I SUPERCONDUCTIVITY

At the end of the 19th and the beginning of the 20th century, cooling technology made great progress. Liquifying gases, were able to reach temperatures as low as 4 K (the boiling point of Helium). Using that, SC was discovered in mercury in 1911 by Heike Onnes [1]. Superconductivity describes the phenomenon of the electrical resistance of a material suddenly dropping to zero below a critical temperature T_C. Discovery of Meissner effect, perfect expulsion of external magnetic fields in 1933 [2]. This started almost half a century of intensive theoretical research, which culminated in John Bardeen, Leon Cooper and J. Robert Schrieffer developing the microscopic theory now know as BCS theory [3]. 1986 and 1987: discovery of superconductivity with very high T_C found in cuprates [4, 5]. Cuprate superconductors are made up of layers of cooper oxide and charge reservoirs in between [6]. The specific charge reservoir layers determine the properties of the SC and varying them lead to a rich zoo of materials with high T_C, up to

> Citation
>
> Other strongly correlated SC

Most current technical applications of superconductors rely on the fact, that superconducting wires can carry much higher currents that ordinary wires and thus produce much larger magnetic fields. The largest commercial application to date is in magnetic resonance imaging, a medical technique using strong magnetic fields and field gradients [7]. Technical applications in research are much wider, ranging from strong superconducting magnets in the LHC [8] and other particle accelerators over detectors of single photons in astrophysics [9] to extremely sensitive measurement devices for magnetic fields [10, 11].

From a theoretical perspective, superconductivity is fascinating. The superconducting state is a macroscopic quantum state and thus offers fascinating insights into the nature of quantum phenomena. The success of BCS theory combined with GL

> Some words: why is SC interesting from a theoretical perspective?

Theoretical description of HT-SC using BCS-theory based on an attractive interaction mediated by phonons could not explain this. For this reason called

'unconventional superconductors' for them. Some key experimental facts are established in high T_C SC: there is Cooper pairing, ordered normal state is very important for SC, phenomenon of 'Pseudogap' is defining [12].

Today, still many unanswered questions in unconventional SC [13].

This chapter: introduction to the mean-field BCS theory in section I.1, GL-theory in section I.2 and description of a method to treat certain aspects of strongly-correlated superconductivity from a microsopic standpoint away from the mean-field level in section I.3.

> **Unconventional SC, what is the same, what is different, what makes it difficult to grasp**
>
> **Open questions in unconventional SC**
>
> **Connection to my topic: flat bands, strong correlations**

I.1 BCS Theory

Following [14, ch. 14].

Theoretical description of SC: 1957 by John Bardeen, his postdoc Leon Cooper and the graduate in the group, J. Robert Schrieffer [3]. Description is based on the fact, that the Fermi sea is unstable towards development of bound pairs under arbitrarily small attraction [15]. These bound electrons show bosonic behaviour and

> **Why supercurrent?**

This model Hamiltonian

> **BCS hamiltonian, pairing**

It can be connected to

> **Mean-field**
>
> **Phonon interaction**

The final element in this description was the origin of the attractive interaction between electrons, which Bardeen, Cooper and Schrieffer identified as a retarded electron-phonon interaction [3]. This so-called BCS-theory of superconductivity is very successful in explaining experimental results in many compounds, Surprisingly, it

> **Mean field level can already explain a lot**

BCS-theory gave a microscopic explanation to a phenomenological description of superconductivity pioneered by Fritz London in 1937 [16]. This descriptions is based on a one-particle wave function $\phi(x)$

Later, this one-particle wavefunction was identified as the order parameter in the developing GL-theory of phase transitions [17]. GL-theory is discussed in more detail in section I.2. This explains the Meissner effect and in turn the supercurrent.

