

I DRESSED GRAPHENE MODEL

I.1 LATTICE STRUCTURE OF GRAPHENE

Structure of honeycomb lattice following [1].

Monolayer graphene forms a hexagonal lattice.

Primitive lattice vectors of the hexagonal lattice:

$$\mathbf{a}_1 = \frac{a}{2} \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \quad (\text{I.1})$$

$$\mathbf{a}_2 = \frac{a}{2} \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \quad (\text{I.2})$$

with lattice constant $a \approx 2.46 \text{ \AA}$ (distance between unit cells). Have

$$a = \sqrt{3}a_0 \quad (\text{I.3})$$

with the nearest-neighbour distance a_0 .

Vectors to the nearest-neighbor B_i ($i = 1, 2, 3$) atoms from atom A :

$$\delta_{AB,1} = \begin{pmatrix} 0 \\ \frac{a}{\sqrt{3}} \end{pmatrix}, \delta_{AB,2} = \begin{pmatrix} \frac{a}{2} \\ -\frac{a}{2\sqrt{3}} \end{pmatrix}, \delta_{AB,3} = \begin{pmatrix} -\frac{a}{2} \\ -\frac{a}{2\sqrt{3}} \end{pmatrix} \quad (\text{I.4})$$

Vectors to the nearest-neighbor A_i ($i = 1, 2, 3$) atoms from atom B :

$$\delta_{BA,1} = \begin{pmatrix} 0 \\ -\frac{a}{\sqrt{3}} \end{pmatrix}, \delta_{BA,2} = \begin{pmatrix} \frac{a}{2} \\ \frac{a}{2\sqrt{3}} \end{pmatrix}, \delta_{BA,3} = \begin{pmatrix} -\frac{a}{2} \\ \frac{a}{2\sqrt{3}} \end{pmatrix} \quad (\text{I.5})$$

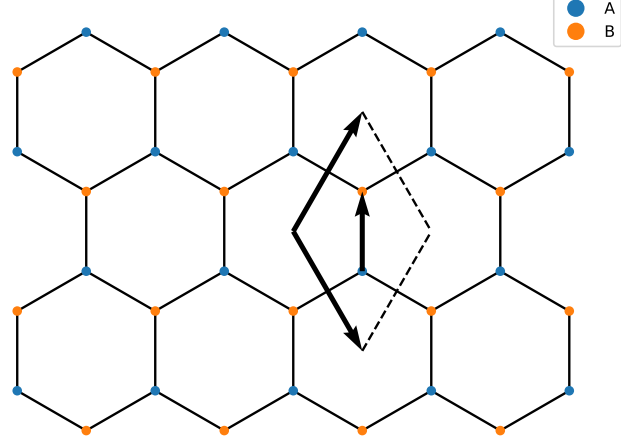


Figure I.1: *Graphene lattice structure*

The vectors between the Graphene A atom and the six neighbours on the same sub lattice can be found by rotating \mathbf{a}_1 six times by $1/6 * 2\pi = \pi/3$:

$$\delta_{AA,1} = \mathbf{a}_1 = \frac{a}{2} \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = a \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = a \begin{pmatrix} \sin(\frac{\pi}{6}) \\ \cos(\frac{\pi}{6}) \end{pmatrix} \quad (\text{I.6})$$

$$\delta_{AA,2} = a \begin{pmatrix} \sin(\frac{3\pi}{6}) \\ \cos(\frac{3\pi}{6}) \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (\text{I.7})$$

$$\delta_{AA,3} = a \begin{pmatrix} \sin(\frac{5\pi}{6}) \\ \cos(\frac{5\pi}{6}) \end{pmatrix} = a \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \quad (\text{I.8})$$

$$\delta_{AA,4} = a \begin{pmatrix} \sin(\frac{7\pi}{6}) \\ \cos(\frac{7\pi}{6}) \end{pmatrix} = a \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \quad (\text{I.9})$$

$$\delta_{AA,5} = a \begin{pmatrix} \sin(\frac{9\pi}{6}) \\ \cos(\frac{9\pi}{6}) \end{pmatrix} = a \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad (\text{I.10})$$

$$\delta_{AA,6} = a \begin{pmatrix} \sin(\frac{11\pi}{6}) \\ \cos(\frac{11\pi}{6}) \end{pmatrix} = a \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \quad (\text{I.11})$$

