§1 Lecture 01-22

§1.1 Finite Dimensional

$B = (v_1, \dots, v_n)$, a basis for V.

If $v \in V$, then $\exists!(x_1, \dots, x_n) \in F^n$ such that $v = x_1 v_1 + \dots + x_n v_n$. (The exclamation points indicates uniqueness).

The n-tuple $(x_1, \dots, x_n)$ are called the coordinates of v in B.

This sets up an isomorphism between $V \simeq_B F^n$.

Any vector space of dimension n "is" F^n (is non-canonically isomorphic to F^n). This non-canonicalness is reflected in the dependence on a basis.

Note 1.1. If $(x_1, \dots, x_n)$ are the coordinates of v relative to B, then

$$v = \begin{pmatrix} v_1 \cdots \cdots v_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix}$$

If $T : V_1 \to V_2$ is a linear transformation, then T can be described by a matrix $M_{T,B_1,B_2} \in M_{m \times n}$.

$$V_1 \to_T V_2$$
$$V_1 \simeq_{B_1} F_1^n$$
$$V_2 \simeq_{B_2} F_2^n$$
$$F_1^n \to_{M_{T,B_1,B_2}} F_2^n$$

Properties:

1. Let $B_1 = (v_1, \dots, v_n)$. $B_2 = (w_1, \dots, w_m)$ be bases for V_1 and V_2.

$$T(v_1) = a11w_1 + a_{21}w_2 + \dots + a_{m1}w_m$$
$$T(v_2) = a12w_1 + a_{22}w_2 + \dots + a_{m2}w_m$$
$$\vdots$$
$$T(v_n) = a1nw_1 + a_{2n}w_2 + \dots + a_{mn}w_m$$
$$M_{T,B_1,B_2} = (a_{ij})_{i \le i \le m, 1 \le j \le n}$$
$$(T(v_1), T(v_2), \dots, T(v_n)) = (w_1, \dots, w_m) M_{T,B_1,B_2}$$
$$T(B_1) = B_2 M_{T,B_1,B_2}$$

2. Effect of T on coordinates

$$v = \begin{pmatrix} v_1 \cdots \cdots v_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + \cdots + x_n v_n$$

$$T(v) = T(x_1 v_1 + \cdots + x_n v_n) = x_1 T(V_1) + \cdots + x_n T(v_n) = \begin{pmatrix} T(v_1) \ldots \ldots T(v_n) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \big((w_1, \cdots, w_m) M_{T,B_1,B_2}\big) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \big(w_1, \cdots, w_m\big) \big(M_{T,B_1,B_2}\big) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

3. Conclusion:

   The column vector

§1.2 Important Special Case of Transformation to itself

$V_1 = V_2 = V$. $T : V \to V$. Choose $B = (v_1, \ldots, v_n)$.

$M_{T,B}$ is the matrix of T relative to $B \in M_{n\times n}(F)$

$$(T(v_1), \ldots, T(v_n)) = (v_1, \ldots, v_n) M_{T,B}$$

This gives an identification

$$\mathrm{Hom}_F(V, V) = \mathrm{End}_F(V) \simeq_B M_n(F)$$

Dependency of $M_{T,B}$ on B. Let B and B' be two bases. Then there eixst unique matrices, P, P' such that $B' = BP$.

$$B = (v_1, \ldots, v_n)$$
$$B' = (v_1', \ldots, v_n')$$
$$T(B) = BM_{T,B}$$
$$T(B') = B'M_{T,B'}$$
$$B' = BP$$
$$T(BP) = BPM_{T,B}$$
$$T((v_1, \ldots, v_n)P) = (T(v_1), \ldots, T(v_n))P$$
$$T(B)P = BPM_{T,B'}$$
$$BM_{T,B}P = BPM_{T,B'}$$
$$M_{T,B} = PM_{T,B'}$$

Note 1.2. P is invertible

Proof.

$$(v_1', \ldots, v_n') = (v_1, \ldots, v_n)P(v_1, \ldots, v_n) = (v_1', \ldots, v_n')P'$$
$$\Rightarrow (v_1', \ldots, v_n') = (v_1', \ldots, v_n')P'P$$
$$\Rightarrow P'P = E_{n\times n}$$

□

So

$$M_{T,B'} = P^{-1} M_{T,B} P$$

Definition 1.3. Matrices v in $M_n(F)$ which are related by $M_1 = P^{-1} M_2 P$ for some $P \in M_n(F)^X$ are conjugate.

Theorem 1.4

If M_1 and M_2 in $M_n(F)$ represent the same linear transformation $T : V \to V$ in different bases, they are conjugate.

Even though the matrices are not unique, they are conjugate to one another based on the basis.

What functions $\varphi M_n(F) \to F$ are invariant under conjugation.

$$\varphi(A) = \varphi(PAP^{-1})$$

for all P invertible.