I.1.1 BCS theory as a mean field-theory (Hubbard model)

Hubbard model is the simplest model for interactions

[18]

[19]

> Some relevance of the repulsive Hubbard model

I.1.2 Multi-band BCS theory on the Hubbard model

> Motivation for taking a negative U

Define sublattice index

$$\alpha = 1, 2, 3 \tag{I.1}$$

> Phase diagram

with $1 \cong \mathrm{Gr}_1, 2 \cong \mathrm{Gr}_2, 3 \cong \mathrm{X}$. Then we can write the non-interacting term as

> BCS mean field theory on the Hubbard model

$$H_0 = - \sum_{\langle i,j \rangle, \alpha, \beta, \sigma} [\mathbf{t}]_{i\alpha, j\beta} \, c_{i\alpha}^\dagger c_{j\beta} \tag{I.2}$$

with the matrix

> Derivation for superfluid weight in this case?

$$\mathbf{t} = \begin{pmatrix} 0 & t_{\mathrm{Gr}} & 0 \\ t_{\mathrm{Gr}} & 0 & -V\delta_{ij} \\ 0 & -V\delta_{ij} & t_{\mathrm{X}} \end{pmatrix} \tag{I.3}$$

Add chemical potential:

> Do general multiband BCS theory

$$-\mu \sum_{i\alpha\sigma} n_{i\alpha\sigma} \tag{I.4}$$

Also write the interaction part with α (with changed signs compared to Niklas, to keep in line with papers about the attractive Hubbard model):

$$H_{int} = - \sum_{i\alpha} U_\alpha c_{i\alpha\uparrow}^\dagger c_{i\alpha\downarrow}^\dagger c_{i\alpha\downarrow} c_{i\alpha\uparrow} \tag{I.5}$$

Fourier transformation:

$$H_{int} = -\frac{1}{N^2} \sum_{\alpha, \mathbf{k}_{1,2,3,4}} U_\alpha e^{i(\mathbf{k}_1 + \mathbf{k}_4 - \mathbf{k}_1 - \mathbf{k}_3) r_{i\alpha}} c_{\mathbf{k}_1\alpha\uparrow}^\dagger c_{\mathbf{k}_3\alpha\downarrow}^\dagger c_{\mathbf{k}_2\alpha\downarrow} c_{\mathbf{k}_4\alpha\uparrow} \tag{I.6}$$

Impose zero-momentum pairing: $\mathbf{k}_1 + \mathbf{k}_3 = 0$ and $\mathbf{k}_2 + \mathbf{k}_4 = 0$:

$$H_{int} = - \sum_{\alpha, \mathbf{k}, \mathbf{k}'} U_\alpha c_{\mathbf{k}\alpha\uparrow}^\dagger c_{-\mathbf{k}\alpha\downarrow}^\dagger c_{-\mathbf{k}'\alpha\downarrow} c_{\mathbf{k}'\alpha\uparrow} \tag{I.7}$$

Mean-field approximation:

$$H_{int} \approx \sum_{\alpha,\mathbf{k}} (\Delta_\alpha c_{\mathbf{k}\alpha\uparrow}^\dagger c_{-\mathbf{k}\alpha\downarrow}^\dagger + \Delta_\alpha^* c_{-\mathbf{k}\alpha\downarrow} c_{\mathbf{k}\alpha\uparrow}) \tag{I.8}$$

with

$$\Delta_\alpha = -U_\alpha \sum_{\mathbf{k}'} \langle c_{-\mathbf{k}'\alpha\downarrow} c_{\mathbf{k}'\alpha\uparrow} \rangle \tag{I.9}$$

$$\Delta_\alpha^* = -U_\alpha \sum_{\mathbf{k}'} \langle c_{\mathbf{k}'\alpha\uparrow}^\dagger c_{-\mathbf{k}'\alpha\downarrow}^\dagger \rangle \tag{I.10}$$

This gives the BCS mean field Hamiltonian:

$$H_{BCS} = \sum_{\mathbf{k}\alpha\beta\sigma} [H_{0,\sigma}(\mathbf{k})]_{\alpha\beta} c_{\mathbf{k}\alpha\sigma}^\dagger c_{\mathbf{k}\beta\sigma} - \mu \sum_{\mathbf{k}\alpha\sigma} n_{\mathbf{k}\alpha\sigma} + \sum_{\alpha,\mathbf{k}} (\Delta_\alpha c_{\mathbf{k}\alpha\uparrow}^\dagger c_{-\mathbf{k}\alpha\downarrow}^\dagger + \Delta_\alpha^* c_{-\mathbf{k}\alpha\downarrow} c_{\mathbf{k}\alpha\uparrow}) \tag{I.11}$$

