§1 Isomorphisms Continued

Theorem 1.1

If G is cyclic and $|G| = n$, then $G \equiv \mathbb{Z}_n$.

Proof. Consider $\phi : \mathbb{Z}_n \to G$ given by $\phi(i) = g^i$, then ϕ is a bijection.
Injective: $\phi(i) = \phi(j) \Rightarrow g^i = g^j \Rightarrow g^{i-j} = g^0 \Rightarrow i - j \equiv_n = 0 \Rightarrow i = j$
Surjective: Let $G = \langle g \rangle$.
$\{g^0, g^1, \ldots, g^{n-1}\} = G$
$\{0, 1, \ldots, n-1\} = \mathbb{Z}_n$ $\square$

Theorem 1.2

Cor 9.9.
If $|G| = p$ and p is prime, then $G \equiv\sim \mathbb{Z}_p$

Proof. We showed that $G = \langle g \rangle$ for any $g \neq e$.
My understanding: if prime order, it must be cyclic. $\square$

Theorem 1.3

Isomorphism is an equivalence relation on a set of groups.
Reflexive: $G \equiv\sim G$ because $1_G : G \to G$ is isomorphism.

$$1_G(ab) = ab = 1_G(a) \cdot 1_G(b)$$

Symmetrical: $G \equiv\sim K \Rightarrow K \equiv\sim G$ because $\phi : G \to K$ isomorphism then $\phi^{-1} : K \to G$ is isomorphism.
Transitive: $f : G \to K$ and $h : K \to J$ are isomorphisms then $h \circ f : G \to J$ is ismorphism.

Theorem 1.4 (Cayley's Theorem)
Every group is isomorphic to a permutation group.

Recall 1.5. A permutation group is a subgroup of S_n

Proof. G is isomorphic to a subgroup of the group of bijections of the set G. You could think of this as S_G.

For $g \in G$, let $\lambda_g : G \to G$ be permutation "left multiply by g" i.e. $\lambda_g(x) = gx$ for all $x \in G$.

Let $\overline{G} = \{\lambda_g : g \in G\}$

Claim: $G \cong \overline{G}$ with $\phi(g) = \lambda_g$

Injectivity: if $\phi(x) = \phi(y)$ then λ_x and λ_y are some bijection of G.

$$x = xe = \lambda_x(e) = \lambda_y(e) = ye = y$$

Surjectivity (immediate). $\overline{G} = \{\lambda_g : g \in G\} = \{\phi(g) : g \in G\} = \phi(G)$

Homomorphism:

$$\begin{gathered}
\phi(xy) = \lambda_{xy} \\
\phi(x)\phi(y) = \lambda_x\lambda_y \\
\lambda_{xy}(z) = (xy)z \text{ for all } z \in G \\
\lambda_x(\lambda_y(z)) = \lambda_x(yz) = x(yz) \\
(xy)z = x(yz) \checkmark
\end{gathered}$$

□

Example 1.6

$$\begin{gathered}
G = \{\pm 1, \pm i\} \\
G \cong G \subset S_G \cong S_4 \\
1 \to \lambda_1 = \begin{bmatrix} 1 & -1 & i & -i \\ 1 & -1 & i & -i \end{bmatrix} = () \\
-1 \to \lambda_{-1} = \begin{bmatrix} 1 & -1 & i & -i \\ -1 & 1 & -i & i \end{bmatrix} = (1\ \ -1)(i\ \ -i) \\
i \to \lambda_i \begin{bmatrix} 1 & -1 & i & -i \\ i & -i & -1 & 1 \end{bmatrix} = (1\ i\ \ -1\ \ -i) \\
-i \to \lambda_{-i} = \begin{bmatrix} 1 & -1 & i & -i \\ -i & i & 1 & -1 \end{bmatrix} = (1\ \ -i\ \ -1\ i)
\end{gathered}$$

Example 1.7

$$Q_8 \cong \overline{Q_8} \subset S_8$$

Example 1.8

$$\mathbb{Z}_6 \hookrightarrow S_{\mathbb{Z}_6} = S_{\{0,1,2,3,4,5\}}$$

$$2 \to_\phi \lambda_2 \qquad \lambda_2 : \mathbb{Z}_6 \to \mathbb{Z}_6 \qquad \lambda_2(x) = 2 + x$$

$$\lambda_2 = (0\ 2\ 4)(1\ 3\ 5)$$

$$\lambda_3 = (0\ 3)(1\ 4)(2\ 5)$$

$$\lambda_5 = (0\ 5\ 4\ 3\ 2\ 1)$$