§1 Sequences

Definition 1.1. Limit. $x_n \to x$ if $\forall \epsilon > 0, \ \exists k \in \mathbb{N}$ such that $|x_n - x| < \epsilon. \ \forall n \geq K$.

Example 1.2

$$\lim(\frac{2n}{n+1}) = 2$$

Let $\epsilon > 0$. Compute (for any $n \in \mathbb{N}$)

$$|\frac{2n}{n+1} - 2| = |\frac{2n - 2n - 2}{n+1}| = \frac{2}{n+1} < \frac{2}{n}$$

By A.P, $\exists k \in \mathbb{N}$ such that $K > \frac{2}{\epsilon}$. Then $\forall n \geq K$:

$$|\frac{2n}{n+1} - 2| < \frac{2}{n} \leq \frac{2}{k} < \epsilon$$

Example 1.3

$$\lim \frac{3n+1}{2n+5} = \frac{3}{2}$$

First, for any $n \in \mathbb{N}$, we have that

$$|\frac{3n+1}{2n+5} - \frac{3}{2}| = |\frac{6n + 2 - 6N - 15}{2(2n+5)}| = \frac{13}{4n+10} \leq \frac{10^6}{n}$$

Note: If unsure, use number much bigger i.e. $10^6 > 13$.

Now, for any $\epsilon > 0$, by A.P, $\exists k \in \mathbb{N}$ such that $k > \frac{10^6}{\epsilon}$. Then, $\forall n \geq K$:

$$|\frac{3n+1}{2n+5} - \frac{3}{2|} \leq \frac{10^6}{n} \leq \frac{10^6}{k} < \epsilon$$

Example 1.4

$$\lim \frac{n^2-1}{2n^2+3} = \frac{1}{2}$$

First, $\forall n \in \mathbb{N}$,

$$|\frac{n^2-1}{2n^2+3} - \frac{1}{2}| = |\frac{2n^2-2-2n^2-3}{2(2n^2+3)}| = \frac{5}{4n^2+6} \leq \frac{5}{n^2}$$

$\forall \epsilon > 0$, $\exists k \in \mathbb{N}$ such that $k > \sqrt{\frac{5}{\epsilon}}$

Then, for any $n \geq k$

$$|\frac{n^2-1}{2n^2+3} - \frac{1}{2}| \leq \frac{5}{n^2} \leq \frac{5}{k^2} < \epsilon$$

Example 1.5

$$\lim \frac{\sqrt{n}}{n+1} = 0$$

For any $n \in \mathbb{N}$:

$$|\frac{\sqrt{n}}{n+1} - 0| = \frac{\sqrt{n}}{n+1} \leq \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$$

So, $\forall \epsilon > 0$, let $k \in \mathbb{N}$ be such that $k > \frac{1}{\epsilon^2} \Rightarrow \epsilon^2 > \frac{1}{k} \Rightarrow \epsilon > \frac{1}{\sqrt{k}}$ Then for any $n \geq k$,

$$|\frac{\sqrt{n}}{n+1} - 0| \leq \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{k}} < \epsilon$$

Note: $\epsilon > \frac{1}{\sqrt{k}} \Leftrightarrow \epsilon^2 > \frac{1}{k} \Leftrightarrow k > \frac{1}{\epsilon^2}$

Proposition 1.6

If $x_n \to x$, then $|x_n| \to |x|$.

Proof. Let $\epsilon > 0$ be arbitrary. We know that $\exists k \in \mathbb{N}$ such that $|x_n - x| < \epsilon \quad \forall n \geq K$.

$$||x_n| - |x|| \leq |x_n - x| < \epsilon \quad \forall n \geq k$$

□

Side proof

Proof.

$$|x_n| = |x_n - x + x| \leq |x_n - x| + |x|$$
$$\Rightarrow |x_n| - |x| \leq |x_n - x|$$
$$\dots$$

□

Proposition 1.7

If $|x_n| \to 0$, then $x_n \to 0$.

Proof. Let $\epsilon > 0$. Then $\exists k \in \mathbb{N}$ such that

$$|x_n - 0| = |x_n| = ||x_n| - 0| < \epsilon \quad \forall n \geq k$$

□

Exercise 1.8. Show that if $a > 1$, then $\frac{1}{a^n} \to 0$.

Proof. If $a > 1$, then $a = 1 + r$ where $r > 0$.

$$a^n = (1+r)^n \geq 1 + rn \text{ Bernoulli}$$
$$\Rightarrow |\frac{1}{a^n} - 0| = \frac{1}{a^n} \leq \frac{1}{1+rn} \leq \frac{1}{rn}$$

For any $\epsilon > 0$, we can pick $K \in \mathbb{N}$ such that $K > \frac{1}{r\epsilon}$. Then $\forall n \geq k$

$$|\frac{1}{a^n} - 0| \leq \frac{1}{rn} \leq \frac{1}{rK} < \epsilon$$

□

Exercise 1.9. Show that if $a \in (-1, 1)$, then $a^n \to 0$.

Proof. First, if $a = 0$, we are done.

If $a > 0$, pick $b = \frac{1}{a}$. $a^n = \frac{1}{b^n} \to 0$.

If $a < 0$, then $0 < |a| < 1 \Rightarrow |a|^n \to 0 \Rightarrow |a^n| \to 0 \Rightarrow a^n \to 0$ □

Note 1.10.

$$\lim_{m\to\infty} \lim_{n\to\infty} a_{n,m} \neq \lim_{n\to\infty} \lim_{m\to\infty} a_{n,m}$$

Definition 1.11. Another definition of limit: We have $x_n \to x$ if and only if for any open set $x \in U$, $\forall \epsilon > 0$, $\exists K \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq K$.

($\Rightarrow$) First, suppose $x_n \to x$. Let $U \ni x$ where U is open. We know that $\exists \epsilon > 0$ such that $V_\epsilon(x) \subseteq U$. This means that $y \in \mathbb{R}$ such that $|x - y| < \epsilon \Rightarrow y \in U$.

$\exists K \in \mathbb{N}$ such that $|x_n - x| < \epsilon \quad \forall n \geq K$. So, if $n \geq K$, then $|x_n - x| < \epsilon \Rightarrow x_n \in V_\epsilon(x) \subseteq U$

($\Leftarrow$) Fix $\epsilon > 0$. We know that $V_\epsilon(x)$ is open. So, $\exists K \in \mathbb{N}$ such that $x_n \in V_\epsilon(x) \forall n \geq K \Rightarrow |x_n - x| < \epsilon \quad \forall n \geq K$

Proposition 1.12

Let x_n be a positive sequence. If lim...