Math #235 Notes

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Contents

§1 Proof of $(A \cup B)' = A' \cap B'$

In order to prove $A = B$, prove that $A \subset B$ and $A \supset B$

§1.1 Proof of $(A \cup B)' \subset A' \cap B'$

Proving that let $x \in (A \cup B)'$

$$\begin{aligned} &\to x \notin A \cup B \\ &\to x \notin A \text{ and } \notin B \\ &\to x \in A' \text{ and } x \in B' \\ &\to x \in A' \cap B' \end{aligned}$$

§1.2 Proof of $(A \cup B)' \supset A' \cap B'$

$$\begin{aligned} \text{let } x \in A' \cap B' \\ \to x \in A \text{ and } x \in B' \\ \to x \notin A \text{ and } x \notin B \\ \to x \notin A \cup B \\ \to x \in (A \cup B)' \end{aligned}$$

§2 Product of Sets

$$\begin{aligned} A \times B = \{(a,b) : a \in A \text{ and } b \in B\} \\ \text{i.e. } A^n = A \times A \times A \ldots \\ \mathbb{R} = R \times R \end{aligned}$$

§3 Relations and Functions

A relation from A to B is a subset of $A \times B$

A map or funciton from A to B is a relation where $f \subset A \times B$ such that for each $a \in A$ there exists a unique $(a, b) \in f$

Notation: $f : A \to B$

Think of it as $f(a) = b$ instead of $(a, b) \in f$

A is domain of f, B is codomain or target of f

image of f is $f(A) = \{f(a) : a \in A\}$

Example: $f(A) = \{(1, y), (2, y), (3, y)\}$

image $= f(A) = \{y, z\}$

Definition: $f : A \to B$ is subjective if $f(A) = B$

Definition: $f : A \to B$ is injective or one-to-one or "into" if there does not exist $a \in A$ and $b \in A$ such that $f(a) = f(b)$

§4 Composite Functions

$f : A \to B$
$g : B \to C$
Composition $g \circ f$ is a function. $g \circ f : A \to C$
$(g \circ f)(a) = g(f(a))$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \quad \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$g \circ f(1) = g(f(1)) = g(y) = r$$
$$g \circ f(2) = g(f(2)) = g(y) = r$$
$$g \circ f(3) = g(f(3)) = g(z) = r$$

Theorem 1.1.8: The quick brown fox jumps right over the lazy dog. the quick brown fox jumps right over the lazy dog. the quick brown fox jumps right over the lazy dog. the quick brown fox jumps right over the lazy dog. the quick brown fox jumps right over the lazy dog. the quick brown fox jumps right over the lazy dog. the quick brown fox jumps right over the lazy dog.

§5 Lecture 2019-09-09

Theorem 5.1
Suppose $f : A \to B$ and $g : B \to C$ are surjective then $g \circ f : A \to C$ is surjective
Proof: $c \in C$
since $g : B \to C$ there exists $b \in B$ s.t. $g(b) = c$ since $f : A \to B$ is surjective, exists $a \in A$ s.t. $f(a) = b$
thus $(g \circ f)(a) = g(f(a)) = g(b) = c$

Definition 5.2. A function $g : b \to A$ is inverse to function $f : A \to B$ if:

$$f \circ g = 1_B$$
$$g \circ f = 1_A$$

$$A \to^f B \to^g C$$
$$A \to^f B \to^g A$$
$$g \circ f = 1_A$$

$$B \to^g A \to^f B$$
$$f \circ g = 1_B$$

Note 5.3. They say f and g are inverible, use notation f^-1 for inverse of f

Theorem 5.4
Let $f : A \to B$ be a map: f is invertible if and only if f is a bijection

$$P \Leftrightarrow Q$$
$$P \Leftarrow Q$$
$$P \Rightarrow Q$$

Proof that f is invertible means f is a bijection:

$$\text{let } g = f^{-1} \text{ f is surjective since for all } b \in B$$
$$\text{we have } \; f(g(b)) = f \circ g(b) = 1_B(b) = b$$

$$\text{f is injective since if } f(a_1) = f(a_2) \Rightarrow g(f(a_1)) = g(f(a_2))$$

injective: if $f(a_1) = f(a_2)$, then $a_1 = a_2$

Proof that f is a bijection means f is invertible define $f^{-1} : B \to A$ thus:

for each $b \in B$, there exists $a \in A$ s.t. $f(a) = b$ and a is unique with this property (by injectivity)

define $f^{-1}(b) = a$ then $f \circ f^{-1}(b) = f(f^{-1}(b)) = f(a) = b$ $f^{-1} \circ f(a) = f^{-1}(f(a)) = f^{-1}(b)$

so $f \circ f^{-1} = 1_B$

Definition 5.5 (Equivalence Relation)**.** Equivalence Relation on a set X is a relation $R \subset X \times X$

$$R \text{ is reflexive } (x, x) \in R \text{ for all } x \in X$$
$$\text{is symmetric } (x, y) \in R \to (y, x) \in R$$
$$\text{is transitive } (x, y) \in R \text{ and } (y, z) \in R \to (x, z) \in R$$

Note 5.6. Usually denote equiv relations by $x \sim y$ instead of $(x, y) \in R$

$$\text{or } x = y$$
$$x \equiv y$$

Definition 5.7. A partition of X is a collection of disjoint nonempty subsets of X whose union is X

Example 5.8
$\{X_k : k \in K\}$ $x_i \cap x_j = \varnothing$ for $i \neq j$
$\{1,2,3,4,5,6,7,8,9,0\} = \{1,4,5\} \cup \{6\} \cup \{9\} \cap \{2,3,7,9,0\}$
$X = X_1 \cup X_2 \cup X_3 \cup X_4$

§5.1 Creating a partition

let x be a set with equivalence relation $\sim$ for $y \in X$, let $[y] = \{x \in X : x \sim y\}$
$[y]$ is the equivalence class represented by y

Theorem 5.9
Theorem 1.25: The equivalence classes of an equivalence relation ($\sim$) form a partition of X.

Proof:

1. each equiv class is nonempty since $y \in [y]$
2. equiv classes are either disjoint or equal since if $y \in [a]$ and $y \in [b]$

then $[a] \subset [b]$ since $c \in [a] \Rightarrow c \sim a \Rightarrow^{transitivity} c \sim y \Rightarrow^{transitivity} c \sim b \Rightarrow c \in [b]$

similarly $[b] \subset [a]$

$3. X = \cup_{x \in X}[x]$

Conversely, given a partition of X you can define an equivalence relation by declaring $x \sim y \Rightarrow x, y$ lie in the same part of the partition

Note 5.10. An equivalence relation is a disguised version of a partition

Definition 5.11. Definition: congruence modulo n equivalence relation on Z

$a \equiv_n b$ if n divides $(b-a)$ i.e. $b - a = mn$ for some $m \in Z$ do NOT use $(a \equiv b(modn)$
EX. $\equiv_2$ partition

$$\{, -4, -2, 0, 2, \cdot\}$$
$$\{, -3, -1, 1, 3, \cdot\}$$

Proof: $\equiv_n$ is equiv relation

$1. a \equiv_n a$ since $n|(a-a)$
$2. (a \equiv_n b) \Rightarrow (b \equiv_n a)$ since $n|(b-a)$ then $n|(a-b)$
$3. a \equiv_n b$ and $b \equiv_n c$ then $a \equiv_n c$

$n|(b-a)$ and $n|(c-b)$ so $n|(b-a)+(c-b)$

§6 Mathematical Induction

Suppose have sequence of statements: $S_1, S_2, \ldots \{S_n : n \in \mathbb{N}\}$
Principle of mathematical induction:
Suppose S_1 is true (base case)
Suppose $S_n \Rightarrow S_{n+1}$ for each n (the induction)
Then S_n is true for each $n \in \mathbb{N}$
ex. $\int_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$
(base case) $S_1 : 1 = \frac{1(1+1)}{2}$
(induction) $S_n \Rightarrow S_{n+1}$
$S_n : 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
$+n + 1$
$S_n : 1 + 2 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1)$

Theorem 6.1

For $a, b \in \mathbb{Z}$ with $a \neq 0$ say a divides b if $b = a \cdot k$ for some $k \in \mathbb{Z}$
in other words b is a multiple of a
notation: $a|b$
d is a common divisor of a and b if $d|a$ and $d|b$
Greatest common divisor if largest integer that is a common divisor. Denoted by gcd (a, b)
a, b are relatively prime if gcd $(a, b) = 1$
ex. gcd $(48, 40) = 8$
gcd $(49, 39) = 1$

Theorem 6.2

Theorem 2.10 - Let $a, b \in \mathbb{Z} : a, b \neq 0$
There exists $r, s \in \mathbb{Z}$ s.t. gcd $(a, b) = ra + sb$

Example 6.3

gcd $(12, 20) = 2 * 12 - 1 * 20$
gcd $(14, 20) = 3 * 14 - 2 * 20$

Proof. let $S = \{ma + nb : m, n \in \mathbb{Z}$ and $ma + nb > 0\}$
$S \neq 0$ since $a^2 + b^2 > 0$
by W.O.P (well ordering property), let $d = ra + sb$ be least element of S
Claim: gcd $(a, b) = d$
First show that $d|a$ and $d|b$
Second: if $d'|a$ and $d'|b$ then $d'|d$

□

Theorem 6.4

2.9 - Division Algorithm - Review

Theorem 6.5

Theorem There are infinitely many primes.

Proof: Argument by contradiction. Suppose finetly many primes - $P_1, P_2, \ldots, P_n$
let $p = p_1p_2p_3 \ldots p_n + 1$
$p > p_n$ which means that p is not prime
but every composite number has prime factor so $p = p_k r$ for some k
impossible!
$p_k r = p_k(p_1 \ldots p_{k+1} \ldots p_n) + 1$
which would require that $p_k|1$ but this is impossible □

Theorem 6.6

Theorem Fundametnal theorem of arithmetic
let $n \in \mathbb{Z}$ with $n > 1$ Then $n = p_1p_2 \ldots p_k$ is a product of primes
This product is unique in a certain sense that:
if $n = q_1q_2 \ldots q_l$, then $k = l$ and sequences are actually the same after reording them
ex. $2 * 2 * 3 * 3 * 3 * 5 * 5$
$5 * 2 * 3 * 2 * 5 * 3 * 3$

Why is this true?

two things going on: exist and unique

proof of existence:

Show by (strong) induction that for $n \geq 2$, $S_n =$ "n is a product of primes"
(base case) n = 2
2 is a product of primes. $2 = 2$ ✓
((strong) induction): Either n+1 is prime, or $n + 1 = ab$ where $2 \leq a, b, \leq n$
by (strong) induction, $a = p_1p_2 \ldots p_k, b = q_1q_2 \ldots q_l$ where a and b are a product of primes. Therefore n+1 is a product of primes.
Proof of uniqueness. Note, new discussion, dosen't realte to previous proof

§6.1 Review proof of uniqueness

suppose $p_1 \ldots p_k$ = n = $q_1 \ldots q_l$
assume $p_1 \leq p_2 \leq \cdots \leq p_k$ and $q_1 \leq q_2 \leq \cdots \leq q_l$
assume $p_1 \leq q_1$
then $p_1|n$ so $p_1|q_k$ for some k
so $p_1 = q_k$ thus $p_1 \leq q_1 \leq q_k$
so $p_1 = q_1$
now $(p_2 \ldots p_k) = (q_2 \ldots q_l)$ by induction $k = l$ and the sequence are the same. n/p has a unique prime factorization and so

§6.2 Definition and example of Groups

a binary operation on a set G is a function $f : G \times G \to G$
math world is built out of binary operation: multiplication, subtraction, addition...
denote $f(a, b)$ by $a \circ b$ or $a \cdot b$ or ab
Def: a group $(G, \circ)$ is a set G with a binary operation $(a, b) \to a \cdot b \in G$
such that
(1) the operation is associative. i.e. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Review: associative, communative...

