§1 Lecture 01-20

Textbook Correction. Zorn's Lemma. It doesn't imply "the maximal" element, but rather "a maximal" element. This translates over to its application of proving that every vector space has a basis, not "the" basis (Multiple vs. Single).

§1.1 Quotients of Vector Spaces

$W \subset V$. W a subspace.

$$V/W = \{v + w : v \in V\}$$
$$\lambda \in F, \quad \lambda(v + W) = \lambda v + W$$

If $v_1 + W = v_2 + W$, then $\lambda v_1 + W = \lambda v_2 + W \; \forall \lambda \in F$. This implies that $(v_1 - v_2) \in W \Rightarrow \lambda(v_1 - v_2) \in W \Rightarrow_1 - \lambda v_2 \in W$.

Theorem 1.1

If V is finite dimensional and $W \subseteq V$ is a subspace, then W and V/W are both finite dimensional.

$$\dim(V) = \dim(W) + \dim(V/W)$$

This makes sense, because (V/W) reduces the dimension by W, because suddenly all elements in W are considered equal to one another. So the dimension behaves like a logarithm in a sense.

Proof: Subspace of finite dimensional vector space is finite dimensional.

Let $d = \dim(W)$. Let $(v_1, \ldots, v_d)$ be a basis for W. Therefore $(v_1, \ldots, v_d)$ is linearly independent in V. We can complete it to a basis for V, $(v_1, \ldots, v_d, v_{d+1} + \cdots + v_n$. Where $n = \dim(V)$.

Basis for V/W. There are associated cosets for $v_{d+1}, \ldots, v_n$. It would be incorrect to say that the basis is $v_{d+1}, \ldots, v_n$, because these elements don't live in the quotient. Claim: The $(n-d)$-tuple $(v_{d+1} + W, \ldots, v_n + W)$ is a basis for V/W.

Proof of linear independence. Let $(\lambda_{d+1}, \ldots, \lambda_n) \in F^{n-d}$.

$$\lambda_{d+1}\overline{v_{d+1}} + \cdots + \lambda_n \overline{v_n} = 0 \text{ in } (V/W).$$
$$\Rightarrow \lambda_{d+1} v_{d+1} + \cdots + \lambda_n v_n \in W$$

$$\text{Hence } \exists(\lambda_1, \ldots, \lambda_d) \in F^d \text{ s.t. } \lambda_{d+1} v_{d+1} \cdots + \lambda_n v_n = \lambda_1 v_1 + \cdots + \lambda_d v_d$$
$$\text{This works because } (v_1, \ldots, v_n) \text{ span } W.$$
$$\text{Because } (v_1, \ldots, v_n) \text{ are linearly independent}$$
$$\Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$$
$$\Rightarrow \lambda_{d+1} = \cdots = \lambda_n = 0$$

□

□

Proof. $(_{d+1}, \dots, \overline{v_n})$ spans V/W.

Let $v + W \in V/W$. Because $v \in V$, $\exists(\lambda_1, \dots, \lambda_n) \in F^n$ such that $v = \lambda_1 v_1 + \cdots + \lambda_d v_d + \cdots + \lambda_n v_n$.

In V/W.

$$\overline{v} = \lambda_1 \overline{v_1} + \cdots + \lambda_d \overline{v_d} + \cdots + \lambda_n \overline{v_n}$$
$$\Rightarrow \overline{v} = \lambda_{d+1} \overline{v_{d+1}} + \cdots + \lambda_n \overline{v_n}$$

because the other vectors are all in W. □

Note 1.2.

$$\overline{v} = v + W$$

$v \in V$, $\overline{v} \in V/W$.

Theorem 1.3 (Isomorphism Theorem)
If $T : V \to W$ is a linear transformation, then T induces an <u>injective</u> linear transformation

$$\overline{T} : V/\ker T \hookrightarrow W$$

In particular, $V/\ker(T) \simeq \mathrm{Im}(T)$.

$$\overline{T}(v + \ker(T)) = T(v)$$

$\overline{T}$ is injective.

$$\overline{T}(v + W) \Leftrightarrow T(v) = 0 \Leftrightarrow v \in \ker(T)$$
$$\Leftrightarrow v + \ker(T) = 0 \text{ in } V/\ker(T)$$

Theorem 1.4 (Rank-nullity theorem)
Let $T : V \to W$ be a linear transofmration with $\dim(V) < \infty$. Then $\dim\ker(T) + \dim\mathrm{Im}(T) = \dim(V)$.

Proof.

$$V/\ker(T) \simeq \mathrm{Im}(T)$$
$$\dim(V/\ker(T)) = \dim\mathrm{Im}(T)$$
$$\dim(V) - \dim\ker(T) = \dim\mathrm{Im}(T)$$

□

Remark 1.5. If $H \subset G$ is a group, $\#\ G < \infty$, then $\#(G/H) = \#G/\#H$

A vector space V is finite as a set $\Leftrightarrow \#F < \infty$ and $\dim_F(V) < \infty$. Let $q = \#F$ and $n = \dim_F(V)$. Then $\#V = q^n$. $\dim(V) = \log_q(\#V)$. $V \simeq F^n$.

Theorem 1.6 (Counting Principle)
If A and B are finite sets of the same cardinality, and $f : A \to B$ is an injective function, then f is surjective.