§1 Lecture 01-17

§1.1 Constructions of Vector Spaces

Given vector spaces V_1, V_2, we can construct new vector spaces.

1. Direct sum, or cartesian product. $V_1 \times V_2 = V_1 \oplus V_2 = \{(v_1, v_2) : v_1 \in V_1, v_2 \in V_2\}$.

$$(v_1, v_2) + (v_1', v_2') = (v_1 + v_1', v_2 + v_2')$$
$$\lambda(v_1, v_2) = (\lambda v_1, \lambda v_2)$$

   Note that $\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2)$

2. Subspace. If V is a vector space over F, then $W \subseteq V$ is a subspace if it is closed under addition and scalar multiplication.

$$w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$$
$$\lambda \in F,\ w \in W \Rightarrow \lambda w \in W$$

   Conclusion is that W is a vector space. Other properties are inherited from V.

3. Homs. $\hom_F(V_1, V_2)$ is a vector space. If V_1 and V_2 are finite dimensional, dimensions n_1 and n_2, then $\dim_F(\hom_F(V_1, V_2)) = n_1 n_2$.

   Let $(e_1, \ldots, e_n)$ be a basis for V_1.

   Key remark: A linear transformation $T : V_1 \to V_2$ is completely determined by $(T(e_1), \ldots, T(E_{n_1})$.

   Why? If $v \in V_1$, then $v = \lambda_1 e_1 + \cdots + \lambda_{n_1} e_{n_1}$. So $T(v) = T(\lambda_1 e_1 + \cdots + \lambda_{n_1} e_{n_1} = \lambda_1 T(e_1) + \cdots + \lambda_{n_1} T(e_{n_1})$

$$\hom(V_1, V_2) = \underbrace{V_2 \oplus \cdots \oplus V_2}_{n_1 \text{ times}}$$

4. Dual space: $V^* = \hom_F(V, F)$. If B is a basis for V, then $V \simeq F_0(B, F)$. $V^* \simeq F(B, F)$.

   The choice of B determines an injection $V \hookrightarrow V^*$. When B is finite, i.e. $\dim(V) = n < \infty$, then $V \simeq V^*$.

   There is a canonical inclusion of $V \hookrightarrow V^{**}$.

$$V \to V^{**}$$
$$v \mapsto v^{**}(l) = l(v)$$
$$l \in V^*,\ l : V \to F$$

   v^{**} is a linear functional on V^*. I.e. a linear transformation from $F^* \to F$. The rule $v \mapsto v^{**}$ is itself linear. i.e. $(v_1 + v_2)^{**} = v_1^{**} + v_2^{**}$.

5. The tensor product of V_1 and V_2. $V_1 \otimes V_2 = \hom_F(V_1^*, V_2)$

   If V_1 and V_2 are finite dimensional, then $\dim(V_1 \otimes V_2) = \dim(V_1)\dim(V_2)$.