§1 Kernal

Definition 1.1. The kernel of the homomorphism $\phi : G \to K$ is the pre image of the identity element. i.e. $\phi^{-1}(\{e\})$.

Theorem 1.2

The kernel of $\phi : G \to H$ is a normal subgroup of G.

Proof. Special case of Thm 11.4 □

Example 1.3

$\ker(\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^*) = \mathrm{SL}_n(\mathbb{R})$

Example 1.4

Let $g \in G$ be an element of order n. Let $\phi : \mathbb{Z} \to G$ be $\phi(p) = g^p$.

Which integers are going to map to the identity? Any integers that are multiplies of n. So $\ker(\phi) = n\mathbb{Z}$.

Example 1.5

Let N be a normal subgroup of G. The map $\phi : G \to G/N$ given by $\phi(g) = gN$ is a homomorphism. Indeed, $\phi(ab) = (ab)N = aNbN = \phi(a)\phi(b)$.

Note 1.6. This is the natural or canonial homomorphism.

Theorem 1.7 (First isomorphism theorem.)

Let $\psi G \to H$ be a homomorphism. Let N be the kernel of ϕ. Let $\phi : G \to G/N$ be canonical homomorphism. Then there exists an isomorphism $f : G/N \to \psi(G)$ such hat $\psi = f \circ \phi$.

$f(xN) = \psi(x)$. (f is well defined).

Example 1.8

Let $g \in G$, and $\psi(p) = g^p$. We know that if $|g| = n$, then $\ker(\psi) = n\mathbb{Z}$.

$\langle g \rangle = \psi(\mathbb{Z}) \subset G$.

$Z/n\mathbb{Z}$ is a cyclic group of order n and is therefore isomorphic to $\langle g \rangle = \psi(\mathbb{Z})$ which is also a cyclic group of order n.

Lemma 1.9
If $f : A \to B$ and $g : B \to C$ are homomorphisms, then $g \circ f : A \to C$ is a homomorphism.

Proof.

$$g \circ f(a_1 a_2) = g(f(a_1 a_2)) = g(f(a_1)f(a_2)) = g(f(a_1))g(f(a_2))$$

□

Example 1.10
$\phi : A \times B \to A$. $\phi((a, b)) = a$ is a homomorphism.

Check: $\phi((a_1, b_1)(a_2, b_2)) = \phi((a_1 a_2, b_1 b_2)) = a_1 a_2 = \phi(a_1, b_1)\phi(a_2, b_2)$

$\ker(\phi) = \{(e, b) : b \in B\} = \{e\} \times B \subset A \times B$.

Example 1.11

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$$

$\phi(a, b, c) = (a + b + c) \mod 2$. Kernel of ϕ is $\{(0, 0, 0), (1, 0, 1), (0, 1, 1), (1, 1, 0)\} = N$.

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 / N \cong \mathbb{Z}_2$$

Example 1.12
ϕ :Isometries of a cube$\to S_3 =$ permutations (x, y, z)

$\phi(g) =$ permutations of axes determined by ϕ.

Kernel of $\phi \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.