with Nambu spinor

$$\Psi_{\mathbf{k}} = \begin{pmatrix} c_{1,\mathbf{k}\uparrow} \\ c_{2,\mathbf{k}\uparrow} \\ c_{3,\mathbf{k}\uparrow} \\ c_{1,-\mathbf{k}\downarrow}^\dagger \\ c_{2,-\mathbf{k}\downarrow}^\dagger \\ c_{3,-\mathbf{k}\downarrow}^\dagger \end{pmatrix} \tag{I.12}$$

we have:

$$H_{MF} = \sum_{\mathbf{k}} \Psi_{\mathbf{k}}^\dagger \mathcal{H}(\mathbf{k}) \Psi_{\mathbf{k}} \tag{I.13}$$

with

$$\mathcal{H}(\mathbf{k}) = \begin{pmatrix} H_{0,\uparrow}(\mathbf{k}) - \mu & \Delta \\ \Delta^\dagger & -H_{0,\downarrow}^*(-\mathbf{k}) + \mu \end{pmatrix} \tag{I.14}$$

with $H_{0,\sigma}$ being the F.T. of the kinetic term and $\Delta = diag(\Delta_1, \Delta_2, \Delta_3)$.

I.1.3 BdG Hamiltonian in band basis

Use transformation

$$c_{\mathbf{k}\alpha\sigma}^\dagger = \sum_n [\mathbf{G}]_{\alpha n}^* d_{n\mathbf{k}\sigma}^\dagger \tag{I.15}$$

where the columns are made up of the eigenvectors of $\mathbf{H}_{0,\sigma}$ for a given $\mathbf{k}$:

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_1 & \mathbf{G}_2 & \mathbf{G}_3 \end{pmatrix} \tag{I.16}$$

with that:

$$\mathbf{G}_\sigma^\dagger(\mathbf{k})\mathbf{H}_{0,\sigma}(\mathbf{k})\mathbf{G}_\sigma(\mathbf{k}) = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \tag{I.17}$$

So the kinetic part of the BdG Hamiltonian becomes:

$$\sum_{\mathbf{k}\alpha\beta\sigma} [H_{0,\sigma}(\mathbf{k})]_{\alpha\beta} \sum_n [\mathbf{G}(\mathbf{k})]_{\alpha n}^* d_{n\mathbf{k}\sigma}^\dagger \sum_m [\mathbf{G}(\mathbf{k})]_{\beta m} d_{m\mathbf{k}\sigma} - \mu \sum_{\mathbf{k}\alpha\sigma} n_{n\mathbf{k}\sigma} \tag{I.18}$$

$$= \sum_{mn\mathbf{k}\sigma} d_{n\mathbf{k}\sigma}^\dagger d_{m\mathbf{k}\sigma} \sum_{\alpha\beta} [\mathbf{G}(\mathbf{k})]_{\alpha n}^* [H_{0,\sigma}(\mathbf{k})]_{\alpha\beta} [\mathbf{G}(\mathbf{k})]_{\beta m} - \mu \sum_{\mathbf{k}\alpha\sigma} n_{n\mathbf{k}\sigma} \tag{I.19}$$

$$= \sum_{mn\mathbf{k}\sigma} d_{n\mathbf{k}\sigma}^\dagger d_{m\mathbf{k}\sigma} \epsilon_n \delta_{nm} - \mu \sum_{\mathbf{k}\alpha\sigma} n_{n\mathbf{k}\sigma} \tag{I.20}$$

$$= \sum_{n\mathbf{k}\sigma} \epsilon_n d_{n\mathbf{k}\sigma}^\dagger d_{n\mathbf{k}\sigma} - \mu \sum_{\mathbf{k}\alpha\sigma} n_{n\mathbf{k}\sigma} \tag{I.21}$$

$$=: \sum_{n\mathbf{k}\sigma} \xi_{\mathbf{k}} d_{n\mathbf{k}\sigma}^\dagger d_{n\mathbf{k}\sigma} \tag{I.22}$$

with $\xi_{\mathbf{k}} := \epsilon_{\mathbf{k}} - \mu$. The pairing terms become:

