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NumO-Übungsblatt 5

Aufgabe 1:

a) $f'(x) = 3x^2 - 2x$ $\quad f''(x) = 6x - 4$ $\quad f''(0) = 2$ ↯ kein nat. Spline

b) $f(x) = x^3 - 6x^2 - (x^3 - 6x^2 + 12x - 8)$
$= -12x + 8$

$f'(x) = -12$ $\quad f''(x) = 0$

Natürlicher Spline, da $f''(0) = f''(2) = 0$

c) $f_{0,1} = -\frac{x^3}{2}$ $\quad f_{1,2} = (x-1)^3 - \frac{x^3}{2}$

$f'_{0,1} = -\frac{3}{2}x^2$ $\quad f'_{1,2} = 3x^2 - 6x + 3 - \frac{3}{2}x^2$

$f''_{0,1} = -3x$ $\quad f''_{1,2} = 3x - 6$

$f''(2) = 0$ $\quad f''(0) = 0$

Natürlicher Spline, da $f''(2) = f''(0) = 0$

Aufgabe 2:

Zentraler Differentialquotient $\dfrac{f(x+h)-f(x-h)}{2h}$ $\quad x=0{,}6$

$h_0=0{,}08$: $\dfrac{f(0{,}68)-f(0{,}52)}{0{,}16} \approx 0{,}637333$

$h_1=0{,}04$ $\dfrac{f(0{,}64)-f(0{,}56)}{0{,}08} \approx 0{,}636824$

h_i	$a(h_i)=a_{i,0}$	$a_{i,k}$
0,08	0,637333	
0,04	0,636824	

$$a_{1,1} = a_{1,0} + \frac{a_{1,0}-a_{0,0}}{\left(\frac{h_0}{h_1}\right)^2-1} = 0{,}636824 + \frac{0{,}636824-0{,}637333}{\left(\frac{0{,}08}{0{,}04}\right)^2-1}$$

$$= 0{,}636824 + \frac{0{,}636824-0{,}637333}{3}$$

$$\approx 0{,}636654$$ (Taschenrechner rundet...)

Aufgabe 3:

$$x_0 = \frac{0 \cdot \pi}{2} = 0 \qquad f(x_0) = \pi$$

$$x_1 = \frac{1 \cdot \pi}{2} = \frac{1}{2}\pi \qquad f(x_1) = \frac{1}{2}\pi$$

$$x_2 = \frac{2\pi}{2} = \pi \qquad f(x_2) = \pi$$

$$x_3 = \frac{3\pi}{2} = \frac{3}{2}\pi \qquad f(x_3) = \frac{3}{2}\pi$$

$$x_4 = \frac{4\pi}{2} = 2\pi \qquad f(x_4) = \pi$$

Koeffizienten:

$$a_0 = \frac{2}{5}\sum_{j=0}^{4} y_j \cos(0) = \frac{2}{5}\left(\pi + \frac{1}{2}\pi + \pi + \frac{3}{2}\pi + \pi\right)$$

$$= \frac{2}{5}(5\pi)$$

$$= 2\pi$$

$$a_1 = \frac{2}{5}\sum_{j=0}^{4} y_j \cos(j\tfrac{1}{2}\pi) = \frac{2}{5}\left(\pi + \frac{1}{2}\pi\cos(\tfrac{1}{2}\pi) + \pi\cos(\pi) + \frac{3}{2}\pi\cos(\tfrac{3}{2}\pi) + \pi\cos(2\pi)\right)$$

$$= \frac{2}{5}(\pi + 0 - \pi + 0 + \pi)$$

$$= \frac{2}{5}\pi$$

$$a_2 = \frac{2}{5}\sum_{j=0}^{4} y_j \cos(j\pi) = \frac{2}{5}\left(\pi + \frac{1}{2}\pi\cos(\pi) + \pi\cos(2\pi) + \frac{3}{2}\pi\cos(3\pi) + \pi\cos(4\pi)\right)$$

$$= \frac{2}{5}\left(\pi - \frac{1}{2}\pi + \pi - \frac{3}{2}\pi + \pi\right)$$

$$= \frac{2}{5}(3\pi - 2\pi)$$

$$= \frac{2}{5}\pi$$

$$b_1 = \frac{2}{5}\sum_{j=0}^{4} y_j \sin(j\tfrac{1}{2}\pi) = \frac{2}{5}\left(0 + \tfrac{1}{2}\pi\sin(\tfrac{1}{2}\pi) + \pi\sin(\pi) + \tfrac{3}{2}\pi\sin(\tfrac{3}{2}\pi) + \pi\sin(2\pi)\right)$$

$$= \frac{2}{5}\left(0 + \tfrac{1}{2}\pi + 0 - \tfrac{3}{2}\pi + 0\right)$$

$$= -\frac{2}{5}\pi$$

$$b_2 = \frac{2}{5}\sum_{j=0}^{4} y_j \sin(j\pi) = \frac{2}{5}\left(0 + \tfrac{1}{2}\pi\sin(\pi) + \pi\sin(2\pi) + \tfrac{3}{2}\pi\sin(3\pi) + \pi\sin(4\pi)\right)$$