(2) there exists an identity element $e \in G$ s.t. $e \cdot x = x = x \cdot e$ for all $x \in G$
(3) Each element $x \in G$ has an inverse $y \in G$ s.t. $x \cdot y = e$
x^{-1} Often denotes inverse
We are blessed with a group theorist :)

example

ex. $(\mathbb{Z}, +)$ is a group
(1) $(a + b) + c = a + (b + c)$
(2) $e = 0, a + 0 = a = 0 + a$
(3) inverse of x denoted by $-x$

idea

$(G, \circ)$ is commutative or abelian if $a \circ b = b \circ a$ for all $a, b \in G$

examples of commutative groups

ex. $(\mathbb{Z}, \cdot)$, $\cdot$ = "times"/multiplication is NOT a group
(1) yes associative $(a * b) * c = a * (b * c)$
(2) has identity element $e = 1$
(3) BUT inverses don't always exist. $2^{-1} =?$. No integer inverse of 2
On the other hand: $(\mathbb{Q}*, \cdot)$ is a commutative group. Note: $\mathbb{Q}* = \mathbb{Q} - \{0\}$
identity (better word for e) is 1

ex. $(\mathbb{Q}, +)$ is a commutative group.

inverse of $\frac{2}{3}$ is $-\frac{2}{3}$

definition: $(G, \circ)$ is a finite group if G is a finite set.

otherwise we call G an infinite group.
What is more important when talking about a group. G or $\circ$? The $\circ$, everything is built into the $\circ$. i.e. $G \times G \to^f G$ and $(a, b) \to a \circ b$.
$|G|$ represents the number of elements in G
Let us now get familiar with Finite cyclic group $\mathbb{Z}_n$
Let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n - 1\}$
Define binary operation $a + b = c$ where $a + b \equiv_n c$ (called addition modulo n)
Turns out that this is a commutative group. $(\mathbb{Z}_n, +)$ is a commutative group.
ex. in $\mathbb{Z}_n$
$2 + 2 = 4$, $3 + 3 = 1$, $4 + 1 = 0$, $4 + 4 = 3$
Requirements:

(1) associative ✓
(2) 0 is the identity element
(3) Inverse exists. i.e. inverse of 3 = 2, inverse of 4 = 1, inverse of 1 = 4

Starting discussions on wednesday with Cayley table

I'm not gonna be able to type this lmao

Grid like a multiplication table, but more general. "The Cayley table of a group". Summary of a binary operation.

§6.3 Proposition 3.21

Proposition 3.21: Let G be a group, let $a, b \in G$.

Then the equations $ax = b$ and $xa = b$ have unique solutions.

§6.3.1 Proof of existence:

Let $x = a^{-1}b$. This is a solution for the first equation. $a * a^{-1}b = b = b$ ✓

Let $x = ba^{-1}$. This is a solution for the second equation.

§6.3.2 Proof of uniqueness:

To do this we will show that two solutions are always the same. Suppose c and d are solutions to $ax = b$. Therefore $a * c = b$ and $a * d = b$.

Therefore $a * c = a * d \Rightarrow a^{-1} * a * c = a^{-1} * a * d \Rightarrow c = d$ ✓. This is the proof for the first equation; the same steps can be used for the second equation.

§6.4 Proposition 3.22

Let G be a group. $(ba = ca) \Rightarrow (b = c)$. $(ab = ac) \Rightarrow b = c$. The idea behind this is left right cancellation.

$ba = ca \Rightarrow baa^{-1} = caa^{-1} \Rightarrow b = c$

§6.5 Notation

$g^n = g \circ g \circ g \circ \cdots \circ g$ where the number of g equals n - 1

$g^0 = e$

$g^{-1} = g^{-1} \circ g^{-1} \circ \cdots \circ g^{-1}$ where the number of g equals n - 1

From this: $g^m \circ g^n = g^{m+n}$ and $(g^m)^n = g^{m*n}$

Careful: $(ab)^m \neq a^m b^n$. Not necessarily commutative so you can't pass them through one another. Review a conceptually understanding of this.

For commutative $(G, +)$

$-g$ is notation for inverse of g.

§6.6 3.3 a subgroup H of group G is a subset of G

st $(H, \circ)$ is itself a group.

§6.6.1 ex. of the above.

${}_3\mathbb{Z} = \{\dots, -9, -6, -3, 0, 3, 6, \dots\}$

Perform checks. every element has an inverse. Review the group requirements. Every element has inverse. There is a unique identity element. Associative.

ex. The trivial subgroup $\{e\} \subset G$

ex. $(\mathbb{C}^*, \circ)$. Let $H = \{1, -1, i, -i\}$

ex. $SL_2(R) \subset GL_2(R)$.

Subgroup of 2×2 real invertible matricies but this time determ inent must equal 1. This works because determ inent of inverse of matrix is multiplicative inverse of determ inent; which in this case is also 1.

ex. $SL_2(Z) \subset SL_2(R) \subset GL_2(R)$.

§6.7 Proposition 3.30: Criterian for subgroup

A subset $H \subset G$ of a group $(G, \circ)$ is a subgroup iff:

(1) $e \in H$

(2) $h_1, h_2 \in H \Rightarrow h_1 h_2, \in H$

3) $h \in H \Rightarrow h^{-1} \in H$

§6.7.1 Proof that being a group gives these requirements

(1) Let e' be identity element of H. Then $e' = e'e = ee' = e \Rightarrow e = e'$

(2) Holds because H is a group. We like to say: H is "closed under multiplication".

(3) Since H is a group, h must have an inverse.

§6.7.2 Proof that these requirements means it must be a subgroup

Conditions (1), (2), (3), and associativity $\Rightarrow$ H is a group using operation of G. Must be associative because G is associative so any subset of G is also associative.

§6.8 Proposition 3.31

H is a subgroup $\Leftrightarrow H \neq \varnothing$

Easy to understand that $g, h \in H \Rightarrow gh^{-1} \in H$

A little harder to see that $gh^{-1} \in H$

$H \neq \varnothing \Rightarrow \exists x \in H$

§7 Cyclic Groups

§7.1 Cyclic Subgroup

Let $g \in (G, \circ)$. Notation: $< g >= \{g^n : n \in \mathbb{Z}\}$

Let $g \in (G, +)$. Notation: $< g >= \{ng : n \in \mathbb{Z}\}$

§7.2 Examples

$5 \in \mathbb{Z}$. $< 5 >= \{\dots, -10, -5, 0, 5, \dots\}$

$2 \in \mathbb{Z}$. $< 2 >= \{\text{even integers}\}$

$5 \in \mathbb{Z}_{10}$. $< 5 >= \{0, 5\}$

$6 \in \mathbb{Z}_{10}$. $< 6 >= \{6, 2, 8, 4, 0\}$

$2 \in \mathbb{Z}_{10}$. $< 2 >= \{2, 4, 6, 8, 0\}$

$3 \in \mathbb{Z}_{10}$. $< 3 >= \{3, 6, 9, 2, 5, 8, 1, 4, 7, 0\}$

Note: $< 1 >=< 3 >=< 7 >=< 9 >= \mathbb{Z}_{10}$. These capture the whole group.

Theorem 4.3 - Let G be a group. Let $x \in G$, then $< x >$ is a subgroup of G. Another way of thinking about it: $< x >$ is the smallest subgroup containing x.

Definition / Notation: $< x >$ is the cyclic subgroup generated by x. If $G =< x >$, then G is a cyclic group and x is a generator of G.

Detecting whether or not a subset is a subgroup.

Criteria

(0) Identity element.

(1) Inverse of each element is inside.

(2) Two elements inside, their product is inside.

§7.3 Proof

(0) $x^0 \in < x >$ so $e \in < x >$.

(1) If $g \in < x >$ then $g = x^m$ for some $m \in \mathbb{Z}$. $g^{-1} = x^{-m}$ because $x^{-m} * x^m = x^0 = e$. Therefore $g^{-1} \in < x >$

(2) Let $g, k \in < x >$, then $g = x^m$ and $k = x^n$ for some $m, n \in \mathbb{Z}$so $g \circ h = x^m \circ x^n = x^{m+n} \in < x >$.

Note: Finite groups are really complicated.

The order of x in G equals the smallest $n > 0$ such that $x^n = e$. If $x^n \neq e$ for all $n > 0$ we declare x in G to have infinite order.

Definition / Notation: $|x|$ represents the order of x.

§7.4 Examples

In $\mathbb{Z}_{10}$: $|5| = 2$, $|3| = 10$, $|0| = 1$

3 in $\mathbb{Z}$ has infinite order. All x in Z have infinite order except the identity element.

$2 \in \mathbb{R}^*$. $< 2 >= \{2^n : n \in \mathbb{Z}\} = \{\ldots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, \ldots\}$. Infinite order.

Theorem 4.9 - Every cyclic group is abelian (commutative).

§7.5 Proof

Suppose $G = < x >$. For each $g, k \in G$ there exist $m, n \in \mathbb{Z}$ such that $g = x^m$ and $k = x^n$

$g \circ k = x^m * x^n = x^{m+n} = x^{n+m} = x^n + x^m = k \circ g$

§7.6 Practice

$\mathbb{Q}_8$. Quaternians. I'm not sure what the "8" is for.

$< i >= \{1, i, -1, -i\}$

$< -i >= \{1, -i, -1, i\}$

$< 1 >= \{1\}$

$< -1 >= \{-1, 1\}$

$< j >= \{1, j, -1, -j\}$

Note to self: Groups are not necessarily commutative, but cyclic groups are always commutative. Review: Abelian.

§7.7 The group of units modulo n

$U_n = \{m : 1 \leq m < n, \gcd(m, n) = 1\}$

Binary operation: Multiply elements of U_n by computing remainder of xy modulo n.

§7.8 Examples

$U_{10} = \{1, 3, 7, 9\}$

Cayley Table: Can't make the table fast enough. Notes: each element appears once per row.

Changin to U_{15} :

$U_{15} = \{1, 2, 4, 7, 8, 11, 13, 14\}$

$U_8 = \{1, 3, 5, 7\}$

$< 1 >= \{1\}$

$< 3 >= \{1, 3\}$

$< 5 >= \{1, 5\}$

$< 7 >= \{1, 7\}$

U_8 is not cyclic. It is commutative because the cayley table is symmetric across $y = -x$.

Remember: U_n is abelian because xy mod n equals yx mod n ((because multiplication in integers is commutative)).

$U_3 = \{1, 2\}$. Is it cyclic. Yes because $< 2 >$ generates it. $< 2 >= \{1, 2\}$

$U_4 = \{1, 3\} =< 3 >$

$U_5 = \{1, 2, 3, 4\} =< 2 >= \{1, 2, 4, 3\} = \{2^0, 2^1, 2^2, 2^3\}$

> **Theorem 7.1**
> Every subgroup of a cyclic group G is cyclic

§7.9 Proof

Let $G = \langle x \rangle$. Let $H \subset G$ be a subgroup.

Show that $H = \langle y \rangle$ for some y.

If $H = \{e\}$ then $H = \langle e \rangle$ ✓.

Wha t is H contains an element $g \neq e$

Let $S = \{n > 0 : x^n \in H\}$

$S \neq \varnothing$ because it at least must include (review this proof that $S \neq \varnothing$.

Next step:

By W.O.P, let m be the least element of S.

Claim: $H = \langle y \rangle$ where $y = x^m$

Proof of claim: Let $h \in H$ Show that $h \in \langle y \rangle$ which means $h = y^q$ for some q.

Since $h \in G = \langle x \rangle$, $h = x^a$ for some $a \in \mathbb{Z}$

By the division algorithm: $a = mq + r$ where $0 \leq r < m$

If $r = 0$ then $h = x^a = x^{mq} = x^{m^q} = y^q$ ✓

If $r > 0$ the $hy^{-q} = x^{mq+r}x^{m^{-q}} = x^{mq+r}x^{-mq} = x^r$, but this would contradict that m is least element of S because... i'm not sure why review this

Cor 4.11. Subgroups of $\mathbb{Z}$ are $\langle n \rangle = n\mathbb{Z}$ (notation)

Prop 4.12. Let G by cyclic of order n $\Rightarrow x^n = e$. Suppose $G = \langle x \rangle$. This means that $(x^k = e) \Leftrightarrow n|k$.

Proof "$\Leftarrow$" If $k = n * l$, then x^k

§8 Cyclic Subgroups

Note 8.1 (Generator Group Notation)**.**
Let $g \in (G, \circ)$. Notation: $\langle g \rangle = \{g^n : n \in \mathbb{Z}\}$
Let $g \in (G, +)$. Notation: $\langle g \rangle = \{ng : n \in \mathbb{Z}\}$

Example 8.2 (Generator Groups)

$5 \in \mathbb{Z}$. $\quad \langle 5 \rangle = \{\dots, -10, -5, 0, 5, \dots\}$

$2 \in \mathbb{Z}$. $\quad \langle 2 \rangle = \{\text{Even integers.}\}$

$5 \in \mathbb{Z}_{10}$. $\quad \langle 5 \rangle = \{0, 5\}$

$6 \in \mathbb{Z}_{10}$. $\quad \langle 6 \rangle = \{6, 2, 8, 4, 0\}$

Note: $\mathbb{Z}_{10} = \langle 1 \rangle = \langle 3 \rangle = \langle 7 \rangle = \langle 9 \rangle$

Theorem 8.3
Let G be a group. Let $x \in G$, then $\langle x \rangle$ is a subgroup of G. Also, $\langle x \rangle$ is the smallest subgroup containing x.

