§1 10-30

§1.1 Classification of finitely generated abelian groups

Definition 1.1. A group G is generated by a subset of its elements $\{g_1, g_2, \dots\}$ if $\{g_1, g_2, \dots\}$ is not contained in any proper subgroup.

Equivalently, every element $h \in G$ can be expressed as $h = x_1 x_2 x_3 \dots x_n$ where each $x_i \in \{g_1^{\pm 1}, g_2^{\pm 2}, \dots\}$.

Note 1.2. We then write $G = \langle g_1, g_2, \dots \rangle$

Example 1.3

1. $\mathbb{Z} = \langle 1 \rangle = \langle 2, 3 \rangle$
2. $S_n = \langle (ij) : i \neq j \rangle$

Definition 1.4. G is finitely generated if $G = \langle g_1, \dots, g_n \rangle$ for some finite set $\{g_1, \dots, g_n\}$.

Example 1.5

1. Every finite group is finitely generated because it is generated by the group itself.
2. $\mathbb{R}$ is not finitely generated because it is uncountable.
   Finitely generated $\Rightarrow$ countable.
3. Any finitely generated subgroup of $\mathbb{R}$ is isomorphic to $\mathbb{Z}^m$ for some $m \geq 0$.
$$\langle \sqrt{2}, \pi, e \rangle \cong \mathbb{Z}^3$$
4. $\mathbb{Q}$ is not finitely generated. Every subgroup of $\mathbb{Q}$ is infinite cyclic.
5. $\underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots}_{\infty}$ is not finitely generated.

Theorem 1.6 (Fundamental theorem of finitely generated abelian groups.)
Let G be a finitely generated abelian group. Then G is isomorphic to a product of infinitely many cyclic groups (finite or infinite).

Example 1.7

$$G \cong \underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots}_{m \geq 0} \times \underbrace{\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}}_{k \geq 0}$$

Moreover,

$$G \cong \mathbb{Z}_0^m \times \underbrace{\mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \cdots \times \mathbb{Z}_{p_k^{m_k}}}_{k \geq 0}$$

where each p_i is prime. Moreover, decomposition is <u>unique</u> up to permuting factors.

Example 1.8
What are all possible abelian groups of order $1000 = 2^3 \cdot 5^3$.

$$\mathbb{Z}_{5^3} \times \mathbb{Z}_{2^3}$$
$$\mathbb{Z}_{5^3} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^1}$$
$$\mathbb{Z}_{5^3} \times \mathbb{Z}_{2^1} \times \mathbb{Z}_{2^1} \times \mathbb{Z}_{2^1}$$
$$\mathbb{Z}_{5^2} \times \mathbb{Z}_{5^1} \times \mathbb{Z}_{2^3}$$
$$\mathbb{Z}_{5^2} \times \mathbb{Z}_{5^1} \times \mathbb{Z}_{2^2} \times \mathbb{Z}^{2^1}$$
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$$\mathbb{Z}_{5^1} \times \mathbb{Z}_{5^1} \times \mathbb{Z}_{5^1} \times \mathbb{Z}_{2^3}$$
$$\mathbb{Z}_{5^1} \times \mathbb{Z}_{5^1} \times \mathbb{Z}_{5^1} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^1}$$
$$\mathbb{Z}_{5^1} \times \mathbb{Z}_{5^1} \times \mathbb{Z}_{5^1} \times \mathbb{Z}_{2^1} \times \mathbb{Z}_{2^1} \times \mathbb{Z}_{2^1}$$