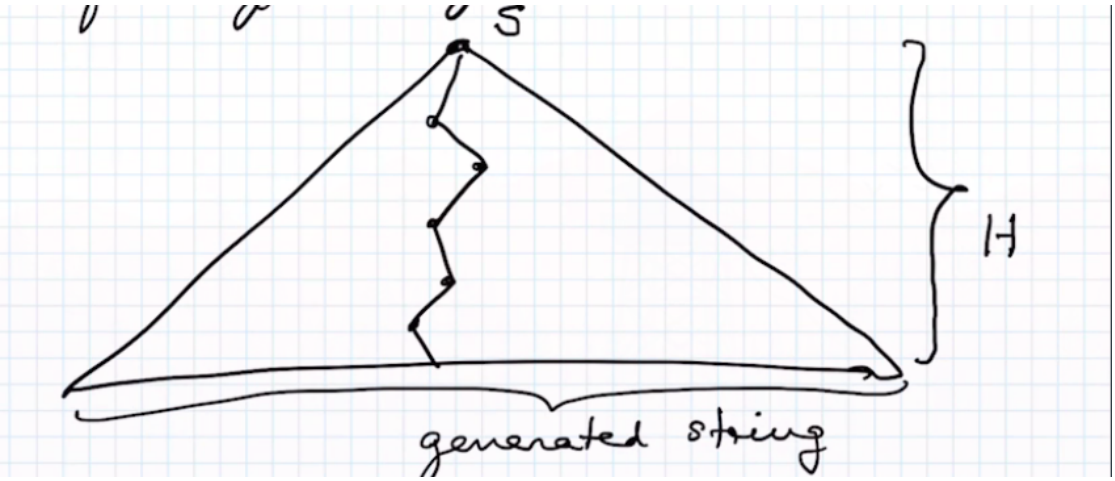
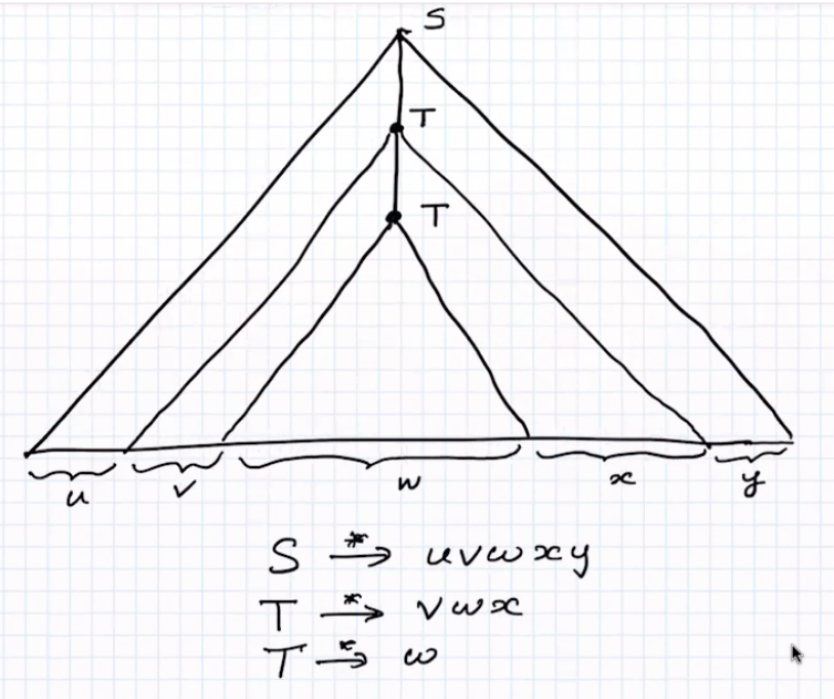
§1 05-20

Proving pumping lemma we used the way that DFA has finitely many states. But the size of the stack can be infinite. Instead we use the fact that a grammar for a CFL has only finitely many variables.



Height of parse tree increases less quickly than the length of the word. For a long enough word, $H >$ # of non terminals in the grammar.

This means that at some point there has to be a repeated non-terminal.



It is not possible that both v and x are empty. This is true if we assume Noam Chomsky form. But then we can change the second T into generating vwx which would mean thta $uv^2wx^2y \in L$. And you can do this again which would mean $uv^3wx^3y \in L$. We conclude

that

$$\forall i \geq 0 \; uv^i wx^i \in L$$

Lemma 1.1 (Pumping Lemma for Context Free Languages)

$$\forall CFL, L, \; \exists p > 0$$
$$\forall s \in L \; |s| \geq p$$
$$\exists u, v, w, x, y \in \Sigma^* \text{ s.t. } s = uvwxy, \; |vx| > 0 \; , |vwx| \leq p$$
$$\forall i \geq 0, \; uv^i wx^i y \in L$$

No longer guaranteeing that is happens early on in the string.

