(\alpha)}{\partial\alpha}\cdot A(\alpha)^{-1} + A\frac{\partial A(\alpha)^{-1}}{\partial\alpha} = 0$$

$$A(\alpha)\cdot\frac{\partial A(\alpha)^{-1}}{\partial\alpha} = -\frac{\partial A(\alpha)}{\partial\alpha}A(\alpha)^{-1}$$

$$\frac{\partial A(\alpha)^{-1}}{\partial\alpha} = -A(\alpha)^{-1}\cdot\frac{\partial A(\alpha)}{\partial\alpha}\cdot A(\alpha)^{-1}$$

Figura 4: Demostracion punto 4