§1 Lecture 12-02

Theorem 1.1
Let $f : A \to \mathbb{R}$ be uniformly continuous on A.

Let (x_n) be a cauchy sequence in A. Then $(f(x_n))$ is also a cauchy sequence.

Proof. Let $\epsilon > 0$. Then $\delta > 0$ such that $|x - \mu| < \delta \Rightarrow |f(x) - f(\mu)| < \epsilon$

(x_n) cauchy then $\exists N \in \mathbb{N}$ such that $\forall n, m \geq N : |x_n - x_m| < \delta \Rightarrow |f(x_n) - f(x_m)| < \epsilon$.

i.e. $(\exists N \in \mathbb{N})(\forall n, m \geq N : |f(x_n) - f(x_m)| < \epsilon \Rightarrow (f(x_n)$ is a cauchy sequence. □

Remark 1.2. This result is, in general, false, if f is just continuous on A.

Example 1.3
$f :]0, \infty[\to \mathbb{R}, x \to 1/x$.

f is continuous but <u>not</u> uniformally continuous on $]0, \infty[$.

Consider $x_n := 1/n$. Then (x_n) is a cauchy sequence but $(f(x_n)) = (n)$ which diverges.

$$\Rightarrow (f(x_n)) \text{ is } \underline{\text{not}} \text{ a cauchy sequence}$$

However: if $f : A \to \mathbb{R}$ is continuous, (x_n) is a convergent sequence in A such that $\lim(x_n) \in A$. Then:

$\lim(x_n) := x \in A$. Then f is continuous at x. Thus let $\lim(f(x_n)) = f(x)$ be the sequence of continuity. Especially, $(f(x_n))$ is cauchy sequence in this case.

This can be turned into another criterion for non-uniform continuous functions.

Theorem 1.4 (One sequence criterion for a non-uniform continuous function)
Let $f : A \to \mathbb{R}$. If (x_n) is cauchy sequence in A such that $(f(x_n))$ is not cauchy, then f is not uniformally continuous on A.

Example 1.5
$f :]0, \infty[\to \mathbb{R}, x \to 1/x$.

$$x_n := \frac{1}{n}$$

cauchy but $(f(x_n)) = (n)$ is not cauchy.

$$\Rightarrow f \text{ is not uniformly continuous on }]0, \infty[$$

Theorem 1.6

Let $f : A \to \mathbb{R}$, A bounded, f a uniformly continuous on A, then f is bounded (i.e. $f(A)$ is bounded.

Proof. Assume that f is unbounded. Then $\forall n \in \mathbb{N}$, $\exists x_n \in A : |f(x_n)| \geq n$.

Consider (x_n). Since A is bounded, (x_n) is bounded and thus has a convergent subsequence (x_{n_k}). Thus (x_{n_k}) is cauchy $\Rightarrow$ $(f(x_{n_k}))$ is cauchy and thus especially bounded. But $|f(x_{n_k})| \geq n_k \geq k$ for all $k \in \mathbb{N}$.

This implies that $f(x_{n_k})$ is unbounded. Contradiction!

Thus f is bounded. $\square$

Example 1.7

$f :]0, 1[\to \mathbb{R}, x \to 1/x$. Then f is unbounded on the bounded domain $]0, 1[\Rightarrow f$ is <u>not</u> continuous on $]0, 1[$.