Definition 8.4. $\langle x \rangle$ is the cyclic subgroup generated by x. If $G = \langle x \rangle$, then G is a cyclic group and x is a generator of G.

Definition 8.5. Detecting wheter or not a subset is a subgroup.

1. The identity Element is in the subgroup.
2. Inverse of each element is inside.
3. If two elements are inside, their product is inside as well.

§9 Chapter 5 09-27

Definition 9.1. A permutation of set X is a bijection $f : X \to X$.

Example 9.2
$x = \{1, 2, 3, 4\} \to \{3, 1, 4, 3\}$

I'm not fast enough to write this

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 3 \end{bmatrix}$$

General notation: $\begin{bmatrix} 1 & 2 & & n \\ f(1) & f(2) & f(\quad) & f(n) \end{bmatrix}\dots$

Definition 9.3. The symmetric group of degree n (on n objects) is group S_N consisting of all permutations of $X = \{1, 2, \dots n\}$

Theorem 9.4
S_n is a group whose binary operation is composition of functions.

Proof.

1. Composition of functions is associative.
2. Inverses exist because inverses of bijections are bijections. f^{-1} is inverse of f.

□

Example 9.5

$$\begin{bmatrix}1&2&3&4\\4&3&1&2\end{bmatrix}\begin{bmatrix}1&2&3&4\\2&1&4&3\end{bmatrix}=\begin{bmatrix}1&2&3&4\\3&4&2&1\end{bmatrix}$$

Note 9.6. S_n has $n!$ elements.

Example 9.7

Consider the following function.

$$\begin{bmatrix}1&2&3&4&5&6\\1&3&4&5&2&6\end{bmatrix}$$

Definition 9.8. A cycle is a permutation with property that there is a subset $\{a_1, a_2, \ldots, a_m\} \subset \{1, 2, \ldots, n\}$ such that $f(a_i) = a_{i+1}$ for $1 \leq i < m$, and $f(a_m) = a_1$, and $f(x) = x$ when $x \notin \{a_1, \ldots, a_m\}$.

Example 9.9

Consider the following function.

$$\begin{bmatrix}1&2&3&4&5&6&7\\4&2&3&7&5&6&1\end{bmatrix}$$

$1 \to 4 \to 7 \to 1$. $(1, 4, 7)$ are being cycled.
$(2, 3, 5, 6)$ are fixed.

Note 9.10. Use notation $(a_1, a_2, \ldots, a_m)$ for the cycle. All other elements are fixed.

Example 9.11

$(3\ 7\ 5\ 1) \in S_7$ contains the same information as:

$$\begin{bmatrix}1&2&3&4&5&6&7\\3&2&7&4&1&6&5\end{bmatrix}$$

But the former is easier to understand.

Note 9.12. $(a_1, a_2, \ldots, a_m)$ and $(b_1, b_2, \ldots, b_l)$ are disjoint if $a_i \neq b_j$ for i, j.

Example 9.13

(3 7 5 1) is disjoint from (64), but note that there are multiple ways of representing the same cycle.
For example. $(3\ 7\ 5\ 1) = (5\ 1\ 3\ 7) = (7\ 5\ 1\ 3)$

Theorem 9.14

Disjoint cycles commute.

$$(a_1\ \dots\ a_m)(b_1\ \dots\ b_l) = (b_1\ \dots\ b_l)(a_1\ \dots\ a_m)$$

if $c \notin \{a_1\ \dots\ a_m, b_1\ \dots\ b_l\}$

Theorem 9.15

Every permutation is a product of disjoint cycles.

Example 9.16

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 6 & 5 & 9 & 8 & 2 & 4 & 1 & 7 \end{bmatrix} = (3\ 5\ 8)(2\ 6)(7\ 4\ 9) \in S_9$$

More Practice:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 8 & 2 & 1 & 5 & 3 & 6 & 10 & 9 & 4 \end{bmatrix} = (1\ 7\ 6\ 3\ 2\ 8\ 10\ 4)(5)(9) \in S_{10}$$

Practice in the other direction:

$$((1\ 3\ 5)(2\ 7\ 6\ 4))((1\ 2)(3\ 4)(5\ 6\ 7)) = (1\ 7)(2\ 3)(4\ 5)(6)$$

The (6) at the end is unnecessary because it is an identity element.

He just drew a pictorial circle on the board. I am just going to watch and absorb.

Theorem 9.17

Every permutation is a product of transpositions because:

> **Theorem 9.18**
> Every n-cycle is a product of $(n-1)$ transpositions.

Proof. $(a_1\ a_2\ \dots\ a_m) = (a_1\ a_m)(a_1\ a_{m-1})\dots(a_1\ a_3)(a_1\ a_2)$ □

§10 Math 235 Tutorial — 09-27

§10.1 Groups & Subgroups

Definition 10.1. A group $(G, \circ)$ is a set G together with an operation $\circ$ such that:

1. The elements are associative: $(a \circ b) \circ c = a \circ (b \circ c)$
2. There exists an identity element: $\exists e \in G$ such that $\forall g \in G$, $ge = eg = g$.
3. All elements contain an inverse within the group. $\forall g \in G \ \ \exists g^{-1}$ such that $g \circ g^{-1} = g^{-1}g = e$

Example 10.2

1. $(\mathbb{Z}, +)$. It is associative. Identity element is 0. $a^{-1} = -a$.
2. $(\mathbb{R} \setminus \{0\}, *)$. Is is associatve. Identity element is 1. All elements have an inverse (we removed 0 because 0 doesn't have an inverse).
3. $(\mathbb{Z}_n, +)$. It is associative. Identity element is 0. $a^{-1} = -a = n - a$.
4. $(\mathbb{Z}, *)$ is NOT a group because many integers do not have inverses that belong to the integers. $5^{-1} = \frac{1}{5} \notin \mathbb{Z}$
5. $(\mathbb{R}, *)$ is NOT a group because 0 does not have an inverse.

Note 10.3.

1. Multiplicative Notation: g^n means use the operation n times. $(G, \cdot)$
2. Additive Notation: ng means use the operation n times. $(G, +)$

Example 10.4

$$\begin{aligned} 11x + 2 &\equiv 16 \pmod{26} \\ 11x &\equiv 14 \pmod{26} \\ 11^{-1}11x &\equiv 11^{-1}14 \pmod{26} \\ x &\equiv 11^{-1}14 \pmod{26} \end{aligned}$$

If we were in $\mathbb{R}$, $(11)^{-1} = \frac{1}{11}$, but $\frac{1}{11} \notin \mathbb{Z}_{26}$ We need to find $(11)^{-1} \in \mathbb{Z}_{26}$. We know it exists because gcd $(11, 26) = 1$

Euclidean Algorithm

$$\begin{aligned} 26 &= 2 * 11 + 4 \\ 11 &= 2 * 4 + 3 \\ 4 &= 1 * 3 + 1 \\ 3 &= 3 * 1 + 0 \end{aligned}$$

$$\begin{aligned} 1 &= 4 - 3 \\ &= 4 - (11 - 24) \\ &= 3 * 4 - 11 \\ &= 3(26 - 2 * 11) - 11 \\ &= 3 * 26 - 7 * 11 \end{aligned}$$

gcd $(11, 26) = 1$ means there is a linear combination of 11 and 26 that equals 1. Taking the mod of both sides, mod of 26 is 0 and mod of 1 is 1 so it means that there is a multiple of 11 equal to 1 in mod 26. This means that it has an inverse. It's inverse is $-7 = 26 - 7 = 19$.

Back to equation:

$$\begin{aligned} 11x &= 14 \\ 19 \cdot 11x &= 19 \cdot 14 \\ x &= 266 \\ x &= 6 \end{aligned}$$

Solution: $\{x \in \mathbb{Z} : 26 \cdot n + 6 \quad n \in \mathbb{Z}\}$

Example 10.5

When presented with a Cayley table, how can we tell whether or not we are looking at a group.

$$\begin{bmatrix} a & b & c & d \\ b & b & c & d \\ c & d & a & b \\ d & a & b & c \end{bmatrix}$$

We must check identity element, inverses, and associativity,.

1. Identity element is a. ✓
2. b doesn't have an inverse so this is not a group.
3. Whether or not associativity fails, this is not a group. In order to see associativity in a cayley table, it must by symmetric along the line $y = -x$.

Advice: When checking if two groups are the same with cayley tables, look at the inverses and see if they match perfectly.

Exercise 10.6. Let G be a group such that $g^2 = e \quad \forall g \in G$. Show that G is abelian. In other words, $\forall a, b \in G \quad ab = ba$.

Solution. Let $a, b \in G$. We want to show that $ab = ba$. Note: $e = a^2 = b^2 = (ab)^2 = (ba)^2$

$$\begin{aligned} ab &= a \cdot e \cdot b \\ ab &= a \cdot (ab)(ab) \cdot b \\ ab &= (aa)(ba)(bb) \\ ab &= e \cdot ba \cdot e \\ ab &= ba \end{aligned}$$

□

Advice: when proving that a group is abelian, play around with the identity matrix.

Definition 10.7. H is a subgroup of G if $H \subset G$ and H is a group with the inherited operation from G.

Example 10.8

$(\mathbb{Z}, +)$. Even integers are a subgroup of $\mathbb{Z}$ with the + operation.
$(\mathbb{Z}, +)$. Odd integers are NOT a subgroup of $\mathbb{Z}$ with the + operation because they don't have closure. $1 + 3 = 4$ and 4 is not an element of the odd integers.

Exercise 10.9. H_1 and H_2 are subgroups of G. Prove or disprove the following:

1. $H_1 \cap H_2$ is a subgroup of G.

   This is TRUE because it has the identity element, it has the inverses, and there is closure. There is no need to prove associativity because it is inherited from the binary operation.

   a) (Identity) $e \in H_1$ and $e \in H_2$ because H_1 and H_2 are subgroups.

b) (Inverses) $a \in H_1 \cap H_2$. In particular, $a \in H_1 \Rightarrow a^{-1} \in H_1$ and $a \in H_2 \Rightarrow a^{-1} \in H_2$. So $a^{-1} \in H_1 \cap H_2$. Note: This works because inverses are unique.

c) (Closure) $a, b \in H_1 \cap H_2$. $a, b \in H_1 \Rightarrow ab \in H_1$. Same for H_2. Therefore $ab \in H_1 \cap H_2$.

2. $H_1 \cup H_2$ is a subgroup of G?

   This is FALSE. Counter example: Let $A = \{n \in \mathbb{Z} : \text{n is a multiple of 2}\}$. Let $B = \{n \in \mathbb{Z} : \text{n is a multiple of 5}\}$.

   a) Identity ✓

   b) Inverses ✓

   c) Closure ✗

§11 Lecture 09-30

§11.1 Review Complex Numbers

Recall $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$

Come equipped with:

1. Addition: $(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$
2. Multiplication:

Complex numbers are associative and commutative under these operations.

There exists a complex conjugate.

Example 11.1

Complex conjugate of $(a + bi) = (a - bi)$

Note 11.2. $(a + bi)(a - bi) = a^2 + b^2$

$\mathbb{C}^*$ is a multiplicative group of complex numbers. All imaginary numbers have an inverse.

Rectangular and Polar coordinates.

Rectangular: x axis is real component, y axis is imaginary component.

Polar: radius is $\sqrt{a^2 + b^2}$

$\sqrt{a^2 + b^2} \cos\theta + \sqrt{a^2 + b^2} \sin\theta i = \sqrt{a^2 + b^2} cis\theta = re^{i\theta}$

We have that $(r_1 cis\theta_1)(r_2 cis\theta_2) = r_1 r_2 cis(\theta_1 + \theta_2)$.

r is the "scale factor"

$cis\theta$ is the "rotation"

When $r = 1$, we get the subgroup of unit length com plex numbers.

$cis\theta = 1 =$ identity. $(cis\theta)^{-1} = cis(\theta)$ $cis\theta_1(cis\theta_2) = cis(\theta_1 + \theta_2)$

The nth roots of unity are the solutions to $x^n = 1$ in $\mathbb{C}^*$.

They form a cyclic subgroup of order n. Form verticies of a polygon with n verticies.

Recall. Given a geometric object X, its group of isometries or symmetries Isom(X) is a group with multiplication as composition of functions.

Recall. An isometry $f : X \to X$ is a distance preserving function.

dist $(p, q) =$ dist $(f(p), f(q) \forall p, q \in X$.

Definition 11.3. Dinedral group - D_n is the group of isometries of regular n-gon.

Example 11.4

$|D_n| = 2n$. It has n reflections and n rotations (counting the identity).

Consider $n = 3$. This gives a regular triangle. There are three reflections and three rotations. In this example, reflections fix 1 vertex and midpoing of opposite edge. Rotations fix the center. Group of isometries is not abelian because if you do the same thing in different orders you get different results.

Lemma. The set of rotations forms a subgroup.

