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0.1 Homework: Read 2.1 on your own

1 Absolute Values

Definition: Let $x \in \mathbb{R}$, then

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0 \end{cases}$$

Note that $|x| = \sqrt{x^2}$

1.1 Discussing properties of absolute value

Theorem
(a) $\forall x, y \in \mathbb{R} : |x * y| = |x| * |y|$
(b) Let $a > 0$. Then $|x| \leq a \Leftrightarrow -a \leq x \leq a$
(c) $\forall x \in \mathbb{R} : -|x| \leq x \leq |x|$
Review: Math notation so that I can confidently write it myself instead of copying from the board. This will propably improve my retention.

1.2 Proof

(a) $|xy| = \sqrt{(xy)^2} = \sqrt{x^2y^2} = \sqrt{x^2}\sqrt{y^2} = |x||y|$ ✓

(b) "$\Rightarrow$ " Let $|x| \leq a$. First case: $x \geq 0$. If this is true, then if follows that $x = |x| \leq a \Rightarrow x \leq a \text{ and } -a \leq 0 \leq x \Rightarrow -a \leq x \leq a$✓. Second case: $x < 0$. If this is true then $-x = |x| \leq a \Rightarrow x \geq -a \Rightarrow -a \leq x$ and $x \leq 0 \leq a \Rightarrow x \leq a \Rightarrow -a \leq x \leq a$. Combining these cases gives that $-a \leq x \leq a$ in all cases.

(b) "$\Leftarrow$ ". Let $-a \leq x \leq a \Rightarrow a \geq -x \geq -a \Rightarrow -a \leq -x \leq a$. Because $|x| = x$ or $|x| = -x$, it follows that $-a \leq |x| \leq a \Rightarrow |x| \leq a$

(c) Let $a \equiv |x| \geq 0$, then $|x| \leq a = |x|$. Also, it follows from (b) that $-a \leq x \leq a$ which can also be seen as $-|x| \leq x \leq |x|$.

1.3 The triangle inequality

About estimating absolute values of sums. Very important to analysis. Possibly most important in all of mathematics.

$\forall x, y \in \mathbb{R} : |x + y| \leq |x| + |y|$

1.4 Proof

By Previous theorem part c we have $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. The trick to the proof involves adding these inequalities together.

This gives $-(|x| + |y|)_{\text{Let this be "-a"}} \leq x + y \leq (|x| + |y|)_{\text{Let this be "a"}}$. It follows from previous theorem part b that $|x + y| \leq a = |x| + |y| \Rightarrow |x + y| \leq |x| + |y|$. This theorem (the triangle inequality) is used to find the upper bounds of sums.

Next theorem helps with lower bounds:

Theorem - $\forall x, y \in \mathbb{R} : |x - y| \geq |x| - |y|$ and $|x - y| \geq |y| - |x|$. This one is called the triangle inequality for sums.

1.5 Proof

$|x| = |x - y + y| \leq |x - y| + |y| \Rightarrow |x| - |y| \leq |x - y| \Rightarrow |x - y| \geq |x| - |y|$.

Interchange x and y (to avoid redoing the proof): $|y - x| = |x - y| \geq |y| - |x| \Rightarrow |x - y| \geq |y| - |x|$

Remark: $|x - y| \geq |x| - |y|$ and $|x - y| \geq |y| - |x|$ can be combined to $\Rightarrow |x - y| \geq ||x| - |y||$. This final equation looks nice but can be hard to but into practice. It is normally easier to pick the correct of the other two equations.

Theorem - Generalized Triangle Inequality. Let $x_1, \ldots, x_n \in \mathbb{R}$, then $|x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n|$

Proof of this is on assignment 3.

1.6 Moving on. Absolute values are needed in the following definition:

Definition: ϵ neighborhood

Let $\epsilon > 0$ and let $a \in \mathbb{R}$, the ϵ neighborhood of a is defined as $V_\epsilon(a) \equiv \{x \in \mathbb{R} : |x - a| \leq \epsilon\}$

$|x - a| < \epsilon \Leftrightarrow -\epsilon < x - a < \epsilon \Leftrightarrow a - \epsilon < x < a + \epsilon.$. This leads to $V_\epsilon(a) =]a - \epsilon, a + \epsilon[$

Theorem - if $x \in V_\epsilon(a)$ for all $\epsilon > 0$, then $x = a$

1.7 Proof

Assume that $x \neq a$ and find a contradiction.

First case: $x > a$. Let $\epsilon = x - a > 0$, then $a + \epsilon = x \Rightarrow x \ni]a - \epsilon, a + \epsilon[= V_\epsilon(a)$

Second case: $x < a$. Let $\epsilon = a - x$. Prove the rest yourself.

This theorem implies the following.

$$\bigcap_{\epsilon > 0} V_\epsilon(a) = \{a\}$$

2 Supremum and Infimum

Def: Let $s \subset \mathbb{R}, s \neq \varnothing$. We say that:

S is bounded from above if $\exists u \in \mathbb{R}$ such that $\forall s \in S : s \leq u$. Upper bound follows same idea.

2.1 Examples

(1) $S = [0, 1[$.

Then 1, 2, π, 1.5 are all upper bounds for S, and 0, -1, ... are lower bounds for S.(This answers my question about whether or not an upper or lower bound has to be right at the bound.)

(2) $A = [1, \infty[$ is not bounded from above.

Definition: Let $S \subset \mathbb{R}, S \neq \varnothing$, S is bounded from above. $s \in \mathbb{R}$ is called the <u>SUPREMUM</u> or <u>Least upper bound</u> of S. Symbolically: $s = \text{supr}S$ if:

(1) s is an upper bound for S. (2) $\forall t$ upper bounds of S, $s \leq t$.

Similary for Infimum. Definition: Let $S \subset \mathbb{R}, S$ is bounded from below. A number $u \in \mathbb{R}$ is called the infimum of S if u is a lower bound of S and $\forall t$ lower bounds of S, $u \geq t$

2.2 Examples

$S = [0, 1[$. Claim that $\inf S = 0$. Proof: 0 is indeed lower bound of S ✓. Let v be any lower bound for S. This lower bound cannot be positive because if it was $0 < v$ and so it wouldn't be a lower bound. $\Rightarrow v \leq 0 \Rightarrow 0$ is the infimum of S.

No supremum in this case (THIS IS WHAT I THOUGHT INITIALLY BUT I WAS WRONG). Claim: $\text{supr}S = 1$. Proof: 1 is an upperbound of S ✓. Let v be any upper bound of S. If we assume that v is less than 1, we get contradiction that v is not an upper bound of S. Therefore $v \geq 1$. Therefore $1 = \text{supr}(S)$.

Questions: Given any non empty set $S \subset \mathbb{R}$ bounded from above, must there be a supremum? Same idea of question for bounded below infimum. Complicated answers to these questions. Postpone this to next class.