Friction between latch and projectile
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This file only includes the derivations for while the projectile is in contact with the latch.

Using $\Sigma F = ma$

Latch moves in x direction, projectile in y direction.

For the latch,

$$\hat{x} : F_l + N \sin\theta - \mu_k N \cos\theta = m_l a_l \tag{1}$$

Solving for N

$$N = \frac{m_l a_l - F_l}{\sin\theta - \mu_k \cos theta} \tag{2}$$

For the projectile,

$$\hat{y} : F_{spr} + \mu_k N \cos\theta - N \cos\theta = ma \tag{3}$$

Solving for N

$$N = \frac{F_{spr} - ma}{\cos\theta + \mu_k \sin\theta} \tag{4}$$

Putting accelerations a, a_l in terms of θ

$$x = r \sin\theta \tag{5}$$

$$a_l = r \cos\theta \; \ddot{\theta} - r \sin\theta \; \dot{\theta}^2 \tag{6}$$

$$y = r(1 - \cos\theta) \tag{7}$$

$$a = r \sin\theta \; \ddot{\theta} + r \cos\theta \; \dot{\theta}^2 \tag{8}$$

Eliminating normal force N

Plugging equations (6), (8) into (2), (4) (respectively), we find

$$N = \frac{m_l (r \cos\theta \; \ddot{\theta} - r \sin\theta \; \dot{\theta}^2) - F_l}{\sin\theta - \mu_k \cos\theta} \tag{9}$$

$$N = \frac{F_{spr} - m(r \sin\theta \; \ddot{\theta} + r \cos\theta \; \dot{\theta}^2)}{\cos\theta + \mu_k \sin\theta} \tag{10}$$

$$\tag{1}$$

Note that $N \geq 0$ while the projectile is latched.

Setting (9) and (10) equal to each other,

$$\begin{aligned}(\cos\theta + \mu_k \sin\theta)\big(m_l(r\cos\theta\ \ddot{\theta} - r\sin\theta\ \dot{\theta}^2) - F_l\big) \\ = (\sin\theta - \mu_k \cos\theta)\big(F_{spr} - m(r\sin\theta\ \ddot{\theta} + r\cos\theta\ \dot{\theta}^2)\big)\end{aligned} \tag{11}$$

Dividing by $\cos\theta$, we can place this into the form $A + B\dot{\theta}^2 + C\ddot{\theta}$

$$\begin{aligned}&(\tan\theta - \mu_k)F_{spr} + F_l(1 + \mu_k \tan\theta) \\ &\quad + [\mu_k m r\cos\theta - m r\sin\theta + m_l r\sin\theta + \mu_k m_l r\sin\theta\tan\theta]\ \dot{\theta}^2 \\ &\quad + [\mu_k m r\sin\theta - m r\sin\theta\tan\theta - m_l r\cos\theta - \mu_k m_l r\sin\theta]\ \ddot{\theta} = 0\end{aligned} \tag{12}$$

Substitutions to put eqn (12) in terms of y

Using a rounded latch,

$$y = r - \sqrt{r^2 - x_l^2} \tag{13}$$

$$y' = \frac{dy}{dx_l} = \frac{x_l}{\sqrt{r^2 - x_l^2}} \tag{14}$$

Since $x_l = r\sin\theta$,

$$y' = \frac{r\sin\theta}{\sqrt{r^2(1 - \sin^2\theta)}} = \frac{r\sin\theta}{r\cos\theta}$$

$$y' = \tan\theta \tag{15}$$

which matches our expectation that $\frac{dy}{dx_l} = \tan\theta$.

$$
\begin{aligned}
y'' &= \frac{d}{dx_l}\left(x_l(r^2 - x_l^2)^{-1/2}\right) \\
&= (r^2 - x_l^2)^{-1/2} + x_l\left(\frac{-1}{2}(r^2 - x_l^2)^{-3/2}(-2x_l)\right) \\
&= \frac{1}{r\sqrt{1-\sin^2\theta}} + \frac{x_l^2}{(r^2 - x_l^2)^{3/2}} \\
&= \frac{1}{r\cos\theta} + \frac{1}{r}\ \frac{r^2\sin^2\theta}{r^2(1-\sin^2\theta)^{3/2}} \\
&= \frac{1}{r\cos\theta} + \frac{1}{r}\ \frac{\sin^2\theta}{\cos^3\theta} \\
&= \frac{1}{r\cos\theta}(1+\tan^2\theta) \\
y'' &= \frac{1}{r}\sec^3\theta
\end{aligned} \tag{16}
$$

