§1 Lecture 11-13

Definition 1.1. Let I be an ideal of R. Then $\phi : R \to R/I$ is a canonical homomorphism associated to I.

$$\phi(r) = r + I$$
$$\phi(xy) = xh + I$$
$$\phi(x)\phi(y) = (x + I)(y + I)$$

§1.1 Maximal and Prime Ideals

Definition 1.2. An ideal $M \subseteq R$ is maximal if the only ideal larger than M is R itself.

i.e. There does not exist ideal I with $M \subsetneq I \subsetneq R$.

Definition 1.3. An ideal $P \subsetneq R$ where R is commutative is prime if for all $a, b \in R$, $ab \in P \Rightarrow [a \in P \text{ or } b \in P]$.

Example 1.4

A proper ideal $n\mathbb{Z} \subset \mathbb{Z}$ is maximal $\Leftrightarrow$ $n\mathbb{Z} \subset \mathbb{Z}$ is prime $\Leftrightarrow$ n is a prime number. Reasoning:

$n\mathbb{Z} \subsetneq m\mathbb{Z} \subsetneq \mathbb{Z}$ if and only if $m|n$ but $m \neq 1$ and $m \neq n$ i.e. n is not prime.

$(ab \in n\mathbb{Z}) \Leftrightarrow n|(ab)$ but if n is prime then $n|(ab) \Leftrightarrow n|a$ or $n|b$. Hence $a \in n\mathbb{Z}$ or $b \in n\mathbb{Z}$.

If n is not prime, then $n = xy$ where $1 < x, y < n$ and $xy \in n\mathbb{Z}$ but $x \notin n\mathbb{Z}$ and $y \notin n\mathbb{N}$. This would mean that $n\mathbb{Z}$ is not prime.

Example 1.5

In $\mathbb{Z}[x]$, the ideal $\langle x \rangle$ is prime but not maximal.

Maximal Proof:

$\langle x \rangle$ is not maximal because $\langle x \rangle \subsetneq \langle x, 2 \rangle \subsetneq \mathbb{Z}[x]$

$\langle x, 2 \rangle$ consists of all polynomials of the form $f \cdot x + g \cdot 2$ (where $f, g \in \mathbb{Z}[x]$). i.e. all polynomials whose consant term is even.

$\langle x \rangle$ consists of all polynomials of the form $f \cdot x$. i.e. all polynomials whose constant term is zero.

Prime Proof:

$\langle x \rangle$ is prime because $f \cdot g \in \langle x \rangle \Rightarrow (f \in \langle x \rangle$ or $g \in \langle x \rangle$ because if both f and g have non zero constant term than $f \cdot g$ has a non zero constant term.

Theorem 1.6

Let R be a commutative ring with 1. Let $I \subsetneq R$ be a proper ideal. Then:

I is maximal $\Leftrightarrow R/I$ is a field.

I is prime $\Leftrightarrow R/I$ is an integral domain.

Example 1.7

Let $R = \mathbb{R}[x]$, and $I = \langle x^2 + 1 \rangle$. Then $R/I \cong \mathbb{C}$. Note the following for gaining an intuition:

$$(x + I)(x + I) = (x^2 + I)$$
$$(x^2 + I) + (1 + I) = (x^2 + 1 + I) = 0 + I \Rightarrow (x^2 + I) = (-1 + I)$$

$$i \leftrightarrow x + I$$
$$1 \leftrightarrow 1 + I$$

Going through an a demonstration:

$$7x^3 - 3x^2 + x + 9 + I \leftrightarrow ? \in \mathbb{C}$$
$$(7x^3 + I) + (-3x^2 + I) + (x + I) + (9 + I)$$
$$(7x + I)(x^2 + I) + (-3 + I)(x^2 + I) + (x + I) + (9 + I)$$
$$(7x + I)(-1 + I) + (-3 + I)(-1 + I) + (x + I) + (9 + I)$$
$$(-7x + I) + (3 + I) + (x + I) + (9 + I)$$
$$(-6x + I) + (12 + I)$$