§1 10-28

Theorem 1.1
Let (x_n) be a convergent sequence, then every subsequence of (x_n) also converges to the same limit. i.e. $\lim(x_{n_k}) = \lim(x_n)$.

Lemma 1.2
If $n_1 < n_2 < n_3 < \dots$ where $n_k \in \mathbb{N}$ for all k, then $n_k \geq k$ for all $k \in \mathbb{N}$.

Proof. By induction.

$k = 1$: Base case where $n_k \geq k$.

$k \to k+1$: Assume that $n_k \geq k$. Then

$$n_{k+1} > n_k \geq k \Rightarrow n_{k+1} > k \Rightarrow n_{k+1} \geq k+1$$

Thus $n_k \geq k$ for all $k \in \mathbb{N}$. □

Proof. Let $x := \lim(x_n)$. Let $\epsilon > 0$, then $\exists N \in \mathbb{N} \quad \forall n \geq N : |x_n - x| < \epsilon$.

Since $n_k \geq k$, by the lemma, we also have that $|x_{n_k} - x| < \epsilon$ for all $k \geq N$, since $n_k \geq k \geq N$.

Thus (x_{n_k}) converges to x. □

§1.1 Criterion for the divergence of sequences

Theorem 1.3 (1)
Let (x_n) be a sequence such that (x_n) has a subsequence (x_{n_k}) that diverges.

Proof. If (x_n) were convergent, (x_{n_k}) would converge, but it doesn't. Thus (x_n) diverges. □

Theorem 1.4
Let (x_n) be a sequence such that there exists two subsequences (x_{n_k}) and (x_{n_j}) that converge to different limits, then (x_n) diverges.

Proof. If (x_n) was convergent to x_1, then (x_{n_k}) and (x_{n_j}) would converge to x_1; but they don't. Thus (x_n) diverges. □

Example 1.5

$x_n = (-1)^n$. Consider the subsequences of the even and odd terms (x_{2n}) and (x_{2n-1}).

$x_{2n} = (-1)^{2n} = 1^{2n} = 1$. i.e. (x_{2n}) is a constant sequence and $\lim(x_{2n}) = 1$.

Similarly, $x_{2n-1} = (-1)(-1)^{2n} = -1$. i.e. (x_{2n-1}) is a constant sequence and $\lim(x_{2n-1}) = -1$.

According to one of the criterion for the divergence of sequences theorems, (x_n) diverges.

Example 1.6

$x_n : 1, 1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}$. Then $x_{2n-1} : 1, 2, 3, 4, \ldots$. Which diverges, thus (x_n) diverges.

Example 1.7

$x_n = \sqrt[n]{n}$; Prove that (x_n) converges to 1.

1st step: (x_n) is eventually decreasing.

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)^{\frac{1}{n+1}}}{n^{\frac{1}{n}}}$$
$$\Rightarrow (\frac{x_{n+1}}{x_n})^{n(n+1)} = \frac{1}{n} \cdot \frac{n+1}{n}^n = \frac{1}{n} \cdot (1 + \frac{1}{n})^n \leq \frac{1}{n} \cdot e < \frac{3}{n} \leq 1$$

As long as $n \geq 3$. Thus (x_n) is decreasing for all $n \geq 3$.

Furthermore, (x_n) is bounded from below by 1. Thus (x_n) is bounded and eventually decreasing $\Rightarrow (x_n$ converges by monotone convergence theorem. Let $x := \lim(x_n)$.

Second step: Show that $x = 1$.
Consider the subsequence (x_{2n}) of even terms.

$$x_{2n} = \sqrt[2n]{2n} \Rightarrow x_{2n}^2 = \sqrt[n]{2n} = \sqrt[n]{2} \cdot \sqrt[n]{n} = \sqrt[n]{2} \cdot x_n$$

Thus

$$\lim(x_{2n}^2) = \lim(\sqrt[n]{2} \cdot x_n) = \underbrace{\lim(\sqrt[n]{2})}_{=1} \cdot \lim(x_n)$$
$$\lim(x_{2n}^2) = (\lim(x_{2n}))^2$$
$$\Rightarrow x^2 = x \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0$$
$$\Rightarrow x = 0 \lor x = 1. \text{ but } x_n \geq 1 \quad \forall n \in \mathbb{N}$$
$$\Rightarrow x = 1$$

Theorem 1.8 (Bolzano - Weirstrass)
Let (x_n) be a bounded sequence. Then (x_n) has a convergent subsequence.

Proof. Since (x_n) is bounded, $\exists \mu > 0$ such that $x_n \in \underbrace{[-M, M]}_{=I_1}$ for all $n \in \mathbb{N}$.

