

I Ginzburg-Landau theory of superconductivity

I.1 Coherence length and penetration depth in strongly correlated superconductors

From [wittBypassingLatticeBCSBEC2024].

In most materials: Cooper pairs do not carry finite center-of-mass momentum. In presence of e.g. external fields or magnetism: SC states with FMP might arise.

Theory/procedure in the paper: enforce FMP states via constraints on pair-center-of-mass momentum \mathbf{q} , access characteristic length scales ξ_0, λ_L through analysis of the momentum and temperature-dependent OP. Constrain for FF-type pairing:

$$\psi_{\mathbf{q}}(\mathbf{r}) = |\psi_{\mathbf{q}}|e^{i\mathbf{q}\mathbf{r}} \quad (\text{I.1})$$

I.2 Phase transitions and broken symmetry

Following [colemanIntroductionManyBodyPhysics2015].

I.2.1 Order parameter concept

Landau theory: phase transitions (e.g. iron becomes magnetic, water freezes, superfluidity/superconductivity) are associated with the development of an order parameter when the temperature drops below the transition temperature T_C .

$$|\psi| = \begin{cases} 0, & T > T_C \\ |\psi_0| > 0, & T < T_C \end{cases} \quad (\text{I.2})$$

Landau theory does not need microscopic expression for order parameter, it provides coarse-grained description of the properties of matter. The order parameter description is good at length scales above ξ_0 , the coherence length (e.g. size of Cooper pairs for SC).

1.2.2 Landau theory

Basic idea of Landau theory: write free energy as function $F[\psi]$ of the order parameter. Region of small ψ , expand free energy of many-body system as simple polynomial:

$$f_L = \frac{1}{V}F[\psi] = \frac{r}{2}\psi^2 + \frac{u}{4}\psi^4 \quad (\text{I.3})$$

Provided r and u are greater than 0: minimum of $f_L[\psi]$ lies at $\psi = 0$. Landau theory assumes: at phase transition temperature r changes sign, so:

$$r = a(T - T_C) \quad (\text{I.4})$$

Minimum of free energy occurs for:

$$\psi = \begin{cases} 0 \\ \pm \sqrt{\frac{a(T_C - T)}{u}} \end{cases} \quad (\text{I.5})$$

Could put a bit more into here about second order phase transition

Two minima for free energy function for $T < T_C$.

Going from a one to a n -component order parameters, OP acquires directions and magnitude. Particularly important example: complex or two component order parameter in superfluids and superconductors:

$$\psi = \psi_1 + i\psi_2 = |\psi|e^{i\phi} \quad (\text{I.6})$$

The Landau free energy takes the form:

$$f[\psi] = r(\psi^*\psi) + \frac{u}{2}(\psi^*\psi)^2 \quad (\text{I.7})$$

As before:

$$r = a(T - T_C) \quad (\text{I.8})$$

Figure [I.1](#) shows the Landau free energy as function of ψ .

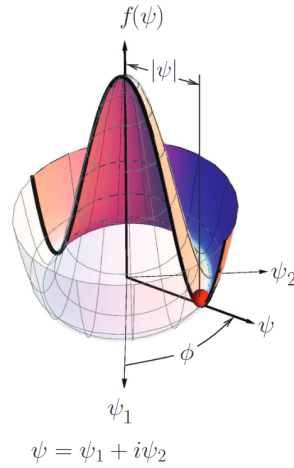


Figure I.1: Mexican hat potential

Rotational symmetry, because free energy is independent of the global phase of the OP:

$$f[\psi] = f[e^{ia}\psi] \quad (\text{I.9})$$

In this ‘Mexican hat’ potential: order parameter can be rotated continuously from one broken-symmetry state to another. If we want the phase to be rigid, we need to introduce an There is a topological argument for the fact that the phase is rigid. This leads to Ginzburg-Landau theory. Will see later: well-defined phase is associated with persistent currents or superflow.

I.2.3 Ginzburg-Landau theory I: Ising order

Landau theory: energy cost of a uniform order parameter, more general theory needs to account for inhomogenous order parameters, in which the amplitude varies or direction of order parameter is twisted -> GL theory. First: one-component, ‘Ising’ order parameter. GL introduces additional energy $\delta f \propto |\Delta\psi|^2$, $f_{GL}[\psi, \Delta\psi] = \frac{s}{2}|\Delta\psi|^2 + f_L[\psi(s)]$, or in full:

$$f_{GL}[\psi, \Delta\psi, h] = \frac{s}{2}(\Delta\psi)^2 + \frac{r}{2}\psi^2 + \frac{u}{4}\psi^4 - h\psi \quad (\text{I.10})$$

GL theory is only valid near critical point, where OP is small enough to

What is the h here?

length scale/correlation length

permit leading-order expansion. Dimensional analysis shows: $\frac{c}{r} = L^2$ has dimension of length squared. Length scale introduced by

What is c?

