§1 Lecture 02-19

Vector spaces. Basis. Let V be a vector space over F. Then there exists a basis B of V such that all $v \in V$ can be written uniquely as a sum

$$\sum_{w \in B} \lambda_w w \quad \lambda_w \in F$$

$\lambda_w = 0$ for all but finitely many $w \in B$. Therefore

$$\begin{aligned} V &= F_0(B, F) \\ v &\mapsto (w \mapsto \lambda_w) \\ F_0(B, F) &\subseteq F(B, F) \end{aligned}$$

Exercise 1.1. Show that there are vector spaces that are not isomorphic to $F(X, F)$ for any set X.

Attempt 1:

$$\begin{aligned} V = F[[x]] &= \{a_0 + a_1 x + a_2 x^2 + \cdots\} \\ &= \{(a_0, a_1, a_2, \dots) \quad a_j \in F\} \\ &= \{a : \{0, 1, 2, \dots\} \to F\} \\ &= Func(\mathbb{N}, F) \end{aligned}$$

Attempt 2:

$$F[x] = \{a_0 + a_1 x + \cdots + a_n x^n \quad a_i \in F\}$$

So $F[x]$ has a countable basis.

1. If X is finite, then $\dim(Func(X, F))$ is also finite.
2. If X is infinite, then $Func(X, F)$ does not have a countable basis.

If $F = \mathbb{Q}$, we observe that $Func(X, \mathbb{Q})$ is uncountable.

$$\begin{array}{ll} f_1 & f_1(x_1), f_1(x_2) \dots \\ f_2 & f_2(x_1), f_2(x_2) \dots \\ f_n & f_n(x_1), f_n(x_2) \dots \end{array}$$

Define $f(x_n) = f_n(x_n) + 1$

1. TOH, $\mathbb{Q}[x]$ is countable.
2. $F = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. $Func(X, \mathbb{F}_2) = P(X) = \{A \subseteq X\}$.

   $Func(X, \mathbb{F}_2)$ is either finite, or uncountable. $\mathbb{F}_2[x]$ is countable.

Exercise 1.2. Show that, for any F, that $Func(X, F)$ does not have a countable basis.