$$\sum_{\mathbf{k}\alpha} \Delta_\alpha c_{\mathbf{k}\alpha\uparrow}^\dagger c_{-\mathbf{k}\alpha\downarrow}^\dagger = \sum_{\mathbf{k}\alpha} \Delta_\alpha \sum_n [\mathbf{G}_\uparrow(\mathbf{k})]_{\alpha n}^* d_{n\mathbf{k}\uparrow}^\dagger \sum_m [\mathbf{G}_\downarrow(-\mathbf{k})]_{\beta m}^* d_{m-\mathbf{k}\downarrow}^\dagger \tag{I.23}$$

$$= \tag{I.24}$$

So that:

$$\mathcal{H}(\mathbf{k}) = \begin{pmatrix} \epsilon_{\mathbf{k}} - \mu & G^\dagger \Delta G \\ G^\dagger \Delta^\dagger G & -\epsilon_{\mathbf{k}} + \mu \end{pmatrix} \tag{I.25}$$

with

$$\epsilon_{\mathbf{k}} = \begin{pmatrix} \epsilon_1(\mathbf{k}) & 0 & 0 \\ 0 & \epsilon_2(\mathbf{k}) & 0 \\ 0 & 0 & \epsilon_3(\mathbf{k}) \end{pmatrix} \tag{I.26}$$

Concrete example for transformation of gaps from orbital to band basis at $K = \frac{4\pi}{3a}\begin{pmatrix}1\\0\end{pmatrix}$. There, the non-interacting part becomes simply:

$$\mathcal{H}_0 = \begin{pmatrix} 0 & 0 & V \\ 0 & 0 & 0 \\ V & 0 & 3t_X \end{pmatrix} \tag{I.27}$$

The eigenvalue problem can be solved e.g. via sympy:

$$G = \begin{pmatrix} \dfrac{-3t_X - \sqrt{4V^2+9t_X^2}}{\sqrt{4V^2+\left(3t_X+\sqrt{4V^2+9t_X^2}\right)^2}} & 0 & \dfrac{-3t_X+\sqrt{4V^2+9t_X^2}}{\sqrt{4V^2+\left(3t_X-\sqrt{4V^2+9t_X^2}\right)^2}} \\ 0 & 1 & 0 \\ \dfrac{2V}{\sqrt{4V^2+\left(3t_X+\sqrt{4V^2+9t_X^2}\right)^2}} & 0 & \dfrac{2V}{\sqrt{4V^2+\left(3t_X-\sqrt{4V^2+9t_X^2}\right)^2}} \end{pmatrix} \tag{I.28}$$

So for $V \to 0$:

$$G = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{I.29}$$

but for $V > 0$, there are off-diagonal elements, e.g. $V = 0.1$:

$$G = \begin{pmatrix} -0.7578 & 0 & 0.6526 \\ 0 & 1 & 0 \\ 0.6526 & 0 & 0.7578 \end{pmatrix} \tag{I.30}$$

So the transformation of the gap from orbital to band space reads:

$$G^\dagger \Delta G = \begin{pmatrix} \dfrac{3\Delta_1 t_X - 3\Delta_3 t_X + (\Delta_1+\Delta_3)\sqrt{4V^2+9t_X^2}}{2\sqrt{4V^2+9t_X^2}} & 0 & \dfrac{V(-\Delta_1+\Delta_3)}{\sqrt{4V^2+9t_X^2}} \\ 0 & \Delta_2 & 0 \\ \dfrac{V(-\Delta_1+\Delta_3)}{\sqrt{4V^2+9t_X^2}} & 0 & \dfrac{-3\Delta_1 t_X + 3\Delta_3 t_X + (\Delta_1+\Delta_3)\sqrt{4V^2+9t_X^2}}{2\sqrt{4V^2+9t_X^2}} \end{pmatrix} \tag{I.31}$$

So in particular there is no interband pairing for $V \to 0$:

$$G^\dagger \Delta G = \begin{pmatrix} \Delta_1 & 0 & 0 \\ 0 & \Delta_2 & 0 \\ 0 & 0 & \Delta_3 \end{pmatrix} \tag{I.32}$$

But for $V > 0$, there is interband pairing (e.g. $V = 0.1$):

$$G^\dagger \Delta G = \begin{pmatrix} 0.5742\Delta_1 + 0.4258\Delta_3 & 0 & -0.4945\Delta_1 + 0.4945\Delta_3 \\ 0 & \Delta_2 & 0 \\ -0.4945\Delta_1 + 0.4945\Delta_3 & 0 & 0.4258\Delta_1 + 0.5742\Delta_3 \end{pmatrix} \tag{I.33}$$

I.1.4 Grand potential

See [20], especially supplementary material, notes 1 and 3.