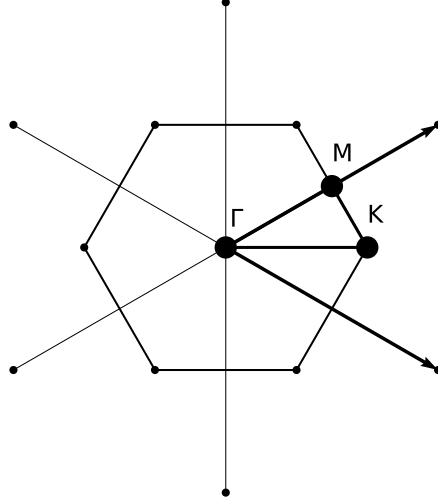


Figure I.2: *Graphene Brillouin Zone*

The primitive reciprocal lattice vectors $\mathbf{b}_1, \mathbf{b}_2$ fulfill

$$\mathbf{a}_1 \cdot \mathbf{b}_1 = \mathbf{a}_2 \cdot \mathbf{b}_2 = 2\pi \quad (\text{I.12})$$

$$\mathbf{a}_1 \cdot \mathbf{b}_2 = \mathbf{a}_2 \cdot \mathbf{b}_1 = 0, \quad (\text{I.13})$$

so we have:

$$\mathbf{b}_1 = \frac{2\pi}{a} \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \quad (\text{I.14})$$

$$\mathbf{b}_2 = \frac{2\pi}{a} \begin{pmatrix} 1 \\ -\frac{1}{\sqrt{3}} \end{pmatrix} \quad (\text{I.15})$$

Points of high symmetry in the Brillouin zone are:

$$\Gamma = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (\text{I.16})$$

$$\text{M} = \frac{\pi}{a} \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \quad (\text{I.17})$$

$$\text{K} = \frac{4\pi}{3a} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (\text{I.18})$$

I.2 EG-X MODEL

Graphene lattice and a site X. Real-life motivation: layer of graphene on top

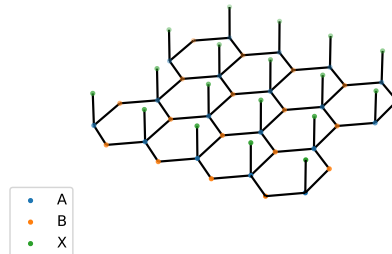


Figure I.3: *EG-X model*

of a substrate of another material (which provides the additional X atoms). There is no spin-orbit coupling considered in the model (but when according to Niklas: when mapping to substrates Sn or Pb, it could be necessary (but does not the qualitative result?)).

Spin-orbit coupling,
drop second spin
index?

Without interaction :

$$H_0 = -t_X \sum_{\langle ij \rangle, \sigma \sigma'} d_{i, \sigma}^\dagger d_{j, \sigma'} + \text{h.c.} - t_{\text{Gr}} \sum_{\langle ij \rangle, \sigma \sigma'} \left(c_{i, \sigma}^{(A), \dagger} c_{j, \sigma'}^{(B)} + c_{j, \sigma'}^{(B), \dagger} c_{i, \sigma}^{(A)} + \text{h.c.} \right) \quad (\text{I.19})$$

$$+ V \sum_{i, \sigma \sigma'} \left(d_{i, \sigma}^\dagger c_{i, \sigma'}^{(A)} + c_{i, \sigma}^{(A), \dagger} d_{i, \sigma'} \right) \quad (\text{I.20})$$

with:

- d operators on the X atom
- $c^{(\epsilon)}$ operators on the graphene site ($\epsilon = A, B$)
- t_X NN hopping for X
- t_{Gr} NN hopping of Gr
- V hybridization between X and Graphene B sites

We can also introduce an onsite Hubbard interaction:

$$H_{\text{int}} = U_X \sum_i d_{i, \uparrow}^\dagger d_{i, \downarrow}^\dagger d_{i, \downarrow} d_{i, \uparrow} + U_{\text{Gr}} \sum_{i, \epsilon=A, B} c_{i, \uparrow}^{(\epsilon), \dagger} c_{i, \downarrow}^{(\epsilon), \dagger} c_{i, \downarrow}^{(\epsilon)} c_{i, \uparrow}^{(\epsilon)} \quad (\text{I.21})$$

