§1 Lecture 11-13

§1.1 Limits and Inequalities

Theorem 1.1 (Bounded Limit Theorem for Functions)
Let $f : A \to \mathbb{R}$, and x_0 bea cluster point of A. Assume that $\lim_{x\to x_0} f(x)$ exists.

Furthermore, assume that $\exists a, b \in \mathbb{R}$ such that $a \le f(x) \le b$ for all $x \in A \setminus \{x_0\}$. Then $a \le \lim_{x\to x_0} f(x) \le b$.

Proof. Let $\lim_{x\to x_0} f(x) = L$. Then $\forall (x_n)$ in $A \setminus \{x_0\}$ with $\lim(x_n) = x_0$, it holds that $\lim(f(x_n)) = L$.

Since $\forall n \in \mathbb{N} : x_n \in A \setminus \{x_0\}$, we have that

$$a \le f(x_n) \le b \qquad \underbrace{\Rightarrow}_{\text{Theorem from Chapter 3}} \qquad a \le L = \lim(f(x_n)) \le b$$

$$\Rightarrow a \le \lim_{x\to x_0} f(x) \le b$$

□

Theorem 1.2 (Squeeze Theorem for Functions)
Let $f, g, h : A \to \mathbb{R}$, and let x_0 be a cluster point of A. Assume that

$$g(x) \le f(x) \le h(x)$$

For all $x \in A \setminus \{x_0\}$.

Furthermore, assume that

$$L := \lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x)$$

Then the limit of $f(x)$ as $x \to x_0$ <u>exists</u> and equals L.

Proof. Let (x_n) be a sequence in $A \setminus \{x_0\}$ such that $\lim(x_n) = x_0$. Then $\lim(g(x_n)) = L$ and $\lim(h(x_n)) = L$.

And since $\forall n \in \mathbb{N} : x_n \in A \setminus \{x_0\}$, we know that

$$g(x_n) \le f(x_n) \le h(x_n)$$

By the squeeze theorem for sequences it now follows that $(f(x_n)$ converges to L. Since this holds for <u>any</u> (x_n) in $A \setminus \{x_0\}$ with $\lim(x_n) = x_0$, it follows from sequence criterion that

$$\lim_{x\to x_0} f(x) = L$$

□

Example 1.3
Consider the following function :

$$f(x) : \mathbb{R} \setminus \{0\} \text{ where } x \to x \cdot \sin(\frac{1}{x})$$

Solution.

$$|x \cdot \sin(\frac{1}{x})| = |x| \cdot |\sin(\frac{1}{x})| \leq |x|$$
$$\Rightarrow -|x| \leq x \sin(\frac{1}{x}) \leq |x|$$

for all $x \in \mathbb{R} \setminus \{0\}$.

Note that

$$\lim_{x \to x_0} |x| = 0$$
$$\lim_{x \to x_0} (-|x|) = -\lim_{x \to x_0} |x| = 0$$

Therefore, by squeeze theorem we have that

$$-|x| \leq x \sin(\frac{1}{x}) \leq |x| \quad \underbrace{\Rightarrow}_{\text{Squeeze Theorem}} \quad \lim_{x \to x_0} (x \sin(\frac{1}{x})) = 0$$

□

Example 1.4
Let $f : \mathbb{R}^+ \to \mathbb{R}$ and $x \to x^{3/2}$. We want to find $\lim_{x \to 0} x^{3/2}$.

Restrict f to the interval $[0, 1]$. On this interval we have that

$$0 \leq x \leq x^{1/2}$$
$$\Rightarrow 0 \leq x^{3/2} \leq x$$

and $\lim_{x \to 0} x = 0$.

Therefore, by squeeze theorem,

$$\underbrace{0}_{=0} \leq x^{3/2} \leq \underbrace{x}_{=0} \Rightarrow \lim_{x \to 0} x^{3/2} = 0$$

§1.2 Criteria for non-existence of limits of functions

Theorem 1.5 (Non-existence criteria where $(f(x_n))$ diverges.)
Let $f : A \to \mathbb{R}$ and x_0 be a cluster point of A. If $\exists(x_n)$ in $A \setminus \{0\}$ such that $\lim(x_n) = x_0$ but such that $\lim(f(x_n))$ diverges, then $\lim_{x\to x_0} f(x)$ DNE.

