

I Green's Function Formalism

Following [Bruus_Flensberg_2004]

Green's functions: method to encode influence of many-body effects on propagation of particles in a system.

Have different kinds of Green's functions, for example the retarded Green's function:

$$G^R(\mathbf{r}\sigma t, \mathbf{r}'\sigma' t') = -i\Theta(t - t') \langle \{c_{\mathbf{r}\sigma}(t), c_{\mathbf{r}'\sigma'}^\dagger(t')\} \rangle \quad (\text{I.1})$$

They give the amplitude of a particle inserted at point \mathbf{r}' at time t' to propagate to position \mathbf{r} at time t . For time-independent Hamiltonians and systems in equilibrium, the GFs only depend on time differences:

$$G^R(\mathbf{r}\sigma t, \mathbf{r}'\sigma' t') = G^R(\mathbf{r}\sigma, \mathbf{r}'\sigma', t - t') \quad (\text{I.2})$$

So we can take $t' = 0$ and consider t as the only free variable:

$$G^R(\mathbf{r}\sigma, \mathbf{r}'\sigma', t) = -i\Theta(t) \langle \{c_{\mathbf{r}\sigma}(t), c_{\mathbf{r}'\sigma'}^\dagger(0)\} \rangle \quad (\text{I.3})$$

In a translation invariant system: can use \mathbf{k} as a natural basis set:

$$G^R(\mathbf{k}, \sigma, \sigma', t) = -i\Theta(t) \langle \{c_{\mathbf{k}\sigma}(t), c_{\mathbf{k}\sigma'}^\dagger(0)\} \rangle \quad (\text{I.4})$$

Define Fourier-transform:

$$G^R(\mathbf{k}, \sigma, \sigma', \omega) = \int_{-\infty}^{\infty} dt G^R(\mathbf{k}, \sigma, \sigma', t) \quad (\text{I.5})$$

Can define the spectral function from this:

$$A(\mathbf{k}\sigma, \omega) = -2\Im G^R(\mathbf{k}\sigma, \omega) \quad (\text{I.6})$$

Looking at the diagonal elements of G^R here. The spectral function can be thought of as the energy resolution of a particle with energy ω . This mean,

for non-interacting systems, the spectral function is a delta-function around the single-particle energies:

$$A_0(\mathbf{k}\sigma, \omega) = 2\pi\delta(\omega - \epsilon_{\mathbf{k}\sigma}) \quad (\text{I.7})$$

Show GFs can be related to observables

For interacting systems this is not true, but A can still be peaked.

Mathematical technique to calculate retarded GFs involves defining GFs on imaginary times τ :

$$t \rightarrow -i\tau \quad (\text{I.8})$$

where τ is real and has the dimension time. This enables the simultaneous expansion of exponential $e^{-\beta H}$ coming from the thermodynamic average and e^{-iHt} coming from the time evolution of operators.

Define imaginary time/Matsubara GF $\mathcal{C}_{AB}(\tau, 0)$:

$$\mathcal{C}_{AB}(\tau, 0) = -\langle T_\tau(A(\tau)B(0)) \rangle \quad (\text{I.9})$$

with time-ordering operator in imaginary time:

$$T_\tau(A(\tau)B(\tau')) = \Theta(\tau - \tau')A(\tau)B(\tau') \pm \Theta(\tau' - \tau)B(\tau')A(\tau) \quad (\text{I.10})$$

so that operators with later 'times' go to the left.

Can prove from properties of Matsubara GF, that they are only defined for

$$-\beta < \tau < \beta \quad (\text{I.11})$$

Due to this, the Fourier transform of the Matsubara GF is defined on discrete values:

$$\mathcal{C}_{AB}(i\omega_n) = \int_0^\beta d\tau \quad (\text{I.12})$$

with fermionic/bosonic Matsubara frequencies

$$\omega_n = \begin{cases} \frac{2n\pi}{\beta} & \text{for bosons} \\ \frac{(2n+1)\pi}{\beta} & \text{for fermions} \end{cases} \quad (\text{I.13})$$

How to resolve ambiguity at borders of integral

It turns out that Matsubara GFs and retarded GFs can be generated from a common function $\mathcal{C}_{AB}(z)$ that is defined on the entire complex plane except for the real axis. So we can get the retarded GF $\mathcal{C}_{AB}^R(\omega)$ by analytic continuation:

$$\mathcal{C}_{AB}^R(\omega) = \mathcal{C}_{AB}(i\omega_n \rightarrow \omega + i\eta) \quad (\text{I.14})$$

So in particular the extrapolation of the Matsubara GF to zero is proportional to the density of states at the chemical potential. Gapped: density is zero (Matsubara GF goes to 0), metal: density is finite (Matsubara GF goes to finite value) [Bruus_Flensberg_2004].

I.1 Perturbation theory, Dyson equation

Dyson equation:

$$\mathcal{G}_\sigma(\mathbf{k}, i\omega_n) = \frac{\mathcal{G}_\sigma^0(\mathbf{k}, i\omega_n)}{1 - \mathcal{G}_\sigma^0(\mathbf{k}, i\omega_n)\Sigma_\sigma(\mathbf{k}, i\omega_n)} = \frac{1}{i\omega_n - \xi_{\mathbf{k}-\Sigma_\sigma(\mathbf{k}, i\omega_n)}} \quad (\text{I.15})$$

I.2 Nambu-Gorkov GF

Order parameter can be chosen as the anomalous GF:

$$\Psi = F^{\text{loc}}(\tau = 0^-) \quad (\text{I.16})$$

or the superconducting gap

$$\Delta = Z\Sigma^{\text{AN}} \quad (\text{I.17})$$

that can be calculated from the anomalous self-energy Σ^{AN} and quasiparticle weight Z

single-particle Matsubara GF

equations of motion for Matsubara GF

Short introduction to diagrams

Self energy

Dyson equation

More general introduction into NG GFs, how they look like, what they describe etc.

Sources for these?

How to get quasiparticle weight?