$$= \frac{2}{5}\cdot 0$$

$$= 0$$

<u>Interpolationspolynom:</u>

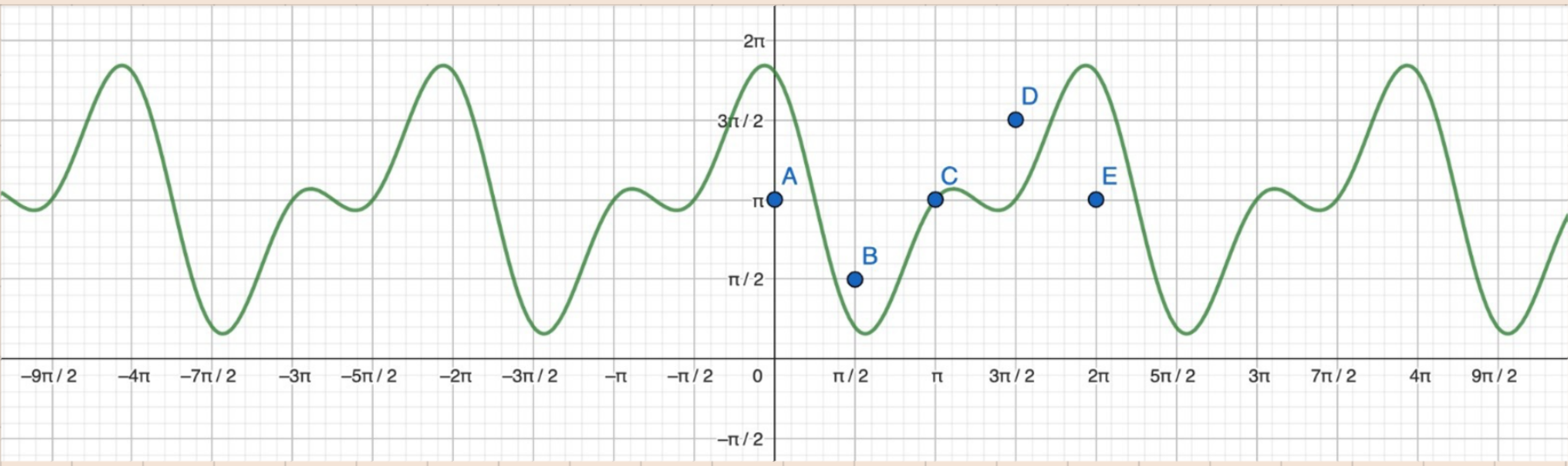
$\theta = 0$ und $m = \frac{1}{2}4 = 2$ da $n = 4$ gerade.

$$t_4(x) = \frac{1}{2}a_0 + \sum_{k=1}^{2}(a_k\cos(kx) + b_k\sin(kx))$$

$$= \pi + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x)$$

$$= \pi + \frac{2}{5}\pi\cos(x) - \frac{2}{5}\pi\sin(x) + \frac{2}{5}\pi\cos(2x)$$

$$= \pi + \frac{2}{5}\pi(\cos(x) - \sin(x) + \cos(2x))$$



2π
3π / 2
π
π / 2
−π / 2
−9π / 2
−4π
−7π / 2
−3π
−5π / 2
−2π
−3π / 2
−π
−π / 2
0
π / 2
π
3π / 2
2π
5π / 2
3π
7π / 2
4π
9π / 2
A
B
C
D
E

Aufgabe 4:

a) $P_n(x) = \sum_{i=0}^{n} y[x_0,\dots,x_i] N_i(x)$ $\qquad y[x_i] =: f[x_i] = y_i$

$\Rightarrow P_n(x) = \sum_{i=0}^{n} f[x_0,\dots,x_i] N_i(x)$

$$\frac{P_{n+1}(x) - P_n(x)}{\prod_{j=0}^{n}(x-x_j)} = \frac{\sum_{i=0}^{n+1} f[x_0,\dots,x_i] N_i(x) - \sum_{i=0}^{n} f[x_0,\dots,x_i] N_i(x)}{\prod_{j=0}^{n}(x-x_j)}$$

$$\Rightarrow \frac{f[x_0,\dots,x_n,x] N_{n+1}(x)}{\prod_{j=0}^{n}(x-x_j)} = \frac{f[x_0,\dots,x_n,x] \prod_{j=0}^{n}(x-x_j)}{\prod_{j=0}^{n}(x-x_j)}$$

$$= f[x_0,\dots,x_n,x] \qquad \square$$

b) Beweis per Induktion

Induktionsschritt: $n=0$

$f(x) - P_0(x) = f(x) - f(x_0) = f[x_0,x](x-x_0)$ ✓

Der letztere Ausdruck lässt sich ebenfalls als Integral schreiben:

$$(x-x_0) \cdot \int_0^1 f'(x_0 + t(x-x_0))\,dt$$

Induktionsannahme: $\forall n \geq 1$ gelte $f[x_0, \dots, x_n, x] = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}$

Induktionsschritt: $n \to n+1$

$$f(x) - p_{n+1}(x) = f(x) - \sum_{i=0}^{n+1} f[x_0, \dots, x_i] \prod_{j=0}^{i-1} (x - x_j)$$

$$= \underbrace{f(x) - p_n(x)}_{IA} - f[x_0, \dots, x_n] \prod_{j=0}^{n} (x - x_j) \quad \text{(aus der a))}$$

$$= f[x_0, \dots, x_n, x] \prod_{j=0}^{n} (x - x_j) - f[x_0, \dots, x_n] \prod_{j=0}^{n} (x - x_j)$$

$$= \frac{f[x_0, \dots, x_n, x] - f[x_0, \dots, x_n]}{x - x_n} \prod_{j=0}^{n+1} (x - x_j)$$

□