§1 Lecture 02-05

Minimal polynomial. If $T : V \to V$, $\dim V = n$, then there exists polynomial $p(x)$ such that

$$p(T) = 0$$
$$\deg(p(x)) \leq n$$

Clear for $n = 1$.

If $\exists v \in V$ such that $(v, Tv, T^2v, \ldots, T^{n-1}v)$ are linearly independent, then it is fine.

$$\begin{aligned} \exists p(x) \text{ such that } p(T)(v) &= 0 \\ \Rightarrow p(T)(Tv) &= 0 \\ p(T)(T^2v) &= 0 \\ &\vdots \\ p(T)(T^{n-1}v) &= 0 \\ \Rightarrow p(T) &= 0 \end{aligned}$$

Suppose that there is no cyclic vector. Let $v \neq 0$ in V. Then

$$\operatorname{span}(v, T(v), \ldots, T^{n-1}(v)) = W \subsetneq V$$

W is preserved by T.

$$T_W : W \to W$$
$$\dim W = d < n$$

The induction hypothesis implies that there is a polynomial $p_W(x)$ such that $p_W(T_W) = 0$.

It would be nice if we could write $V = W \oplus W'$. Problem is that there need not be a T-stable complementary space W'.

Solution: Define $W' = V/W$. W' is naturally equipped with

$$\overline{T} : W' \to W'$$
$$\overline{T}(v + W) = T(v) + W$$

Checking well-defined:

$$\begin{aligned} v_1 + W = v_2 + W \Rightarrow v_1 - v_2 \in W \Rightarrow T(v_1 - v_2) \in T(W) \subseteq W \\ \Rightarrow T(v_1) - T(v_2) \in W \end{aligned}$$

There is a polynomial $p_{W'}(x)$ such that $p_{W'}(\overline{T}) = 0$ where $\deg p_{W'}(x) \leq n - d$.

Claim: $p(x) \coloneqq p_W(x) \cdot p_{W'}(x)$ satisfies $p(T) = 0$.

Proof. $p_{W'}(\overline{T}) = 0$. Image $p_{W'}(T) \subseteq W$.

$$\begin{aligned} p_{W'}(\overline{T}) = 0 &\Rightarrow p_{W'}(\overline{T})(v + W) = 0 \\ &\Rightarrow p_{W'}(T)(v) + W = 0 + W \ \ \forall x \in V \\ &\Rightarrow p_{W'}(T)(v) \in W \ \ \forall v \in V \\ p_W(T)(W) &= 0 \\ p_W(T)p_{W'}(T) &= p_W(T) \circ p_{W'}(T) = 0 \end{aligned}$$

□

Goal of linear algebra:

1. Given $T : V \to V$, classify all possible T.
2. Find bases for V which are "convenient" to study T.

Structural invariants attached to T.

1. Minimal polynomial $p_T(x)$
2. Characteristic polynomial $f_T(x) = \det(xI - T)$. $\deg(f_T(x)) = n = \dim V$.

Definition 1.1 (Eigenvalue)**.** An element $\lambda \in F$ is an eigenvalue for T if $\exists$ a non-zero $v \in V$ such that $T(v) = \lambda v$. A vector v with this property is called an eigenvector of T, with eigenvalue λ. Note that the zero vector is never considered an eigenvector.

Definition 1.2 (Eigenspace)**.** The set $V_\lambda = \{v \in V \text{ such that } T(v) = \lambda v\}$ is called the eigenspace for T.

Definition 1.3 (Spectrum)**.** The spectrum of T is the collection of eigenvalues of T. $spec(T) \subseteq F$.

Proposition 1.4

If $\lambda_1 \neq \lambda_2 \in spec(T)$, then V_{λ_1} and V_{λ_2} are linearly disjoint, i.e. $V_{\lambda_1} \cap V_{\lambda_2} = (0)$.

Proof. If $v \in V_{\lambda_1} \cap V_{\lambda_2}$, then $T(v) = \lambda_1 v, T(v) = \lambda_2 v \Rightarrow (\lambda_1 - \lambda_2)v = 0$. $\lambda_1 - \lambda_2 \neq 0 \Rightarrow v = 0$. ☐

Definition 1.5 (Diagonalizable)**.**

$$\text{If } \oplus_{\lambda \in spec(T)} V_\lambda = V$$

then T is diagonalizable.

Equivalently, T is diagonalizable if V has a basis of eigenvectors for T.

Example 1.6
1. $V = \mathbb{R}^2$, where T is a rotation by $\pi/2$.