Mean-Field Hamiltonian (with the last two terms due to exchange of anticommuting fermion operators and the term quadratic in the expectation value from the mean-field decoupling respectively):

$$H_{MF} = \sum_{\mathbf{k}} \Psi_{\mathbf{k}}^\dagger \mathcal{H}(\mathbf{k}) \Psi_{\mathbf{k}} + \sum_{\mathbf{k}} \mathrm{Tr}(H_{\mathbf{k}}^\downarrow) + \sum_{\mathbf{k}\alpha} \frac{|\Delta_\alpha|^2}{U} \tag{I.34}$$

The second term is the trace of the non-interacting Hamiltonian.

Thermodynamic grand potential (which at zero temperature is equivalent to the mean-field energy):

$$\Omega(T, \Delta) = -\frac{1}{\beta} \ln Z_\Omega = -\frac{1}{\beta} \ln \mathrm{Tr}(e^{-\beta H_{MF}}) \tag{I.35}$$

$$= \sum_{\mathbf{k}} \mathrm{Tr}(H_{\mathbf{k}}^\downarrow) + \sum_{\mathbf{k}\alpha} \frac{|\Delta_\alpha|^2}{U} - \frac{1}{\beta} \ln \mathrm{Tr}(e^{-\beta \Psi_{\mathbf{k}}^\dagger \mathcal{H}(\mathbf{k}) \Psi_{\mathbf{k}}}) \tag{I.36}$$

Zero temperature limit:

$$\Omega(\Delta) = \sum_{\mathbf{k}} \mathrm{Tr}(H_{\mathbf{k}}^\downarrow) + \sum_{\mathbf{k}\alpha} \frac{|\Delta_\alpha|^2}{U} - \frac{1}{2} \sum_{\mathbf{k}} \mathrm{Tr}([|\mathcal{H}_{\mathbf{k}}|]) \tag{I.37}$$

where a function of a matrix H (such as taking the absolute value of the BdG Hamiltonian $\mathcal{H}_{\mathbf{k}}$) is defined for the diagonal matrix of eigenvalues D and the unitary matrix U that diagonalizes H:

$$f(H) = U f(D) U^\dagger \tag{I.38}$$

The route to finding the value of the order parameter for a fixed interaction U is minimizing the grand potential with respect to Δ.

I.2 GINZBURG-LANDAU THEORY OF SUPERCONDUCTIVITY, PHASE TRANSITIONS AND BROKEN SYMMETRY

Following [14, ch. 11].

I.2.1 ORDER PARAMETER CONCEPT

Landau theory: phase transitions (e.g. iron becomes magnetic, water freezes, superfluidity/superconductivity) are associated with the development of an order parameter when the temperature drops below the transition temperature T_C.

$$|\psi| = \begin{cases} 0 \, , \, T > T_C \\ |\psi_0| > 0 \, , \, T < T_C \end{cases} \tag{I.39}$$

Landau theory does not need microscopic expression for order parameter, it provides corse-grained description of the properties of matter. The order parameter description is good at length scales above ξ_0, the coherence length (e.g. size of Cooper pairs for SC).