Remark. Each reflection is its own inverse. Remark. Product of two reflections is a rotation.Generally it is by twice the angle between the axes of reflection.

When n is even, reflections occur in two ways.

1. Fix two vertices
2. Fix two midpopints of opposite edges.

Interpretation for $n = 1$ and $n = 2$.

1. $n = 2$ represented by isometry of "bigon" (looks like a lemon).
2. $n = 1$ represented by isometry of a water droplet. $reflection, identity$.

These two groups are the only abelian dinedral groups.

Let X be a rigid object. Its group of rigit motions consists of isometries that can be "physically realised". They are all rotations.

Example 11.5

Let X be a "brick". Dimensions: $2 \times 3 \times 5$

1. Isom X consists of 16 elements. We have 3 π rotations, the identity, and reflections.
2. Rigid motions of X consists of 8 elements.

§12 10-02

Theorem 12.1

Any n-cycle is the product of $(n-1)$ transpositions.

Proof.

$$(a_1\ a_2\ \dots\ a_n) = (a_1\ a_n)(a_1\ a_{n-1})\dots(a_1\ a_3)(a_1\ a_2)$$

An alternative proof:

$$(a_1\ a_2\ \dots\ a_n) = (a_1\ a_2)(a_2\ a_3)\dots(a_{n-1}\ a_n)$$

□

Definition 12.2. An element $\sigma \in S_n$ is:

1. Even if σ is the product of an even number of transpositions
2. Odd if σ is the product of an odd number of transpositions.

Theorem 12.3

No $\sigma \in S_n$ is both even and odd.

Note 12.4. This means that any odd σ can only be expressed as a product of an odd number of transpositions.

Proof. Matricies over $\mathbb{R}$ have det positive or negative. Positive determinent maintains orientation. Negative determinent inverses orientation. An even element can be likened to a matrix with a positive determinent, while an odd element can be likened to a matrix with a negative determinent. □

Example 12.5

(2 3 5 7 9) is even because it is the product of four transpositions $(n-1)$.
(1 8 6 2) is odd because it is the product of three transpositions $(n-1)$.

Theorem 12.6

The set of even permutations of S_n is a subgroup. $A_n \subset S_n$, alternating group.

Proof. Identity Element: $() = (1\ 2)(1\ 2)$
Inverse: $() \in A_n$. Need to prove that $\sigma \in A_n \Rightarrow \sigma^{-1} \in A_n$.Indeed:

$$\sigma = (a_1\ b_1)(a_2\ b_2)\dots(a_k\ b_k)$$

$$\sigma = (a_k\ b_k)\dots(a_2\ b_2)(a_1\ b_1)$$

Closure: $\sigma, \phi \in A_n \Rightarrow \sigma \cdot \phi \in A_n$. Even number of permutations times even number of permutations gives an even number of permuatations which is in A_n.

Note 12.7. $|A_n| = \frac{1}{2}|S_n|$.

□

Understanding $A_4 \subset S_4$.
Listing elements in S_4. $S_4 =$

$$\{()\}$$

$$\{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$$

$$\{(1\ 2\ 3), (1\ 3\ 2)$$
$$(1\ 2\ 4), (1\ 4\ 2)$$
$$(1\ 3\ 4), (1\ 4\ 3)$$
$$(2\ 3\ 4), (2\ 4\ 3)\}$$

Note 12.8. S_4 "is" an isometry group of tetrahedron.
A_4 "is" the subgroup of its rigit motions.

§12.1 Cosets and Lagrange's Theorem

Let $H \subset G$ be a subgroup.
Let $g \in G$.
The left coset of H represented by g is $gH = \{gh : h \in H\}$.
The right coset of H represented by g is $Hg = \{hg : h \in H\}$.
Usually $gH \neq Hg$. (If equal just call them cosets if you'd like).

Example 12.9

Misleading but simple example.
$G = \mathbb{Z}_{12}, H = \langle 4 \rangle = \{0, 4, 8\}$.

$$0 + H = 4 + H = 8 + H = \{0, 4, 8\}$$
$$1 + H = 5 + H = 9 + H = \{1, 5, 9\}$$
$$2 + H = 6 + H = 10 + H = \{2, 6, 10\}$$
$$3 + H = 7 + H = 11 + H = \{3, 7, 11\}$$

Note 12.10. Notation can be confusing. gh means binary operation between g and h so when binary operation is $+$ it means $g + h$.

Cosets formed a partition of the group.
Review: What is a partition? Disjoint subsets that unionize to form a set.

Example 12.11

$$H = \{1, -1, i, -i\} \subset \mathbb{Q}_8$$
$$1 \cdot H = \{1 * 1, 1 * -1, 1 * i, 1 * -i\}$$
$$jH = \{j * 1, j * -1, j * i, j * -i\} = \{j, -j, -k, k\} = Hj$$

Note 12.12. These form a partition of the quaternions.

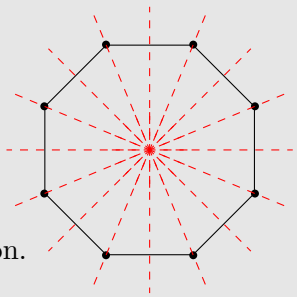
Example 12.13

$$K \subset \mathbb{Q}_8$$
$$K = \{1, -1\}$$

$$1K = \{1, -1\}$$
$$iK = \{i, -i\}$$
$$jK = \{j, -j\}$$
$$kK = \{k, -k\}$$

Note 12.14. For all of these, a left coset is a right coset.

Example 12.15



D_8. Symmetries of a regular octogon.

$$D_8 = \{r_i, \frac{i}{8}2\pi : 0 \leq i \leq 7\}$$

Subgroup (Same as the isometry of a rectangle living inside):

$$H = \{r_1, r_5, 0, \pi\}$$

Note 12.16. Product of rotation with rotation is a rotation. Product of a rotation with a reflection is a reflection. Product of a reflection with a reflection is a rotation.

$$0H = H$$
$$\frac{\pi}{8}H = \{r_2, r_6, \frac{\pi}{8}, \frac{5\pi}{8}\}$$
$$\frac{2\pi}{8}H = \{r_3, r_7, \frac{2\pi}{8}, \frac{6\pi}{8}\}$$
$$\frac{3\pi}{8}H = \{r_4, r_0, \frac{3\pi}{8}, \frac{7\pi}{8}\}$$
$$H\frac{\pi}{8} = \{r_0, r_4, \frac{\pi}{8}, \frac{5\pi}{8}\}$$

Note 12.17. Finding the product of a reflection and a rotation can be tricky. See how the composition affects a single point, and then identity a single rotation that affects the point in the same way.

Theorem 12.18

Lem 6.2. Let $g_1, g_2 \in G$ and $H \subset G$ be a group.
TFAE (similar for right cosets)

1. $g_1 H = g_2 H$
2. $Hg_1^{-1} = Hg_2^{-1}$. There is a bijection from a group to itself. Most obvious is identity bijection, but another one is every element to its inverse. In order to prove that statements 1 and 2 imply one another, use $\phi : G \to G, \phi(g) = g^{-1}.\phi$ is a bijection.$\phi(g_1 h) = h^{-1} g_1^{-1} \Rightarrow \phi(gH) \subset Hg^{-1}$.
3. $g_1 H \subset g_2 H$
4. $g_2 \in g_1 H$
5. $g_1^{-1} g \in H$. Reading this statement: "The difference between g_1 and g_2 lies in H.

Proof. $1 \Leftrightarrow 5$.
$\Rightarrow$. Suppose $g_1^{-1} g_2 \in H$, then $g_2^{-1} g_1 = (g_1^{-1} g_2)^{-1} \in H$.
Therefore $g_1 H \subset g_2 H$ because $g_1 h = (g_2 g_2^{-1}) g_1 h = g_2 (g_2^{-1} g_1) h \in g_2 H$. ($g_2 h'$ with $h' \in H$).
$\Leftarrow$. ($g_1 H = g_2 H$) $\Rightarrow$ ($g_1 e \in g_2 H$) $\Leftrightarrow$ ($g_1 \in g_2 H$) $\Rightarrow$ ($g_1 = g_2 h$ for some $h \in H$) $\Rightarrow g_2^{-1} g_1 = h \in H$. □

Theorem 12.19

Lem 6.4. Let $H \subset G$ be a subgroup. The left (or right) cosets of H form a partition of G.

Proof. Look:

$$G = \bigcup_{g \in G} gH \text{ because } g = ge \in gH$$

If $g_1 H \cap g_2 H \neq \varnothing$, then $g_1 H = g_2 H$ because if $g_1 h_1 = g_2 h_2$, then $g_1^{-1} g_2 = h_1 h_2^{-1} \in H$, hence $g_1 H = g_2 H$. □

Definition 12.20. Let $[G : H]$ be the index of H in G denote the number of left cosets of H in G.

Example 12.21

$$[D_8 : \{r_1, r_5, 0, \pi\}] = 4$$
$$[\mathbb{Z}_{12} : \{0, 4, 8\}] = 4$$
$$[\mathbb{Q}_8 : \{-1, 1\}] = 4$$
$$[\mathbb{Q}_8 : \{-1, 1, i, -i\}] = 2$$
$$[\mathbb{Z} : n\mathbb{Z}] = n$$
$$[G : G] = 1$$
$$[G : \{e\}] = |G|$$

Theorem 12.22

6.4. Let $H \subset G$. The number of left cosets equals the number of right cosets.

Proof. The inversion map on G sends left cosets to right cosets and right cosets to left cosets.

Let L be the collection of left cosets and R be the collection of right cosets.

Define bijection $\phi : L \to R$ by $\phi(gH) = Hg^{-1}$. Now to check that this function is well defined

Definition 12.23. . Well defined: independent of choice of representative.

Check that if $gH = kH \Rightarrow \phi(gH) = \phi(kH)$.

$$gH = kH \Rightarrow Hg^{-1} = Hk^{-1} \Rightarrow \phi(gH) = \phi(kH).$$

Now check that ϕ is injective.

$$[\phi(gH) = \phi(kH)] \Rightarrow [Hg^{-1} = Hk^{-1}] \Rightarrow [gH = kH]$$

Now check that ϕ is surjective.

$Hx = H(x^{-1})^{-1} = \phi(x^{-1}H)$ □

§13 Tutorial 5: Cyclic Groups - 10-04

Theorem 13.1

Every cyclic group is abelian.

Theorem 13.2

Every subgroup of a cyclic group is cyclic.

Let G be a cyclic group and let $a \in G$ be of order n.

Theorem 13.3

$a^m = e \Leftrightarrow n|m$

Theorem 13.4
$b = a^k \in G$, then $|b| = \frac{n}{\gcd(n,k)}$
Corollary: In additive notation

- $\mathbb{Z}_n = \langle 1 \rangle$ with $|1| = n$.
- $k = k \cdot 1$, then $|k| = \frac{n}{\gcd(n,k)}$
- Generators of $\mathbb{Z}_n$ are the integers k such that $1 \leq k < n$ and $\gcd(k, n) = 1$.

Example 13.5
Subgroups of $(\mathbb{Z}_8, +)$. Observe that $1, 2, 4, 8$ divide 8. We have to find $k \in \mathbb{Z}_8$ such that:

- $\gcd(8, k) = 1$

  $\{1, 3, 5, 7\}$. These generate subgroups of order $\frac{8}{\gcd} = 8$. There is only one such subgroup of $\mathbb{Z}_8$ so they must all be the same.

- $\gcd(8, k) = 2$

  $\{2, 6\}$. These generate subgroups of order 4: $\{0, 2, 4, 6\}$

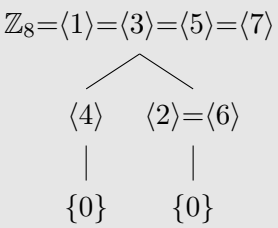
  A question arises: Do 2 and 6 generate the same subgroup? $\langle 2 \rangle = \{0, 2, 4, 6\}$. $6 \in \langle 2 \rangle$ so $\langle 2 \rangle = \langle 6 \rangle$.