I.2.1 REVIEW: HUBBARD MODEL ON THE HONEYCOMB LATTICE

Write review for Hubbard model on the honeycomb lattice

I.2.2 BAND STRUCTURE OF THE NON-INTERACTING EG-X MODEL

To treat eq. I.20, we first write out the sums over nearest neighbours $\langle i, j \rangle$ explicitly, writing $\delta_X, \delta_\epsilon$ ($\epsilon = A, B$) for the connections to the nearest neighbours of the X atoms and Graphene A, B sites. Doing the calculation for the example of the X atoms:

$$-t_X \sum_{\langle ij \rangle, \sigma \sigma'} (d_{i, \sigma}^\dagger d_{j, \sigma'} + d_{j, \sigma}^\dagger d_{i, \sigma'}) \quad (\text{I.22})$$

$$= -\frac{t_X}{2} \sum_{i, \sigma, \sigma'} \sum_{\delta_X} d_{i, \sigma}^\dagger d_{i+\delta_X, \sigma'} - \frac{t_X}{2} \sum_{j, \sigma, \sigma'} \sum_{\delta_X} d_{j, \sigma}^\dagger d_{j+\delta_X, \sigma'} \quad (\text{I.23})$$

$$= -t_X \sum_{i, \sigma, \sigma'} \sum_{\delta_X} d_{i, \sigma}^\dagger d_{i+\delta_X, \sigma'} \quad (\text{I.24})$$

(The factor $1/2$ is to account for double counting when going to the sum over all lattice sites i)

Now we can input the discrete Fourier transform (for both graphene and X operators) into eq. I.24

$$c_i = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}_i} c_{\mathbf{k}} \quad (\text{I.25})$$

$$c_i^\dagger = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{r}_i} c_{\mathbf{k}}^\dagger \quad (\text{I.26})$$

with the completeness relation:

$$\sum_i e^{i\mathbf{k}\mathbf{r}_i} e^{-i\mathbf{k}'\mathbf{r}_i} = N \delta_{\mathbf{k},\mathbf{k}'} . \quad (\text{I.27})$$

We get:

$$-t_X \frac{1}{N} \sum_{i,\sigma,\sigma'} \sum_{\delta_X} d_{i,\sigma}^\dagger d_{i+\delta_X,\sigma'} = -t_X \frac{1}{N} \sum_{i,\sigma,\sigma'} \sum_{\delta_X} \sum_{\mathbf{k},\mathbf{k}'} e^{-i\mathbf{k}\mathbf{r}_i} d_{\mathbf{k},\sigma}^\dagger e^{i\mathbf{k}'\mathbf{r}_i} e^{i\mathbf{k}'\delta_X} d_{\mathbf{k}',\sigma'} \quad (\text{I.28})$$

$$= -t_X \frac{1}{N} \sum_{\mathbf{k},\mathbf{k}',\sigma,\sigma'} \sum_{\delta_X} d_{\mathbf{k},\sigma}^\dagger e^{i\mathbf{k}'\delta_X} d_{\mathbf{k}',\sigma'} \sum_i e^{-i\mathbf{k}\mathbf{r}_i} e^{i\mathbf{k}'\mathbf{r}_i} \quad (\text{I.29})$$

$$= -t_X \frac{1}{N} \sum_{\mathbf{k},\mathbf{k}',\sigma,\sigma'} \sum_{\delta_X} d_{\mathbf{k},\sigma}^\dagger e^{i\mathbf{k}'\delta_X} d_{\mathbf{k}',\sigma'} N \delta_{\mathbf{k},\mathbf{k}'} \quad (\text{I.30})$$

$$= -t_X \sum_{\mathbf{k},\sigma,\sigma'} d_{\mathbf{k},\sigma}^\dagger d_{\mathbf{k},\sigma'} \sum_{\delta_X} e^{i\mathbf{k}\delta_X} \quad (\text{I.31})$$

