§1 Lecture 01-31

Calculate (k, n) = collection of k-dimensional subspaces in a vector space of dim n over F.

Strategy: First count the number of k-tuples $(v_1, \ldots, v_k)$ of linearly independent vectors.

Possibilities for $v_1 = q^n - 1$. Possibilities for $v_2 = q^n - q$. Possibilities for $v_3 = q^n - q^2$. Possibilities for $v_k = q^n - q^{k-1}$.

Let $\Sigma :=$ the set of ordered k-tuples of linearly independent vectors.

Σ injects into (k, n) with the function $(v_1, \ldots, v_k) \mapsto (v_1, \ldots, v_k)$.

Given a $W \subset V$ of dim k, how many $(v_1, \ldots, v_k)$ span W. i.e. how many bases does W have?

Choices for $v_1 = q^k - 1$. Choises for $v_2 = q^k - q$. Choices for $v_k = q^k - q^{k-1}$. Therefore $p^{-1}(W) = (q^k - 1)(q^k - q)\cdots(q^k - q^{k-1})$.

$$\begin{aligned}
\#(k,n) &= \frac{(q^n - 1)(q^n - q)(\cdots)(q^n - q^{k-1})}{(q^k - 1)(q^k - q)(\cdots)(q^k - q^{k-1})} \\
= \#(k,n) &= \frac{(q^n - 1)(q^{n-1} - 1)(\cdots)(q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1)(\cdots)(q^1)} \\
&= \binom{n}{k}_q \\
\binom{n}{k} &= \frac{n * (n-1) * \cdots * (n-k+1)}{k * (k-1) * \cdots * 1}
\end{aligned}$$

Note 1.1.

$$\begin{aligned}
\frac{q^j - 1}{q - 1} &= [j]_q = 1 + q + q^2 + \cdots + q^{j-1} \\
[j]_q! &= [1]_q[2]_q\cdots[j]_q \\
\binom{n}{k}_q &= \frac{[n]_q!}{[k]_q![n-k]_q}
\end{aligned}$$

How many subsets of size k are there in a set of size n? This question is linked to how many subspaces of dimension k are there in a vector space of dimension n.

$$\binom{n}{k} \quad \binom{n}{k}_q$$

$$x \mapsto \mathcal{F}(x, F),\ \mathcal{F}_0(x, F)$$

$$\lim_{q \to 1} \binom{n}{k}_q = \binom{n}{k}$$

$$\binom{n}{k}_q = \frac{(q^n - 1)(q^{n-1} - 1)(\cdots)(q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1)(\cdots)(q^1)}$$