§1 Lecture 02-28

An action of G on X is a function

$$G \times X \to X$$
$$(g, x) \mapsto gx$$

satisfying $1_G \cdot x = x$ and $g_1(g_2x) = (g_1g_2)x$.

Equivalently, an action of G on X is a homomorphism

$$\varphi : G \to S_x = perm(X)$$
$$\varphi \rightsquigarrow g \cdot x = \varphi(g)(x)$$
$$\text{Action } G \times X \to X \rightsquigarrow \varphi(g)(x) = gx$$

Terminology: A set X endowed with an action of G is called a G-set.

If X_1 and X_2 are two G-sets, a homomorphism $f : X_1 \to X_2$ is a function satisfying $f(gx_1) = g \cdot f(x_1)$

If X_1 and X_2 are G-sets, so it $X_1 \sqcup X_2$.

Definition 1.1 (Transitive G-set)**.** A G-set X is transitive if it cannot be expressed as a disjoin union of non-empty G-sets. If X is transitive, choose $x_0 \in X$.

$$Gx_0 = \{gx_0,\ g \in G\}$$

is called the orbit of x_0 under actions of G. Then $X = Gx_0,\ \forall x_0 \in X$. More generally,

$$\exists x_i,\ (i \in I)\ X = \sqcup_{i \in I} G_{xi}$$

Example 1.2

$X = G$. $G \times X \to X$ is left multiplication. X is transitive. If $g \in G$, and $gx = x \forall x \in X \Rightarrow g = id$.

$$\varphi : G \hookrightarrow S_G$$

Cayley's theorem: Every G is a subgroup of S_n. So $G = S_n$, $\varphi : G \hookrightarrow S_{S_n} = S_{n!}$

Example 1.3

If H is a subgroup of G, then G/H is a G-set.

$$
\begin{gathered}
(g, aH) \rightsquigarrow gaH \\
\ker(G \to S_{G/H}) = \{g \in G \text{ such that } gaH = aH\} \\
gaH = aH, \ \forall a \in G \\
a^{-1}gaH = H, \ \forall a \in G \\
a^{-1}ga \in H, \ \forall a \in G \\
g \in aHa^{-1}, \ \forall a \in G \\
g \in \cup_{a \in G} aHa^{-1}
\end{gathered}
$$

$\ker(G \to S_{G/H})$ is the largest normal subgroup of G conained in H. In particular, if H contains no non-trivial normal subgroups, then $G \hookrightarrow S_{G/H}$ is injective.

Example 1.4

$$
\begin{aligned}
X = G, \ g * x &= gxg^{-1} \\
1_G * x &= 1x1^{-1} = x \\
(g_1g_2) * x = g_1g_2x(g_1g_2)^{-1} &= g_1(g_2xg_2^{-1})g_1^{-1} = g_1(g_2 * x)g_1^{-1} \\
&= g_1 * (g_2 * x)
\end{aligned}
$$

$$
\begin{gathered}
G = S_3 = \{1, (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2)\} \\
\text{Orbits: } \{1\}, \{(1\,2\,3), (1\,3\,2)\}, \{(1\,2), (1\,3), (2\,3)\}
\end{gathered}
$$

Proposition 1.6

If X is a transitive G-set, then it is isomorphic to G/H for some subgroup H.

Proof. Let $x_0 \in X$. We know that $Gx_0 = X$. Consider the function

$$\begin{aligned} G &\to X \\ g &\mapsto gx_0 \end{aligned}$$

This function is a homomorphism of G-sets. It is surjective, by transitivity.

The map ζ is not injective in general. $\zeta^{-1}(x_0)$ = the preimage of x_0 is

$$Stab_G(x_0) = G_{x_0} = \{g \in G \text{ such that } gx_0 = x_0\}$$

Set $H = G_{x_0}$. We defined $\overline{\zeta} : G/H \to X$ by $\overline{\zeta}(gH) = gx_0$.

Claim: $\overline{\zeta}$ is a bijection of G-sets.

1. $\overline{\zeta}$ is well-defined.

   If $g_1 H = g_2 H$, then $g_2 = g_1 h,\ h \in H$. $g_2 x_0 = (g_1 h) x_0 = g_1 (h x_0) = g_1 x_0$

2. $\overline{\zeta}$ is surjective $\Leftarrow$ transitivity.

3. $\overline{\zeta}$ is injective.

$$\begin{aligned} \overline{\zeta}(g_1 H) = \overline{\zeta}(g_2 H) &\Rightarrow g_1 x_0 = g_2 x_0 \Rightarrow g_2^{-1} g_1 x_0 = x_0 \\ &\Rightarrow g_2^{-1} g_1 \in H \Rightarrow g_1 H = g_2 H \end{aligned}$$

□

Corollary 1.7

If G is finite, then any transitive G-set X is also finite, and

$$\#X = \frac{\#G}{\#Stab_G(x_0)}$$

Orbit stabiliser theorem.

Proof. $X \simeq G/Stab_G(x_0)$ as a G-set. Hence

$$\#X = \#(G/Stab_G(x_0)) = \frac{\#G}{\#Stab_G(x_0)}$$

□