The nearest neighbours for X atoms are the vectors $\delta_{AA,i}$ from section I.1. With that, we can calculate:

$$f_X(\mathbf{k}) = -t_X \sum_{\delta_X} e^{i\mathbf{k}\delta_X} \quad (\text{I.32})$$

$$= -t_X \left(e^{ia(\frac{k_x}{2} + \frac{\sqrt{3}k_y}{2})} + e^{iak_x} + e^{ia(\frac{k_x}{2} - \frac{\sqrt{3}k_y}{2})} \right) \quad (\text{I.33})$$

$$+ e^{ia(-\frac{k_x}{2} - \frac{\sqrt{3}k_y}{2})} + e^{-iak_x} + e^{ia(-\frac{k_x}{2} + \frac{\sqrt{3}k_y}{2})} \quad (\text{I.34})$$

$$= -t_X \left(2 \cos(ak_x) + 2e^{ia\frac{\sqrt{3}k_y}{2}} \cos(\frac{a}{2}k_x) + 2e^{-ia\frac{\sqrt{3}k_y}{2}} \cos(\frac{a}{2}k_x) \right) \quad (\text{I.35})$$

$$= -2t_X \left(\cos(ak_x) + 2 \cos(\frac{a}{2}k_x) \cos(\sqrt{3}\frac{a}{2}k_y) \right) \quad (\text{I.36})$$

We can do the same for the hopping between Graphene sites, for example :

$$-t_{Gr} \sum_{\langle ij \rangle, \sigma \sigma'} c_{i, \sigma}^{(A), \dagger} c_{j, \sigma'}^{(B)} = -t_{Gr} \sum_{i, \sigma \sigma'} \sum_{\delta_{AB}} c_{i, \sigma}^{(A), \dagger} c_{i + \delta_{AB}, \sigma'}^{(B)} \quad (I.37)$$

$$= -t_{Gr} \sum_{\mathbf{k}, \sigma, \sigma'} c_{\mathbf{k}, \sigma}^{(A), \dagger} c_{\mathbf{k}, \sigma'}^{(B)} \sum_{\delta_{AB}} e^{i\mathbf{k} \delta_{AB}} \quad (I.38)$$

We note

$$\sum_{\delta_{AB}} e^{i\mathbf{k} \delta_{AB}} = \left(\sum_{\delta_{BA}} e^{i\mathbf{k} \delta_{BA}} \right)^* = \sum_{\delta_{BA}} e^{-i\mathbf{k} \delta_{BA}} \quad (I.39)$$

and calculate

$$f_{Gr} = -t_{Gr} \sum_{\delta_{AB}} e^{i\mathbf{k} \delta_{AB}} \quad (I.40)$$

$$= -t_{Gr} \left(e^{i \frac{a}{\sqrt{3}} k_y} + e^{i \frac{a}{2\sqrt{3}} (\sqrt{3} k_x - k_y)} + e^{i \frac{a}{2\sqrt{3}} (-\sqrt{3} k_x - k_y)} \right) \quad (I.41)$$

$$= -t_{Gr} \left(e^{i \frac{a}{\sqrt{3}} k_y} + e^{-i \frac{a}{2\sqrt{3}} k_y} \left(e^{i \frac{a}{2} k_x} + e^{-i \frac{a}{2} k_x} \right) \right) \quad (I.42)$$

$$= -t_{Gr} \left(e^{i \frac{a}{\sqrt{3}} k_y} + 2e^{-i \frac{a}{2\sqrt{3}} k_y} \cos\left(\frac{a}{2} k_x\right) \right) \quad (I.43)$$

All together, we get:

$$H_0 = \sum_{\mathbf{k}, \sigma, \sigma'} \begin{pmatrix} c_{\mathbf{k}, \sigma}^{A, \dagger} & c_{\mathbf{k}, \sigma}^{B, \dagger} & d_{\mathbf{k}, \sigma}^{\dagger} \end{pmatrix} \begin{pmatrix} 0 & f_{Gr} & V \\ f_{Gr}^* & 0 & 0 \\ V & 0 & f_X \end{pmatrix} \begin{pmatrix} c_{\mathbf{k}, \sigma}^A \\ c_{\mathbf{k}, \sigma}^B \\ d_{\mathbf{k}, \sigma} \end{pmatrix} \quad (I.44)$$

The band structure for the non-interacting EG-X model is easily obtained by diagonalising the matrix in eq. I.44. This was done in fig. I.4.