I.2.4 Ginzburg-Landau theory II: complex order and superflow

Now: G-L theory of complex or two-component order parameters, so superfluids and superconductors. Heart of discussion: emergence of a ‘macroscopic wavefunction’, where the microscopic field operators $\psi(\hat{x})$ acquire an expectation value:

$$\langle \psi(x) \rangle = \psi(x) = |\psi(x)|e^{i\theta(x)} \quad (\text{I.11})$$

Magnitude determines density of particles in the superfluid:

$$|\psi(x)|^2 = n_s(x) \quad (\text{I.12})$$

Twist/gradient of phase determines superfluid velocity:

$$\mathbf{v}_s(x) = \frac{\hbar}{m} \Delta \phi(x) \quad (\text{I.13})$$

We will derive this later in the chapter. Counterintuitive from quantum mechanics: GL suggested that $\Phi(x)$ is a macroscopic manifestation of a macroscopic number of particles condensed into precisely the same quantum state. Emergent phenomenon, collective properties of matter not a-priori self-evident from microscopic physics.

GL free energy density for superfluid (with one added term in comparison to Landau energy):

$$f_{GL}[\psi, \Delta\psi] = \frac{\hbar^2}{2m} |\Delta\psi|^2 + r|\psi|^2 + \frac{u}{2} |\psi|^4 \quad (\text{I.14})$$

Interpreted as energy density of a condensate of bosons in which the field operator behaves as a complex order parameter. Gives interpretation of gradient term as kinetic energy:

$$s|\Delta\psi|^2 = \frac{\hbar^2}{2m} \langle \Delta\hat{\psi}^\dagger \Delta\hat{\psi} \rangle \implies s = \frac{\hbar^2}{2m} \quad (\text{I.15})$$

As in Ising order: correlation length/GL-coherence length governs characteristic range of amplitude fluctuations of the order parameter:

$$\xi = \sqrt{\frac{s}{|r|}} = \sqrt{\frac{\hbar^2}{2m|r|}} = \xi_0 \left(1 - \frac{T}{T_C}\right)^{-\frac{1}{2}} \quad (\text{I.16})$$

What exactly are field operators again?

More info on that? Does that come later in chapter?

energy density of bosonic field? -> for comparison!

where $\xi_0 = \xi(T = 0) = \sqrt{\frac{\hbar^2}{2maT_C}}$ is the coherence length. Beyond this length: only phase fluctuations survive. Freeze out fluctuations in amplitude (no x -dependence in amplitude) $\psi(x) = \sqrt{n_s}e^{i\phi(x)}$, then $\Delta\psi = i\Delta\phi\psi$ and $|\Delta\psi|^2 = n_s(\Delta\phi)^2$, dependency of kinetic energy on the phase twist is (bringing it into the form $\frac{m}{2}v^2$):

$$\frac{\hbar^2 n_s}{2m} (\Delta\phi)^2 = \frac{mn_s}{2} \left(\frac{\hbar}{m} \Delta\phi\right)^2 \quad (\text{I.17})$$

So twist of phase results in increase in kinetic energy, associated with a superfluid velocity:

$$\mathbf{v}_s = \frac{\hbar}{m} \Delta\phi \quad (\text{I.18})$$

For interpretation of superfluid states: coherent states. These are eigenstates of the field operator

$$\hat{\psi}(x) |\psi\rangle = \psi(x) |\psi\rangle \quad (\text{I.19})$$

and don't have a definite particle number. Importantly, this small uncertainty in particle number enables a high degree of precision in phase (which is the property of a condensate).

Phase rigidity and superflow: in GL theory, energy is sensitive to a twist of the phase. Substitute $\psi = |\psi|e^{i\phi}$ into GL free energy, gradient term is:

$$\Delta\psi = (\Delta|\psi| + i\Delta\phi|\psi|)e^{i\phi} \quad (\text{I.20})$$

So:

$$f_{GL} = \frac{\hbar}{2m} |\psi|^2 (\Delta\phi)^2 + \left[\frac{\hbar}{2m} (\Delta|\psi|)^2 + r|\psi|^2 + \frac{u}{2} |\psi|^4 \right] \quad (\text{I.21})$$

The second term resembles GL functional for an Ising order parameter, describes energy cost of variations in the magnitude of the order parameter.

Compare with Ising order, especially dependence on T

Compare with Ising order. Is that derived or postulated?

Here: particle-current operator, especially for coherent state, connection with phase twist