

## Friction between latch and projectile

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This file only includes the derivations for while the projectile is in contact with the latch.

**Using**  $\Sigma F = ma$

Latch moves in x direction, projectile in y direction.

For the latch,

$$\hat{x} : F_l + N \sin \theta - \mu_k N \cos \theta = m_l a_l \quad (1)$$

Solving for N

$$N = \frac{m_l a_l - F_l}{\sin \theta - \mu_k \cos \theta} \quad (2)$$

For the projectile,

$$\hat{y} : F_{spr} + \mu_k N \cos \theta - N \sin \theta = ma \quad (3)$$

Solving for N

$$N = \frac{F_{spr} - ma}{\cos \theta + \mu_k \sin \theta} \quad (4)$$

**Putting accelerations  $a$ ,  $a_l$  in terms of  $\theta$**

$$x = r \sin \theta \quad (5)$$

$$a_l = r \cos \theta \ddot{\theta} - r \sin \theta \dot{\theta}^2 \quad (6)$$

$$y = r(1 - \cos \theta) \quad (7)$$

$$a = r \sin \theta \ddot{\theta} - r \cos \theta \dot{\theta}^2 \quad (8)$$

**Eliminating normal force N**

Plugging equations (6), (8) into (2), (4) (respectively), we find

$$N = \frac{m_l(r \cos \theta \ddot{\theta} - r \sin \theta \dot{\theta}^2) - F_l}{\sin \theta - \mu_k \cos \theta} \quad (9)$$

$$N = \frac{F_{spr} - m(r \sin \theta \ddot{\theta} - r \cos \theta \dot{\theta}^2)}{\cos \theta + \mu_k \sin \theta} \quad (10)$$

(1)

Setting (9) and (10) equal to each other,

$$\begin{aligned} & (\cos \theta + \mu_k \sin \theta) (m_l (r \cos \theta \ddot{\theta} - r \sin \theta \dot{\theta}) - F_l) \\ & = (\sin \theta - \mu_k \cos \theta) (F_{spr} - m (r \sin \theta \ddot{\theta} - r \cos \theta \dot{\theta})) \end{aligned} \quad (11)$$

Dividing by  $\cos \theta$ , we can place this into the form  $A + B\dot{\theta} + C\ddot{\theta}$

$$\begin{aligned} & (\tan \theta - \mu_k) F_{spr} + F_l (1 + \mu_k \tan \theta) \\ & + [\mu_k m r \cos \theta - m r \sin \theta + m_l r \sin \theta + \mu_k m_l r \sin \theta \tan \theta] \dot{\theta} \\ & + [\mu_k m r \sin \theta - m r \sin \theta \tan \theta - m_l r \cos \theta - \mu_k m_l r \sin \theta] \ddot{\theta} = 0 \end{aligned} \quad (12)$$

### Substitutions to put eqn (12) in terms of $y$

Using a rounded latch,

$$y = r - \sqrt{r^2 - x_l^2} \quad (13)$$

$$y' = \frac{dy}{dx_l} = \frac{x_l}{\sqrt{r^2 - x_l^2}} \quad (14)$$

Since  $x_l = r \sin \theta$ ,

$$\begin{aligned} y' &= \frac{r \sin \theta}{\sqrt{r^2 (1 - \sin^2 \theta)}} = \frac{r \sin \theta}{r \cos \theta} \\ y' &= \tan \theta \end{aligned} \quad (15)$$

which matches our expectation that  $\frac{dy}{dx_l} = \tan \theta$ .

$$\begin{aligned}
y'' &= \frac{d}{dx_l} \left( x_l (r^2 - x_l^2)^{-1/2} \right) \\
&= (r^2 - x_l^2)^{-1/2} + x_l \left( \frac{-1}{2} (r^2 - x_l^2)^{-3/2} (-2x_l) \right) \\
&= \frac{1}{r \sqrt{1 - \sin^2 \theta}} + \frac{x_l^2}{(r^2 - x_l^2)^{3/2}} \\
&= \frac{1}{r \cos \theta} + \frac{1}{r} \frac{r^2 \sin^2 \theta}{r^2 (1 - \sin^2 \theta)^{3/2}} \\
&= \frac{1}{r \cos \theta} + \frac{1}{r} \frac{\sin^2 \theta}{\cos^3 \theta} \\
&= \frac{1}{r \cos \theta} (1 + \tan^2 \theta) \\
y'' &= \frac{1}{r} \sec^3 \theta
\end{aligned} \tag{16}$$

Since  $y = r - r \cos \theta$ ,  $\frac{dy}{d\theta} = r \sin \theta$  and

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{d\theta} \frac{d\theta}{dt} = r \sin \theta \dot{\theta} \tag{17}$$

Similarly,

$$\begin{aligned}
\ddot{y} &= \frac{d}{dt} \left( r \sin \theta \frac{d\theta}{dt} \right) \\
\ddot{y} &= r \cos \theta \dot{\theta}^2 + r \sin \theta \ddot{\theta}
\end{aligned} \tag{18}$$

**Compare to diff. eqn. w/r/t  $y$**

Prof. Ilton's derivation:

$$\begin{aligned}
&\left[ F_{spr}(\mu_k - y_l') - F_l(\mu_k y_l' + 1) \right] - \left[ m_l(1 + \mu_k y_l') \frac{y_l''}{y_l'^3} \right] \dot{y} \\
&+ \left[ m(y_l' - \mu_k) + \frac{m_l}{y_l''} (1 + \mu_k y_l') \right] \ddot{y} \\
&= A + B\dot{\theta} + C\ddot{\theta} = 0
\end{aligned} \tag{19}$$

Starting with the A component of eqn (19) and making the substitutions from above,

$$\begin{aligned} F_{spr}(\mu_k - y'_l) - F_l(\mu_k y'_l + 1) \\ = F_{spr}(\mu_k - \tan \theta) - F_l(\mu_k \tan \theta + 1) \end{aligned}$$

which matches the A component of eqn (12), off by a factor of  $-1$ .