§1 Lecture 03-09

A quadratic space is a pair $(V, <, >)$ where V is a vector space, and $<, >: V \times V \to F$ which is bilinear.
The pairing $(<, >)$ is non-degenerate if it induces an injection

$$V \to V^*$$
$$v \mapsto (w \mapsto < v, w >)$$
$$(\text{when dim V } < \infty, \text{ then } V \simeq V^*)$$

The adjoint of $T : V \to V$ is the map satisfying

$$T* : V \to V$$
$$< v, Tw > = < T^*(v), w >$$

Question: Where do non-degenerate bilinear forms arise "in nature"?
Answer: Geometry, distance.
From now on, $F = \mathbb{R}$ or $\mathbb{C}$.

Definition 1.1. A <u>real inner product</u> on V is a bilinear form satisfying:

1. $$< v, w > = < w, v >, \quad \forall v, w \in V$$

2. $$< v, v > \geq 0, \quad < v, v > = 0 \text{ iff } v = 0$$

Example 1.2

1. $V = \mathbb{R}^n$

$$< (x_1, \ldots, x_n), (y_1, \ldots, y_n) > = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$
$$< (x_1, \ldots, x_n), (x_1, \ldots, x_n) > = x_1^2 + x_2^2 + \cdots + x_n^2$$

2. $V = e([0, 1])$ represents continuous real-valued functions on $[0, 1]$.

$$< f, g > = \int_0^1 f(t)g(t)dt, \quad < f, f > = \int_0^1 f(t)^2 dt$$

Definition 1.3 (Complex Inner Product)**.** A <u>complex inner product</u> on V is a hermition-bilinear form satisfying

Note 1.4. It would become problamatic to try and declare it as a standard bilinear form

1. $$< v, \lambda w_1 + w_2 > = \overline{\lambda} < v, w_1 > + < v, w_2 >$$

2.

$$< v, w >= \overline{< w, v >}$$

3.

$$< v, v > \in \mathbb{R} \geq 0, \quad < v, v >= 0 \Leftrightarrow v = 0$$

Example 1.5

Reviewing the previous examples with the new complex inner product

1. $V = \mathbb{C}^n$

$$< (x_1, \ldots, x_n), (y_1, \ldots, y_n) >= x_1\overline{y_1} + x_2\overline{y_2} + \cdots + x_n\overline{y_n}$$
$$< (x_1, \ldots, x_n), (x_1, \ldots, x_n) >= |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2$$

2. $V = e([0, 1])$ represents continuous complex-valued functions on $[0, 1]$.

$$< f, g >= \int_0^1 f(t)\overline{g(t)}dt, \quad < f, f >= \int_0^1 |f(t)|^2 dt$$

Note 1.6. Caveat: A complex inner product space is not (quite) a quadratic space as defined before.
We define the norm of v to be $||v|| = \sqrt{< v, v >}$. "Length of v".

Example 1.7

1. $V = \mathbb{R}^n$.

$$||(x_1, \ldots, x_n)|| = \sqrt{x_1^2 + \cdots + x_n^2}$$

2. $V = \mathbb{C}^n$.

$$|(z_1, \ldots, z_n)|| = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}$$

Definition 1.8 (Properties of $||\ ||$)**.** Always easier to think about the square of the norm.

1.

$$||v + w||^2 =< v + w, v + w >=< v, v > + < v, w > + < w, v > + < w, w >$$
$$= ||v||^2 + 2 \text{ Real part of } < v, w > + ||w||^2$$

Definition 1.9. Two vectors v, w are orthogonal if $< v, w >= 0$.

Theorem 1.10 (Pythagorean Theorem)

$$||v + w||^2 + ||v||^2 + ||w||^2$$

Theorem 1.11 (Parallelogram Law)

$$||v+w||^2 + ||v-w||^2 = 2(||v||^2 + ||w||^2)$$

$$\begin{aligned} ||v+w||^2 + ||v-w||^2 &= ||v||^2 + 2Re < v,w > + ||w||^2 + ||v||^2 - 2Re(< v,w >) + ||w||^2 \\ &= 2(||v||^2 + ||w||^2) \end{aligned}$$

$\square$

Theorem 1.12 (Polarization Formula)
The function $v \mapsto < v,v >$ is enough to recover $(v,w) \mapsto < v,w >$.

1. If $F = \mathbb{R}$

$$< v,w >= 1/2(< v+w, v+w > - < v,v > - < w,w >)$$
$$< v,w >= \frac{1}{4}(< v+w, v+w > - < v-w, v-w >)$$

2. If $F = \mathbb{C}$

$$\begin{aligned} < v,w >=&< v+w, v+w > \\ &+ i < v+iw, v+iw > \\ &+ -1 < v-w, v-w > \\ &+ -i < v-iw, v-iw > \end{aligned}$$

Theorem 1.13 (Cauchy Schwarz Inequality)

$$| < v,w > |^2 \leq ||v||^2 ||w||^2$$