§1 Lecture 12-03

Lipschitz Continuous.

Example 1.1

Last class: $\sqrt{x}$ is not lipschitz on $[0,\infty[$, however $\sqrt{x}$ is lipschitz on $[a,\infty[$ for any $a>0$.

Proof. Let $x,\mu \in [a,\infty[$. Then

$$\begin{aligned}|\sqrt{x}-\sqrt{\mu}| &= |\frac{(\sqrt{x}-\sqrt{\mu})(\sqrt{x}+\sqrt{\mu})}{\sqrt{x}+\sqrt{u}}| \\ &\leq \frac{1}{2a}|x-\mu|\end{aligned}$$

i.e. $\sqrt{x}$ is lipschitz continuous on $[a,\infty[$ with lipschitz constant $k=\frac{1}{2\sqrt{a}}$ □

Example 1.2

Last class: x^2 is lipschitz on $]-a,a[,\ a>0$.

However, x^2 is not lipschitz on $\mathbb{R}$.

Proof. x^2 isn't even uniformly continuous on $\mathbb{R}$ and thus cannot be lipschitz. □

Definition 1.3 (Geometric interpretation of lipschitz continuous)**.** Geometric interpretation of lipschitz continuous:

$f: A \to \mathbb{R}$ is lipschitz if

$$\begin{aligned}\exists k>0 \ :\ \forall x,\mu \in A \ :\ |f(x)-f(\mu)| \leq k\cdot|x-\mu| \\ \text{if } x\neq\mu \Leftrightarrow \underbrace{|\frac{f(x)-f(\mu)}{x-\mu}|}_{\text{Difference Quotient}} \leq k\end{aligned}$$

i.e. f is lipschitz if and only if the average slope of f is bounded on A.

§1.1 Another method for proving that $\sqrt{x}$ is uniformly continuous on $[0,\infty[$.

Idea: If $x\geq 1$, $\sqrt{x}$ is lipschitz on $[1,\infty[$ and thus uniformly continuous. And: if $0\leq x\leq 1$: $\sqrt{x}$ is uniformly continuous since it is continuous and $[0,1]$ is compact.

Q: $if \sqrt{x}$ is uniformly continuous on $[0,1]$ and $[1,\infty[$, does it follow that f is uniformly continuous on $[0,\infty[$.

A: Yes; this requries proof!

Theorem 1.4

Let f be uniformly continuous on intervals I_1, I_2 where I_1 is closed on the right with $\sup I_1 = \max I_1 = b$. And I_2 is closed on the left with $\inf I_2 = \min I_2 = b$, then f is uniformly continuous on $I = I_1 \cup I_2$.

Proof. Let $\epsilon > 0$, f uniformly continuous on I_1, thus $\exists \delta_1 > 0$ such that $|x - \mu| < \delta_1 \Rightarrow |f(x) - f(\mu)| < \epsilon/2$.

f is uniformly continuous on I_2. Thus $\exists \delta_2 > 0$ such that $|x - \mu| < \delta_2 \Rightarrow |f(x) - f(\mu)| < \epsilon/2$.

Let $\delta := \min\{\delta_1, \delta_2\}$.

1. Case $x, \mu \in I_1$
$$|x - \mu| < \delta \leq \delta_1 \Rightarrow |f(x) - f(\mu)| < \epsilon/2 < \epsilon$$
2. Case $x, \mu \in I_2$
$$|x - \mu| < \delta \leq \delta_2 \Rightarrow |f(x) - f(\mu)| < \epsilon/2 < \epsilon$$
3. Case $x \in I_1, \mu \in I_2$
$$\begin{gathered} |x - \mu| < \delta \Rightarrow |x - b|\delta \wedge |u - b| < \delta \\ \text{Thus } |f(x) - f(b)| < \frac{\epsilon}{2} \text{ and } |f(\mu) - f(b)| < \frac{\epsilon}{2} \\ \text{Now: } |f(x) - f(\mu)| = |[f(x) - f(b)] - [f(\mu) - f(b)]| \\ \leq |f(x) - f(b)| + f(\mu) - f(b)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\ \text{i.e. } |x - \mu| < \delta \Rightarrow |f(x) - f(\mu)| < \epsilon \\ \Rightarrow f \text{ is uniformly continous on } I = I_1 \cup I_2 \end{gathered}$$

□

Application: $\sqrt{x}$ is uniformly continuous on $[0, 1]$ and $[1, \infty[\Rightarrow \sqrt{x}$ is uniformly continuous on $[0, \infty[$.

§1.2 Differentiation

Definition 1.5 (Differentiable Definition)**.** Let $f : I \to \mathbb{R}$, I be an interval, $x_0 \in I$.

We say that f is differentiable at x_0, if
$$\lim_{x \to x_0} \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\text{Difference Quotient}} \quad \text{exists.}$$
If the limit exists, we call its value the derivative of f at x_0, denoted by
$$f'(x_0) = \frac{df}{dx}(x_0)$$
If f is differentiable at all $x_0 \in I$, we say that f is differentiable on I.

