§1 Lecture 02-14

$$T : V \to V$$

If $p_T(x)$ factors into linear factors, (for example if the field F is algebraically closed, then every irreducible polynomial is linear), then

$$V = \oplus_{\lambda \in spec(T)} V_{[\lambda]}$$
$$\text{Generalized eigenspace for } \lambda\text{:} \quad V_{[\lambda]} = \{v : (T - \lambda)^j(v) = 0\}$$
$$\text{Eigenspace for } \lambda\text{:} \quad V_\lambda = \{v : (T - \lambda)(v) = 0\}$$

$V_\lambda = V_{[\lambda]} \Leftrightarrow (x - \lambda) | p_T(x)$ but $(x - \lambda)^2 \not| p_1(x)$

T is diagonalizable $\Leftrightarrow p_T(x)$ factors into distinct linear factors.

Theorem 1.1
T diagonalizable $\Leftrightarrow p_T(x)$ factors into distinct linear factors.

Example 1.2
$F = \mathbb{Z}/p\mathbb{Z}$. T satisfies $T^p = T \Rightarrow T$ satisfies $x^p - x \Rightarrow p_T(x)$ divides $x^p - x$.

This implies $p_T(x) = (x - \lambda_1)(\cdots)(x - \lambda_r)$, $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_r$. Therefore T is diagonalizable.

Example 1.3

$T^n = 1 \Rightarrow p_T(x)$ divides $x^n - 1$.

If $x^n - 1$ factors into distinct linear factors in F, then T is diagonalizable.

Conversely, if all T satisfying $T^n = 1$ are diagonalizable, then $x^n - 1$ factors into distinct linear factors.

In order to prove the converse, we need to show that $\exists T$ such that $p_T(x) = x^n - 1$.

$$V = F^n = Fe_1 \oplus \cdots \oplus Fe_n$$
$$T(e_j) = e_{j+1} \quad (j = 1, \ldots, n-1)$$
$$T_(e_n) = e_1$$

Proposition 1.4

If $p(x) \in F[x]$, then $\exists$ a vector space V over F, and $T : V \to V$ such that $p_T(x) = p(x)$.

Proof. Let $V = F[x]/(p(x))$. $\dim V = n$.

$$T(g(x) + (p(x)) = xg(x) + (p(x))$$
$$f(T)(g(x) + (p(x)) = f(x)g(x) + (p(x))$$

If we want $f(T) = 0$ then $f(T)(1 + (p(x)) = 0 \Rightarrow f(x) + (p(x)) \Rightarrow p(x)|f(x) \Rightarrow p(x) = p_T(x)$ □

Example 1.5

When is it possible to factor $x^n - 1$ in the following fields? $F = \mathbb{Q}$.Then $n \leq 2$. $F = \mathbb{R}$, then $n \leq 2$. $F = \mathbb{C}$, then any n. $F = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.