§1 10-02

Theorem 1.1
Any n-cycle is the product of $(n-1)$ transpositions.

Proof.

$$(a_1\ a_2\ \dots\ a_n) = (a_1\ a_n)(a_1\ a_{n-1})\dots(a_1\ a_3)(a_1\ a_2)$$

An alternative proof:

$$(a_1\ a_2\ \dots\ a_n) = (a_1\ a_2)(a_2\ a_3)\dots(a_{n-1}\ a_n)$$

□

Definition 1.2. An element $\sigma \in S_n$ is:

1. Even if σ is the product of an even number of transpositions
2. Odd if σ is the product of an odd number of transpositions.

Theorem 1.3
No $\sigma \in S_n$ is both even and odd.

Note 1.4. This means that any odd σ can only be expressed as a product of an odd number of transpositions.

Proof. Matricies over $\mathbb{R}$ have det positive or negative. Positive determinent maintains orientation. Negative determinent inverses orientation. An even element can be likened to a matrix with a positive determinent, while an odd element can be likened to a matrix with a negative determinent. □

Example 1.5
(2 3 5 7 9) is even because it is the product of four transpositions $(n-1)$.
(1 8 6 2) is odd because it is the product of three transpositions $(n-1)$.

Theorem 1.6
The set of even permutations of S_n is a subgroup. $A_n \subset S_n$, alternating group.

Proof. Identity Element: $() = (1\ 2)(1\ 2)$
Inverse: $() \in A_n$. Need to prove that $\sigma \in A_n \Rightarrow \sigma^{-1} \in A_n$.Indeed:

$$\sigma = (a_1\ b_1)(a_2\ b_2)\dots(a_k\ b_k)$$

$$\sigma = (a_k\ b_k)\dots(a_2\ b_2)(a_1\ b_1)$$

Closure: $\sigma, \phi \in A_n \Rightarrow \sigma \cdot \phi \in A_n$. Even number of permutations times even number of permutations gives an even number of permuatations which is in A_n.

Note 1.7. $|A_n| = \frac{1}{2}|S_n|$.

□

Understanding $A_4 \subset S_4$.
Listing elements in S_4. $S_4 =$

$$\{()\}$$
$$\{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$$

$$\begin{gathered}\{(1\ 2\ 3), (1\ 3\ 2)\\ (1\ 2\ 4), (1\ 4\ 2)\\ (1\ 3\ 4), (1\ 4\ 3)\\ (2\ 3\ 4), (2\ 4\ 3)\}\end{gathered}$$

Note 1.8. S_4 "is" an isometry group of tetrahedron.
A_4 "is" the subgroup of its rigit motions.

§1.1 Cosets and Lagrange's Theorem

Let $H \subset G$ be a subgroup.
Let $g \in G$.
The left coset of H represented by g is $gH = \{gh : h \in H\}$.
The right coset of H represented by g is $Hg = \{hg : h \in H\}$.
Usually $gH \neq Hg$. (If equal just call them cosets if you'd like).

Example 1.9

Misleading but simple example.
$G = \mathbb{Z}_{12}, H = \langle 4 \rangle = \{0, 4, 8\}$.

$$0 + H = 4 + H = 8 + H = \{0, 4, 8\}$$
$$1 + H = 5 + H = 9 + H = \{1, 5, 9\}$$
$$2 + H = 6 + H = 10 + H = \{2, 6, 10\}$$
$$3 + H = 7 + H = 11 + H = \{3, 7, 11\}$$

Note 1.10. Notation can be confusing. gh means binary operation between g and h so when binary operation is + it means $g + h$.

Cosets formed a partition of the group.
Review: What is a partition? Disjoint subsets that unionize to form a set.

Example 1.11

$$H = \{1, -1, i, -i\} \subset \mathbb{Q}_8$$
$$1 \cdot H = \{1 * 1, 1 * -1, 1 * i, 1 * -i\}$$
$$jH = \{j * 1, j * -1, j * i, j * -i\} = \{j, -j, -k, k\} = Hj$$

Note 1.12. These form a partition of the quaternions.

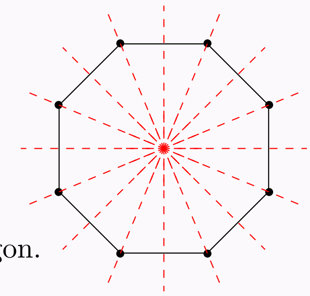
Example 1.13

$$K \subset \mathbb{Q}_8$$
$$K = \{1, -1\}$$

$$1K = \{1, -1\}$$
$$iK = \{i, -i\}$$
$$jK = \{j, -j\}$$
$$kK = \{k, -k\}$$

Note 1.14. For all of these, a left coset is a right coset.