- $\gcd(8, k) = 4$

  $\{4\}$. Generates a group of order 2: $\{0, 4\}$

- $\gcd(8, k) = 8$

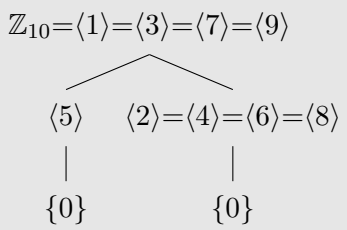
  $\{0\}$. Generates a sugroup of order 1: $\{0\}$



Example 13.6

List all the subgroups of $\mathbb{Z}_{10}$. Observe that 1, 2, 5, and 10 divide 10. Find $k \in \mathbb{Z}_{10}$ such that:

- $\gcd(k, 10) = 1$.
  $k = 1, 3, 7, 9$. $\mathbb{Z}_{10} = \langle 1 \rangle = \langle 3 \rangle = \langle 7 \rangle = \langle 9 \rangle$
- $\gcd(k, 10) = 2$.
  $k = 2, 4, 6, 8$. These generate subgroups of order 5.
  $\langle 2 \rangle = \langle 4 \rangle = \langle 6 \rangle = \langle 8 \rangle = \{0, 2, 4, 6, 8\}$
- $\gcd(k, 10) = 5$.
  $k = 5$. $\langle 5 \rangle = \{0, 5\}$.
- $\gcd(k, 10) = 10$.
  $k = 0$. $\langle 0 \rangle = \{0\}$



Example 13.7

Let G be a group. Assume $a \in G$ such that $a^{24} = e$. What are the possible orders of a?

Recall that when $a^n = e$, the possible orders of a are those which divide n. Possible orders are therefore 1, 2, 3, 4, 6, 8, 12, 24.

$|a| = n \Rightarrow a^n = e$. NOT $\Leftarrow$

Example 13.8

Let $a, b \in G$. Prove the following statements:

(a) $|a| = |a^{-1}|$

Proof. $|a| = n$. $|a^{-1}| = m$

$$\begin{aligned} & a^n = e \\ \Rightarrow & (a^n)^{-1} \cdot a^n = (a^n)^{-1} \cdot e \\ \Rightarrow & e = (a^n)^{-1} \\ \Rightarrow & e = (a^{-1})^n \Rightarrow m|n \end{aligned}$$

You can show similarly that $n|m$. By proving that $m|n$ and that $n|m$, we have proven that $n = m$. □

(b) $\forall g \in G, |a| = |g^{-1}ag|$

Proof. Let $g \in G$, $|a| = n$, $|g^{-1}ag| = m$. Observe that:

$$\begin{aligned} & (g^{-1}ag)^m = e \\ \Rightarrow & (g^{-1}ag)(g^{-1}ag)\dots(g^{-1}ag) = e \\ \Rightarrow & g^{-1}a^m g = e \\ \Rightarrow & g \cdot g^{-1}a^m g \cdot g^{-1} = g \cdot e \cdot g^{-1} \\ \Rightarrow & a^m = e \end{aligned}$$

Therefore $n|m$ because $|a| = n$. Similarly $m|n$. Therefore $m = n$. □

(c) $|ab| = |ba|$

Proof. By (b), $|ab| = |a^{-1}(ab)a| = |a^{-1}aba| = |ba|$ □

Exercise 13.9. Show that if G has no proper non-trivial subgroups, then G is a cyclic group of prime orders.

Proof.

(a) Showing that G is cyclic. Let $g \in G : g \neq e$. $\langle g \rangle$ is a non-trivial subgroup of G because $g \in \langle g \rangle$ and $g \neq e$. By assumption that G has no proper non-trivial subgroups, $\langle g \rangle = G$.

(b) Showing that G must be of prime order.

a) Case where $|G| = \infty$. Let G

Observe that $\langle g^2 \rangle$ is a non-trivial subgroup of G. Observe that $\langle g^2 \rangle \neq G$ because $g \notin \langle g^2 \rangle$. "If the order of a group is infinity, we will always be able to generate non-trivial proper subgroups."

b) Case where $|G| = n < \infty$

Assume that $n = d \cdot m$ for some d, m. Since $d|n$, then G must have a subgroup H of order d. This would mean that H is non-trivial and $H \neq G$. This is a contradiction $\Rightarrow |G| = p$ for some prime number.

□

Exercise 13.10. An infinite cyclic group G has exactly 2 generators.

$G = \langle a \rangle = \langle b \rangle$. This would mean that $a = b^k$ for some k, and that $b = a^l$ for some l.

$$a = b^k = (a^l)^k = a^{lk}$$
$$\Rightarrow a^{-1} \cdot a = a^{-1}a^{lk}$$
$$\Rightarrow e = a^{lk-1}$$

We know that $|a| = \infty$, therefore $lk - 1 = 0 \Rightarrow lk = 1$. This gives two possible cases: $l = k = 1$ or $l = k = -1$ because l and k must be integers. Therefore either $b = a$ or $b = a^{-1}$. This means that the only generators of G are a and a^{-1}.

§14 Lecture 10-07

Theorem 14.1

Proposition 6.9. Let $H \subset G$ and $g \in G$.

There is a bijection $\phi : H \to gH$ defined by $\phi(h) = gh$

Proof. This is injective because $(gh_1 = gh_2) \Rightarrow h_1 = h_2$

This is surjective because $gH = \{gh : h \in H\}$

$\{\phi(h) : h \in H\} = \phi(H)$ □

Theorem 14.2

Lagrange's Theorem

Let G be a finite group and $H \subset G$ a subgroup.

Then $\frac{|G|}{|H|} = [G : H]$

Proof. $G = g_1H \cup g_2H \dots g_hH$ by theorem 6.4

Each of $|g_iH| = |H|$ by proposition 6.9

$|G| = n|H| = [G : H]\,|H| \Rightarrow \frac{|G|}{|H|} = [G : H]$ □

Theorem 14.3

Corollary 6.11.

Let G be finite and $g \in G$. Then $|g|$ divides $|G|$.

Proof. $|g| = |\langle g \rangle|$ which represents a subgroup of G which divides $|G|$ by Lagrange's Theorem. □

Theorem 14.4

Corollary. If $g \in G$ is finite, then $g^{|G|} = e$. Intuitively this makes sense because the order of g divides the order of G.

$g^{|G|} = g^{|g| \cdot [G:\langle g \rangle]} = g^{|g|^{[G:\langle g \rangle]}} = e^{[G:\langle g \rangle]} = e$

Example 14.5

For $\sigma \in S_n, \sigma^{n!} = e$. But this is very inefficient.

Theorem 14.6

Corollary. If $|G| = p$ with p prime, then $G = \langle g \rangle$ for each $g \in G - \{e\}$.

Proof. $1 \neq |g|$ Nd $|g|$ divides $|G| = p (by 6.11)$. Therefore $|\langle g \rangle| = p$ so $\langle g \rangle = G$. □

Theorem 14.7

Corollary. If $K \subset H \subset G$ is a finite group, then $[G : K] = [G : H]\,[H : K]$

Proof. $[G : K] = \frac{|G|}{|K|} = \frac{|G|}{|H|} \cdot \frac{|H|}{|K|} = [G : H]\,[H : K]$. □

Definition 14.8. Euler ϕ function. $\phi : \mathbb{N} \to \mathbb{N}$
$|U_n| = \phi(n)$

Example 14.9

$\phi(1) = 1$
$\phi(9) = |\{1, 2, 4, 7, 8\}| = 5$
$\phi(8) = |\{1, 3, 5, 7\}| = 4$

Theorem 14.10

6.18. Euler's Theorem
Let $a, n \in \mathbb{Z}$ with $n > 0$ and $\gcd(a, n) = 1$, then $a^{\phi(n)} = 1$ (mod n)

Proof. Regard $a \in U_n$.
$a^{\phi(n)} = a^{|U_n|} = 1$ (mod n)
i.e. $g^{|G|} = e$
At the time of righting this it makes perfect sense, but we will see how it goes when I revisit it haha. □

Theorem 14.11

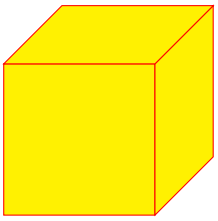
6.19. Fermat Little Theorem.
Let p be prime and p does not divide a. (If p divided a, then a wouldn't be in the group of units U_p.)
Then $a^{p-1} = 1$ (mod p)

Proof. with $p = n$ prime, $a^{p-1} = a^{\phi(p)} \equiv_p 1$. This works because when p is prime, $\phi(n) = p - 1$.
moreover, for any a, $a^p \equiv_p a$. If $p|a$ this is $0 \equiv 0$. □

Definition 14.12. Conjugacy: Let $x, y \in G$. x is conjugate to y if there exists $g \in G$ such that $x = gyg^{-1}$. We use the notation $x \sim y$.
Lem: Conjugacy is an equivalence relation. Transitivity, Reflexivity, Symmetry.

Proof. Reflexivity: $x \sim x$ because $x = exe^{-1}$.
Symmetry: $(x \sim y) \Rightarrow (x = gyg^{-1}) \Rightarrow y = g^{-1}xg \Rightarrow y = gg^{-1}xg^{-1}g^{-1}$
Transitivity: □

§15 Case Study: The Cube



8 vertices, 6 faces, 12 edges.

Let G denote the symmetry group of the cube (the same as the isometry or motion group).

This group has 48 elements.

Indeed, there are 8 isometries that map each

§16 10-11

Lemma 16.1

Conjugacy is an equivalence relation.

$x \sim y$ if $x = gyg^{-1}$ for some $g \in G$

Theorem 16.2

Any two k-cycles in S_n are conugate. Moreover, any conjugate of a k-cycle is a k-cycle.

Proof. $\alpha = (a_1 a_2 \dots a_k) \quad \beta = (b_1 b_2 \dots b_k)$

Let σ be a bijection with $\sigma(b_i) = a_i$ for $1 \leq i \leq k$

Then $\sigma\beta\sigma^{-1} = \alpha$. Claiming that β and α are conjugate to one another.

If $x \notin \{a_1 \ \dots \ a_k\}$, then $\sigma\alpha\sigma^{-1}(x) = x$

Example 16.3

$\sigma\beta\sigma^{-1}(a_i) = \sigma\beta b_i = \sigma b_{i+1} = a_{i+1} = \alpha(a)$

□

§16.1 A_4 is the group of rigid motions of a tetrahedron

Note 16.4. A_4 is the subgroup of even elements in S_4.

$$\text{identity: } [e] = \{()\}$$
$$[(12)(34)] = \{(12)(34), (13)(24), (14)(23)\}$$
$$\text{Clockwise rotations about a face: } [(123)] = \{(123), (134), (142), (243)\}$$
$$\text{Counter clockwise rotations about a face: } [(132)] = \{(132), (143), (124), (234)\}$$

Note 16.5. A_4 has no 6 element subgroup even though 6 divides $|A_4|$

§17 Isomorphisms

Definition 17.1. $(G, \cdot)$ and $(H, \circ)$ are isomorphic if there exists a bijection $\phi : G \to H$ such that $\phi(a \cdot b) = \phi(a) \circ \phi(b)$ for all $a, b \in G$.

This would make ϕ an isomorphism.

$G"equalsignwithsimontop"H$

Note 17.2. Let $H \subset G$ be a subgroup. It is possible for $x \not\sim y$ in H while $x \sim y$ in G. This is a distinguishing between $\sim_H$ and $\sim_G$.

Example 17.3

$H \subset S_{17}$

$H = \langle (1\ 2 \dots 17) \rangle$

H is cyclic and abelian.

It has 17 conjugacy classes.

All nontrivial elements of H are conjugate in S_{17}.

Note 17.4. In an abelian group, two elements are conjugate iff they are equal.

If H is abelian, then $(x \sim y) \Leftrightarrow x = aya^{-1} = y$

§18 Tutorial 6: Permutation Groups

Recall 18.1.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix} = (1\ 3\ 5\ 4\ 2)$$

Note 18.2. What is the order of a cycle σ of length r?

Answer: $|\sigma| = r$

Reasoning: $\sigma = (x_0\ x_1\ \dots\ x_{r-1})$

$\sigma^m(x_i) = x_{i+m \pmod r}$

So if $m = r$, $\sigma^m(x_i) = x_i$ ✓

Example 18.3

Show that A_{10} contains an element of order 15.

Recall 18.4. A_n = even permutations of S_n
Every permutation σ can be written as a product of transpositions.
$(14356) = (16)(15)(13)(14)$.
There are an even number of transpositions. So $(14356) \in A_n$

First trying with A_{15}. $(1\ 2\ \dots\ 14\ 15)$ is an example of an element with order 15. This is easy because we have access to 15 elements.

What about A_{10}?
$\sigma = (1\ 2\ 3)(4\ 5\ 6\ 7\ 8)$
$\sigma^m = ((1\ 2\ 3)(4\ 5\ 6\ 7\ 8))^m$

Note 18.5. $((1\ 2)(3\ 4))^2 = ((1\ 2)(3\ 4)(1\ 2)(3\ 4)) = (1\ 2)^2(3\ 4)^2$
Only because the cycles are disjoint do they have this property.