I.2.2 LANDAU THEORY

Basic idea of Landau theory: write free energy as function $F[\psi]$ of the order parameter. Region of small ψ, expand free energy of many-body system as simple polynomial:

$$f_L = \frac{1}{V} F[\psi] = \frac{r}{2} \psi^2 + \frac{u}{4} \psi^4 \tag{I.40}$$

Provided r and u are greater that 0: minimum of $f_L[\psi]$) lies at $\psi = 0$. Landau theory assumes: at phase transition temperature r changes sign, so:

$$r = a(T - T_C) \tag{I.41}$$

Minimum of free energy occurs for:

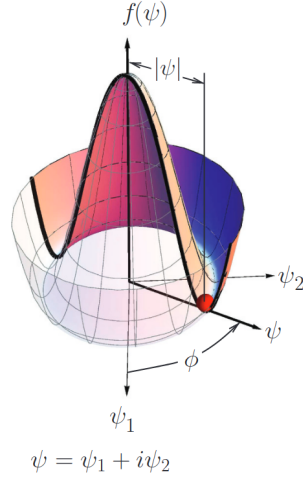$$\psi = \begin{cases} 0 \\ \pm\sqrt{\frac{a(T_C - T)}{u}} \end{cases} \tag{I.42}$$

Figure I.1: *Mexican hat potential*

Two minima for free energy function for $T < T_C$. With this, we can extract T_C from the knowledge of the dependence of $|\psi|^2$ on T via a linear fit. This is only valid for an area near T_C (where Landau theory holds), but can be used to get T_C from microscopic theories.

Going from a one to a n-component order parameters, OP acquires directions and magnitude. Particularly important example: complex or two component order parameter in superfluids and superconductors:

$$\psi = \psi_1 + \mathrm{i}\psi_2 = |\psi|e^{\mathrm{i}\phi} \tag{I.43}$$

The Landau free energy takes the form:

$$f[\psi] = r(\psi^*\psi) + \frac{u}{2}(\psi^*\psi)^2 \tag{I.44}$$

As before:

$$r = a(T - T_C) \tag{I.45}$$

Figure I.1 shows the Landau free energy as function of ψ.

Rotational symmetry, because free energy is independent of the global phase of the OP:

$$f[\psi] = f[e^{\mathrm{i}a}\psi] \tag{I.46}$$

In this 'Mexican hat' potential: order parameter can be rotated continuously from one broken-symmetry state to another. If we want the phase to be rigid, we need to introduce an There is a topological argument for the fact that the phase is rigid. This leads to Ginzburg-Landau theory. Will see later: well-defined phase is associated with persistent currents or superflow.

I.2.3   Ginzburg-Landau theory I: Ising order

Landau theory: energy cost of a uniform order parameter, more general theory needs to account for inhomogenous order parameters, in which the amplitude varies or direction of order parameter is twisted -> GL theory. First: one-component, 'Ising' order parameter. GL introduces additional energy $\delta f \propto |\Delta\psi|^2$, $f_{GL}[\psi, \Delta\psi] = \frac{s}{2}|\Delta\psi|^2 + f_L[\psi(s)]$, or in full:

$$f_{GL}[\psi, \Delta\psi, h] = \frac{s}{2}(\Delta\psi)^2 + \frac{r}{2}\psi^2 + \frac{u}{4}\psi^4 \tag{I.47}$$

GL theory is only valid near critical point, where OP is small enough to permit leading-order expansion. Length scale introduced by the gradient term: correlation length

$$d \tag{I.48}$$

> length scale/correlation length

I.2.4   Ginzburg-Landau theory II: complex order and superflow

Now: GL theory of complex or two-component order parameters, so superfluids and superconductors. Heart of discussion: emergence of a 'macroscopic wavefunction', where the microscopic field operators $\hat{\psi}(x)$ acquire an expectation value:

$$\langle\hat{\psi}(x)\rangle = \psi(x) = |\psi(x)|e^{i\theta(x)} \tag{I.49}$$

Reminder: Field operators are the real space representations of creation/annihilation operators. They can be thought of the super position of all ways of creating a particle at position x via the basis coefficients.

Magnitude determines density of particles in the superfluid:

$$|\psi(x)|^2 = n_s(x) \tag{I.50}$$

Density operator is

$$\hat{\rho} = \hat{\psi}(x)\hat{\psi}^{\dagger}(x) \tag{I.51}$$

so expectation value of that is the formula above.

Twist/gradient of phase determines superfluid velocity:

$$\mathbf{v}_s(x) = \frac{\hbar}{m}\Delta\phi(x) \tag{I.52}$$

We will derive this later in the chapter. Counterintuitive from quantum mechanics: GL suggested that $\Phi(x)$ is a macroscopic manifestation of a macroscopic number of particles condensed into precisely the same quantum state. Emergent phenomenon, collective properties of mater not a-priori self-evident from microscopic physics.