Theorem 1.6 (Caratheodory Alternative Description of Differentiability)
Let $f : I \to \mathbb{R}$, $x_0 \in I$, then f is differentiable at x_0 if and only if there exists a function $\phi : I \to \mathbb{R}$ continuous at x_0 such that

$$\forall x \in I \quad f(x) = f(x_0) + \phi(x)(x - x_0)$$

If ϕ exists, it holds that $\phi(x_0) = f'(x_0)$.

Proof. "$\Rightarrow$" Let f be differentiable at x_0. Let

$$\phi(x) := \begin{cases} \frac{f(x)-f(x_0)}{x-x_0}, & x \neq x_0 \\ f'(x_0), & x = x_0 \end{cases}$$

Then

$$\lim_{x \to x_0} \phi(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) = \phi(x_0)$$
$$\Rightarrow \phi \text{ is continuous at } x_0$$

"$\Leftarrow$" Let $\phi : I \to \mathbb{R}$, continuous at x_0 such that

$$f(x) = f(x_0) + \phi(x)(x - x_0)$$

Let $x \neq x_0$. $\Rightarrow \phi(x) = \frac{f(x)-f(x_0)}{x-x_0}$

ϕ continuous at x_0 $\Rightarrow$ $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists and equals $\phi(x_0)$ $\Rightarrow$ f is differentiable at x_0 and $f'(x_0) = \phi(x_0)$

□

Applications: Differentiable implies continuous. i.e. if $f : I \to \mathbb{R}$ is differentiable at $x_0 \in I$, then f is continuous at x_0.

Proof. f differentiable at x_0 $\Rightarrow$ $\exists \phi : I \to \mathbb{R}$, continuous at x_0 such that $\forall x \in I$, $f(x) = \underbrace{f(x_0) + \phi(x) \cdot (x - x_0)}_{\text{continuous at } x_0}$ □

Theorem 1.7 (Product Rule)
Let $f, g : I \to \mathbb{R}$ be differentiable at x_0. Then $f \cdot g$ is differentiable at x_0 and $(f \cdot g)'(x_0) = f'(x_0)g(x_0) - f(x_0) \cdot g'(x_0)$.

Proof. f, g differentiable at $x_0 \Rightarrow \exists \phi, : I \to \mathbb{R}$ continuous at x_0 such that

$$f(x) = f(x_0) + \phi(x)(x - x_0)$$
$$g(x) = g(x_0) + \psi(x)(x - x_0)$$
$$\Rightarrow (f \cdot g)(x) = f(x) \cdot g(x)$$
$$= f(x_0)g(x_0) + f(x_0)(x)(x - x_0) + g(x_0)(x)(x - x_0) + \phi(x)\psi(x)(x - x_0)^2$$
$$\Rightarrow (f \cdot g)(x) = f(x_0)g(x_0) + [f(x)g(x_0) + f(x_0)\psi(x) + \phi(x)\psi(x)(x - x_0)] \cdot (x - x_0)$$

□

§1.3 Relationship Between Lipschitz Continuity and Differentiability

Recall 1.9 (Mean Value Theorem)**.** The mean value theorem. Let $I = [a, b]$, $f : I \to \mathbb{R}$ differentiable on $]a, b[$ and continuous on the entire interval. Then there exists $c \in]a, b[$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Theorem 1.10

Let $f : I \to \mathbb{R}$ be differentiable. Then f is lipschitz on I if and only if f' is bounded on I.

Proof. "$\Rightarrow$" Let f be lipschitz with lipchitz constant k.

$$\begin{gathered} x, \mu \in I_\mu \ (x \neq \mu) \ \text{ then } |f(x) - f(\mu)| \leq k|x - \mu| \\ \Rightarrow \left| \frac{f(x) - f(\mu)}{x - \mu} \right| \leq k \\ \Rightarrow -k \leq \frac{f(x) - f(\mu)}{x - \mu} \leq k \\ \\ \Rightarrow -k \leq \lim_{x \to \mu} \frac{f(x) - f(\mu)}{x - \mu} \leq k \\ \Rightarrow -k \leq f'(\mu) \leq k \\ \Rightarrow |f'(\mu)| \leq k \ \forall \ \mu \in I \\ \Rightarrow f' \text{ is bounded on } I \end{gathered}$$

"$\Leftarrow$" Assume that f' is bounded on I.

Let $k > 0$ such that $|f'(x)| \leq k$ for all $x \in I$.

Let $x < \mu$, $x, \mu \in I$. Apply mean value theorem to f on $[x, \mu]$ then $\exists c \in]x, \mu[$ such that

$$\begin{gathered} \frac{f(x) - f(\mu)}{x - \mu} = f'(c) \Rightarrow \frac{|f(x) - f(\mu)|}{|x - \mu|} = |f'(c)| \leq k \\ \Rightarrow |f(x) - f(\mu)| \leq k|x - \mu| \\ \Rightarrow f \text{ is lipschitz on } I \end{gathered}$$

□