§1 Lecture 01-15

Assignment 1 due today. Burnside Hall 10th floor mail slot.
Basis for a vector space.

Theorem 1.1
If V is a vector space over F, then V has a basis. i.e. $\exists B \subset V$ which is linearly independent and spans V.

Proof. Let B be a maximal linearly independent subset of V. This ensures that is spans V. □

Example 1.2
$V = F[x]$. $B = \{1, x, x^2, x^3, \dots\}$. The fact that this is a basis is the statement that every polynomial can be written as a finite combination of powers of x.

Example 1.3
$V = F[[x] = \{\sum_{i=0}^{\infty} a_i x^i,\ a_i \in F$. Infinite linear combination of powers of x. No one has ever written down a basis for this vector space. Although there muts be one according to the theorem.

Example 1.4
$V = \mathbb{R}$ as a vector space over the rationals. B is called a Hanel basis. Source of counter examples in measure theory. Gives rise to non measurable set. Pathalogical set.

Example 1.5
Take $V = F^n = \{(a_1, \dots, a_n),\ a_i \in F\}$. You can take

$$B = \{(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 0, 1)\} = \{e_1, e_2, \dots, e_n\}$$

The "standard basis". Bases are typically most useful when they are finite.

Definition 1.6 (Finite-dimensional)**.** A vector space which has a finite-basis is said to be <u>finite-dimensional</u>.

§1.1 The Dimension

Theorem 1.7
If V is a finite dimensional vector space, and B_1, B_2 are two basesfor V, then the conclusion is that B_1 and B_2 are both finite and have the same cardinality.

Recall 1.8. If B is a basis for V, then V is isomorphic to the space of functions $F_0(B, F) = \{$Space of functions $f : B \to F$ such that $f(x) = 0\ \forall$ but finitely many

$x \in B$.

$$\varphi : F_0(B, F) \to V$$
$$f \mapsto \sum_{x \in B} f(x) \cdot x$$

If $B < \infty$, then $F_0(B, F) = F(B, F) = F^N,\ N = B$. (i.e., can assume $B = \{1, \dots, N\} \to \{v_1, \dots, v_N\}$.

$$\varphi : F^n \to V$$
$$(a_1, \dots, a_n) \mapsto a_1 v_1 + \dots + a_N v_N$$

Reformulation of theorem.
If $F^{n_1} isomorphic F^{n_2}$, then $n_1 = n_2$.

Lemma 1.9
Let $v_1, \dots, v_m$ be a collection of linearly independent vectors in F^n. Then $m \leq n$.

Proof. If $v_1 = (a_{1_1}\ a_{1_2}\ \dots\ a_{1_n})\ \dots\ v_m = (a_{m_1}\ a_{m_2}\ \dots\ a_{m_n})$ are linearly independent.

$$x_1 v_1 + \dots + x_m v_m = 0 \Leftrightarrow (x_1, \dots, x_m) = 0$$

Gives rise to homogenous system of linear equations. There are n linearly equation with m unknowns.

The system must have a non-trivial solution if $n < m$. Since we are told that there is only a trivial solution, it must be that $m \leq n$. $\square$

Example 1.10
If $F^{n_1} isomorphic F^{n_2}$. Let

$$\varphi : F^{n_1} \to F^{n_2}$$
Let $e_1, \dots, e_{n1}$ be the standard basis of F^{n_1}.
$\varphi(e_1), \dots, \varphi(e_{n1})$ are linearly independent in $F^{n_2} \Rightarrow n_1 \leq n_2$.
By symmetry $n_2 \leq n_1$
$$\Rightarrow n_1 = n_2$$

Definition 1.11 (Dimension)**.** The dimension of V is the cardinality of a basis for V.

Convention:

$$\dim(V) \in \{0, 1, 2, 3, \dots, \} \cup \{\infty\}.$$

$\dim(V) = \infty$ if V contains an infinite collection of linearly independent vectors.

§1.2 Completing to a basis

Proposition 1.12

If S_0 is a collection of linearly independent vectors in V, then $\exists$ a basis such that $S \supseteq S_0$.

Proof. Let L be the set of linearly independent subsets of V containing S_0. iLet B be a maximal element of L. □

Example 1.13

Let X be a set.

1. $\dim_F F_0(X, F) = X$
2. $\dim_F(F^n) = n$. Dimension is kind of like the logarithm base F of the cardinality.
3. $\dim(V_1 \times V_2) = \dim(V_1) + \dim(V_2)$