§1 Lecture 02-10

$T : V \to V$. $T \in \mathrm{End}_F(V)$.

Key Invariants

1. Minimal polynomial $p_T(x)$. Defining property: for all $g(x) \in F[x]$, $g(T) = 0 \Rightarrow p_T(x) | g(x)$.

2. Characteristic polynomial $f_T(x) = \det(xI_V - T)$. $f_T(x)$ is a monic polynomial of $d = n = \dim V$.

$spec(T) = \{eigenvalues\}$. $\lambda \in spec(T), 0 \neq V_\lambda \subseteq V$

Eigenvalue decomposition

$$\oplus_{\lambda \in spec(T)} V_\lambda \subseteq V$$

If $\oplus_{\lambda \in spec(T)} V_\lambda = V$, we say that T is diagonalizable.

Theorem 1.1
The spectrum of T is exactly the set of roots of the characteristic polynomial or of the minimal polynomial of T. This means that the characteristic and minimal polynomial have the same roots.

Note 1.2. Very often, polynomials need not have root in F.

Example 1.3
1. $F = \mathbb{R}$. $p(x) = x^2 + 1$
2. $F = \mathbb{Q}$. $p(x) = x^2 - 2$.

In $F[x]$, every polynomial can be written uniquely as $p(x) = p_1(x)^{e_1} p_2(x)^{e_2}_r(x)$, where $p_j(x)$ distinct, monic, irreducible polynomials.

1. Given (T, V), can $\overline{V}$ be broken into a direct sum of (proper) T-stable subspaces.

2. Give simple criteria for T to be diagonalizable.

Proposition 1.4

Suppose that $p_T(x) = p_1(x)p_2(x)$ with $\gcd(p_1(x), p_2(x)) = 1$. Then $V = V_1 \oplus V_2$ where V_1 and V_2 are preserved by T, and $T_j = T|V_j$ has minimal polynomial $p_j(x)$.

Proof. $P_T(x) = p_1(x)p_2(x)$. $0 = p_1(T) \circ p_2(T)$. Define

$$V_1 = \ker(p_1(T))$$
$$V_2 = \ker(p_2(T))$$

Now to show that $T(V_1) \subseteq V_1$. Let $w \in V_1$. We want to check if $T(w) \in ker(p_1(T)) \Rightarrow p_1(T)(T(w)) = 0$

$$p_1(T)(T(w)) = p_1(T) \circ T(w) = T \circ p_1(T)(w) = T(p_1(T)(w)) = T(0)$$

We can do this because T commutes with itself, and $p_1(T)(w) = 0$. □

$$\{a(x)p_1(x) + b(x)p_2(x),\ a, b \in F[x]\} = F[x]$$
$$\Rightarrow \exists a(x), b(x) \in F[x] \text{ such that } a(x)p_1(x) + b(x)p_2(x) = 1$$
$$\Rightarrow a(T) \circ p_1(T) + b(T)p_2(T) = 1_V \text{ the identity from } V \text{ to } V$$
$$\text{Evaluating at } w \in V\ p_1(T)(a(T)(w)) + p_2(T)(b(T)(w)) = w$$
$$w_2 + w_1 = w$$
$$w_1 \in \operatorname{Im}(p_2(T)) \subseteq \ker(p_1(T)) = V_1$$
$$w_2 \in \operatorname{Im}(p_1(T)) \subseteq \ker(p_2(T)) = V_2$$
$$\Rightarrow \operatorname{span}(V_1, V_2) = V$$

Remains to show that $V_1 \cap V_2 = \{0\}$. Suppose we have $\ker(p_1(T)) \cap \ker(p_2(T))$. Evaluating $(*)$ at w_1 we get $0 + 0 = w \Rightarrow w = 0$.

Theorem 1.5

If $p_T(v) = p_1(x)p_2(x)\cdots p_r(x)$, where $\gcd(p_1(x), p_j(x)) = 1\ \forall i \neq j$, then $\exists\ V_1, \ldots, V_r$ such that

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$$

where

$$V_j = \ker(p_j(T))$$

i.e. $T|V_j$ has minimal polynomial $p_j(x)$.

Proof. Induction on r. □