§1 Lecture 11-25

Definition 1.1. Let $A \subseteq \mathbb{R}$ and let $c \coloneqq \{U_i : i \in I\}$, where I is an index set, U_i is open for all $i \in I$.

Then c i scalled an open cover of A if $A \subseteq U_{i \in I} U_i$. i.e. every $x \in A$ is contained.

If $y \subseteq I$ such that $\{U_j : j \in J\} coloneqq \varphi$ is still a cover of A, we say that φ' is a finite subcover of φ.

Example 1.2

Let $A = [0, 1]$ and let $\varphi \coloneqq \{V_{1/2}(x) : x \in [0, 1]\}$.

Then φ is an open cover of $[0, 1]$ because

$$[0,1] \subseteq \cup_{x \in [0,1]} V_{1/2}(x) : x \in [0,1] \subseteq \left]-1/2, 3/2\right[$$

Theorem 1.3 (Heine-Borel)
$A \subseteq \mathbb{R}$ is compact (closed and bounded) if and only if every open cover of A has a finite subcover.

Proof.

- $\Rightarrow$ Special Case: A is a closed and bounded interval $[a, b] := I_0$. Assume that c is an open cover of I_0 that doesn't have a finite subcover. Divide I_0 into two closed subintervals of equal width $[a, c]$ and $[c, b]$ where $c = \frac{a+b}{2}$.

  For at least one of these subintervals, φ does not have a finite subcover. Otherwise, φ would have a finite subcover φ' of $[a, \varphi]$ and φ'' of $[\varphi, b]$. Then $\varphi' \cup \varphi''$ would be a finite open cover of I_0, which doesn't exist.

  Let I_1 be (one of) the subinterval(s) without finite subcover. Divide I_1 into 2 closed subintervals of equal width. At least one of them doesn't have A.

  We obtain a nested sequence $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ of closed and bounded intervals. Then
  $$\cap_{n \in \mathbb{N}_0} I_n \neq \varnothing$$
  by the nested interval property.

  Let $x_0 \in \cap_{n \in \mathbb{N}_0} I_n$. Then $x_0 \in I_0$, thus $\exists i \in I$ such that $x_0 \in U_i$ which is open. Thus, $\exists \epsilon > 0 : V_\epsilon(x_0) \subseteq U_i$.

  Claim: $\exists n \in \mathbb{N}_0 : I_n \subseteq V_\epsilon(x_0)$.

  Proof. $|I_n| = 1/2^n |I_0|$. Let $n \in \mathbb{N}_0$ such that $1/2^n |I_0| < \epsilon$.

  Let $x \in I_n$ be arbitrary. Then $|\underbrace{x}_{\in I_n} - \underbrace{x_0}_{\in I_n}| \leq 1/2^n |I_0| < \epsilon \Rightarrow x \in V_\epsilon(x_0)$.

  $\Rightarrow I_n \subseteq V_\epsilon(x_0)$. Now we have:
  $$I_n \subseteq V_\epsilon(x_0) \subseteq U_i$$
  i.e. $\{U_i\}$ covers I_n

  φ has a finite (of length 1) subcover for I_n. CONTRADICTION.

  $\Rightarrow \varphi$ does have a finite subcover. $\square$

  General Case; $A \subseteq \mathbb{R}$ compact. φ open cover. Since A is bounded, $\exists M > 0$ such that $A \subseteq [-M, M]$. Let $U := \mathbb{R}/A$ which is open.
  Consider $\varphi' := \varphi \cap \{U\}$. Then φ' covers $\mathbb{R}$. Thus φ' covers $[-M, M]$ which is closed and bounded interval by special case.
  By special case, φ' has a finite subcover φ''. φ'' may not be a subcover of φ because φ'' may contain U. However, if φ'' should contain U, we can simply remove it.
  i.e. if $U \in \varphi''$, let $\varphi''' = \varphi''/\{U\}$. If $U \notin \varphi''$, let $\varphi''' := \varphi''$.
  Since $U = \mathbb{R}/A$, φ''' will still cover A. Thus we've obtained a finite subcover of A. $\square$

Theorem 1.4
$A \subseteq \mathbb{R}$ is compact (closed and bounded) if and only if every open cover of A has a finite subcover.

Proof.

$\Leftarrow$ Let A not be compact. We need to find an open cover of A without a finite subcover. A not closed: assignment 12.

A **unbounded**
Let $\varphi := \{U_n : n \in \mathbb{N}\}$ where $U_n :=] - n, n[$. Then φ covers $\mathbb{R}$ and thus A. Consider any finite subset $m\{U_{n_1}, \cdots, U_{n_k}$.

□

Remark 1.5. THe "classical" definition of compacness is closed and bounded, however this definition doesn't generalize will beyond $\mathbb{R}^n$ since there isn't even a notion of boundedness on general "topological spaces" However, open covers still make perfect sense on topological spaces. Thus, the def of compactness was revised to

Definition 1.6 (Modern definition of compactness)**.** A is called compact if every open cover of A has a finite subcover.

"Modern" heine borel becomes:

Definition 1.7. $A \subseteq \mathbb{R}$ is compact if and only if A is closed and bounded.

Applications of heine borel: It can often be useful to generalize "local" properties of functions to "global" properties if the domain is compact.

Definition 1.8. $f : A \to \mathbb{R}$ is called locally bounded if $\forall x_0 \in A$, $\exists \epsilon > 0 : f$ is bounded on the domain $V_\epsilon(x_0)$.

Example 1.9
$f :]0, \infty[\to \mathbb{R}$, $x \to 1/x$.

f is bounded on any neighborhood about x_0 that does not contain 0 is in its boundary. Thus f is locally bounded, but not (globally) bounded!

However, this can't happen if the domain is compact

Theorem 1.10

Let $A \subseteq \mathbb{R}$ be compact. $f : A \to \mathbb{R}$ be locall bounded. Then f is bounded (on A).

Proof. Let $x \in A$ be arbitrary. f locally bounded $\Rightarrow \exists \epsilon_x > 0$ such that f is bounded on interval $V_{\epsilon_x}(x)$.

Then $\varphi \coloneqq \{V_{\epsilon_x} : x \in A$ is an open cover of A. Since A is compact, φ has a finite subcover $\{V_{\epsilon_{x_1}}, \cdots, V_{\epsilon_{x_n}}(x_n)\}$.

On each of these n neighborhoods, f is bounded.

$$\Rightarrow \exists M_1, \cdots, M_n \geq 0$$

such that $|f|(x) \leq M_1, \cdots, |f|(x) \leq M_n$ bounded on $V_{\epsilon_n}(x_n)$.

Let $M \coloneqq \max\{M_1, \cdots, M_n\}$. Then $|f|(x) \leq M, \cdots, |f| \leq M$. $\square$