Lemma 1.2 (Contrapositive of Pumping Lemma for Context Free Languages)
Fix some language L.

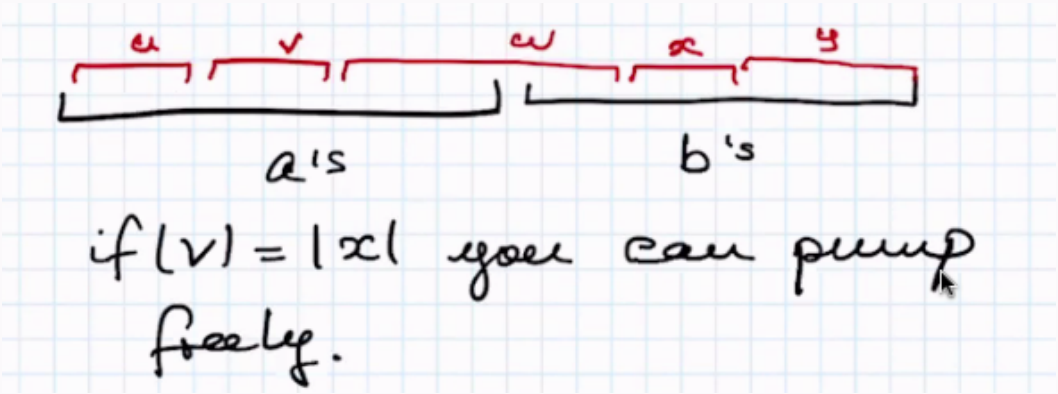
$$\forall p > 0, \; \exists s \in L, \; |s| \geq p$$
$$\forall u, v, w, x, y \in \Sigma^*$$
$$s = uvwxy, \; |vx| > 0, |vwx| \leq p$$
$$\exists i \geq 0, \; uv^i wx^i y \notin L$$
$$\Rightarrow L \text{ is not context free}$$

Example 1.3
$L = \{a^n b^n a^n \; | n \geq 0\}$.

1. Demon chooses p
2. We choose $a^p b^p a^p$.
3. Possibilities for $uvwxy$.
   a) If v or x stradles a block boundary, the demon loses with $i = 2$ because the a's and b's will be out of order.
   b) If v and x both contain only a's they must be in the same block. The demon loses with $i = 2$ because blocks will no longer be of the same size.
   c) If v and x contain only b's, the demon loses with $i = 2$ because the b block is too long.
   d) If v contains only a's and x contains only b's, then demon loses with $i = 2$ because the untouched block will be of different length.
4. Therefore the language is not context free because we have a winning strategy.

Example 1.4

$L = \{a^n b^n \mid n \geq 0\}$. In the regular case, the y consists of only a's so we win. But now in the context free case, there isn't a winning strategy.



Example 1.5

$L = \{a^{i+j} b^{j+k} c^{i+k} \mid i, j, k \geq 0\}$. Try to recognize instictively whether or not the language is context free before proving it.

This is context free! Trying to reason about why. $i + j$ is unbounded, but the stack is also unbounded. A lot more tricky to check CFGness.

Exercise: Cook up a CFG for this.

Example 1.6

$|\Sigma| \geq 2$. $L = \{ww \mid w \in \Sigma^*\}$, $L' = \{ww^{REV} \mid w \in \Sigma^*\}$. The first is not context free while the second is. The first is not possible because it's difficult to check if the strings are the same. The second is context free because the stack is good for checking the reversed string.

Fact 1.7. If L is CF and R is regular, then $L \cap R$ is a CFL.

Example 1.8

$L = \{ww \mid w \in \Sigma^*\}$. We use $R = a^*b^*a^*b^*$ and show $L \cap a^*b^*a^*b^*$ is not a CFL so L cannot be a CFL.

1. Demon chooses p.
2. We choose $a^p b^p a^p b^p$.
3. Possibilities for $uvwxy$.
   a) If v, x straddle block boundaries, the demon will lose
   b) If v, x are in the same block the demon loses because words aren't the same length.
   c) v and x have to be in consecutive blocks because of the bounded size of vwx. Note that the first copy of w starts with a so the second word cannot start with b. Also that the second copy ends with b so the first copy must end with b. So the boundary between 2 copies of w has to be at the end of the 2nd block. So although the two words could be the same size, they won't be the same words.
4. Therefore there is a strategy to always beat the devil so the language is not context free.

Example 1.9

$L = \{0^i 1^j \mid j = i^2\}$. This is not a CFL.

Let A be the adversary and I be us.