Therefore $\sigma^m = ((1\ 2\ 3)(4\ 5\ 6\ 7\ 8))^m = (1\ 2\ 3)^m(4\ 5\ 6\ 7\ 8)^m$
When does $(1\ 2\ 3)^m = (1)$? So long as $3 \mid m$
When does $(4\ 5\ 6\ 7\ 8)^m = (1)$? So long as $5 \mid m$
Therefore $\sigma^m = (1) \Leftrightarrow\ 5 \mid m$ and $3 \mid m$
Smallest such m is $m = 15$. Therefore $|\sigma| = 15$.
$\sigma = (1\ 3)(1\ 2)(4\ 8)(4\ 7)(4\ 6)(4\ 5)$

Exercise 18.6. Find $(a_1\ a_2\ \dots\ a_n)^{-1}$

Answer 18.7. $(a_1\ a_2\ \dots\ a_n)^{-1} = (a_n\ a_{n-1}\ \dots\ a_1)$

Example 18.8

1. $(1\ 7\ 5\ 4\ 3)^{-1} = (34571)$
2. $[(1\ 4\ 2\ 3)(5\ 6)(1\ 3\ 2)]^{-1} = (1\ 3\ 2)^{-1} \cdot (5\ 6)^{-1} \cdot (1\ 4\ 2\ 3)^{-1}$
3. $(1\ 3\ 5\ 4\ 7)^{702} = (1\ 3\ 5\ 4\ 7)^{700} \cdot (1\ 3\ 5\ 4\ 7)^2 = (1) \cdot (1\ 3\ 5\ 4\ 7)^2 = (1\ 5\ 7\ 3\ 4)$

Exercise 18.9. $t = (a_1\ a_2\ \cdots\ a_k)$ is a cycle of length k

1. Show that for any permutation σ, $\sigma t \sigma^{-1} = (\sigma(a_1)\ \sigma(a_2)\ \cdots\ \sigma(a_k))$

   This is equivalent to showing that $\sigma t = (\sigma(a_1)\ \sigma(a_2)\ \cdots\ \sigma(a_k))\sigma$

   a) Case where $x \notin \{a_1, \cdots, a_k\}$

      $\sigma t(x) = \sigma(x)$ because t fixes x

      On the other hand, $(\sigma(a_1)\ \sigma(a_2)\ \cdots\ \sigma(a_k))\sigma(x) = \sigma(x)$ because $\sigma(x) \neq (a_i)$ for all i.

      $\sigma(x) = \sigma(x)$ ✓

   b) Case where $x = a_i$ for some i.

      If $i \neq k$,

      $\sigma t(a_i) = \sigma(a_{i+1})$

$$(\sigma(a_1)\ \sigma(a_2)\ \cdots\ \sigma(a_k))\sigma(a_i) = \sigma(a_{i+1})$$

If $i = k$,

$\sigma t(a_k) = \sigma(a_1)$

$(\sigma(a_1)\ \sigma(a_2)\ \cdots\ \sigma(a_k))\sigma(a_k) = \sigma(a_1)$

2. Let μ be a cycle of length k.

   $t = (a_1\ a_2\ \cdots\ a_k)$

   Show that $\exists \sigma \in S_n$ such that $\sigma t \sigma^{-1} = \mu$

   We just showed that $\sigma t \sigma^{-1} = (\sigma(a_1)\ \sigma(a_2, \cdots \sigma(a_k))$

   Assume that $\mu = (b_1\ b_2\ \cdots\ b_k)$

   Let σ be such that $\sigma(a_i) = b_i$ and $\sigma(x) = x$ if $x \neq a_i$ for all i, then the result follows.

Exercise 18.10. Let there be group G and fix $g \in G$. Define $\lambda_g : G \to G$ where $\lambda(a) = ga$. Show that λ_g is a permutation.

Definition 18.11. A permutation of a set S is a bijection $\pi : S \to S$

1. Showing that λ_g is injective.

   Let $a, b \in G$ such that $\lambda_g(a) = \lambda_g(b)$.

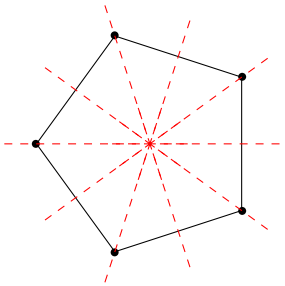
   $\Rightarrow ga = gb \Rightarrow a = b$ ✓

2. Showing that λ_g is surjective.

   For all $b \in G$, let $a = g^{-1}b \in G$. $\lambda_g(a) = gg^{-1}b = b$.

   Therefore the image of λ_g is G so λ_g is surjective. ✓

Exercise 18.12 (Dihedral Group)**.** Using the cyclic notation, list the elements in D_5.
$r = (1\ 5\ 4\ 3\ 2)$ $r^2 = (1\ 4\ 2\ 5\ 3)$ $r^3 = \ldots$
$s = (2\ 5)(3\ 4)$ $sr = \ldots$



Exercise 18.13. Show that S_n is non-abelian for $n \geq 3$. We need to find $\sigma, \tau \in S_n$ such that $\sigma\tau \neq \tau\sigma$
$\sigma = (1\ 2)$ $\sigma = (1\ 3)$
$\sigma\tau = (1\ 3\ 2)$ $\tau\sigma = (1\ 2\ 3)$
$(1\ 3\ 2) \neq (1\ 2\ 3)$

§19 Isomorphisms

Definition 19.1. $(G, \cdot)$ and $(H, \circ)$ are isomorphic if there exists a bijection $\phi : G \to H$ such that $\phi(a \cdot b) = \phi(a) \circ \phi(b)$ for all $a, b \in G$.

Then ϕ is an isomorphism and we write $G \equiv\sim H$

Example 19.2

$\phi : \mathbb{Z}_4 \to U_5$

$$0 \to 1$$
$$1 \to 2$$
$$2 \to 4$$
$$3 \to 3$$
$$\phi(3+2) = \phi(1) = 2 = \phi(3) \cdot \phi(2) = 3 \cdot 4 = 2 \checkmark$$

$(\mathbb{Z}_4, +)$:

$\circ$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$(U_5, \cdot)$:

$\circ$	1	2	4	3
1	1	2	4	3
2	2	4	3	1
4	4	3	1	2
3	3	1	2	4

Note 19.3. G and H are isomorphic if by "reordering the elements of H", they have the same caylety table - the only difference is notation.

"bijection between groups extends to a bijection between multiplication tables. Multiplication tables are the same, the difference being notation. A different language"

Example 19.4

$\phi : \mathbb{Z}_4 \to \{\pm 1, \pm i\} \subset \mathbb{C}^*$

$$0 \to 1$$
$$1 \to i$$
$$2 \to -1$$
$$3 \to -i$$

$$\phi(n) = i^n$$
$$\phi(a+b) = i^{a+b} = i^a \cdot i^b = \phi(a) \cdot \phi(b)$$

$(\{\pm 1, \pm i\} \subset \mathbb{C}^*, \cdot)$:

$\circ$	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	$-i$

Example 19.5

$\phi : \mathbb{Z}_4 \to \mathbb{Z}_4$

$$0 \to 0$$
$$1 \to 3$$
$$2 \to 2$$
$$3 \to 1$$

This is an isomorphism.

Theorem 19.6

G is abelian if and only if the map $\phi : G \to G$ given by $\phi(a) = a^{-1}$ for all $a \in G$ is an isomorphism.

Proof. .

($\Leftarrow$)

$$ba = (a^{-1}b^{-1})^{-1} = \phi(a^{-1}b^{-1}) = \phi(a^{-1})\phi(b^{-1}) = (a^{-1})^{-1}(b^{-1})^{-1} = ab$$

($\Rightarrow$)

$$\phi(a \cdot b) = (ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1} = \phi(a) \cdot \phi(b)$$

□

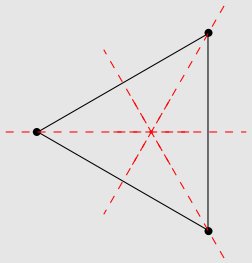
Example 19.7

$$Q_8 \equiv\sim \left\{ \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \pm\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \pm\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \right\}$$
$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

These two representations of the quaternions are isomorphic to one another.

Example 19.8

$D_3 \equiv\sim S_3$



$$0 \to ()$$
$$\frac{2\pi}{3} \to \{1\ 3\ 2\}$$
$$\frac{4\pi}{3} \to \{1\ 2\ 3\}$$
$$\alpha \to \{2\ 3\}$$
$$\beta \to \{1\ 2\}$$
$$\gamma \to \{1\ 3\}$$

There are many isomorphisms from D_3 to S_3.

Theorem 19.9

If $\phi : G \to H$ is an isomorphism, then $\phi^{-1} : H \to G$ is an isomorphism.

Proof. phi^{-1} is a bijection since ϕ is (ϕ^{-1} exists because ϕ is a bijection)

$$\phi^{-1}(a \cdot b) = \phi^{-1}\Big(\phi(\phi^{-1}(a))\phi(\phi^{-1}(b))\Big) = \phi^{-1}\Big(\phi(\phi^{-1}(a)\phi^{-1}(b))\Big) = \phi^{-1}(a)\phi^{-1}(b)$$

□

Theorem 19.10
Any "property" of G is a "property" of H.

Example 19.11
$|G| = |H|$
G is abelian $\Leftrightarrow$ H is abelian
G is cyclic $\Leftrightarrow$ H is cyclic
$G = \langle g \rangle \Leftrightarrow H = \langle \phi(g) \rangle$

Theorem 19.12
If G is cyclic and $|G| = \infty$ the $G \equiv\sim \mathbb{Z}$
If G is cyclic and $|G| = n$ the $G \equiv\sim \mathbb{Z}_n$

Proof. Let $G = \langle g \rangle$. Consider map $\phi : \mathbb{Z} \to G$ given by $\phi(i) = g^i$
Claim that ϕ is a bijection.
Surjective because each $x \in G$ is $g = g^i$ for some i so $\phi(i) = x$ where x is arbitrary.
Injective because $\phi(i) = \phi(j) \Rightarrow g^i = g^j \Rightarrow g^i g^{-j} = e \Rightarrow g^{i-j} = e \Rightarrow i = j = 0 \Rightarrow i = j$
Therefore ϕ is an isomorphism because $\phi(i + j) = g^{i+j} = g^i g^j = \phi(i)\phi(j)$ □

§20 Isomorphisms Continued

Theorem 20.1
If G is cyclic and $|G| = n$, then $G \equiv \mathbb{Z}_n$.

Proof. Consider $\phi : \mathbb{Z}_n \to G$ given by $\phi(i) = g^i$, then ϕ is a bijection.
Injective: $\phi(i) = \phi(j) \Rightarrow g^i = g^j \Rightarrow g^{i-j} = g^0 \Rightarrow i - j \equiv_n = 0 \Rightarrow i = j$
Surjective: Let $G = \langle g \rangle$.
$\{g^0, g^1, \ldots, g^{n-1}\} = G$
$\{0, 1, \ldots, n-1\} = \mathbb{Z}_n$ □

Theorem 20.2
Cor 9.9.
If $|G| = p$ and p is prime, then $G \equiv\sim \mathbb{Z}_p$

Proof. We showed that $G = \langle g \rangle$ for any $g \neq e$.
My understanding: if prime order, it must be cyclic. □

Theorem 20.3

Isomorphism is an equivalence relation on a set of groups.

Reflexive: $G \equiv\sim G$ because $1_G : G \to G$ is isomorphism.

$$1_G(ab) = ab = 1_G(a) \cdot 1_G(b)$$

Symmetrical: $G \equiv\sim K \Rightarrow K \equiv\sim G$ because $\phi : G \to K$ isomorphism then $\phi^{-1} : K \to G$ is isomorphism.

Transitive: $f : G \to K$ and $h : K \to J$ are isomorphisms then $h \circ f : G \to J$ is ismorphism.

Theorem 20.4 (Cayley's Theorem)

Every group is isomorphic to a permutation group.

Recall 20.5. A permutation group is a subgroup of S_n

Proof. G is isomorphic to a subgroup of the group of bijections of the set G. You could think of this as S_G.

For $g \in G$, let $\lambda_g : G \to G$ be permutation "left multiply by g" i.e. $\lambda_g(x) = gx$ for all $x \in G$.