Since $y = r - r\cos\theta$, $\frac{dy}{d\theta} = r\sin\theta$ and

$$
\dot{y} = \frac{dy}{dt} = \frac{dy}{d\theta}\frac{d\theta}{dt} = r\sin\theta\ \dot{\theta} \tag{17}
$$

Similarly,

$$
\begin{aligned}
\ddot{y} &= \frac{d}{dt}\left(r\sin\theta\frac{d\theta}{dt}\right) \\
\ddot{y} &= r\cos\theta\ \dot{\theta}^2 + r\sin\theta\ \ddot{\theta}
\end{aligned} \tag{18}
$$

Compare to diff. eqn. w/r/t y

Prof. Ilton's derivation:

$$
\begin{aligned}
&\Big[F_{spr}(\mu_k - y_l') - F_l(\mu_k y_l' + 1)\Big] - \left[m_l(1+\mu_k y_l')\frac{y_l''}{y_l'^3}\right]\dot{y}^2 \\
&\quad + \left[m(y_l' - \mu_k) + \frac{m_l}{y_l'}(1+\mu_k y_l')\right]\ddot{y} \\
&\quad = \bar{A} + \bar{B}\dot{y}^2 + \bar{C}\ddot{y} = 0
\end{aligned} \tag{19}
$$

Starting with the $\bar{A}$ component of eqn (19) and making the substitutions from above,

$$\begin{aligned}
\bar{A} &= F_{spr}(\mu_k - y_l') - F_l(\mu_k y_l' + 1) \\
&= F_{spr}(\mu_k - \tan\theta) - F_l(\mu_k \tan\theta + 1) \\
&= -A
\end{aligned}$$

which matches the A component of eqn (12), off by a factor of -1.

Repeating for the $\bar{B}$ and $\bar{C}$ terms (considering both together due to the substitution $\ddot{y}$ containing both $\dot{\theta}$ and $\ddot{\theta}$),

$$\begin{aligned}
\bar{B}\dot{y}^2 + \bar{C}\ddot{y} =& - \left[m_l(1+\mu_k y_l')\frac{y_l''}{y_l'^3}\right]\dot{y}^2 + \left[m(y_l' - \mu_k) + \frac{m_l}{y_l'}(1+\mu_k y_l')\right]\ddot{y} \\
=& - \left[\frac{m_l}{r\cos^3\theta\tan^3\theta} + \frac{\mu_k m_l}{r\cos^3\theta\tan^2\theta}\right] r^2\sin^2\theta\ \dot{\theta}^2 \\
&+ \left[m\tan\theta + \mu_k(m_l - m) + \frac{m_l}{\tan\theta}\right]\left[r\cos\theta\ \dot{\theta}^2 + r\sin\theta\ \ddot{\theta}\right] \\
=& - \left[\frac{m_l r}{\sin\theta} + \frac{\mu_k m_l r}{\cos\theta}\right]\dot{\theta}^2 + \left[mr\sin\theta + \mu_k(m_l - m)r\cos\theta + \frac{m_l r\cos^2\theta}{\sin\theta}\right]\dot{\theta}^2 \\
&+ \left[\frac{mr\sin^2\theta}{\cos\theta} + \mu_k(m_l - m)r\sin\theta + m_l r\cos\theta\right]\ddot{\theta} \\
=& \left[\frac{-m_l r(1-\cos^2\theta)}{\sin\theta} + \frac{-\mu_k m_l r(1-\cos^2\theta)}{\cos\theta} + mr\sin\theta - \mu_k mr\cos\theta\right]\dot{\theta}^2 \\
&+ \left[mr\sin\theta\tan\theta - \mu_k mr\sin\theta + \mu_k m_l r\sin\theta + m_l r\cos\theta\right]\ddot{\theta} \\
=& \left[-m_l r\sin\theta - \mu_k m_l r\sin\theta\tan\theta + mr\sin\theta - \mu_k mr\cos\theta\right]\dot{\theta}^2 \\
&+ \left[mr\sin\theta\tan\theta - \mu_k mr\sin\theta + \mu_k m_l r\sin\theta + m_l r\cos\theta\right]\ddot{\theta} \\
=& - B\dot{\theta}^2 - C\ddot{\theta}
\end{aligned}$$

Therefore, $\bar{A}+\bar{B}\dot{y}^2+\bar{C}\ddot{y} = -(A+B\dot{\theta}^2+C\ddot{\theta})$, and eqns (12) and (19) are equivalent.