§1 10-30

Theorem 1.1
A bounded sequence converges if and only if it has exactly one accumulation point.

Proof.

($\Rightarrow$) Let (x_n) be convergent. $x := \lim(x_n)$. Then every subsequence (x_{n_k}) of (x_n) converges to x. Thus x is the only accumulation point of (x_n).

($\Leftarrow$) Let (x_n) be a bounded sequence which has only one accumulation point x. We will show that (x_n) converges to x. Assume that this is not the case.

Convergence: $\forall \epsilon > 0,\ \exists N \in \mathbb{N} : \forall n \geq N,\ |x_n - x| < \epsilon$
Negation: $\exists \epsilon > 0 : \forall N \in \mathbb{N},\ \exists n \geq N : |x_n - x| \geq \epsilon$

Thus $\exists$ infinitely many $n \in \mathbb{N}$ such that $|x_n - x| \geq \epsilon_0$.
Let $n_1 < n_2 < n_3 < \dots$ such that $\forall k \in \mathbb{N} : |x_{n_k} - x| \geq \epsilon_0$.

Consider the subsequence (x_{n_k}) of $(x_n) \Rightarrow (x_{n_k})$ is bounded because (x_n) is bounded.

By Bolzano-weierstrass, (x_{n_k}) has a convergent subsequence $(x_{n_{k_j}})$. Let $\sim x := \lim(x_{n_{k_j}})$. Since it is a subsequence of (x_n) which has only one accumulation point. It follows that $\sim x = x$.

Thus $\lim(it) = x$ and $\forall j \in \mathbb{N}, |it - x| \geq \epsilon_0 CONTRADICTION$
Thus our assumption was wrong which proves that (x_n) converges to x.

□

Theorem 1.2

Let (x_n) be a bounded sequence and let A be the set of all accumulation points of (x_n). Then $A \neq \varnothing$ and A is compact (i.e. A is closed and bounded).

Proof. By BOLZANO-WEIERSTRASS, (x_n) has at least one convergent subsequence. Its limit is an accumulation point of $(x_n) \Rightarrow A \neq \varnothing$.

A is bounded: (x_n) is bounded i.e. $\exists M > 0$ such that $\forall n \in \mathbb{N},\ -M \leq x_n \leq M$.

Let $x \in A$ be arbitrary. Then $\exists$ subsequence (x_{n_k}) of (x_n) with $x = \lim(x_{n_k})$.
We have that $\forall k \in \mathbb{N} : -M \leq x_{n_k} \leq M \Rightarrow -M \leq x \leq M$.
$\Rightarrow x \in [-M, M]$ for all accumulation points x of (x_n).
$\Rightarrow A \subseteq [-M, M] \Rightarrow A$ is bounded.

A is closed: Let $x \in \mathbb{R} \setminus A$ i.e. x is not an accumulation point. Thus $\exists \epsilon > 0 : V_\epsilon(x)$ contains at most finitely many terms of (x_n).
Let $t \in V_\epsilon(x)$. $V_\epsilon(x)$ is open. Thus $\exists \tilde{\epsilon} > 0 : V_{\tilde{\epsilon}(t)} \subseteq V_\epsilon(x)$.
Thus $V_{\tilde{\epsilon}(t)}$ contains at most finitely many terms of (x_n). Thus t is not an accumulation point $\Rightarrow$ no point in $V_\epsilon(x)$ is an accumulation point of (x_n) $\Rightarrow V_\epsilon(x) \subseteq \mathbb{R} \setminus A$.
Thus $\mathbb{R} \setminus A$ is open $\Rightarrow A$ is closed.

□

We've just seen that the set of all accumulation points of a bounded sequence (x_n) is $\neq 0$, closed, and bounded.

Since A is bounded, it has a supremum and an infimum. Both sup and inf are boundary points. A is closed so it contains sup and inf. Therefore $\sup(A)$ is the Maximum of A and $\inf(A)$ is the minimum of A. i.e. $\sup(A)$ is an accumulation point of (x_n), the greatest accumulation point of (x_n). Similarly $\inf(A)$ is the least accumulation point of (x_n).

Definition 1.3.

1. Let (x_n) be a bounded sequence. Then the greatest accumulation point of (x_n) is called the LIMES SUPERIOR of (x_n). In symbols: $\limsup(x_n)$.
2. The least accumulation point of (x_n) is called the LIMES INFERIOR of (x_n). In symbols: $\liminf(x_n)$.

Theorem 1.4

Let (x_n) be a bounded sequence. Then (x_n) is convergent if and only if

$$\liminf(x_n) = \limsup(x_n)$$

Proof.

($\Rightarrow$) Let $x \coloneqq \lim(x_n)$. Then every subsequence (x_{n_k}) of (x_n) converges to x.

$$\Rightarrow A = \{x\} \Rightarrow \liminf(A) = x = \limsup(A)$$

($\Leftarrow$) Assume that $\liminf(x_n) = \limsup(x_n) \coloneqq x$.

$$A = \{x\}$$

i.e. (x_n) has only one accumulation point. By previous theorem, (x_n) converges.