1. $A \to p$
2. $I \to 0^p 1^{p^2}$
3. $A \to uvwxy = 0^p 1^{p^2},\ |vwx| \leq p,\ |vw| > 0.$
4. $I \to$ case analysis
   a) vwx all zeros pick any $i \neq 1$.
   b) vwx all ones, pick any $i \neq 1$.
   c) v, x straddles the boundary, $i = 2$ gives 0's and 1's out of order.
   d) v is all zeros and x is all ones. We pick $i = 2$. Only non trivial case is when $|v| = m > 0,\ |x| = q > 0$. Pumping to $i = 2$ gives $0^{p+m}1^{p^2+q}$. How do we know that $(p+m)^2 \neq p^2 + q$?

$$(p+m)^2 = p^2 + 2pm + m^2$$
$$2pm > p \Rightarrow 2pm + m^2 > p > q \Rightarrow p^2 + 2pm + m^2 > p^2 + q$$

Example 1.10

$L = \{a^q \mid q \text{ a prime}\}$. Even with a stack you cannot check this.

1. $A \to p$
2. $I \to a^q \ |q > p$, prime
3. $A \to uvwxy$ s.t. $|uvwxy| = q,\ |vwx| \leq p,\ |vw| > 0$.
4. Let $vx = r > 0$. Let $i = q + 1$. Then $|uv^i wx^i y| = q + qr = q(1 + r)$. Can't be a prime because it's the product of two number where each is greater than 1.

Theorem 1.11

Over a one-letter alphabet a language is context-free if and only if it is regular.

Example 1.12

$L = \{a^{2^n} \mid n \geq 0\}$ is not a CFL. We showed that this language was not regular.

Exercise 1.13. $L = \{a^k b^j c^k \mid 0 < i < j < k\}$. Show that this is not a CFL.

Definition 1.14 (DPDA)**.** PDA's feature non determinism. This cannot be eliminated so DPDA's are strictly less powerful than PDA's.

If a CFL can be recognized by a DPDA we call it a Deterministic CFL. All modern languages are DCFL's. This makes things fast.

DPDA is a DFA with a stack.

Fact 1.15. DCFL's are closed under complement and therefore under intersection.

Example 1.16

$L = \{ww \mid w \in \Sigma^*\}$, $|\Sigma| \geq 2$. This is not a CFL, but $\overline{L}$ is a CFL.

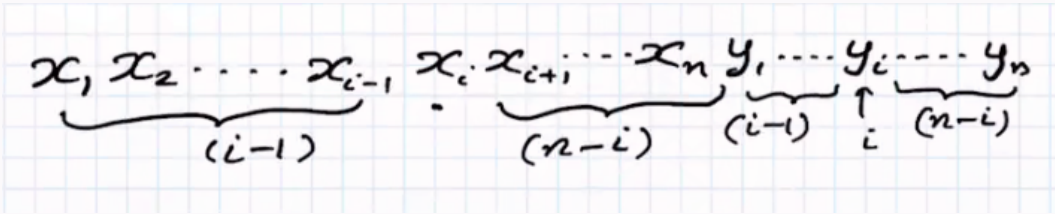
$\overline{L}$ an be recognized by a PDA. What is in $\overline{L}$.

1. Strings of odd length.
2. Might be even length. Then it can be written as concatenation of two equal length words where the words are not equal to one another. $|w_1| = |w_2|$ but $w_1 \neq w_2$. Two words are not the same if there exists i such that $w_1[i] \neq w_2[i]$.

Crucial difference is that instead of checking every position, you only have to guess one position. If there is an existing path, it works.

The idea is guess where i is and guess where the middle of the word is. It is okay that this i is unbounded because of the non determinism.

How the PDA works:



Push $x_1 \dots x_{i-1}$ onto the stack. Then remember what x_i is in the state. Start popping the stack until it is empty. Then start pushing letters onto the stack. It guesses where y_i is. It looks at y_i and compares it with x_i. It then pops the stack as it reads the remeaining letters. If the stack is empty at the end then x_i and y_i are indeed at the right positions.

$(i-1)$ symbols on the stack. $(n-i)+(i-1) = n-1$ between x_i and y_i.

After checking y_i there are $(n-i)$ letters on the stack. This exactly matches the $(n-i)$ letters after y_i.

Key point is didn't have to look at all the symbols, but used them to count.

Example 1.17

Equivalent CFG for $\overline{L}$. Let $V = \{S, A, B, C\},\ \Sigma = \{a, b\}$.

$$S \to$$

A produces odd length words with a in the middle.

B produces odd length words with b in the middle.

$S \to AB$ ultimately generates $xay\ ubv$. $n = |x| = |y|$ and $m = |u| = |v|$.

There is guaranteed to be one place where the corresponding positions are not the same.

Coming up.

1. Thursday. Concept of computability. Models of computation. Unsolvable problems. CE functions. Dovetailing.
2. Monday. Reduction.
3. Tuesday. FOC.
4. Wednesday. Lambda-calculus.
5. Thursday. Godel's Theorem. Recursion Theorem.