§1 11-06

§1.1 Divergence to infinity

Definition 1.1. Let (x_n) be a sequence. We say that (x_n) diverges to $+\infty$ if

$$\forall M > 0,\ \exists N \in \mathbb{N},\ \forall n \geq N : x_n > M$$

In symbols:

$$\lim(x_n) = +\infty$$

(x_n) diverges to $-\infty$ if

$$\forall M > 0(\exists N \in \mathbb{N})(\forall \geq N) : x_n < -M$$

In symbols:

$$\lim(x_n) = -\infty$$

Remark 1.2. If $\lim(x_n) = +\infty$ or $\lim(x_n) = -\infty$, then the sequence diverges. The limit laws thus do NOT apply.

> **Example 1.3**
> $\lim(n^2) = +\infty$. Let $M > 0$. Then $n^2 > M \Leftrightarrow n > \sqrt{M}$.
>
> Let $N > \sqrt{M}$. Then $\forall n \geq N : n^2 \geq N^2 > M \Rightarrow n^2 > M$ for all $n \geq M \Rightarrow (n^2)$ diverges to $+\infty$.

> **Example 1.4**
> Let $a > 1$. Show that $\lim(a^n) = +\infty$.
>
> Since $a > 1$, $b := a - 1 > 0$. Then $a = 1 + b$ and $a^n = (1 + b)^n$. Applying bernoulli's:
> $$(1+b)^n \geq 1 + nb > nb > M \Leftrightarrow n > \frac{M}{b}$$
> Let $N > \frac{M}{b}$. Then $\forall n \geq N$, we know that $a^n > nb \geq Nb > M$. Thus a^n diverges to $+\infty$.

§1.2 Chapter 4: Limits of functions

Preparatory definition:

Definition 1.5 (In A)**.** Let $A \subseteq \mathbb{R}$. A sequence (x_n) is said to be in A if $\forall n \in \mathbb{N} : x_n \in A$.

Definition 1.6 (Cluster point)**.** Let $A \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is called a cluster point of A if:

$$\forall \epsilon > 0 : \underbrace{V_\epsilon(x) \setminus \{x\}}_{\text{Punctured neighborhood}} \cap A \neq \varnothing$$

Note 1.7. Notation for punctered neighborhoods:

$$V_\epsilon^*(x) \coloneqq V_\epsilon(x) \setminus \{x\}$$

i.e. x is a cluster point of A if $\forall \epsilon > 0 : V_\epsilon^*(x) \cap A \neq \varnothing$.

Remark 1.8. Cluster points of A are not necessarily elements of A.

Definition 1.9 (Isolated Point)**.** Let $A \subseteq \mathbb{R}$. $x \in A$ is called an isolated point of A if $\exists \epsilon > 0 : V_\epsilon^*(x) \cap A = \varnothing$.

i.e. x is the only element of A that is in $V_\epsilon(x)$.

Example 1.10

$S \coloneqq \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$.

Claim: 0 is the only cluster point of S. All points $\frac{1}{n} : n \in \mathbb{N}$ are isolated points of S.

0 is a cluster point. Let $\epsilon > 0$. Then $V_\epsilon(0)$ contains infinitely many numbers of the form $\frac{1}{n}$ because $\lim(\frac{1}{n}) = 0$. Thus 0 is a cluster point of S.

Let $x \neq 0$. Then $\exists \epsilon > 0 : V_\epsilon^*(x) \cap S = \varnothing$ (left as exercise). Especially, such $\epsilon > 0$ exists for all $x = \frac{1}{n}$. Thus every $\frac{1}{n}$ is an isolated point of S. □

Example 1.11

Let $A \coloneqq \mathbb{Q}$. Then every real number is a cluster point of A.

Proof. Let $x \in \mathbb{R}$ be arbitrary and let $\epsilon > 0$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $_\epsilon(x)$ contains infinitely many rational numbers. Thus $V_\epsilon^*(x)$ contains at least one (in fact infinitely many) rational numbers. i.e.

$$V_\epsilon^*(x) \cap A \neq \varnothing \Rightarrow x \text{ is a cluster point of } A$$

□

Exercise 1.12. Let I be an interval. Then the set of all cluster points of I is $\overline{I}$

Theorem 1.13
Let $A \subseteq \mathbb{R}$. Then $x \in \mathbb{R}$ is a cluster point of A if and only if there exists a sequence (x_n) in $A \setminus \{x\}$ with $\lim(x_n) = x$.

Proof.

($\Rightarrow$) Let x be a cluster point of A.

Let $\epsilon := 1$. Then $V_\epsilon^*(x) \cap A \neq \varnothing$. Let $x_1 \in V_1^*(x) \cap A$.

Let $\epsilon := \frac{1}{2}$. Then $V_\epsilon^*(x) \cap A \neq \varnothing$. Let $x_2 \in V_{\frac{1}{2}}^*(x) \cap A$.