$\square$

Example 1.5

1. $$x_n = (-1)^n$$

   Accumulation points are -1 and $1 \Rightarrow \liminf(x_n) = -1$ and $\limsup = 1$. Especially, $(-1)^n$ diverges because $\liminf \neq \limsup$.

2. Let (x_n) be an enumeration of $\mathbb{Q} \cap [a,b]$ where $a < b$. We'll show that $\liminf = a$ and that $\limsup = b$.

   Proof. Let $x > b$. Let $\epsilon := b - x > 0$. Then $\forall n \in \mathbb{N}$, $x_n \notin V_\epsilon(x) \Rightarrow$ x is <u>not</u> an accumulation point of (x_n).

   Let $x \in [a,b]$ and let $\epsilon > 0$; consider $V_\epsilon(x) =]x - \epsilon, x + \epsilon[$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, $V_\epsilon(x)$ contains infinitely many rational numbers, especially, $V_\epsilon(x_n)$ contains infinitely many terms of $(x_n) \Rightarrow x$ is an accumulation point of (x_n).

   <u>x = a</u>: By density of $\mathbb{Q}$ in $\mathbb{R}$, $]a, a+\epsilon[$ contains infinitely many terms of $(x_n) \Rightarrow a$ is an accumulation point of (x_n). Similarly for $x = b$.

   Therefore $A := [a,b] \Rightarrow \liminf(x_n) = a$ and $\limsup(x_n) = b$. $\square$

3. Find all accumulation points of the following sequence.

   $$x_n : 1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

   Claim: $A = \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

   Proof. For every $k \in \mathbb{N}$, the constant sequence $\frac{1}{n}, \frac{1}{n}, \frac{1}{n}$ is a subsequence of (x_n). Thus
   $$\frac{1}{n} = \lim(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots) \in A$$
   and $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is a subsequence of (x_n). Thus

   $$0 = \lim(1, \frac{1}{2}, \frac{1}{3}, \ldots) \in A$$

   Now let $x > 1$, $\epsilon := x - 1 > 0$. Then $\forall n \in \mathbb{N} : x_n \notin V_\epsilon(x) \Rightarrow x \notin A$.

   Similarly, $x \notin A$ for all $x < 0$. Let $0 < x < 1$; $x \notin A$. Then $\exists n \in \mathbb{N} : \frac{1}{n+1} < x < \frac{1}{n}$.

   Let $\epsilon := \min\{x - \frac{1}{n+1}, \frac{1}{n} - x\} > 0$. Then $\frac{1}{n+1} \notin V_\epsilon(x) \vee \frac{1}{n} \notin V_\epsilon(x)$

   $$\Rightarrow x_n \notin V_\epsilon(x) \ \forall n \in \mathbb{N}$$
   $$x \text{ is } \underline{\text{not}} \text{ an accumulation point of } (x_n)$$

   Thus $A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ $\square$

§1.1 Properties of $\limsup, \liminf$

Theorem 1.6

Let (x_n) be a bounded sequence and let $\epsilon > 0$. Then $\exists N \in \mathbb{N}\ \forall n \geq N : x_n \in]\liminf(x_n), \limsup(x_n) + \epsilon[$. i.e. at most finitely many terms of (x_n) have the property that $x_n > \limsup(x_n) + \epsilon$ or $x_n < \liminf(x_n) - \epsilon$

Proof. assignment 8 □

Theorem 1.7

Let (x_n) be a bounded sequence. Then $\limsup(x_n) = \lim(\sup\{x_k : k \geq n\})$ and $\liminf(x_n) = \lim(\inf\{x_k : k \geq n\})$.

Remark 1.8. It is not clear initially whether this is well defined. We'll prove this.

Let $y_n := \sup\{x_k : k \geq n\}$. Then (y_n) is bounded because (x_n) is bounded.

Let A, B be bounded with $A \subseteq B$. Then $\sup(A) \leq \sup(B)$.

Note 1.9. $\{x_k : k \geq n+1\} \subseteq \{x_k : k \geq n\}$.

Therefore $\sup\{x_k : k \geq n+1\} \leq \sup\{x_k : k \geq n\}$.

Therefore (y_n) is bounded and decreasing and therefore converges.

Thus $\lim(\sup\{x_k : x \geq n\})$ exists. A similar argument applies to $\lim(\inf\{x_k : k \geq n\})$.

Proof. Examination material. This is the cutoff for the midterm exam. Next week coshy sequences. 3.4 in the textbook. Important: This doesn't mean that you don't have to remember the stuff from before. If you don't know stuff from before you will be closed. I used open and closed todayand left it to you to know what open and closed means. It did not contain interior and closure so that is midterm 2 material. And you need to know what boundary sets are in order to make sense of these things but I won't ask a separate question on these things. □