Values used for calculation:

- $a_0 = 1$
- $t_{Gr} = 1$
- $t_X = 0.01$

V is the control parameter. (According to Niklas), a range from $V = 0.1$ to $V = 2$ can be mapped onto materials in experiment.

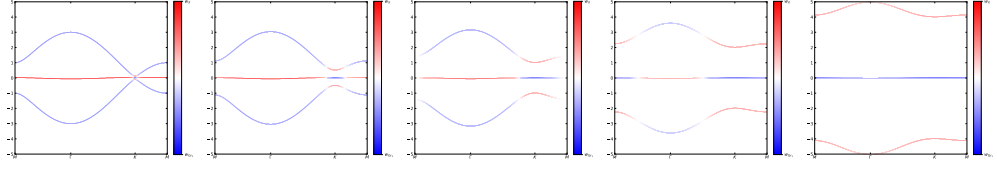


Figure I.4: Bands of the non-interacting EG-X model. All the bands are spin-degenerate.

I.3 MULTIBAND BCS?

Define sublattice index

$$\alpha = 1, 2, 3 \quad (\text{I.45})$$

with $1 \cong \text{Gr}_1, 2 \cong \text{Gr}_2, 3 \cong \text{X}$. Then we can write the non-interacting term as

$$H_0 = - \sum_{\langle i,j \rangle, \alpha, \beta, \sigma} [\mathbf{t}]_{i\alpha, j\beta} c_{i\alpha}^\dagger c_{j\beta} \quad (\text{I.46})$$

with the matrix

$$\mathbf{t} = \begin{pmatrix} 0 & t_{\text{Gr}} & 0 \\ t_{\text{Gr}} & 0 & -V\delta_{ij} \\ 0 & -V\delta_{ij} & t_{\text{X}} \end{pmatrix} \quad (\text{I.47})$$

Add chemical potential:

$$-\mu \sum_{i\alpha\sigma} n_{i\alpha\sigma} \quad (\text{I.48})$$

Also write the interaction part with α (with changed signs compared to Niklas, to keep in line with papers about the attractive Hubbard model):

$$H_{\text{int}} = - \sum_{i\alpha} U_\alpha c_{i\alpha\uparrow}^\dagger c_{i\alpha\downarrow}^\dagger c_{i\alpha\downarrow} c_{i\alpha\uparrow} \quad (\text{I.49})$$

Fourier transformation:

$$H_{\text{int}} = - \frac{1}{N^2} \sum_{\alpha, \mathbf{k}_1, 2, 3, 4} U_\alpha e^{i(\mathbf{k}_1 + \mathbf{k}_4 - \mathbf{k}_1 - \mathbf{k}_3) \cdot \mathbf{r}_{i\alpha}} c_{\mathbf{k}_1\alpha\uparrow}^\dagger c_{\mathbf{k}_3\alpha\downarrow}^\dagger c_{\mathbf{k}_2\alpha\downarrow} c_{\mathbf{k}_4\alpha\uparrow} \quad (\text{I.50})$$

Impose zero-momentum pairing: $\mathbf{k}_1 + \mathbf{k}_3 = 0$ and $\mathbf{k}_2 + \mathbf{k}_4 = 0$:

$$H_{\text{int}} = - \sum_{\alpha, \mathbf{k}, \mathbf{k}'} U_\alpha c_{\mathbf{k}\alpha\uparrow}^\dagger c_{-\mathbf{k}\alpha\downarrow}^\dagger c_{-\mathbf{k}'\alpha\downarrow} c_{\mathbf{k}'\alpha\uparrow} \quad (\text{I.51})$$

Mean-field approximation:

$$H_{int} \approx \sum_{\alpha, \mathbf{k}} (\Delta_{\alpha} c_{\mathbf{k}\alpha\uparrow}^{\dagger} c_{-\mathbf{k}\alpha\downarrow}^{\dagger} + \Delta_{\alpha}^{*} c_{-\mathbf{k}\alpha\downarrow} c_{\mathbf{k}\alpha\uparrow}) \quad (\text{I.52})$$

with

$$\Delta_{\alpha} = -U_{\alpha} \sum_{\mathbf{k}'} \langle c_{-\mathbf{k}'\alpha\downarrow} c_{\mathbf{k}'\alpha\uparrow} \rangle \quad (\text{I.53})$$

$$\Delta_{\alpha}^{*} = -U_{\alpha} \sum_{\mathbf{k}'} \langle c_{\mathbf{k}'\alpha\uparrow}^{\dagger} c_{-\mathbf{k}'\alpha\downarrow}^{\dagger} \rangle \quad (\text{I.54})$$