Let $\overline{G} = \{\lambda_g : g \in G\}$

Claim: $G \cong \overline{G}$ with $\phi(g) = \lambda_g$

Injectivity: if $\phi(x) = \phi(y)$ then λ_x and λ_y are some bijection of G.

$$x = xe = \lambda_x(e) = \lambda_y(e) = ye = y$$

Surjectivity (immediate). $\overline{G} = \{\lambda_g : g \in G\} = \{\phi(g) : g \in G\} = \phi(G)$

Homomorphism:

$$\phi(xy) = \lambda_{xy}$$
$$\phi(x)\phi(y) = \lambda_x\lambda_y$$
$$\lambda_{xy}(z) = (xy)z \text{ for all } z \in G$$
$$\lambda_x(\lambda_y(z)) = \lambda_x(yz) = x(yz)$$
$$(xy)z = x(yz) \checkmark$$

□

Example 20.6

$$G = \{\pm 1, \pm i\}$$
$$G \cong G \subset S_G \cong S_4$$
$$1 \to \lambda_1 = \begin{bmatrix} 1 & -1 & i & -i \\ 1 & -1 & i & -i \end{bmatrix} = ()$$
$$-1 \to \lambda_{-1} = \begin{bmatrix} 1 & -1 & i & -i \\ -1 & 1 & -i & i \end{bmatrix} = (1\ -1)(i\ -i)$$
$$i \to \lambda_i \begin{bmatrix} 1 & -1 & i & -i \\ i & -i & -1 & 1 \end{bmatrix} = (1\ i\ -1\ -i)$$
$$-i \to \lambda_{-i} = \begin{bmatrix} 1 & -1 & i & -i \\ -i & i & 1 & -1 \end{bmatrix} = (1\ -i\ -1\ i)$$

Example 20.7

$$Q_8 \cong \overline{Q_8} \subset S_8$$

Example 20.8

$$\mathbb{Z}_6 \subset\to S_{\mathbb{Z}_6} = S_{\{0,1,2,3,4,5\}}$$
$$2 \to_\phi \lambda_2 \qquad \lambda_2 : \mathbb{Z}_6 \to \mathbb{Z}_6 \qquad \lambda_2(x) = 2 + x$$
$$\lambda_2 = (0\ 2\ 4)(1\ 3\ 5)$$
$$\lambda_3 = (0\ 3)(1\ 4)(2\ 5)$$
$$\lambda_5 = (0\ 5\ 4\ 3\ 2\ 1)$$

§21 Cosets

Definition 21.1. Let G be a group. Let H be a subgroup of G. $g \in G$
$gH = \{gh : h \in H\}$ (left coset)
$Hg = \{hg : h \in H\}$ (right coset)

Theorem 21.2
Left (or right) cosets of H in G partition G.

Theorem 21.3 (Lagrange-theorem)
Let G be a finite group. Let H be a subgroup of G.

$$[G : H] = \frac{|G|}{|H|}$$

Note 21.4. $[G : H]$ is the number of cosets of H in G.

Remark 21.5. Right and left cosets are not necessarily equal.

Example 21.6
Let $H = \{id, (1\ 2)\}$ be a subgroup of S_3.

$$(1\ 2\ 3)H = \{(1\ 2\ 3), (1\ 2\ 3)(1\ 2)\} = \{(1\ 2\ 3), (1\ 3)\}$$
$$H(1\ 2\ 3) = \{(1\ 2\ 3), (1\ 2)(1\ 2\ 3)\} = \{(1\ 2\ 3), (2\ 3)\}$$

Exercise 21.7.

(a) What is the index of $\langle 6 \rangle$ in $\mathbb{Z}_{24}$.

By lagrange:

$$[\mathbb{Z}_{24}, \langle 6 \rangle] = \frac{|\mathbb{Z}_{24}|}{|\langle 6 \rangle|} = \frac{24}{4} = 6$$
$$\langle 6 \rangle = \{0, 6, 12, 18\} \qquad |\langle 6 \rangle| = 4$$

(b) Let $\sigma = (1\,2\,5\,4)(2\,3)$ in S_5. What is the index of $\langle \sigma \rangle$ in S_5?

$$\sigma = (1\,2\,5\,4)(2\,3) = (2\,3\,5\,4\,1) \text{ because sigma is not disjoint}$$
$$|\sigma| = 5 \qquad |\langle \sigma \rangle| = 5$$
$$[S_5 : \langle \sigma \rangle] = \frac{5!}{5} = 4! = 24$$

Exercise 21.8. Find the left cosets of $H = \{id, \mu\}$ in D_4. (D_4 is the symmetries of a square.)

Recall 21.9. $\mu = (1\,2)(3\,4)$

$D_4 = \{id, (1\,2\,3\,4), (1\,3)(2\,4), (1\,2)(3\,4), (1\,4\,3\,2), (1\,4)(2\,3), (1\,3), (2\,4)\}$
By lagrange: $[D_4, H] = \frac{|D_4|}{|H|} = \frac{8}{2} = 4$

Example 21.10
Let $H = \{id, (1\,2)(3\,4)\}$

$$(1\,3)H = \{(1\,3), (1\,3)(1\,2)(3\,4)\} = \{(1\,3), (1\,2\,3\,4)\}$$
$$(2\,4)H = \{(2\,2), (2\,4)(1\,2)(3\,4)\} = \{(2\,4), (1\,4\,3\,2)\}$$
$$(1\,3)(2\,4)H = \{(1\,3)(2\,4), (1\,3)(2\,4)(1\,2)(3\,4)\} = \{(1\,3)(2\,4), (1\,4)(2\,3)\}$$

Conceptually it makes sense that the size of a subgroup must divide the size of the group so that the cosets of the subgroup can partition the group into subgroups of equal sizes.

Recall 21.11. S_n represents all possible bijections between $\mathbb{N}_n$ and $\mathbb{N}_n$

Exercise 21.12. Let G be a group and H a subgroup of G such that $[G : H] = 2$ and $a, b \in G \setminus H$. Show that $ab \in H$.

We know that $a^{-1} \notin H$. Therefore $a^{-1}H \neq H$.

Recall 21.13. $g_1H = g_2H \Leftrightarrow g_1 \in g_2H$. This implies that $g_1 \notin g_2H \Leftrightarrow g_1H \neq g_2H$.

Similarly $b \notin H$ implies that $bH \neq H$.

There are only two cosets, so $a^{-1}H = bH$. Therefore

$$a^{-1}h = bh'$$

for some $h, h' \in H$. Reordering we get that $ab = h(h')^{-1} \in H$

Exercise 21.14. Is it possible to have a group G of order 6 such that all of its elements have order 1 or 2? NO

Proof by contradiction. $G = \{e, g_1, g_2, g_3, g_4, g_5\}$ such that $|g_i| = 2$ for all $i \in \{1, 2, 3, 4, 5\}$

Claim: With this construction, G must be abelian.

$$a, b \in G$$
$$ab = id \cdot ab = (ba)^2ab \text{ because } ba \in G \Rightarrow (ba)^2 = e$$
$$\text{Expanding: } ab = babaab = bab^2 = ba$$

Consider the subgroup $H = \{1, g_1, g_2, g_1g_2\}$. This is a subgroup because G is abelian and all its elements have order 2.

By lagrange: $[G : H] = \frac{|G|}{4}$. This is a contradiction because $|G| = 6$ and 4 does not divide 6.

Basically, with the assumption that G is of order 6 with all elements being of order 1 or 2, we can build a subgroup of order 4 which doesn't make sense because 4 doesn't divide 6.

Exercise 21.15. Let H, K be subgroups of G. Show that for $x, y \in G$, either $xH \cap yK = \varnothing$, or $xH \cap yK$ is a coset of $H \cap K$. (Recall that $H \cap K$ is a subgroup).

Pick $x, y \in G$. If $xH \cap yK = \varnothing$, we are done with this case.

Assume that $xH \cap yK \neq \varnothing$. Pick $g \in xH \cap yK$. Then $g \in xH \Rightarrow xH = gH$ and $g \in yK \Rightarrow yK = gK$.

Therefore $xH \cap yK = gH \cap gK$.

Claim: $gH \cap hK = g(H \cap K)$

Proof. ($\subseteq$). Pick $z \in gH \cap gK$.

$$\begin{gathered}
\Rightarrow z = gh = gk \text{ for some } \ h \in H, k \in K \\
\Rightarrow h = k \text{ so } h \in H \cap K \\
\text{So } z = gh \quad h \in H \cap K \\
\text{Then } z \in g(H \cap K)
\end{gathered}$$

$$\begin{gathered}
(\supseteq)\text{Let } z \in g(H \cap K) \\
\Rightarrow z = gl \text{ for some } l \in H \cap K \\
l \in H \Rightarrow z \in gH \\
l \in K \Rightarrow z \in gK \\
z \in gH \cap gK
\end{gathered}$$

Therefore $gH \cap hK = g(H \cap K)$ □

§22 Direct Products

Let $(G, \cdot), (H, \cdot)$ be groups. The external direct product of $G \times H$.

$$G \times H = \{gh : g \in G, h \in H\}$$

with binary opteration $(g_1, h_1)(g_2, h_2) = (g_1 \cdot g_2, h_1 \cdot h_2$.

Note 22.1. Associative. Proof.

Note 22.2. Identity $= (1_G, 1_H)$.

We define the external direct product of $G_1 \times G_2 \times \cdots \times G_k$

Note 22.3.

$$|G| = \prod_{i=1}^{k} |G_i|$$

Definition 22.4. $G^n = G \times \cdots \times G$.

> **Example 22.5**
> $\mathbb{R}^n$ and $\mathbb{Z}_2^3$.

We have 5 groups of order 8. $Q_8, D_4, \mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2^3$.

1. cyclic with 3 subgroups of order 4. nonabelian
2. cyclic with 1 subgroup of order 4. nonabelian
3. cyclic. abelian
4. not cyclic with cosets of order 4 abelian
5. each element has order 2 abelian

Theorem 22.6
9.17. Let $(g,h) \in G \times H$. $|(g,h)| = \text{lcm}(|g|,|h|)$

Example 22.7

Theorem 22.8
$\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn} \Leftrightarrow \gcd(m,n) = 1$

Proof. □

§23 Normal Subgroups and Factor Groups

Definition 23.1. A subgroup $H \subset G$ is normal if $gH = Hg$ for all $g \in G$.

Example 23.2
1. Every subgroup of H is normal if G is abelian.
2. If $[G:H] = 2$, then H is normal. This is because $gH \cup H = G = H \cup Hg$.
3. Let $H \subset D_h$ be a subgroup of rotations. Then H is normal (because $[D_h : H] = 2$). However, let $R = \langle r \rangle$ where r is reflection, then R is not normal in D_n.
4. $\{e\} \subset G$ and $G \subset G$ are normal.

Theorem 23.3
Let $N \subset G$ be a subgroup. TFAE

1. N is normal in G.
2. $gNg^{-1} \subset N$ for all $g \in G$.
3. $gNg^{-1} = N$ for all $g \in G$.

Note 23.4. For $S \subset G$ and $x, y \in G$, $xSy = \{xsy : s \in S\}$

Proof.

$(1 \Rightarrow 2)$ We must show that $gng^{-1} \in N$ for all $n \in N$.

$$gN = Ng \Rightarrow \exists n' \in N \text{ such that } gn = n'g$$
$$\text{Hence:} \quad (gn)g^{-1} = (n'g)g^{-1} = n' \in N$$

$(2 \Rightarrow 3)$ Suffices to show that $N \subset gNg^{-1}$.

$$g^{-1}ng \in g^{-1}N(g^{-1})^{-1} \subset N \Rightarrow g^{-1}ng = n' \text{ for some } n' \in N$$
$$\text{So } n = gn'g^{-1}$$

$(3 \Rightarrow 1)$ Right multiply by g. $gNg^{-1} = N$ gives $gN = Ng$.

□

§23.1 Factor Group or Quotient Group

Definition 23.5. Let $N \subset G$ be a normal subgroup of G. The left cosets of N in G form a group whose operation is $(aN)(bN) = (abN)$. This is the quotient group of G and N, denoted by G/N.

Theorem 23.6

G/N is really a group!

Proof.

1. To show: Operation is well defined. If $aN = a'N$ and $bN = b'N$, then $abN = a'b'N$.

   We know that $a' = an_1$ and $b' = bn_2$ where $n_1, n_2 \in N$. Hence $a'b' = (an_1)(bn_2)$. Because $Nb = bN$, we hvae that $n_1 b = bn_3$ for some $n_3 \in N$. Therefore $a'b' = a(n_1 b)n_2 = a(bn_3)n_2 = abn_3 n_2$.

   Thus $a'b'N = abN$ since $(ab)^{-1}(a'b') = b^{-1}a^{-1}abn_3 n_2 = n_3 n_2 \in N$.

2. To show: Associativity.

$$aN(bNcN) = aN(bcN) = a(bc)N = abcN$$
$$(aNbN)cN = (abN)cN = (ab)cN = abcN$$

To show: Identity. $eNxN = exN = xN = xeN = xNeN$

To show: Inverses. $(xN)(x^{-1}N) = xx^{-1}N = eN = x^{-1}xN = x^{-1}NxN$ □

Recall 23.7. If G is finite, $|G/N| = [G : N] = |G|/|N|$

Example 23.8

$\mathbb{Z}_n$ is just notation for $\frac{\mathbb{Z}}{n\mathbb{Z}}$

Quotient Group $\mathbb{Z}/4\mathbb{Z}$:

$\circ$	$0+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$
$0+4\mathbb{Z}$	$0+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$
$1+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$	$4+4\mathbb{Z}$
$2+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$	$4+4\mathbb{Z}$	$5+4\mathbb{Z}$
$3+4\mathbb{Z}$	$3+4\mathbb{Z}$	$4+4\mathbb{Z}$	$5+4\mathbb{Z}$	$6+4\mathbb{Z}$

Example 23.9

$H \subset D_n$ be subgroup of rotations. $D_n/H \cong \mathbb{Z}_2$ since $[D_n : H] = 2$.

