Math 100 **Homework 7**

1- **Acknowledgment Statement:** Please write the following statement as the answer to Exercise 1 and place your signature right below the statement.

"I acknowledge that it is my responsibility to carefully read the class notes before attempting the homework problems. I understand that what is in the class notes is the minimum I should know, and I should not expect to pass this course if I do not fully understand the material covered in the class notes."

2- Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two nonzero vectors in $\mathbb{R}^2$. Which of the following statements is true?

(a) $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent if and only if neither of the vectors is a multiple of the other.

(b) $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent if and only if they do NOT lie on the same line through the origin.

(c) Answer (a) and Answer (b) are both correct.

3- Are the vectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ \sqrt{3} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -\sqrt{3} \\ 0 \\ 3 \end{bmatrix}$ linearly dependent? Justify your answer.

independent not more vectors than entries

$\begin{bmatrix} -1 & -\sqrt{3} \\ 0 & 0 \\ \sqrt{3} & 3 \end{bmatrix} \xrightarrow{\text{matlab}} \begin{bmatrix} 1 & \sqrt{3} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ No Free variable

x_1 RHS

4- Determine if the vectors are linearly dependent. Justify your answer.

4-1) $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5 \\ -10 \\ -8 \end{bmatrix}$

$\begin{bmatrix} 2 & 3 & 5 & 0 \\ -1 & -4 & -10 & 0 \\ 1 & 2 & -8 & 0 \end{bmatrix} \xrightarrow{\text{matlab}} \begin{array}{cccc} x_1 & x_2 & x_3 & \text{RHS} \end{array} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 - 2 = 0$
$x_1 = 2$
$x_2 + 3 = 0$
$x_2 = -3$

Free variable so independent

(*Hint:* Consider the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = 0$.)

x_3 is free basic $= x_1, x_2$

4-2) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ -6 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5 \\ 3 \\ -7 \\ 4 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 & 5 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 2 & -6 & -7 & 5 & 0 \\ 5 & 0 & 4 & 4 & 0 \end{bmatrix} \xrightarrow{\text{matlab}} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \text{RHS} \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

independent all variables are basic

$x_1 = 0$
$x_2 = 0$ trivial solution
$x_3 = 0$
$x_4 = 0$

(*Hint:* Consider the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = 0$.)

Note: The next two exercises are NOT directly analogous to an example or an exercise solved in class. Nevertheless, you should be able to solve them using the concepts learned in class. Remember, as discussed on the course homework page, for some problems, you may need to apply what you have learned in class to something new, make connections between various concepts discussed in class, and think about concepts in different ways and from different angles. It is absolutely false (and dangerous) to think that you should be able to solve every homework problem in under 10 minutes by directly applying algorithms/procedures taught in class.

5- **Permutation Matrices:**

Answers on Back of this page

Consider the 3×3 **identity** matrix: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Let $\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ denote a general vector in $\mathbb{R}^3$.

5-1) Compute $I\mathbf{z}$. What is the effect of multiplying the matrix I by the vector $\mathbf{z}$?

5-2) Let P be the matrix that is obtained by applying the elementary row operation $R_1 \longleftrightarrow R_2$ to I. Compute $P\mathbf{z}$. What is the effect of multiplying the matrix P by the vector $\mathbf{z}$?

5-3) Let P be the matrix that is obtained by applying the elementary row operation $R_2 \longleftrightarrow R_3$ to I. Compute $P\mathbf{z}$. What is the effect of multiplying the matrix P by the vector $\mathbf{z}$?

5-4) Let P be the matrix that is obtained by applying the elementary row operation $R_1 \longleftrightarrow R_3$ to I and then applying $R_2 \longleftrightarrow R_1$ to the resulting matrix. Compute $P\mathbf{z}$. What is the effect of multiplying the matrix P by the vector $\mathbf{z}$?

Matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

5-1). $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ z_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

5-2). $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; R_1 \leftrightarrow R_2 \; P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Now what is Pz

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P} \cdot \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}}_{\vec{z}} = \begin{bmatrix} 0 \\ z_1 \\ 0 \end{bmatrix} + \begin{bmatrix} z_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z_3 \end{bmatrix} = \begin{bmatrix} z_2 \\ z_1 \\ z_3 \end{bmatrix}$$

5-3). $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; R_2 \leftrightarrow R_3$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ z_3 \\ 0 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_3 \\ z_2 \end{bmatrix}$$

5-4). $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; R_1 \leftrightarrow R_3$

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{P} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z_1 \end{bmatrix} + \begin{bmatrix} 0 \\ z_2 \\ 0 \end{bmatrix} + \begin{bmatrix} z_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} z_3 \\ z_2 \\ z_1 \end{bmatrix}$$

5-5). $y = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \; R_1 \leftrightarrow R_2 \; \begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix} \; R_3 \leftrightarrow R_5 \; \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} b \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ c \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ d \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ a \\ e \\ d \\ c \end{bmatrix}$$

this 5x5 matrix with listed row operation done onit. $R_1 \leftrightarrow R_2$ then exchange $R_3 \leftrightarrow R_5$

5-5) Now consider the vector $\mathbf{y} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$. Using the ideas you learned from previous items, can you find a 5×5 matrix P such that $P\mathbf{y} = \begin{bmatrix} b \\ a \\ e \\ d \\ c \end{bmatrix}$? (Hint: Note that $\begin{bmatrix} b \\ a \\ e \\ d \\ c \end{bmatrix}$ is obtained from $\mathbf{y}$ by first exchanging R_1 and R_2 and then exchanging R_3 and R_5.)

6- The Link Matrix: (How does Google rank webpages?)

Suppose there is a group of four people: Justin, Chris, Miriam, and Hannah, and we are interested in the following question:

Who is the most popular?

Our goal is to assign to each person in this group a number that we will call the **popularity**: p_J, p_C, p_M, and p_H.

Approach 1: Ask everyone to identify their friends. Suppose they respond as follows:

Justin $\longrightarrow$ C, M (that is, Justin lists Chris and Miriam as his friends)

Chris $\longrightarrow$ J, M, H Chris lists Justin, Miriam & Hannah as friends

Miriam $\longrightarrow$ J,C, H Miriam Lists Justin, Chris, Hannah as Friends

Hannah $\longrightarrow$ J, M Hannah Lists Justin & Miriam as Friends

We can represent this data in a 4×4 matrix of zeros and ones, where a 1 indicates that the person listed in the top row considers the person named on the left to be a friend:

$$\begin{array}{c} \\ J \\ C \\ M \\ H \end{array} \begin{array}{c} \begin{array}{cccc} \textit{Justin} & \textit{Chris} & \textit{Miriam} & \textit{Hannah} \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Now we notice that the sum of each row tells us the number of times the corresponding person is listed as a friend, and we can consider this number as the measure of someone's popularity. So, for instance, using this approach, the popularity of Justin is 3 ($p_J = 3$) and the popularity of Chris is 2 ($p_C = 2$). ($P_M = 3$) ($P_H = 2$)

Approach 2:

The first approach is not the best approach to measure popularity.

1. Some people will tend to list everyone they know, and others will write only their closest friends. One way to deal with this issue, and possibly improve our method, is to divide the numbers in each column by the number of people in it. (So, for example, if a person has listed 10 friends and you are one of them, your score will be $\frac{1}{10}$, whereas, if a person has listed only two friends and you are one of them, your score will be $\frac{1}{2}$.). This idea leads to considering the following *normalized* matrix, which we will refer to as the **link matrix**:

$$
\begin{array}{c|cccc}
 & \textit{Justin} & \textit{Chris} & \textit{Miriam} & \textit{Hannah} \\
\hline
J & & & & \\
C & & & & \\
M & & & & \\
H & & & &
\end{array}
\qquad
\begin{bmatrix}
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{3} & 0 \\
\frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{2} \\
0 & \frac{1}{3} & \frac{1}{3} & 0
\end{bmatrix}
$$

