PHY-MV-FN-T28 – Condensed-matter theory - special topics

Lecture held summer term 2022 by Prof. Dr. Tim Wehling, Universität Hamburg

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Part I.

Entrée – Revisiting Basics of Many-Body Physics

1. Second Quantization

We want to describe a system of identical particles (electrons, photons, neutrons, ...). Each particle has a corresponding single-particle Hilbert space: Lecture 1–04.04.2022

$$\mathcal{H}_1^i = \operatorname{span}\{|\phi_\alpha^i\rangle\} \tag{1.1}$$

With the single-particle basis for the ith particle $|\phi_\alpha^i\rangle$. The index α is a complete system of single-particle quantum numbers, such as $\alpha = \{\vec{k}, n, \sigma\}$. A general N particle Hilbert space is then constructed as

$$\mathcal{H}_N = \mathcal{H}_1^1 \otimes \mathcal{H}_1^2 \otimes \ldots \otimes \mathcal{H}_1^N\,. \tag{1.2}$$

In quantum mechanics, identical particles are indistinguishable, so there should be no measurable consequences of particle permutations (swapping two particles). This means, that only states invariant under particle permutations reflect physical reality.

A more formal treatment: Define the permutation operator (on the basis of $\mathcal{H}_N$):

$$P_{ij}\,|\phi_{\alpha_1}^1\rangle \ldots |\phi_{\alpha_i}^i\rangle \ldots |\phi_{\alpha_j}^j\rangle \ldots |\phi_{\alpha_N}^N\rangle = |\phi_{\alpha_1}^1\rangle \ldots |\phi_{\alpha_j}^i\rangle \ldots |\phi_{\alpha_i}^j\rangle \ldots |\phi_{\alpha_N}^N\rangle\,, \tag{1.3}$$

with the quantum number indeces i, j swapped.

Now for the correct description of identical particles we restrict to eigenstates of P_{ij}. Properties of P_{ij}:

- $P_{ij}^2 = \mathbb{1} \implies P_{ij} = P_{ij}^\dagger \implies$ eigenvalues real
- P_{ij} is unitary $\implies$ eigenvalues $|\epsilon| = 1$

So P_{ij} has eigenvalues $\epsilon = \pm 1$.

From that follows two species of particles:

Fermions	Bosons
$\epsilon = -1$	$\epsilon = +1$
$P_{ij}\,\lvert\Psi\rangle = -\lvert\Psi\rangle$	$P_{ij}\,\lvert\Psi\rangle = \lvert\Psi\rangle$
half-integer spin	integer spin

Due to this fact, we need to define fermionic and bosonic N particle Hilbert spaces $\mathcal{H}_N^{(\epsilon)} \subset \mathcal{H}_N$:

$$|\Psi\rangle \in \mathcal{H}_N^{(\epsilon)} \implies P_{ij}\,|\Psi\rangle = \epsilon\,|\Psi\rangle \;\; \forall\, i \neq j \tag{1.4}$$

2. Interaction effects in solids and electronic correlations

This section will deal with a

$$H = \sum_{i\sigma} \epsilon_0 c^\dagger_{i\sigma} c_{i\sigma} - t \sum_{\langle i,j \rangle \sigma} c^\dagger_{i\sigma} c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} \tag{2.1}$$

with parameters $\epsilon_0, t, U \in \mathbb{R}$. Explanation of the individual terms:

- onsite energy $\sum_{i\sigma} \epsilon_0 c^\dagger_{i\sigma} c_{i\sigma}$: the term $c^\dagger_{i\sigma} c_{i\sigma}$ counts the number of particles lattice site i with spin σ, attaches an energy ϵ_0 to each lattice site if the site (with spin σ) is occupied
- hopping term $t \sum_{\langle i,j \rangle \sigma} c^\dagger_{i\sigma} c_{j\sigma}$: pairs different lattice sites
- double occupancy $U \sum_i n_{i\uparrow} n_{i\downarrow}$: energy prize U when lattice site i has two electrons (one with spin up and one with spin down)

Part II.

