§1 Direct Products

Let $(G, \cdot), (H, \cdot)$ be groups. The external direct product of $G \times H$.

$$G \times H = \{gh : g \in G, h \in H\}$$

with binary opteration $(g_1, h_1)(g_2, h_2) = (g_1 \cdot g_2, h_1 \cdot h_2$.

Note 1.1. Associative. Proof.

Note 1.2. Identity $= (1_G, 1_H)$.

We define the external direct product of $G_1 \times G_2 \times \cdots \times G_k$

Note 1.3.

$$|G| = \prod_{i=1}^{k} |G_i|$$

Definition 1.4. $G^n = G \times \cdots \times G$.

Example 1.5
$\mathbb{R}^n$ and $\mathbb{Z}_2^3$.

We have 5 groups of order 8. $Q_8, D_4, \mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2^3$.

1. cyclic with 3 subgroups of order 4. nonabelian
2. cyclic with 1 subgroup of order 4. nonabelian
3. cyclic. abelian
4. not cyclic with cosets of order 4 abelian
5. each element has order 2 abelian

Theorem 1.6
9.17. Let $(g, h) \in G \times H$. $|(g, h)| = \operatorname{lcm}(|g|, |h|)$

Example 1.7

Theorem 1.8
$\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn} \Leftrightarrow \gcd(m, n) = 1$

Proof. □