Part I

Functions and limits

1 Continuity

12/01/2022

A function is continuous if $\lim_{x \to a} f(x) = f(a)$
(It can be written as $\lim_{x \to a} f(x) = f(\lim_{x \to a} x)$)

List of continuous functions:

- polynomials (everywhere)
- rational functions
- exponential functions (everywhere)
- logarithms
- trigonometric functions ($sinx$, $cosx$, etc)
- the inverse tangent functions

Any function that could be written as a combination of above continuous functions using arithmetic operations is continuous

14/01/2022

№ 1

- $\lim\limits_{x\to 1}(4x^2 - 7x + 5) = 2$
- $\lim\limits_{x\to 0} e^x(3 + x^2 + sinx) = 3$
- $\lim\limits_{t\to 2} \sqrt{\frac{2t^2+1}{3t-2}} = \frac{3}{2}$

2 Piecewise functions

№ 2

$$f(x) = \begin{cases} x^2 + 3, \quad x < -2 \\ 5, \quad x = -2 \\ -3x + 1, \quad -2 < x < 1 \\ 4 - x^2, \quad x \geq 1 \end{cases}$$

$f(2) = 4 - 2^2 = 0$

$a = -4, \quad a = -2, \quad a = 1$ is f(x) continuous at x=1?

1. $a = -4$

$$\left.\begin{array}{l} f(-4) = 19 \\ \lim\limits_{x\to -4^-} f(x) = \lim\limits_{x\to -4^-} (x^2 + 3) = 19 \\ \lim\limits_{x\to -4^+} f(x) = \lim\limits_{x\to -4^+} (x^2 + 3) = 19 \end{array}\right\} \implies \lim\limits_{x\to -4} f(x) = 19$$

2. $a = -2$

$$\left.\begin{array}{l} f(-2) = 5 \\ \lim\limits_{x\to-2^-} f(x) = \lim\limits_{x\to-2^-} (x^2+3) = 7 \\ \lim\limits_{x\to-2^+} f(x) = \lim\limits_{x\to-2^+} (-3x+1) = 7 \end{array}\right\} \implies \lim_{x\to-2} f(x) = 7$$

3. $a = 1$

$$\left.\begin{array}{l} f(1) = 3 \\ \lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^-} (1-3x) = -2 \\ \lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+} (4-x^2) = 3 \end{array}\right\} \implies \lim_{x\to1} f(x) \text{ DNE}$$

18/01/2022

№ 1

Suppose that $g(x) = \begin{cases} sin(\pi x) + c & \text{if } x \leq 2 \\ x^2 - c & \text{if } x > 2 \end{cases}$

For which value of c is g(x) a continuous function?

g(x) is continuous at x=2 $\leftrightarrow$

$$\leftrightarrow \begin{cases} \lim\limits_{x\to2} g(x) = g(2) = sin(\pi\cdot 2) + c = c \\ \lim\limits_{x\to2^-} g(x) = \lim\limits_{x\to2^-} (sin(\pi x) + c) = c \\ \lim\limits_{x\to2^+} g(x) = \lim\limits_{x\to2^+} (x^2 - c) = 4 - c \end{cases} \leftrightarrow 4 - c = c \leftrightarrow$$

$\underline{c = 2}$

№ 2

Prove that there's a real number x in $[1, 3]$ such that
$x^2 + sin(\pi x) = 5$
(How would you approximate this number?)

Solution: Check two things:

1. is $f(x)$ continuous at all points between a and b? YES
2. is c between $f(a)$ and $f(b)$? YES -
   $f(1) = 1 < 5 < f(3) = 9$

By the *Intermediate value theorem*, there is x $\in$ $[1, 3]$ such that
$f(x) = 5$

How could I approximate a solution to f(x)=5?
Bisection method

№ 3

$$\lim_{x\to 2} \frac{x^2+x-6}{x^2-3x+2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+3)}{\cancel{(x-2)}(x-1)} = -5$$

№ 4

$$\lim_{x\to 16} \frac{16-x}{\sqrt{x}-4} = \lim_{x\to 16} -\frac{\cancel{(16-x)}(\sqrt{x}+4)}{\cancel{16-x}} = -8$$

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3 Derivatives

3.1 Exponential functions

$\frac{d}{dx}\left[a^x\right] = \lim\limits_{h\to 0}\frac{a^{x+h}-a^x}{h} = \lim\limits_{h\to 0}\frac{a^h-1}{h}\cdot a^x = C(a)\cdot a^x,\quad$ C(a) - constant
$C(a) = 1$ when $h \approx (1+h)^{\frac{1}{h}} \leftrightarrow a = e$

3.2 Trigonometric functions

$\frac{d}{dx}\left[sin(x)\right] = \lim\limits_{h\to 0}\frac{sin(x+h)-sin(x)}{h} = \lim\limits_{h\to}\frac{sin(x)cos(h)+cos(x)sin(h)-sin(x)}{h} =$
$= \lim\limits_{x\to 0}\left(cos(x)\cdot\frac{sin(h)}{h} + +sinx()\cdot\frac{cos(h)-1}{h}\right) = cos(x)\cdot\lim\limits_{h\to 0}\frac{sinh}{h} + sin(x)\cdot\lim\limits_{h\to 0}\frac{cos(h)-1}{h} = cos(x)$
$\frac{d}{dx}\left[cos(x)\right] = -sin(x)$

3.3 Transformation rules

- $\frac{d}{dx}\left[c\cdot f(x)\right] = c\cdot\frac{d}{dx}\left[f(x)\right]$
- $\frac{d}{dx}\left[f(x)+g(x)\right] = \frac{d}{dx}\left[f(x)\right] + \frac{d}{dx}\left[g(x)\right]$

№ 1

a. $\frac{d}{dx}\left[3x^2 - 2cos(x)\right] = 6x + 2sin(x)$

b. $\frac{d}{dx}\left[\frac{(x-3)^2}{x}\right] = \frac{d}{dx}\left[x - 6 + \frac{9}{x}\right] = 1 - \frac{9}{x^2}$

c. $\frac{d}{dx}\left[\frac{8e^{100}+\pi^{\pi}+7}{\sqrt{cos(\pi^8)-\sqrt{e}}}\right] = 0$