§1 Lecture 03-10

Example 1.1

$f_n(x) = x^n, x \in [0, 1]$. For each n fixed. f_n is continuous.

$$\lim_{n\to\infty} f_n(x) \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$$

Question. When does $\lim_{n\to\infty} f_n(x)$ exist and is continuous.
We need to impose some further conditions to ensure that $\lim f_n(x)$ is continuous.

Suppose $f(x) = \lim_{n\to\infty} f_n(x)$. f is continuous at x_0 iff

$$\begin{aligned} \lim_{x\to x_0} f(x) = f(x_0) &= \lim_{n\to\infty} \lim_{x\to x_0} f_n(x) \\ &= \lim_{x\to x_0} \lim_{n\to\infty} f_n(x) \end{aligned}$$

So interchanging the two limits and it is still equal.

Definition 1.2 (Uniform Convergence)**.** Let $\{f_n\}$ be a sequence of functions on X. We say that $\{f_n\}$ is uniformly convergent in X to f if $\forall \epsilon > 0$, $\exists N$ such that $|f_n(x) - f(x)| < \epsilon$, $\forall n \ge N$.

Note 1.3. The choice of N is independent of $x \in X$. It depends only on $\epsilon > 0$.

Example 1.4

Let's revisit the example $\{x^n\}$.

$$f_n \to f \text{ uniformly continuous } \Rightarrow f_n(x) \to f(x) \text{ pointwise}$$

$|f_n(x) - f(x)| < \epsilon$ if $n \ge N(\epsilon)$.

1. When $x = 1$, this is true $\forall n$.
2. The problem is $0 \le x < 1$.

$$\Rightarrow |x^n| < \epsilon, \ \forall n \ge N(\epsilon)$$

For all N fixed, we can always find x_n, such that $x_n^n \to 1, \ n \to \infty$.

Lemma 1.5

$f_n \to f$ is uniformly continuous iff $\forall \epsilon > 0,\ \exists N$ such that $\forall m > n \geq N,\ |f_m(x) - f_n(x)| < \epsilon, \forall x \in X$.

Proof. $\Rightarrow$. Suppose $f_n \to f$ is uniformly continuous. i.e.

$$\exists N,\ \text{such that } \forall n \geq N,$$
$$|f_n(x) - f(x)| < \frac{\epsilon}{2},\ \forall x \in X$$

Then

$$m > n \geq N$$
$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)| \leq \epsilon/2 + \epsilon/2 = \epsilon$$

$\Leftarrow$. If $(*)$ holds, then $\{f_n(x)\}$ is cauchy. $f_n(x) \to f(x),\ \forall x \in X$ pointwise because $\mathbb{R}$ is complete.

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \leq \epsilon + |f_m(x) - f(x)|$$

Let $m \to \infty$

$$\epsilon + |f_m(x) - f(x)| \leq \epsilon$$

$\square$

Theorem 1.6

Suppose $\{f_n\}$ is a sequence of continuous functions defined in X. Suppose $f_n \to f$ is uniformly convergent in X. Then f is continuous in X.

Theorem 1.7

Suppose $\{f_n\}$ is a sequence of continuous functions uniformly convergent to f in X, $x_0 \in X$ is a limit point of X. Suppose then

$$\lim_{x \to x_0} f_n(x) = A_n$$

$$\lim_{x \to x_0} f(x) = \lim_{n \to \infty} A_n = A$$

Proof. By cauchy $(*)$, $\forall \epsilon > 0, \exists N,\ |f_m(x) - f_n(x)| \leq \epsilon,\ \forall m > n \geq N, \forall x \in X$. $\square$

Let $x \to x_0 \Rightarrow |A_m - A_n| < \epsilon,\ m, \geq N \Rightarrow \{A_n\}$ is cauchy. $A_n \to A$.

$$|f(x) - A| \leq |f(x) - f_n(x)| + |f_n(x) - A_n| + |A_n - A|$$

We need to find $\delta > 0$ such that $\forall x \in B_\delta(x_0),\ |f(x) - A| < 3\epsilon$.
Set $n = N \Rightarrow |f(x) - A| \leq 2\epsilon + |f_N(x) - A_N|$. By the assumption, $\lim_{x \to x_0} f_N(x) = A_n \Rightarrow \exists \delta > 0$ such that $|f_N(x) - A_n| < \epsilon,\ \forall x \in B_\delta(x_0)$. This is the δ we are looking for.

Theorem 1.8 (M-test)
When $f_n \to f$ is uniformly continuous in X.

$|f_n(x)| \leq M_n,\ \forall x \in X$, consider $\sum_{n=1}^{\infty} f_n(x)$ is uniform if $\sum_{n=1} \infty M_n$ is convergent.