Example 1.15



D_8. Symmetries of a regular octogon.

$$D_8 = \{r_i, \frac{i}{8}2\pi : 0 \leq i \leq 7\}$$

Subgroup (Same as the isometry of a rectangle living inside):

$$H = \{r_1, r_5, 0, \pi\}$$

Note 1.16. Product of rotation with rotation is a rotation. Product of a rotation with a reflection is a reflection. Product of a reflection with a reflection is a rotation.

$$0H = H$$
$$\frac{\pi}{8}H = \{r_2, r_6, \frac{\pi}{8}, \frac{5\pi}{8}\}$$
$$\frac{2\pi}{8}H = \{r_3, r_7, \frac{2\pi}{8}, \frac{6\pi}{8}\}$$
$$\frac{3\pi}{8}H = \{r_4, r_0, \frac{3\pi}{8}, \frac{7\pi}{8}\}$$
$$H\frac{\pi}{8} = \{r_0, r_4, \frac{\pi}{8}, \frac{5\pi}{8}\}$$

Note 1.17. Finding the product of a reflection and a rotation can be tricky. See how the composition affects a single point, and then identity a single rotation that affects the point in the same way.

Theorem 1.18

Lem 6.2. Let $g_1, g_2 \in G$ and $H \subset G$ be a group.
TFAE (similar for right cosets)

1. $g_1 H = g_2 H$
2. $Hg_1^{-1} = Hg_2^{-1}$. There is a bijection from a group to itself. Most obvious is identity bijection, but another one is every element to its inverse. In order to prove that statements 1 and 2 imply one another, use $\phi : G \to G, \phi(g) = g^{-1}.\phi$ is a bijection.$\phi(g_1 h) = h^{-1} g_1^{-1} \Rightarrow \phi(gH) \subset Hg^{-1}$.
3. $g_1 H \subset g_2 H$
4. $g_2 \in g_1 H$
5. $g_1^{-1} g \in H$. Reading this statement: "The difference between g_1 and g_2 lies in H.

Proof. $1 \Leftrightarrow 5$.
$\Rightarrow$. Suppose $g_1^{-1} g_2 \in H$, then $g_2^{-1} g_1 = (g_1^{-1} g_2)^{-1} \in H$.
Therefore $g_1 H \subset g_2 H$ because $g_1 h = (g_2 g_2^{-1}) g_1 h = g_2 (g_2^{-1} g_1) h \in g_2 H$. ($g_2 h'$ with $h' \in H$).
$\Leftarrow$. $(g_1 H = g_2 H) \Rightarrow (g_1 e \in g_2 H) \Leftrightarrow (g_1 \in g_2 H) \Rightarrow (g_1 = g_2 h$ for some $h \in H) \Rightarrow g_2^{-1} g_1 = h \in H$. □

Theorem 1.19

Lem 6.4. Let $H \subset G$ be a subgroup. The left (or right) cosets of H form a partition of G.

Proof. Look:

$$G = \bigcup_{g \in G} gH \text{ because } g = ge \in gH$$

If $g_1 H \cap g_2 H \neq \varnothing$, then $g_1 H = g_2 H$ because if $g_1 h_1 = g_2 h_2$, then $g_1^{-1} g_2 = h_1 h_2^{-1} \in H$, hence $g_1 H = g_2 H$. □

Definition 1.20. Let $[G : H]$ be the index of H in G denote the number of left cosets of H in G.

Example 1.21

$$[D_8 : \{r_1, r_5, 0, \pi\}] = 4$$
$$[\mathbb{Z}_{12} : \{0, 4, 8\}] = 4$$
$$[\mathbb{Q}_8 : \{-1, 1\}] = 4$$
$$[\mathbb{Q}_8 : \{-1, 1, i, -i\}] = 2$$
$$[\mathbb{Z} : n\mathbb{Z}] = n$$
$$[G : G] = 1$$
$$[G : \{e\}] = |G|$$

Theorem 1.22
6.4. Let $H \subset G$. The number of left cosets equals the number of right cosets.

Proof. The inversion map on G sends left cosets to right cosets and right cosets to left cosets.

Let L be the collection of left cosets and R be the collection of right cosets.

Define bijection $\phi : L \to R$ by $\phi(gH) = Hg^{-1}$. Now to check that this function is well defined

Definition 1.23. . Well defined: independent of choice of representative.

Check that if $gH = kH \Rightarrow \phi(gH) = \phi(kH)$.

$$gH = kH \Rightarrow Hg^{-1} = Hk^{-1} \Rightarrow \phi(gH) = \phi(kH).$$

Now check that ϕ is injective.

$$[\phi(gH) = \phi(kH)] \Rightarrow [Hg^{-1} = Hk^{-1}] \Rightarrow [gH = kH]$$

Now check that ϕ is surjective.
$Hx = H(x^{-1})^{-1} = \phi(x^{-1}H)$ □