Divide I_1 into two subintervals of equal width. At least one of these subintervals contains infinitely many terms of (x_n). Choose this one of these intervals and call it I_2.

Divide I_2 into 2 subintervals of equal width. At least one of them, called I_3 contains infinitely many terms of (x_n). Etc...

We obtain an infinite sequence $\mathrm{I}_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ of closed and bounded intervals. By the nested interval property of $\mathbb{R}$ we know that the intersection over all of these intervals is not empty. i.e. $\cap_{n\in\mathbb{N}} I_n \neq \varnothing$.

Let $x \in \cap_{n\in\mathbb{N}} I_n$. We will now show that there exists a subsequence (x_{n_k}) of (x_n) with $\lim(x_{n_k}) = x$.

Let $n_1 \in \mathbb{N}$ be arbitrary. We know that $x_{n_1} \in I_1$ because all elements are in I_1. I_2 contains infinitely many terms of (x_n). Thus there exists $n_2 > n_1$ such that $x_{n_2} \in I_2$. The same goes for I_3 ; etc...

We obtain $n_1 < n_2 < n_3 < \ldots$ such that $x_{n_k} \in I_k$ for all $k \in \mathbb{N}$.

We also have that $x \in I_k$ for all $k \in \mathbb{N}$. This gives that $|x_{n_k} - x| \leq |I_k|$ where $|I_1| = 2M, \quad |I_2| = M, \quad |I_3| = \frac{M}{2}, \quad \ldots.$

$$\Rightarrow |I_k| = \frac{2M}{2^{k-1}} = \frac{4M}{2^k} \Rightarrow |x_{n_k} - x| \leq 4M \cdot (\frac{1}{2})^k$$

for all $k \in \mathbb{N}$. By convergence criterion, $\lim(x_{n_k}) = x$; especially, (x_{n_k}) converges. Corner stone of the proof is the nested interval property of $\mathbb{R}$. $\square$

Definition 1.9. Let (x_n) be a sequence and let (x_{n_k}) be a convergent subsequence. Let $x := \lim(x_{n_k})$. Then x is called an accumulation point or a subsequential limit (point) of (x_n).

Example 1.10
$x_n = (-1)^n$. The accumulation points of (x_n) are $+1$ and -1.

Example 1.11
Let x_n be an enumeration of Q. Every real number is an accumulation point because Q is dense in $\mathbb{R}$.

Theorem 1.12

Let (x_n) be a sequence. $x \in \mathbb{R}$ is an accumulation point of (x_n) iff $\forall \epsilon > 0 : V_\epsilon(x)$ contains infinitely many terms of (x_n).

Proof.

($\Rightarrow$) Let x be an accumulation point of (x_n). Thus there exists a subsequence (x_{n_k}) of (x_n) with $\lim(x_{n_k}) = x$. Then $\exists k \in \mathbb{N} : \forall k \geq N x_{n_k} \in V_\epsilon(x)$. Thus $V_\epsilon(x)$ contains infinitely many terms of (x_n).

($\Leftarrow$) Let $x \in \mathbb{R}$ be such that $\forall \epsilon > 0 : V_\epsilon(x)$ contains infinitely many terms of (x_n). Let $\epsilon := 1$. Then $V_1(x)$ contains infinitely many terms of (x_n) .Let $n_1 \in \mathbb{N}$ such that $x_{n_1} \in V_1(x)$.

Let $\epsilon := \frac{1}{2}$. Then $V_{\frac{1}{2}}(x)$ contains infinitely many terms of (x_n). Thus $\exists n_l > n_1$ such that $x_{n_2} \in V_{\frac{1}{2}}(x)$.

$\vdots$

$\epsilon = \frac{1}{k}$. Then $V_{\frac{1}{k}}(x)$ contains infinitely many terms of (x_n) thus $\exists n_k > n_{k-1}$ such that $x_{n_k} \in V_{\frac{1}{k}}(x)$

Since $n_1 < n_2 < n_3 < \ldots$, we obtain a subsequence (x_{n_k}) of (x_n) with $x_{n_k} \in V_{\frac{1}{k}}(x)$. Now let $\epsilon > 0$ and let $k > \frac{1}{\epsilon} \Leftrightarrow \frac{1}{k} < \epsilon \Rightarrow x_{n_k}, x_{n_{k+1}}, x_{n_{k+2}}, \dots \in V_{\frac{1}{k}}(x) \subseteq V_\epsilon(x)$.

$$x_{n_k} \in V_\epsilon(x) \quad \forall k \geq K \Rightarrow x_{n_k} \text{ converges to } x$$

$\square$