I Green's Function Formalism

Following [**Bruus_Flensberg_2004**]

Green's functions: method to encode influence of many-body effects on propagation of particles in a system.

Have different kinds of Green's functions, for example the retarded Green's function:

$$G^R(\mathbf{r}\sigma t, \mathbf{r}'\sigma't') = -\mathrm{i}\Theta(t-t')\left\langle\{c_{\mathbf{r}\sigma}(t), c^\dagger_{\mathbf{r}\sigma}(t')\}\right\rangle \tag{I.1}$$

They give the amplitude of a particle inserted at point $\mathbf{r}'$ at time t' to propagate to position $\mathbf{r}$ at time t. For time-independent Hamiltonians and systems in equilibrium, the GFs only depend on time differences:

$$G^R(\mathbf{r}\sigma t, \mathbf{r}'\sigma't') = G^R(\mathbf{r}\sigma, \mathbf{r}'\sigma', t-t') \tag{I.2}$$

So we can take $t' = 0$ and consider t as the only free variable:

$$G^R(\mathbf{r}\sigma, \mathbf{r}'\sigma', t) = -\mathrm{i}\Theta(t)\left\langle\{c_{\mathbf{r}\sigma}(t), c^\dagger_{\mathbf{r}\sigma}(0)\}\right\rangle \tag{I.3}$$

In a translation invariant system: can use $\mathbf{k}$ as a natural basis set:

$$G^R(\mathbf{k}, \sigma, \sigma't) = -\mathrm{i}\Theta(t-t')\left\langle\{c_{\mathbf{k}\sigma}(t), c^\dagger_{\mathbf{k}\sigma'}(0)\}\right\rangle \tag{I.4}$$

Define Fourier-transform:

$$G^R(\mathbf{k}, \sigma, \sigma', \omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, G^R(\mathbf{k}, \sigma, \sigma't) \tag{I.5}$$

Can define the spectral function from this:

$$A(\mathbf{k}\sigma, \omega) = -2\Im G^R(\mathbf{k}\sigma, \omega) \tag{I.6}$$

Looking at the diagonal elements of G^R here. The spectral function can be thought of as the energy resolution of a particle with energy ω. This mean,

for non-interacting systems, the spectral function is a delta-function around the single-particle energies:

$$A_0(\mathbf{k}\sigma, \omega) = 2\pi\delta(\omega - \epsilon_{\mathbf{k}\sigma}) \tag{I.7}$$

For interacting systems this is not true, but A can still be peaked.

Mathematical technique to calculate retarded GFs involves defining GFs on imaginary times τ:

$$t \to -\mathrm{i}\tau \tag{I.8}$$

where τ is real and has the dimension time. This enables the simultaneous expansion of exponential $e^{-\beta H}$ coming from the thermodynamic average and $e^{-\mathrm{i}Ht}$ coming from the time evolution of operators.

Define imaginary time/Matsubara GF $\mathcal{C}_{AB}(\tau, 0)$:

$$\mathcal{C}_{AB}(\tau, 0) = -\langle T_\tau(A(\tau)B(0))\rangle \tag{I.9}$$

with time-ordering operator in imaginary time:

Can prove from properties of Matsubara GF, that they are only defined for

$$-\beta < \tau < \beta \tag{I.10}$$

Due to this, the Fourier transform of the Matsubara GF is defined on discrete values:

$$\mathcal{C}_{AB}(\mathrm{i}\omega_n) = \int_0^\beta \mathrm{d}\tau \tag{I.11}$$

with fermionic/bosonic Matsubara frequencies

$$\omega_n = \begin{cases} \frac{2n\pi}{\beta} \text{ for bosons} \\ \frac{(2n+1)\pi}{\beta} \text{ for fermions} \end{cases} \tag{I.12}$$

Extrapolation of the Matsubara GF to zero is proportional to the density of states at the chemical potential. Gapped: density is zero (Matsubara GF goes to 0), metal: density is finite (Matsubara GF goes to finite value) [**Bruus_Flensberg_2004**].

Show GFs can be related to observables

ordering operator

How to resolve ambiguity at borders of integral

Connection to real-time GFs

single-particle Matsubara GF

equations of motion for Matsubara GF

I.1 Perturbation theory, Dyson equation

Dyson equation:

$$\mathcal{G}_\sigma(\mathbf{k}, \mathrm{i}\omega_n) = \frac{\mathcal{G}_\sigma^0(\mathbf{k}, \mathrm{i}\omega_n)}{1 - \mathcal{G}_\sigma^0(\mathbf{k}, \mathrm{i}\omega_n)\Sigma_\sigma(\mathbf{k}, \mathrm{i}\omega_n)} = \frac{1}{\mathrm{i}\omega_n - \xi_{\mathbf{k}-\Sigma_\sigma(\mathbf{k}, \mathrm{i}\omega_n)}} \tag{I.13}$$

I.2 Nambu-Gorkov GF

Order parameter can be chosen as the anomalous GF:

$$\Psi = F^{\mathrm{loc}}(\tau = 0^-) \tag{I.14}$$

or the superconducting gap

$$\Delta = Z\Sigma^{\mathrm{AN}} \tag{I.15}$$

that can be calculated from the anomalous self-energy Σ^{AN} and quasiparticle weight Z

introduction
grams

nergy

equation

More general introduction into NG GFs, how they look like, what they describe etc.

Sources for these?

How to get quasiparticle weight?