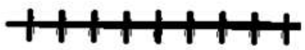
Polychrony as Chinampas.

Introduction

We start with a connected oriented graph with one minimum vertex and one maximum vertex. The direction is from left to right.

In[ ]:= `myline =`

Out[ ]= 

Since this graph has 9 vertex, I want you to think of it as

`x^9 / (1 - x)^10 = sum` $\left(x\text{^}n \binom{n}{9} \right)$

In[ ]:=
```
f[x_] := x^9 / (1 - x)^10
```

Why? well this is the magic of the project.
We are going to ask Mathematica to count all possible ways to label our figure with numbers from 1 to n preserving the order . For the line
with 9 vertex:

We choose a label for ‘a’ between 1 and n-8, the reason is that ‘a<b<c<d<e<f<g<h<i’ implies that ‘a’ cannot be n, or n-1,... n-7.

We choose a label for ‘b’ between ‘a+1’ and n-7, the reason is that ‘a<b<c<d<e<f<g<h<i’ implies that what ever value ‘a’ has, ‘b’ can be at least ‘a+1’ and cannot be n, or n-1,... n-6.
We are computing the sum below:

In[ ]:=
```
ToExpression["\\
\sum_{a=1}^{n-8}\sum_{b=a+1}^{n-7}\sum_{c=b+1}^{n-6}\sum_{d=c+1}^{n-5}\sum_{e=d+1
}^{n-4}\sum_{f=e+1}^{n-3}\sum_{g=e+1}^{n-2}\sum_{h=g+1}^{n-1}\sum_{i=h+1}^{n}
1", TeXForm]
```

Sum: The variable b == a + 1 cannot be localized so that it can be assigned to numerical values.

Out[ ]=
$$\sum_{a=1}^{n-8}\sum_{b==a+1}^{n-7}\sum_{c==b+1}^{n-6}\sum_{d==c+1}^{n-5}\sum_{e==d+1}^{n-4}\sum_{f==e+1}^{n-3}\sum_{g==e+1}^{n-2}\sum_{h==g+1}^{n-1}\sum_{i==h+1}^{n} 1$$

each term assign a value to the letters ‘a,b,c,d,e,f,g,h,i’. In Mathematica this is equivalent to:

In[ ]:=

In[ ]:=
```
Sum[1, {a, 1, n - 8}, {b, a + 1, n - 7}, {c, b + 1, n - 6}, {d, c + 1, n - 5},
 {e, d + 1, n - 4}, {f, e + 1, n - 3}, {g, f + 1, n - 2}, {h, g + 1 , n - 1}, {i, h + 1, n}]
```

Out[ ]=
$$\frac{40320\, n - 109584\, n^2 + 118124\, n^3 - 67284\, n^4 + 22449\, n^5 - 4536\, n^6 + 546\, n^7 - 36\, n^8 + n^9}{362880}$$

In[ ]:=
$$\text{FullSimplify}\left[\frac{40320\, n - 109584\, n^2 + 118124\, n^3 - 67284\, n^4 + 22449\, n^5 - 4536\, n^6 + 546\, n^7 - 36\, n^8 + n^9}{362880}\right]$$

Out[ ]=
$$\frac{(-8+n)\,(-7+n)\,(-6+n)\,(-5+n)\,(-4+n)\,(-3+n)\,(-2+n)\,(-1+n)\, n}{362880}$$

$$\frac{(-8+n)\,(-7+n)\,(-6+n)\,(-5+n)\,(-4+n)\,(-3+n)\,(-2+n)\,(-1+n)\, n}{362880} = \binom{n}{9}$$

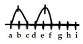
So if we consider a series \$sum x^n a_n\$ where \$a_n\$ stands for the number of labelings of our figure we obtain sum $\left(x^n \binom{n}{9}\right)$=x^9 / (1 - x) ^10

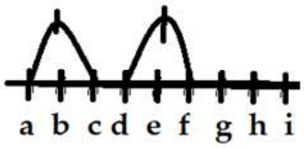
The Mathematica expression Sum[1,{a,1,n-8}, {b,a+1,n-7}, {c,b+1,n-6}, {d,c+1,n-5}, {e,d+1,n-4}, {f,e+1,n-3},{g,f+1,n-2}, {h,g+1 ,n-1},{i,h+1,n}] is just a lot of for loops, a lot of iterated sums, each one for each {x,x_i, x_f }. For example:

```
Sum[1, {i, 3, 8}]
6
```

Main steps.

Now for any new image that we draw over the line with 9 vertex, we are going to create a new power series using the following steps. Start with adding some handles to the line with 9 vertex, remember that our graphs have a direction from left to right.

In[ ]:= `myotherline =` 

Out[ ]= 

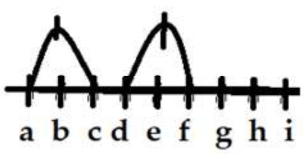
I labeled the vertex to easy our work.

Obtaining the differential equation

Adding handles.

Lets now compute the expression for adding two handles to the line with 9 points

In[ ]:=
```
myotherline
```

Out[ ]= 

We now need to find a label for those points in the handle, but if ‘a’ and ‘c’ have a value then the first handle can have values: a+1, a+2,...,c-1, which are in total (c-a-1) values. The second handle can have (f-d-1) values so in total we are adding (c-a-1)(f-d-1) new labelings.