Ask Prof

2. To measure popularity, it is reasonable to assume that it is better for a widely liked person to think of you as a friend (if someone whom nobody else likes considers you a friend, that does not make you very popular!). For this reason, it makes sense to define a person's popularity as a weighted sum of the popularity of people who reference that person:

$0p_j + \frac{1}{3}p_c + \frac{1}{3}p_m + \frac{1}{2}p_h = 0$

$\frac{1}{2}p_j + 0p_c + \frac{1}{3}p_m + 0p_h = 0$

$\frac{1}{2}p_j + \frac{1}{3}p_c + 0p_m + \frac{1}{2}p_h$

$0p_j + \frac{1}{3}p_c + \frac{1}{3}p_m + 0p_h$

row 1 $\Longrightarrow$ $p_J = \frac{1}{3}p_C + \frac{1}{3}p_M + \frac{1}{2}p_H$ $\quad 0p_j - \frac{1}{3}p_c - \frac{1}{3}p_m - \frac{1}{2}p_h = 0$

row 2 $\Longrightarrow$ $p_C = \frac{1}{2}p_J + \frac{1}{3}p_M$ $\quad -\frac{1}{2}p_j + 0p_c - \frac{1}{3}p_m + 0p_h = 0$

row 3 $\Longrightarrow$ $p_M = \frac{1}{2}p_J + \frac{1}{3}p_C + \frac{1}{2}p_H$ $\quad -\frac{1}{2}p_j - \frac{1}{3}p_c + 0p_m - \frac{1}{2}p_h = 0$

row 4 $\Longrightarrow$ $p_H = \frac{1}{3}p_C + \frac{1}{3}p_M$ $\quad 0p_j - \frac{1}{3}p_c - \frac{1}{3}p_m + 0p_h = 0$

So, we see that computing the popularity numbers amounts to solving a (special kind of) linear system.

Questions to be answered for your homework: matrix $A\vec{x} = b$ unknowns $\vec{x}$ $= b \rightarrow$ RHS bring all to one side

Introduce the popularity vector **p** as follows:

$$
\begin{bmatrix}
0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{2} \\
-\frac{1}{2} & 0 & -\frac{1}{3} & 0 \\
-\frac{1}{2} & -\frac{1}{3} & 0 & -\frac{1}{2} \\
0 & -\frac{1}{3} & -\frac{1}{3} & 0
\end{bmatrix}
\quad
\mathbf{p} = \begin{bmatrix} P_J \\ P_C \\ P_M \\ P_H \end{bmatrix},
\quad
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\quad \text{matlab} \quad
\begin{array}{ccccc}
P_J & P_C & P_M & P_H & \text{RHS}
\end{array}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
$$

Write the above linear system as a matrix equation with the unknown vector **p**. Use the flowchart in Handout 1 to solve the system and write the solution in parametric vector form. Who is the most popular person?

The founders of Google, Larry Page and Sergey Brin, envisioned employing a similar method to rank web pages. When users enter specific keywords, the Google search engine identifies an extensive range of web pages containing those words. However, a significant portion of these pages may be irrelevant or even considered "garbage" from the user's perspective. The primary challenge lies in developing an automated system that effectively filters and identifies a few high-quality, popular web pages.

The Google search engine analyzes each webpage 'w' in the search results and determines the other webpages to which 'w' is linked. Consequently, each webpage generates a binary vector, similar to the lists of friends created by students in the earlier example. Subsequently, the algorithm constructs a matrix featuring the normalized versions of these vectors. The approach then closely follows the steps employed in solving the popularity problem discussed earlier. Google utilizes specialized techniques to very quickly and efficiently compute the popularity of each webpage. The search results are then presented to users in descending order of popularity.

rightmost column Not a pivot column
System is Consistent
No free variables
Unique Solution

$p_j = 0$
$p_c = 0$
$p_m = 0$
$p_h = 0$