We obtain a sequence (x_n) in $A \setminus \{x\}$ with $\forall n \in \mathbb{N} : x_n \in V_{\frac{1}{n}}^*(x) \cap A$.

Let $\epsilon > 0$. Let $N > \frac{1}{\epsilon} \Leftrightarrow \frac{1}{N} < \epsilon$. Then

$$\forall n \geq N : x_n \in V_{\frac{1}{n}}^*(x) \cap A \subseteq V_{\frac{1}{n}}^*(x) \subseteq V_{\frac{1}{n}}(x) \subseteq V_{\frac{1}{N}}(x) \subseteq V_\epsilon(x).$$

i.e. $\forall n \geq N : x_n \in V_\epsilon(x)(x_n)$ converges to x.

($\Leftarrow$) Let (x_n) be a sequence in $A \setminus \{x\}$ such that $\lim(x_n) = x$. Let $\epsilon > 0$. Then $\exists N \in \mathbb{N}$, $\forall n \geq N : x_n \in V_\epsilon(x)$. But since $x_n \in A \setminus \{x\}$, $x_n \neq x$. This means that $x_n \in V_\epsilon^*(x)$ and $x_n \in A$. Thus $\geq N : x_n \in V_\epsilon^*(x) \cap A$. Thus $v_\epsilon^*(x) \cap A \neq \varnothing x$ is a cluster point.

□

Theorem 1.14
Let $A \subseteq \mathbb{R}$. Let x be a cluster point of A. Then $x \in \overline{A}$.

Proof. Let x be a cluster point of A. By previous theorem, $\exists (x_n)$ is $A \setminus \{x\}$ such that $\lim(x_n) = 0$.

Since $\in \mathbb{N} : x_n \in A \setminus \{x\}$. We have that $\forall n \in \mathbb{N} : x_n \in \overline{A} \supseteq A \setminus \{x\}$.

Since $\overline{A}$ is closed, $\lim(x_n) \in \overline{A}$ (see assignment 6). □

Definition 1.15 (The limit of a function: Sequential Definition)**.**
Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$. Let $x_0 \in \mathbb{R}$, we say that L is a limit of f as $x \to x_0$. In symbols:

$$L = \lim_{x \to x_0} f(x)$$

if for all sequences (x_n) in $A \setminus \{x_0\}$ with $\lim(x_n) = x_0$, we have that $\lim(f(x_n)) = L$.

Example 1.16
Let

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, x \to \frac{x^2}{|x|}$$

Note that for $x \neq 0$ we have that

$$\frac{x^2}{|x|} = |x|$$

Claim: $\lim_{x \to 0} f(x) = 0$.

Let (x_n) be a sequence such that $x_n \neq 0$ for all $n \in \mathbb{N}$ and such that $\lim(x_n) = 0$. We need to show that $(f(x_n))$ converges to 0. Note that $f(x_n) = |x_n|$.

Let $\epsilon > 0$. Since $\lim(x_n) = 0$, there exists $(N \in \mathbb{N})(\forall n \geq N) : |x_n - 0| = |x_n| < \epsilon$.

Thus $\forall n \geq N : ||x_n| - 0| = ||x_n|| = |x_n| < \epsilon \Rightarrow \lim(f(x_n)) = 0$. Thus:

$$\lim_{x \to x_0} f(x) = 0$$

Example 1.17
Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ where $x \to \frac{1}{x}$. Let $x_0 \neq 0$. Show that

$$\lim_{x \to x_0} f(x) = \frac{1}{x_0}$$

Proof. Let (x_n) be a sequence in $\mathbb{R} \setminus \{0, x_0\}$ with $\lim(x_n) = x_0$. Then $\lim(f(x_n)) = \lim(\frac{1}{x_n}) = \frac{1}{\lim(x_n)} = \frac{1}{x_0}$. □

Example 1.18
Let $f : \mathbb{Z} \to \mathbb{R}$ where $x \to 0$. Let $L \in \mathbb{R}$ be arbitrary. Then

$$\lim_{x \to 0} f(x) = L$$

Since 0 is an isolated point in $\mathbb{Z}$, there doesn't exist any sequence in $\mathbb{Z} \setminus \{0\}$ that converges to 0. Thus *all* sequences (x_n) in $\mathbb{Z} \setminus \{0\}$ that converge to 0 hvae that property that

$$\lim_{x \to 0} f(x_0) = L$$

Thus $\lim_{x \to 0} f(x) = L$ for any $L \in \mathbb{R}$.

Remark 1.19. This example shows that we should avoid isolated points when considering limits.

Theorem 1.20

Let $f : A \to \mathbb{R}$ where x_0 is a cluster point of A.

Then: if f has a limit as x approaches x_0, then this limit is uniquely determined.

Proof. Let L_1, L_2 be limits of f as x approaches x_0. Then $\exists (x_n)$ is $A \setminus \{x_0\}$ with $\lim(x_n) = x_0$. Because f has a limit at x_0, $\lim(f(x_n))$ exists and $L_1 = \lim(f(x_n)) = L_2$. $\square$