Example 23.10

$S_n/A_n \cong \mathbb{Z}_2$

Example 23.11

$N = \{\pm 1\}$ is normal in Q. It's cosets are:

$$\begin{aligned} 1N &= \{\pm 1\} = N1 \\ jN &= \{\pm j\} = Nj \\ kN &= \{\pm k\} = Nk \\ iN &= \{\pm i\} = Ni \end{aligned}$$

What is Q/N? Note: $|Q/N| = [Q : N] = 4$.

Q/N :

$\circ$	$1N$	iN	jN	kN
$1N$	$1N$	iN	jN	kN
iN	iN	$1N$	kN	jN
jN	jN	kN	$1N$	iN
kN	kN	jN	iN	$1N$

Example 23.12

$(\mathbb{Z}_4, +)$:

$\circ$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(1,0)$	$(1,0)$	$(0,0)$	$(1,1)$	$(0,1)$
$(0,1)$	$(0,1)$	$(1,1)$	$(0,0)$	$(1,0)$
$(1,1)$	$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$

§24 Simple Groups

Definition 24.1. A group G with no normal subgroups except G and $\{1_G\} = \{e\}$ is called simple.

Example 24.2

1. $\mathbb{Z}_p$ with p prime. The only subgroups are G and $\{1_G\}$.
2. $A_n \quad \forall n \geq 5$.

In some sense, simple groups are like the primes. Every group can be built from simple groups.

§25 Homomorphisms

Definition 25.1. A homomorphism from group $(G, \cdot)$ to $(H, \circ)$ is a map $\phi : G \to H$ such that is preserves multiplication. i.e. $\phi(g_1 \cdot g_2) = \phi(g_1) \circ \phi(g_2)$ for all $g_1, g_2 \in G$.

The range $\phi(G) \subset H$ is called the homomorphic image of G.

Remark 25.2. $\phi(G)$ is a subgroup of H.

Note 25.3. All isomorphisms are homomorphisms with the additional property that ϕ is a bijection.

Example 25.4
Let $g \in G$. There is a homomorphism $\phi : \mathbb{Z} \to G$ defined by $\phi(n) = g^n$.

Check: (review how binary operations apply below)

$$\phi(a+b) = g^{a+b} = g^a g^b = \phi(a)\phi(b)$$
$$\phi(\mathbb{Z}) = \langle g \rangle \subset G$$

Example 25.5

$$\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^*$$
$$\det(AB) = \det(A) \cdot \det(B)$$

Example 25.6
Let $G =$ the isometries of a tetrahedron.

$\phi : G \to \{\pm 1\}$. $\phi(g) = \pm 1$ if g preserves orientation. $\phi(g) = -1$ if g reverses orientation.

Theorem 25.7

Let $\phi : G_1 \to G_2$ be a homomorphism.

1. If e_1 is the identity element of G_1, the $\phi(e_1)$ is the identity element of G_2.
2. $\phi(g^{-1}) = [\phi(g)]^{-1}$
3. $H_1 \subset G_1$ is a subgroup $\Rightarrow \phi(H_1) \subset G_2$ is a subgroup
4. $H_2 \subset G_2$ is a subgroup $\Rightarrow \phi^{-1}(H_2) \subset G_1$ is a subgroup
5. $H_2 \subset G_2$ is a normal subgroup $\Rightarrow \phi^{-1}(H_2) \subset G_1$ is a normal subgroup

Note 25.8. Normal groups can be used to build factor and quotient groups.

Proof. Of the above statements.

1. $\phi(e_1) = \phi(e_1 e_1) = \phi(e_1)\phi(e_1)$. Therefore $e_2 = \phi(e_1)$.
2. $e_2 = \phi(e_1) = \phi(g \cdot g^{-1}) = \phi(g) \cdot \phi(g^{-1})$. Therefore $\phi(g)$ and $\phi(g^{-1})$ are inverse to one another.
3. Identity: $e_1 \in H_1 \Rightarrow e_2 = \phi(e_1) \in \phi(H_1)$, so image of ϕ contains identity element.

   Inverses: $g_2 \in \phi(H_1) \Rightarrow g_2 = \phi(g_1)$ for some $g_1 \in H_1 \Rightarrow g_1^{-1} \in H_1 \Rightarrow \phi(g^{-1}) = [\phi(g_1)]^{-1} = g_2^{-1} \in \phi(H_1)$. Therefore image contains inverses.

   Closure: Let $g_2, g_2' \in \phi(H_1)$. Therefore $\exists g_1, g_1' \in H_1$ such that $g_2 = \phi(g_1)$ and $g_2' = \phi(g_1')$. Therefore:

   $$g_1 g_1' \in H_1 \Rightarrow \phi(g_1 g_1') \in \phi(H_1) \Rightarrow g_2 g_2' = \phi(g_1)\phi(g_1') \in \phi(H_1)$$

4. Identity: $e_1 \in \phi^{-1}(H_2)$ because $\phi(e_1) = e_2 \in H_2$.

   Inverses: $g_1 \in \phi^{-1}(H_2) \Rightarrow g_1^{-1} \in \phi^{-1}(H_2)$ because $\phi(g_1^{-1}) = [\phi(g_1)]^{-1} \in H_2$.

   Closure: $g_1, g_1' \in \phi^{-1}(H_2) \Rightarrow g_1 g_1^{-1} \in \phi^{-1}(H_2)$ because $\phi(g_1 g_1') = \phi(g_1)\phi(g_1') \in H_2$
5. Show that for all $g_1 \in G_1$, $g_1 \phi^{-1}(H_2) g_1^{-1} \subset \phi^{-1}(H_2)$

   Let $k \in \phi^{-1}(H_2)$. Then $\phi(g_1 k g_1^{-1}) = \phi(g_1)\phi(k)\phi(g_1^{-1}) = \phi(g_1)\phi(k)[\phi(g_1)]^{-1} \in H_2$. Since we construct with $H_2 \subset G_2$ is normal. Remember that $\phi(k) \in H_2$.

□

§26 Isomorphisms

Definition 26.1. Let G, H be groups and $\phi : G \to H$ where ϕ is bijective and $\phi(ab) = \phi(a)\phi(b)$. Then ϕ is an isomorphism.

Example 26.2

$(\mathbb{Z}_2, +)$:

$\circ$	0	1
0	0	1
1	1	0

$(U(4) = \{1, 3\}, \times)$:

$\circ$	1	3
1	1	3
3	3	1

$$\mathbb{Z}_2 \cong U(4)$$

Example 26.3

Show that $\phi : \mathbb{C} \to \mathbb{C}$ is an isomorphism, where $\phi(a + ib) = a - ib$.

Surjective: Let $a + ib \in \mathbb{C}$, then $\phi(a - ib) = a + ib$. So for every element in $\mathbb{C}$, there exists an element in $\mathbb{C}$ that maps to it via ϕ.

Injective: Let $\phi(a+ib) = \phi(c+id)$. This implies that $a-bi = c-di \Rightarrow a = c$ and $b = d$.

Homomorphism: Let $x, y \in \mathbb{C}$. $x = a + bi$ and $y = c + id$ for some $a, b, c, d \in \mathbb{R}$.

$\phi(x+y) = \phi((a+c)+i(b+d)) = a+c-i(b+d) = (a-ib)+(c-id) = \phi(x)+\phi(y)$

Theorem 26.4

Let G be cyclic such that $G = \langle a \rangle$. Let H be a group isomorphic to G. Then H is cyclic.

Proof. Let $h \in H$. Because ϕ is surjective, $\exists g \in G$ such that $\phi(g) = h$.

$g \in G$, so $g = a^n$ for some n. Therefore $h = \phi(a^n) = (\phi(a)^n)$.

Because h is an arbitrary element in H, $\langle \phi(a) \rangle = H$. □

Let $G = \langle a \rangle$ be cyclic. Let $\phi : G \to H$ be an isomorphism. Then ϕ is completely determined by $\phi(a)$.

Example 26.5

$\phi : \mathbb{Z}_5 \to \mathbb{Z}_5$. In this case we chose $\phi(1) = 2$. The rest is determined by this because if $\phi(a) = b$, then $\phi(a^2) = (b^2)$. Note: Only for cyclic groups.

$$0 \to 0$$
$$1 \to 2$$
$$2 \to 4$$
$$3 \to 1$$
$$4 \to 3$$

Example 26.6
Prove or disprove that $\mathbb{Q}$ is isomorphic to $\mathbb{Z}$.

Answer 26.7. NO. We know that $\mathbb{Q}$ is not cyclic. Because $\mathbb{Z}$ is cyclic, $\mathbb{Z} \ncong \mathbb{Q}$.

Recall 26.8. If $G = \langle b \rangle$, then $|b^k| = \frac{n}{\gcd(k,n)}$.

Exercise 26.9. Find the order of the following.

(a) $(3, 4)$ in $\mathbb{Z}_4 \times \mathbb{Z}_6$.

$|3| = \frac{4}{\gcd(3,4)} = \frac{4}{1} = 4$ in $\mathbb{Z}_4$.

$|4| = \frac{6}{\gcd(4,6)} = \frac{6}{2} = 3$ in $\mathbb{Z}_6$.

$|(3, 4)| = \text{lcm}(4, 3) = 12$.

(b) $(5, 10, 15)$ in $\mathbb{Z}_{25} \times \mathbb{Z}_{25} \times \mathbb{Z}_{25}$.

$|5| = 5$

$|10| = 5$

$|15| = 5$

$|(5, 10, 15)| = \text{lcm}(5, 5, 5) = 5$

Exercise 26.10. Show that G is abelian if and only if $\phi : G \to G$ is an isomorphism where $\phi(x) = x^{-1}$.

($\Rightarrow$) Assume that G is abelian.

Surjectivity: Let $g \in G$. Then $\phi(g^{-1}) = (g^{-1})^{-1} = g$.

Injectivity: Let $x, y \in G$ be such that $\phi(x) = \phi(y) \Rightarrow x^{-1} = y^{-1} \Rightarrow x = y$

Homomorphism: Let $x, y \in G$. $\phi(xy) = (xy)^{-1} = y^{-1}x^{-1} \underbrace{=}_{G \text{ abelian}} x^{-1}y^{-1} = \phi(x)\phi(y)$

($\Leftarrow$) Assume that $\phi : G \to G$ is an isomorphism. Let $a, b, \in G$. We want to show that $ab = ba$.

$$ab = (b^{-1}a^{-1})^{-1} = (\phi(b)\phi(a))^{-1} = (\phi(ba))^{-1} = ((ba)^{-1})^{-1} = ba$$

Exercise 26.11. Show that isomorphism preserves the order of elements. i.e. that if $\phi : G \to H$ is an isomorphism and $a \in G$, then $|a| = |\phi(a)|$.

Proof. Assume that $|a| = n$ and $|\phi(a)| = m$.

We know that $\phi(a)^n = \phi(a^n) = \phi(id) = id$. Then $m|n$, in particular $m \leq n$. Now assuming that $m < n$:

$$id = \phi(a)^m = \phi(a^m) \Rightarrow a^m = id$$

This is a contradiction because $|a| = n > m$. Therefore $m = n$. □

Exercise 26.12. Find an isomorphism between $U(12)$ and a subgroup of S_4.

$$U(12) = \{1, 5, 7, 11\}$$

Begin by learning about the elements in the set: $|5| = 2, \quad |7| = 2, \quad |11| = 2, \quad 5 \cdot 7 = 11.$

Remark 26.13. Isomorphisms are not necessarily unique.

Theorem 26.14
$\mathbb{Z}_{nm} \cong \mathbb{Z}_n \times \mathbb{Z}_m \Leftrightarrow \gcd(n, m) = 1.$

> **Corollary 26.15**
> You can do prime decomposition on $\mathbb{Z}_n$. This decomposes it into simple groups.

Example 26.16
Are the following isomorphic?

(a) $\mathbb{Z}_{14} \times \mathbb{Z}_4 \times \mathbb{Z}_5$ and $\mathbb{Z}_{10} \times \mathbb{Z}_{28}$

(b) $\mathbb{Z}_3 \times \mathbb{Z}_{16} \times \mathbb{Z}_9$ and $\mathbb{Z}_{27} \times \mathbb{Z}_2 \times ZZ_8$