This gives the BCS mean field Hamiltonian:

$$H_{BCS} = \sum_{\mathbf{k}\alpha\beta\sigma} [H_{0,\sigma}(\mathbf{k})]_{\alpha\beta} c_{\mathbf{k}\alpha\sigma}^{\dagger} c_{\mathbf{k}\beta\sigma} - \mu \sum_{\mathbf{k}\alpha\sigma} n_{\mathbf{k}\alpha\sigma} + \sum_{\alpha, \mathbf{k}} (\Delta_{\alpha} c_{\mathbf{k}\alpha\uparrow}^{\dagger} c_{-\mathbf{k}\alpha\downarrow}^{\dagger} + \Delta_{\alpha}^{*} c_{-\mathbf{k}\alpha\downarrow} c_{\mathbf{k}\alpha\uparrow}) \quad (\text{I.55})$$

with Nambu spinor

$$\Psi_{\mathbf{k}} = \begin{pmatrix} c_{1,\mathbf{k}\uparrow} \\ c_{2,\mathbf{k}\uparrow} \\ c_{3,\mathbf{k}\uparrow} \\ c_{1,-\mathbf{k}\downarrow}^{\dagger} \\ c_{2,-\mathbf{k}\downarrow}^{\dagger} \\ c_{3,-\mathbf{k}\downarrow}^{\dagger} \end{pmatrix} \quad (\text{I.56})$$

we have:

$$H_{MF} = \sum_{\mathbf{k}} \Psi_{\mathbf{k}}^{\dagger} \mathcal{H}(\mathbf{k}) \Psi_{\mathbf{k}} \quad (\text{I.57})$$

with

$$\mathcal{H}(\mathbf{k}) = \begin{pmatrix} H_{0,\uparrow}(\mathbf{k}) - \mu & \Delta \\ \Delta^{\dagger} & -H_{0,\downarrow}^{*}(-\mathbf{k}) + \mu \end{pmatrix} \quad (\text{I.58})$$

with $H_{0,\sigma}$ being the F.T. of the kinetic term and $\Delta = \text{diag}(\Delta_1, \Delta_2, \Delta_3)$.

I.3.1 BdG HAMILTONIAN IN BAND BASIS

Use transformation

$$c_{\mathbf{k}\alpha\sigma}^{\dagger} = \sum_n [\mathbf{G}]_{\alpha n}^{*} d_{n\mathbf{k}\sigma}^{\dagger} \quad (\text{I.59})$$

where the columns are made up of the eigenvectors of $\mathbf{H}_{0,\sigma}$ for a given \mathbf{k} :

$$\mathbf{G} = (\mathbf{G}_1 \quad \mathbf{G}_2 \quad \mathbf{G}_3) \quad (\text{I.60})$$

with that:

$$\mathbf{G}_\sigma^\dagger(\mathbf{k})\mathbf{H}_{0,\sigma}(\mathbf{k})\mathbf{G}_\sigma(\mathbf{k}) = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \quad (\text{I.61})$$

So the kinetic part of the BdG Hamiltonian becomes:

$$\sum_{\mathbf{k}\alpha\beta\sigma} [H_{0,\sigma}(\mathbf{k})]_{\alpha\beta} \sum_n [\mathbf{G}(\mathbf{k})]_{\alpha n}^* d_{n\mathbf{k}\sigma}^\dagger \sum_m [\mathbf{G}(\mathbf{k})]_{\beta m} d_{m\mathbf{k}\sigma} - \mu \sum_{\mathbf{k}\alpha\sigma} n_{n\mathbf{k}\sigma} \quad (\text{I.62})$$