Proof. If $\lim_{x\to x_0} f(x)$ would exist, then $\lim(f(x_n) = \lim_{x\to x_0} f(x)$ but $f(x_n))$ diverges $\Rightarrow \lim_{x\to x_0} f(x)$ DNE. □

Theorem 1.6 (Non-existence criteria where $(f(x_n))$ and $(f(t_n))$ converge to different limits)
Let $f : A \to \mathbb{R}$ and x_0 be a cluster point of A. Assume that $\exists(x_n),(t_n)$ in $A \setminus \{x_n\}$ such that $\lim(x_n) = x_0 = \lim(t_n)$ and such that both $(f(x_n))$ and $(f(t_n))$ converge but to <u>different</u> limits. Then $\lim_{x\to x_0} f(x)$ does not exist.

Proof. Assume that $\lim_{x\to x_0} f(x) = L$. Then $\lim(f(x_n)) = L = \lim(f(t_n))$. Contradiction because $\lim(f(x_n)) \neq \lim(f(t_n))$. Thus $\lim_{x\to x_0} f(x)$ diverges. □

Example 1.7
Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ and $x \to \sin(1/x)$. Show that $\lim_{x\to 0} f(x)$ DNE.

1. Solution using the 2-sequence criterion.

   Choose (x_n) where $x_n := \frac{1}{\pi n}$ for all $n \in \mathbb{N}$. Then $f(x_n) = \sin(\pi n) = 0$ for all $n \in \mathbb{N}$. i.e. $\lim(f(x_n)) = 0$.

   Now choose (t_n) where $t_n := \frac{1}{\pi/2+2\pi n}$. Then $f(t_n) = \sin(\pi/2 + 2\pi n) = \sin(\pi/2) = 1$ for all $n \in \mathbb{N}$.

$$\Rightarrow \lim(f(t_n)) = 1 \neq 0 = \lim(f(x_n))$$
$$\Rightarrow \lim_{x\to 0} f(x) \text{ DNE}$$

2. Solution using the 1-sequence criterion.

   Let $x_n := \frac{1}{(2n-1)\pi/2}$. Then $\lim(x_n) = 0$ and $f(x_n) = \sin((2n-1)\pi/2) = (-1)^n$ for all $n \in \mathbb{N}$. i.e. $(f(x_n)) = ((-1)^n)$ which diverges!

$$\Rightarrow \lim_{x\to 0} f(x) \text{ DNE}$$

§1.3 One-sided limits (Brief)

In calculus you've seen

$$\lim_{x\to x_0+} f(x) \text{ and } \lim_{x\to x_0^-} f(x)$$

How do we define these properly?

Definition 1.8 (Definition of limit from left and right)**.** Let $f : A \to \mathbb{R}$ and $x_0 \in \mathbb{R}$.

$$\lim_{x \to x_0^+} f(x) \coloneqq f\big|_{A \cap]x_0, \infty[}(x)$$
$$\lim_{x \to x_0^-} f(x) \coloneqq f\big|_{A \cap]-\infty, x_0[}(x)$$

Example 1.9

$f : \mathbb{R} \to \mathbb{R}$ where $x \to |x|$. Determine $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$.

$$\lim_{x \to 0} x = 0 \Rightarrow \lim_{x \to x^+} |x| = 0$$
$$\lim_{x \to 0} x = 0 \Rightarrow \lim_{x \to x^-} |x| = 0$$

Theorem 1.10 (Limit of function exists iff limits from left and right exists and are equal)

Let $f : A \to \mathbb{R}$ and x_0 be a cluster point of A. Then $\lim_{x \to x_0} f(x)$ exists if and only if $\lim_{x \to x_0^+} f(x)$ and $\lim_{x \to x_0^-} f(x)$ exist and are equal.

Proof. Assignment 11. □

§1.4 Chapter 5: Continuity

Definition 1.11 (Defining a continuous function)**.** Let $f : A \to \mathbb{R}$ and $x_0 \in A$. We say that f is continuous at x_0 if

$$\lim x \to x_0 f(x)$$

exists and is equal to $f(x_0)$. i.e $\lim_{x \to x_0} f(x) = f(x_0)$.

Remark 1.12. In the case that x_0 is an isolated point, this definition should be read as follows: f is continuous at x_0 if it has a limit at x_0 which equals $f(x_0)$. In other words, all functions are continuous at all isolated points. Continuous is thus only interesting at cluster points.