GL free energy density for superfluid (with one added term in comparison to Landau energy):

$$f_{GL}[\psi, \Delta\psi] = \frac{\hbar^2}{2m}|\Delta\psi|^2 + r|\psi|^2 + \frac{u}{2}|\psi|^4 \tag{I.53}$$

Interpreted as energy density of a condensate of bosons in which the field operator behaves as a complex order parameter. Gives interpretation of gradient term as kinetic energy:

> energy density of bosonic field? -> for comparison!

$$s|\Delta\psi|^2 = \frac{\hbar^2}{2m}\langle\Delta\hat{\psi}^{\dagger}\Delta\hat{\psi}\rangle \implies s = \frac{\hbar^2}{2m} \tag{I.54}$$

As in Ising order: correlation length/GL-coherence length governs characteristic range of amplitude fluctuations of the order parameter:

$$\xi = \sqrt{\frac{s}{|r|}} = \sqrt{\frac{\hbar^2}{2m|r|}} = \xi_0(1 - \frac{T}{T_C})^{-\frac{1}{2}} \tag{I.55}$$

where $\xi_0 = \xi(T = 0) = \sqrt{\frac{\hbar^2}{2maT_C}}$ is the coherence length. Beyond this length: only phase fluctuations survive. Freeze out fluctuations in amplitude (no x-dependence in amplitude) $\psi(x) = \sqrt{n_s}e^{i\phi(x)}$, then $\Delta\psi = i\Delta\phi\psi$ and $|\Delta\psi|^2 = n_s(\Delta\phi)^2$, dependency of kinetic energy on the phase twist is (bringing it into the form $\frac{m}{2}v^2$):

> Compare with Ising order, especially dependence on T

> Compare with Ising order. Is that derived or postulated?

$$\frac{\hbar^2 n_s}{2m}(\Delta\phi)^2 = \frac{mn_s}{2}(\frac{\hbar}{m}\Delta\phi)^2 \tag{I.56}$$

So twist of phase results in increase in kinetic energy, associated with a super-fluid velocity:

$$\mathbf{v}_s = \frac{\hbar}{m}\Delta\phi \tag{I.57}$$

For interpretation of superfluid states: coherent states. These are eigenstates of the field operator

$$\hat{\psi}(x)\left|\psi\right\rangle = \psi(x)\left|\psi\right\rangle \tag{I.58}$$

and don't have a definite particle number. Importantly, this small uncertainty in particle number enables a high degree of precision in phase (which is the property of a condensate).

Phase rigidity and superflow: in GL theory, energy is sensitive to a twist of the phase. Substitute $\psi = |\psi|e^{i\phi}$ into GL free energy, gradient term is:

$$\Delta\psi = (\Delta|\psi| + i\Delta\phi|\psi|)e^{i\phi} \tag{I.59}$$

So:

$$f_{GL} = \frac{\hbar}{2m}|\psi|^2(\Delta\phi)^2 + \left[\frac{\hbar}{2m}(\Delta|\psi|)^2 + r|\psi|^2 + \frac{u}{2}|\psi|^4\right] \tag{I.60}$$

The second term resembles GL functional for an Ising order parameter, describes energy cost of variations in the magnitude of the order parameter.

> Here: particle-current operator, especially for coherent state, connection with phase twist

I.3   Coherence length and penetration depth in strongly correlated superconductors

Is that a good fit here?

From [21].

In most materials: Cooper pairs do not carry finite center-of-mass momentum. In presence of e.g. external fields or magnetism: SC states with FMP might arise.

Theory/procedure in the paper: enforce FMP states via constraints on pair-center-of-mass momentum $\mathbf{q}$, access characteristic lenght scales ξ_0, λ_L through analysis of the momentum and temperature-dependent OP. Constrain for FF-type pairing:

$$\psi_{\mathbf{q}}(\mathbf{r}) = |\psi_{\mathbf{q}}|e^{i\mathbf{q}\mathbf{r}} \tag{I.61}$$

> Finish up the discussion of Niklas paper