$$= \sum_{m\mathbf{n}\mathbf{k}\sigma} d_{n\mathbf{k}\sigma}^\dagger d_{m\mathbf{k}\sigma} \sum_{\alpha\beta} [\mathbf{G}(\mathbf{k})]_{\alpha n}^* [H_{0,\sigma}(\mathbf{k})]_{\alpha\beta} [\mathbf{G}(\mathbf{k})]_{\beta m} - \mu \sum_{\mathbf{k}\alpha\sigma} n_{n\mathbf{k}\sigma} \quad (\text{I.63})$$

$$= \sum_{m\mathbf{n}\mathbf{k}\sigma} d_{n\mathbf{k}\sigma}^\dagger d_{m\mathbf{k}\sigma} \epsilon_n \delta_{nm} - \mu \sum_{\mathbf{k}\alpha\sigma} n_{n\mathbf{k}\sigma} \quad (\text{I.64})$$

$$= \sum_{n\mathbf{k}\sigma} \epsilon_n d_{n\mathbf{k}\sigma}^\dagger d_{n\mathbf{k}\sigma} - \mu \sum_{\mathbf{k}\alpha\sigma} n_{n\mathbf{k}\sigma} \quad (\text{I.65})$$

$$=: \sum_{n\mathbf{k}\sigma} \tilde{\xi}_{\mathbf{k}} d_{n\mathbf{k}\sigma}^\dagger d_{n\mathbf{k}\sigma} \quad (\text{I.66})$$

with $\tilde{\xi}_{\mathbf{k}} := \epsilon_{\mathbf{k}} - \mu$. The pairing terms become:

$$\sum_{\mathbf{k}\alpha} \Delta_\alpha c_{\mathbf{k}\alpha\uparrow}^\dagger c_{-\mathbf{k}\alpha\downarrow}^\dagger = \sum_{\mathbf{k}\alpha} \Delta_\alpha \sum_n [\mathbf{G}_\uparrow(\mathbf{k})]_{\alpha n}^* d_{n\mathbf{k}\uparrow}^\dagger \sum_m [\mathbf{G}_\downarrow(-\mathbf{k})]_{\beta m}^* d_{m-\mathbf{k}\downarrow}^\dagger \quad (\text{I.67})$$

$$= \quad (\text{I.68})$$

So that:

$$\mathcal{H}(\mathbf{k}) = \begin{pmatrix} \epsilon_{\mathbf{k}} - \mu & G^\dagger \Delta G \\ G^\dagger \Delta^\dagger G & -\epsilon_{\mathbf{k}} + \mu \end{pmatrix} \quad (\text{I.69})$$

with

$$\epsilon_{\mathbf{k}} = \begin{pmatrix} \epsilon_1(\mathbf{k}) & 0 & 0 \\ 0 & \epsilon_2(\mathbf{k}) & 0 \\ 0 & 0 & \epsilon_3(\mathbf{k}) \end{pmatrix} \quad (\text{I.70})$$

Concrete example for transformation of gaps from orbital to band basis at $K = \frac{4\pi}{3a} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. There, the non-interacting part becomes simply:

$$\mathcal{H}_0 = \begin{pmatrix} 0 & 0 & V \\ 0 & 0 & 0 \\ V & 0 & 3t_X \end{pmatrix} \quad (\text{I.71})$$

The eigenvalue problem can be solved e.g. via sympy:

$$G = \begin{pmatrix} \frac{-3t_X - \sqrt{4V^2 + 9t_X^2}}{\sqrt{4V^2 + (3t_X + \sqrt{4V^2 + 9t_X^2})^2}} & 0 & \frac{-3t_X + \sqrt{4V^2 + 9t_X^2}}{\sqrt{4V^2 + (3t_X - \sqrt{4V^2 + 9t_X^2})^2}} \\ 0 & 1 & 0 \\ \frac{2V}{\sqrt{4V^2 + (3t_X + \sqrt{4V^2 + 9t_X^2})^2}} & 0 & \frac{2V}{\sqrt{4V^2 + (3t_X - \sqrt{4V^2 + 9t_X^2})^2}} \end{pmatrix} \quad (\text{I.72})$$

So for $V \rightarrow 0$:

$$G = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{I.73})$$

but for $V > 0$, there are off-diagonal elements, e.g. $V = 0.1$:

$$G = \begin{pmatrix} -0.7578 & 0 & 0.6526 \\ 0 & 1 & 0 \\ 0.6526 & 0 & 0.7578 \end{pmatrix} \quad (\text{I.74})$$

