§1 Isomorphisms

Definition 1.1. $(G, \cdot)$ and $(H, \circ)$ are isomorphic if there exists a bijection $\phi : G \to H$ such that $\phi(a \cdot b) = \phi(a) \circ \phi(b)$ for all $a, b \in G$.

Then ϕ is an isomorphism and we write $G \equiv\sim H$

Example 1.2

$\phi : \mathbb{Z}_4 \to U_5$

$$\begin{gathered} 0 \to 1 \\ 1 \to 2 \\ 2 \to 4 \\ 3 \to 3 \\ \phi(3+2) = \phi(1) = 2 = \phi(3) \cdot \phi(2) = 3 \cdot 4 = 2 \checkmark \end{gathered}$$

$$(\mathbb{Z}_4, +) : \quad \begin{array}{c|cccc} \circ & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 & 0 \\ 2 & 2 & 3 & 0 & 1 \\ 3 & 3 & 0 & 1 & 2 \end{array}$$

$$(U_5, \cdot) : \quad \begin{array}{c|cccc} \circ & 1 & 2 & 4 & 3 \\ \hline 1 & 1 & 2 & 4 & 3 \\ 2 & 2 & 4 & 3 & 1 \\ 4 & 4 & 3 & 1 & 2 \\ 3 & 3 & 1 & 2 & 4 \end{array}$$

Note 1.3. G and H are isomorphic if by "reordering the elements of H", they have the same caylety table - the only difference is notation.

"bijection between groups extends to a bijection between multiplication tables. Multiplication tables are the same, the difference being notation. A different language"

Example 1.4

$\phi : \mathbb{Z}_4 \to \{\pm 1, \pm i\} \subset \mathbb{C}^*$

$$
\begin{aligned}
0 &\to 1 \\
1 &\to i \\
2 &\to -1 \\
3 &\to -i
\end{aligned}
$$

$$\phi(n) = i^n$$
$$\phi(a+b) = i^{a+b} = i^a \cdot i^b = \phi(a) \cdot \phi(b)$$

$$
(\{\pm 1, \pm i\} \subset \mathbb{C}^*, \cdot) : \quad
\begin{array}{c|cccc}
\circ & 1 & i & -1 & -i \\
\hline
1 & 1 & i & -1 & -i \\
i & i & -1 & -i & 1 \\
-1 & -1 & -i & 1 & i \\
-i & -i & 1 & i & -i
\end{array}
$$

Example 1.5

$\phi : \mathbb{Z}_4 \to \mathbb{Z}_4$

$$
\begin{aligned}
0 &\to 0 \\
1 &\to 3 \\
2 &\to 2 \\
3 &\to 1
\end{aligned}
$$

This is an isomorphism.

Theorem 1.6

G is abelian if and only if the map $\phi : G \to G$ given by $\phi(a) = a^{-1}$ for all $a \in G$ is an isomorphism.

Proof. .

($\Leftarrow$)

$$ba = (a^{-1}b^{-1})^{-1} = \phi(a^{-1}b^{-1}) = \phi(a^{-1})\phi(b^{-1}) = (a^{-1})^{-1}(b^{-1})^{-1} = ab$$

($\Rightarrow$)

$$\phi(a \cdot b) = (ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1} = \phi(a) \cdot \phi(b)$$

$\square$

Example 1.7

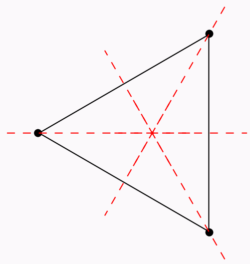
$$Q_8 \equiv\sim \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \pm \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \right\}$$

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

These two representations of the quaternions are isomorphic to one another.

Example 1.8

$D_3 \equiv\sim S_3$



$$\begin{aligned} 0 &\to () \\ \frac{2\pi}{3} &\to \{1\ 3\ 2\} \\ \frac{4\pi}{3} &\to \{1\ 2\ 3\} \\ \alpha &\to \{2\ 3\} \\ \beta &\to \{1\ 2\} \\ \gamma &\to \{1\ 3\} \end{aligned}$$

There are many isomorphisms from D_3 to S_3.

Theorem 1.9

If $\phi : G \to H$ is an isomorphism, then $\phi^{-1} : H \to G$ is an isomorphism.

Proof. phi^{-1} is a bijection since ϕ is (ϕ^{-1} exists because ϕ is a bijection)

$$\phi^{-1}(a \cdot b) = \phi^{-1}\Big(\phi(\phi^{-1}(a))\phi(\phi^{-1}(b))\Big) = \phi^{-1}\Big(\phi(\phi^{-1}(a)\phi^{-1}(b))\Big) = \phi^{-1}(a)\phi^{-1}(b)$$

$\square$

Theorem 1.10
Any "property" of G is a "property" of H.

Example 1.11
$|G| = |H|$
G is abelian $\Leftrightarrow$ H is abelian
G is cyclic $\Leftrightarrow$ H is cyclic
$G = \langle g \rangle \Leftrightarrow H = \langle \phi(g) \rangle$

Theorem 1.12
If G is cyclic and $|G| = \infty$ the $G \equiv\sim \mathbb{Z}$
If G is cyclic and $|G| = n$ the $G \equiv\sim \mathbb{Z}_n$

Proof. Let $G = \langle g \rangle$. Consider map $\phi : \mathbb{Z} \to G$ given by $\phi(i) = g^i$
Claim that ϕ is a bijection.
Surjective because each $x \in G$ is $g = g^i$ for some i so $\phi(i) = x$ where x is arbitrary.
Injective because $\phi(i) = \phi(j) \Rightarrow g^i = g^j \Rightarrow g^i g^{-j} = e \Rightarrow g^{i-j} = e \Rightarrow i = j = 0 \Rightarrow i = j$
Therefore ϕ is an isomorphism because $\phi(i + j) = g^{i+j} = g^i g^j = \phi(i)\phi(j)$ $\square$