Premier plat principal – Green Function Formalism for Many-Body Physics

3. Many-Body Green Functions

3.1. Reminder: Time evolution pictures in Quantum Mechanics

Lecture 3, 14.04.2022

3.1.1. Schrödinger picture

In the Schrödinger picture, the time evolution goes with the wave function, which is governed by the Schrödinger equation

$$i\hbar\partial_t |\Psi(t)\rangle = H |\Psi(t)\rangle \ . \tag{3.1}$$

The time evolution of expectation values is then:

$$\langle A\rangle_t = \langle\Psi(t)| A |\Psi(t)\rangle \tag{3.2}$$

3.1.2. Heisenberg picture

In the Heisenberg picture, the time evolution goes with the operators:

Lecture 4, 21.04.2022

3.2. Green Functions for Many-Body Systems

The fundamental idea of Green functions can be thought of as the process of putting a particle into a system, letting it propagate and then taking it out again. Because the particle is interacting with the full many-body system, this propagation encodes then information about this system.

As an introduction, we look at a single particle system with a Hamilton operator H. We want to understand the following process: an initial state $|\phi_\alpha\rangle$ at time $t_0 = 0$ is prepared, then a measurement of an observable A is taken at time t. The expectation value for the measurement is then:

$$\begin{aligned}
\langle A\rangle_t &= \left\langle \phi_\alpha \,\middle|\, e^{iHt} A e^{-iHt} \,\middle|\, \phi_\alpha \right\rangle & (3.3)\\
&= \langle\phi_\alpha|e^{iHt}(\sum_\beta |\phi_\beta\rangle\langle\phi_\beta|)A(\sum_{\beta'} |\phi_{\beta'}\rangle\langle\phi_{\beta'}|)e^{-iHt}|\phi_\alpha\rangle & (3.4)\\
&= & (3.5)
\end{aligned}$$

4. Matsubara Green Functions

5. Feynman Diagrams

5.1. Wick's Theorem

Lecture 13 – 20.05.2022

In the previous section, the *Kubo formula* was introduced. Higher order correlation functions get introduced in the effort of calculating non-equilibrium properties of interest, for example $\rho * \rho$ has field operators in order 4. Wick's theorem gives a path to deal with those.

The goal is to factorize multi-particle correlation functions (as in, creating/annihilating multiple particles) into single particle correlation functions (creating/annihilating one particle), everything for non-interacting systems.

The starting point is a general, non-interacting Hamiltonian:

$$H_0 = \sum_{\nu,\nu'} h_{0,\nu\nu'} c_\nu^\dagger c_{\nu'} \, . \tag{5.1}$$

Lecture 16 – 13.06.2022

5.2. Diagrammatic Perturbation Theory

The starting point for developing diagrammatic perturbation theory is the Matsubara green function

$$G_{\alpha\beta}(\tau) = -\left\langle T_\tau c_\alpha(\tau) c_\beta^\dagger(0) \right\rangle \tag{5.2}$$

with operators in Heisenberg representation, i.e.

$$c_\tau = e^{\tau H} c_\alpha e^{-\tau H} \, . \tag{5.3}$$

We assume a very general Hamiltonian of the form

$$H = H_0 + V \tag{5.4}$$

with a non-interacting part H_0 and the interaction/perturbation V.

Part III.

Cours intermédiaire – Superconductivity

6. Superconductivity

Part IV.

Deuxième plat principal – Path Integrals in Quantum Many-Body physics

7. Coherent states

Lecture 19 – 23.06.2022

7.1. Coherent states for Bosons

For bosons we have the commutation relation

$$\left[a_i, a_j^\dagger\right] = \delta_{i,j} \tag{7.1}$$

for bosonic creation/annihilation operators. We want to construct (right) eigenvalues of annihilation operators:

$$a_i \left|\alpha\right\rangle = \alpha_i \left|\alpha\right\rangle \ . \tag{7.2}$$

8. Coherent State Path Integrals

8.1. Coherent States

Lecture 19 – 23.06.2022

Our goal in this chapter is to construct many-body quantum mechanics/statistics with path integrals over generalized "transition amplitudes" as central objects rather than direct time-propagation of wave-functions and operators.

8.1.1. Coherent States for Bosons

9. Broken Symmetry and Collective Phenomena