Then the expression

In[ ]:=
```
ToExpression["\\
\sum_{a=1}^{n-8}\sum_{b=a+1}^{n-7}\sum_{c=b+1}^{n-6}\sum_{d=c+1}^{n-5}\sum_{e=d+1
}^{n-4}\sum_{f=e+1}^{n-3}\sum_{g=e+1}^{n-2}\sum_{h=g+1}^{n-1}\sum_{i=h+1}^{n}
(c-a-1)(f-d-1)", TeXForm]
```

Sum: The variable b == a + 1 cannot be localized so that it can be assigned to numerical values.

Out[ ]= $$\sum_{a=1}^{n-8}\sum_{b==a+1}^{n-7}\sum_{c==b+1}^{n-6}\sum_{d==c+1}^{n-5}\sum_{e==d+1}^{n-4}\sum_{f==e+1}^{n-3}\sum_{g==e+1}^{n-2}\sum_{h==g+1}^{n-1}\sum_{i==h+1}^{n}(c-a-1)\,(f-d-1)$$

Counts all possible labelings of this new figure, in Mathematica this is equivalent to:

In[ ]:=
```
Sum[(c - a - 1) * (f - d - 1), {a, 1, n - 8}, {b, a + 1, n - 7}, {c, b + 1, n - 6}, {d, c + 1, n - 5},
  {e, d + 1, n - 4}, {f, e + 1, n - 3}, {g, f + 1, n - 2}, {h, g + 1 , n - 1}, {i, h + 1, n}]
```

Out[ ]=
$$\frac{1}{19\,958\,400}\left(1\,491\,840\, n - 4\,699\,728\, n^2 + 6\,204\,572\, n^3 - 4\,598\,660\, n^4 + 2\,143\,405\, n^5 - 661\,584\, n^6 + 137\,676\, n^7 - 19\,140\, n^8 + 1705\, n^9 - 88\, n^{10} + 2\, n^{11}\right)$$

In[ ]:=
$$\text{FullSimplify}\left[\frac{1}{19\,958\,400}\left(1\,491\,840\, n - 4\,699\,728\, n^2 + 6\,204\,572\, n^3 - 4\,598\,660\, n^4 + 2\,143\,405\, n^5 - 661\,584\, n^6 + 137\,676\, n^7 - 19\,140\, n^8 + 1705\, n^9 - 88\, n^{10} + 2\, n^{11}\right)\right]$$

Out[ ]=
$$\frac{(-8+n)\,(-7+n)\,(-6+n)\,(-5+n)\,(-4+n)\,(-3+n)\,(-2+n)\,(-1+n)\,n\,(37+2\,(-8+n)\,n)}{19\,958\,400}$$

Here the trick is to compare with the previous expression for the line without handles:

In[ ]:=
```
%19
```

Out[ ]=
$$\frac{(-8+n)\,(-7+n)\,(-6+n)\,(-5+n)\,(-4+n)\,(-3+n)\,(-2+n)\,(-1+n)\,n}{362\,880}$$

The difference is (37+2(-8+n)n)/55, this expression is very important so make sure you understands how to obtain it.

obtaining the operator

We rewrite the expression

(37+2(-8+n)n)/55 =(37-16n+2n^2)/55

into a differential operator.

We add f(x) to the constant:

(37*f(x)-16n+2n^2)/55

We replace ‘n’ by ‘x*D[f(x),x]’ and ‘n^2’ by ‘x*D[x*D[f(x),x],x]’, etc.

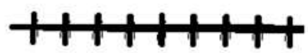
In[ ]:=
```
(37*f(x)–16*x*D[ f(x),x]+2x*D[x*D[ f(x),x],x])/55
(37 * f[x] - 16 * x * D[f[x], x] + 2 * x * D[x * D[f[x], x], x]) / 55
```

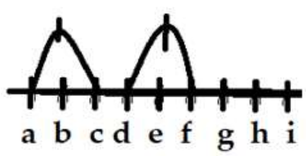
Out[ ]=
$$\frac{1}{55}\left(\frac{37\, x^9}{(1-x)^{10}} - 16\, x\left(\frac{9\, x^8}{(1-x)^{10}} + \frac{10\, x^9}{(1-x)^{11}}\right) + 2\, x\left(\frac{9\, x^8}{(1-x)^{10}} + \frac{10\, x^9}{(1-x)^{11}} + x\left(\frac{72\, x^7}{(1-x)^{10}} + \frac{180\, x^8}{(1-x)^{11}} + \frac{110\, x^9}{(1-x)^{12}}\right)\right)\right)$$

In[ ]:=
$$\text{Simplify}\left[\frac{1}{55}\left(\frac{37\,x^9}{(1-x)^{10}}-16\,x\left(\frac{9\,x^8}{(1-x)^{10}}+\frac{10\,x^9}{(1-x)^{11}}\right)+2\,x\left(\frac{9\,x^8}{(1-x)^{10}}+\frac{10\,x^9}{(1-x)^{11}}+x\left(\frac{72\,x^7}{(1-x)^{10}}+\frac{180\,x^8}{(1-x)^{11}}+\frac{110\,x^9}{(1-x)^{12}}\right)\right)\right)\right]$$

Out[ ]=
$$\frac{x^9\,(1+x)^2}{(-1+x)^{12}}$$

Note that this is =(1+x)/(1-x)^2*(1+x)/(1-x)^2*(1-x)^2*x^9/(1-x)^10

In our language, we started with  = x^9/(1 - x)^10 to create



=x^9/(1-x)^10 * (1+x)^2/(1-x)^2 by adding two lines. So adding two handles is equivalent to multiply by (1+x)^2/(1-x)^2.

The notebook Computation_of_R7 repeats this process for a bigger graph.