So the transformation of the gap from orbital to band space reads:

$$G^\dagger \Delta G = \begin{pmatrix} \frac{3\Delta_1 t_X - 3\Delta_3 t_X + (\Delta_1 + \Delta_3) \sqrt{4V^2 + 9t_X^2}}{2\sqrt{4V^2 + 9t_X^2}} & 0 & \frac{V(-\Delta_1 + \Delta_3)}{\sqrt{4V^2 + 9t_X^2}} \\ 0 & \Delta_2 & 0 \\ \frac{V(-\Delta_1 + \Delta_3)}{\sqrt{4V^2 + 9t_X^2}} & 0 & \frac{-3\Delta_1 t_X + 3\Delta_3 t_X + (\Delta_1 + \Delta_3) \sqrt{4V^2 + 9t_X^2}}{2\sqrt{4V^2 + 9t_X^2}} \end{pmatrix} \quad (\text{I.75})$$

So in particular there is no interband pairing for $V \rightarrow 0$:

$$G^\dagger \Delta G = \begin{pmatrix} \Delta_1 & 0 & 0 \\ 0 & \Delta_2 & 0 \\ 0 & 0 & \Delta_3 \end{pmatrix} \quad (\text{I.76})$$

But for $V > 0$, there is interband pairing (e.g. $V = 0.1$):

$$G^\dagger \Delta G = \begin{pmatrix} 0.5742\Delta_1 + 0.4258\Delta_3 & 0 & -0.4945\Delta_1 + 0.4945\Delta_3 \\ 0 & \Delta_2 & 0 \\ -0.4945\Delta_1 + 0.4945\Delta_3 & 0 & 0.4258\Delta_1 + 0.5742\Delta_3 \end{pmatrix} \quad (\text{I.77})$$

I.3.2 GRAND POTENTIAL

See [2], especially supplementary material, notes 1 and 3.

Mean-Field Hamiltonian (with the last two terms due to exchange of anticommuting fermion operators and the term quadratic in the expectation value from the mean-field decoupling respectively):

$$H_{MF} = \sum_{\mathbf{k}} \Psi_{\mathbf{k}}^\dagger \mathcal{H}(\mathbf{k}) \Psi_{\mathbf{k}} + \sum_{\mathbf{k}} \text{Tr}(H_{\mathbf{k}}^\dagger) + \sum_{\mathbf{k}\alpha} \frac{|\Delta_\alpha|^2}{U} \quad (\text{I.78})$$

The second term is the trace of the non-interacting Hamiltonian.

Thermodynamic grand potential (which at zero temperature is equivalent to the mean-field energy):

$$\Omega(T, \Delta) = -\frac{1}{\beta} \ln Z_\Omega = -\frac{1}{\beta} \ln \text{Tr}(e^{-\beta H_{MF}}) \quad (\text{I.79})$$

$$= \sum_{\mathbf{k}} \text{Tr}(H_{\mathbf{k}}^\dagger) + \sum_{\mathbf{k}\alpha} \frac{|\Delta_\alpha|^2}{U} - \frac{1}{\beta} \ln \text{Tr}(e^{-\beta \Psi_{\mathbf{k}}^\dagger \mathcal{H}(\mathbf{k}) \Psi_{\mathbf{k}}}) \quad (\text{I.80})$$

Zero temperature limit:

$$\Omega(\Delta) = \sum_{\mathbf{k}} \text{Tr}(H_{\mathbf{k}}^\dagger) + \sum_{\mathbf{k}\alpha} \frac{|\Delta_\alpha|^2}{U} - \frac{1}{2} \sum_{\mathbf{k}} \text{Tr}([\mathcal{H}_{\mathbf{k}}]) \quad (\text{I.81})$$

where a function of a matrix H (such as taking the absolute value of the BdG Hamiltonian $\mathcal{H}_{\mathbf{k}}$) is defined for the diagonal matrix of eigenvalues D and the unitary matrix U that diagonalizes H :

$$f(H) = U f(D) U^\dagger \quad (\text{I.82})$$

The route to finding the value of the order parameter for a fixed interaction U is minimizing the grand potential with respect to Δ .