Find the units and zero-divisors of $\mathbb{Z}_5[x]/\langle (x+1)(x+2)\rangle$.

First, since 5 is prime, all $b \neq 0$ are units.

Now consider $ax+b$, $a \neq 0$. Since $a$ is a unit, $ax+b$ is a unit iff $x + b/a$ is a unit, and $ax+b$ is a zero-divisor iff $x+b/a$ is a zero-divisor.

So we only have to consider $x+b$.

$(x+1)(x+2) = 0 \mod (x+1)(x+2)$, so if $b = 1, 2$ then $x+b$ is a zero divisor.

If $b \neq 1,2$, consider $(x+b)(x+c) = x^2 + (b+c)x + bc = (b+c-3)x + (bc-2) \mod (x+1)(x+2)$, since $x^2+3x+2 = (x+1)(x+2) = 0$.

If we let $c = 3-b$, then $b+c-3 = 0$, so $(x+b)(x+c) = bc-2 \mod (x+1)(x+2)$. $bc - 2 = b(3-b) - 2 \neq 0$ since $b \neq 1, 2$. [Check: if $b = 0, 3$ then $b(3-b)-2 = -2$, and if $b=-1$, $b(3-b)-2 = -4-2 = -6 \neq 0$].

In this case, $(x+b)(x+c) = bc-2 \neq 0$ is a unit, so $(x+b)$ is a unit.

So:

| Units | Zero divisors |
| --- | --- |
| $\mathbb{Z}_5^\times = U(5)$ | $a(x+2)$, $a\cdot(x+1)$ for $a \in \mathbb{Z}_5$. |
| $a\cdot(x+b)$ for $a \in \mathbb{Z}_5^\times = U(5)$, $b \neq 1, 2$ | |

Group of units: there are $4 + 4\cdot 3 = 16$ units.
This is an abelian group. Each element of $U(5)$ has order 2 or 4, and $(x+3)$ and $(x+4)$ and $x$ have order 4.

So the group of units is either $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$.

Since there are more than 8 elements of order 4 ($\pm 2, \pm x, \pm(x+3), \pm(x+4)$ and $2x$, for example), this group is $\mathbb{Z}_4 \times \mathbb{Z}_4$.

<u>Alternative solution</u> (easier, uses $1^{st}$ iso thm)

Consider $\phi : \mathbb{Z}_5[x] \to \mathbb{Z}_5 \times \mathbb{Z}_5$, $\phi(f) = (f(-1), f(-2))$. It is easy to check that this is a ring homomorphism.

Clearly $\langle (x+1)(x+2)\rangle \subseteq \operatorname{Ker}\phi$. If $f \in \operatorname{Ker}\phi$, then $f(-1) = f(-2) = 0$, so since $\mathbb{Z}_5$ is a field, $(x+1) \mid f$ and $(x+2) \mid f$, so $(x+1)(x+2) \mid f$ and $f \in \langle (x+1)(x+2)\rangle$.

So $\operatorname{Ker}\phi = \langle (x+1)(x+2)\rangle$. By the $1^{st}$ iso theorem, $\mathbb{Z}_5[x]/\langle (x+1)(x+2)\rangle \cong \operatorname{Im}\phi = \mathbb{Z}_5 \times \mathbb{Z}_5$.

So the units of $\mathbb{Z}_5[x]/\langle (x+1)(x+2)\rangle$ are the units of $\mathbb{Z}_5 \times \mathbb{Z}_5$, which are $U(5) \times U(5) \cong \mathbb{Z}_4 \times \mathbb{Z}_4$, and the zero divisors are $\{0\} \times \mathbb{Z}_5$ and $\mathbb{Z}_5 \times \{0\}$.

Checking $\phi$, we can see that $\phi(f) \in (\{0\} \times \mathbb{Z}_5) \cup (\mathbb{Z}_5 \times \{0\})$ exactly when $f$ is a multiple of $(x+1)$ or $(x+2)$, so this gives all the zero divisors and units we found above.

Find the units and zero divisors of $\mathbb{Z}_4[x]/\langle x^2+2x\rangle$.

Note that $x^2+2x = x(x+2)$, so $2, x, x+2$ are zero divisors, and so are their multiples. This gives us $2, 2x+2, x, 2x, 3x, x+2,$ $3(x+2) = 3x+2$.

Clearly $1, 3$ are units, so we have to consider $ax+b$ for $a=1,3$ and $b \neq 0,2$, and for $a=2$, and $b \neq 0,2$.

For $a=1,3$, $ax+b$ is a unit iff $x+b/a$ is a unit so consider $a=1$.
$(x+1)^2 = x^2+2x+1 = 4x+1 = 1$ since $x^2 = 2x$, so $x+1$ is a unit.
Similarly, $(x+3)^2 = 3^2 = 1$ so $x+3$ is a unit.

For $a=2$, $(2x+1)^2 = 4x^2+4x+1 = 1$ so $2x+1$ is a unit. Similarly, $(2x+3)^2 = 3^2 = 1$ so $2x+3$ is a unit.

So we have:

| <u>Units</u> | <u>Zero divisors</u> |
|---|---|
| $\left(\begin{array}{l} \{1,3\} = U(4) \\ ax+b, \quad b \neq 0,2 \\ \qquad\qquad a \neq 0 \end{array}\right)$ | multiples of $2, x, x+2,$ |
| $\Downarrow$ | $\Downarrow$ |
| $ax+b, \; b \neq 0,2$ | $ax+b \quad b = 0, 2.$ |

Group of units: there are 8 units, this group is abelian, and one can check that $(ax+b)^2 = 1$ whenever $b \neq 0, 2$.

So every element has order 2, and so the group of units is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.