§1 Limit laws

Example 1.1

$$a_n = \frac{n}{4^n}$$

Show that $\lim(a_n) = 0$ Try using bernoulli but here it doesn't help much.

$$4^n = (1+3)^n \geq 1 + 3n$$

$\Rightarrow |a_n - 0| = \frac{n}{4^n} \leq \frac{n}{1+3n} \to \frac{1}{3} \neq 0$

Unfortunately $\frac{n}{1+3n}$ does not converge to 0 so this estimate is too weak to be useful. Note: This argument can be save (see next assignment).

Different approach: We'll show that $4^n \geq n^2$ for all $n \in \mathbb{N}$

Proof by Induction. .
n = 1: $4^1 = 4 \geq 1 = 1^2$
$n \to n+1$: Assume that $4^n \geq n^2$, then

$$\begin{aligned} 4^{n+1} = 4 \cdot 4^n \geq 4 \cdot n^2 = 2n^2 + n^2 + n^2 = 2n^2 + (n+1)^2 + (n-1)^n - 2 \\ = (2n^2 - 2) + (n-1)^2 + (n+1)^2 \geq (n+1)^2 \\ \Rightarrow 4^n \geq n^2 \ \forall n \in \mathbb{N} \end{aligned}$$

Thus $|a_n - 0| = \frac{n}{4^n} \leq \frac{n}{n^2} \leq \frac{1}{n} \to 0$
Therefore $\lim(a_n) = 0$ □

Theorem 1.2
Every convergent sequence is bounded.

Proof. Let (a_n) be a sequence with $\lim(a_n) = L$, and let $\epsilon = 1$.
Then $\exists N \in \mathbb{N} \ \forall n \geq N : |a_n - L| < \epsilon = 1$

$$\Rightarrow |a_n| = |(a_n - L) + L| \leq |a_n - L| + |L| < 1 + |L| \quad \forall n \geq N$$

This proves that when $n \geq N$, a_n is bounded.

Now let $M = \{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |L|\}$
Then $|a_n| \leq M$ for <u>all</u> $n \in \mathbb{N}$. □

Remark 1.3. The convergence condition is essential. The sequence $(n) = (1, 2, 3, \ldots)$ is unbounded.

Theorem 1.4

Let $(a_n), (b_n)$ be convergent sequences. Then $(a_n + b_n)$ is convergent with $\lim(a_n + b_n) = \lim(a_n) + \lim(b_n)$

Proof. Let $a = \lim(a_n), b = \lim(b_n)$. Let $\epsilon > 0$.

$$|a_n + b_n - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$$

Since $\lim(a_n) = a,\ \exists N_1 \in \mathbb{N}\ \forall n \geq N_1 : |a_n - a| < \epsilon/2$

Similarly, because $\lim(b_n) = b,\ \exists N_2 \in \mathbb{N} : \forall n \geq N_2 : |b_n - b| < \frac{\epsilon}{2}$.

Let $N = \max\{N_1, N_2\}$. Then

$$\forall n \geq N : |a_n - a| < \frac{\epsilon}{2} \wedge |b_n - b| < \frac{\epsilon}{2}$$

Therefore

$$|a_n + b_n - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall n \geq N$$

Thus $(a_n + b_n)$ converges and $\lim(a_n + b_n) = a + b = \lim(a_n) + \lim(b_n)$ $\square$

This is supposed to be relatively simple.

Example 1.5

$$\lim(\frac{n+1}{n}) = \lim(1 + \frac{1}{n}) = \lim(1) + \lim(\frac{1}{n}) = 1 + 0 = 1$$

Theorem 1.6

Let $(a_n), (b_n)$ be convergent. Then (a_nb_n) converges and $\lim(a_nb_n) = \lim(a_n)\cdot\lim(b_n)$

Proof. Let $a = \lim(a_n), b = \lim(b_n)$. Let $\epsilon > 0$.

$$\begin{aligned}|a_nb_n - ab| &= |a_nb_n - ab_n + ab_n - ab| \\ &= |(a_n - a)b_n + a(b_n - b)| \\ &\leq |a_n - a||b_n| + |a||b_n - b|\end{aligned}$$

Because (b_n) converges, (b_n) is bounded by a previous theorem. Thus $\exists M_1 > 0$ such that $|b_n| \leq M$ for all $n \in \mathbb{N}$.

$$|a_nb_n - ab| \leq M_1 \cdot |a_n - a| + |a| \cdot |b_n - b|$$

Let $M = \max\{M_1, |a|\}$

$$\leq M|a_n - a| + M|b_n - b| = M\left[|a_n - a| + |b_n - b|\right]$$

Since $\lim(a_n) = a,\ \exists N_1 \in \mathbb{N}\ \forall n \geq N_1 : |a_n - a| < \epsilon/2M$

Similarly, because $\lim(b_n) = b,\ \exists N_2 \in \mathbb{N} : \forall n \geq N_2 : |b_n - b| < \frac{\epsilon}{2M}$.

Let $N = \max\{N_1, N_2\}$. Then

$$\forall n \geq N : |a_n - a| < \frac{\epsilon}{2M} \wedge |b_n - b| < \frac{\epsilon}{2M}$$

Therefore

$$|a_nb_n - ab| \leq M\left[|a_n - a| + |b_n - b|\right] < M(\frac{\epsilon}{2M} + \frac{\epsilon}{2M}) = M \cdot \frac{\epsilon}{M} = \epsilon \quad \forall n \geq N$$

Thus (a_nb_n) converges and $\lim(a_nb_n) = ab = \lim(a_n) \cdot \lim(b_n)$ $\square$

This can be applied to finitely many sequences.

Example 1.7

$\lim(\frac{1}{n^b}) = 0$ for all $k \in \mathbb{N}$

Proof. Because $(\frac{1}{n})$ converges to 0, $\lim(\frac{1}{n^k}) = \lim(\frac{1}{n}) \cdots \lim(\frac{1}{n}) = 0$ $\square$

Note 1.8. Special case where (b_n) is constant. i.e. $b_n = c$ for all $n \in \mathbb{N}$. Let (a_n) be convergent with $\lim(a_n) = a$. Then $\lim(c \cdot a_n) = \lim(c) \cdot \lim(a_n) = c \cdot \lim(a_n)$

Example 1.9

$$\lim(\frac{n-1}{n}) = \lim(1-\frac{1}{n}) = \lim(1+(-\frac{1}{n})) = \lim(1) + \lim(-\frac{1}{n})$$
$$= 1 + \lim(-1\cdot\frac{1}{n}) = 1 + -1\cdot\lim(\frac{1}{n}) = 1 + -1\cdot 0 = 1$$

Theorem 1.10
In general, if $(a_n), (b_n)$ converges, then $(a_n - b_n)$ converges and $\lim(a_n - b_n) = \lim(a_n) - \lim(b_n)$

Proof.

$$\lim(a_n - b_n) = \lim(a_n + (-b_n)) = \lim(a_n) + \lim(-b_n) = \lim(a_n) + -1\lim(b_n) = \lim(a_n) - \lim(b_n)$$

□

Theorem 1.11
Let (a_n) be convergent with $\lim(a_n) \neq 0$ and $a_n \neq 0 \quad \forall n \in \mathbb{N}$. Then $(\frac{1}{a_n})$ converges and $\lim(\frac{1}{a_n}) = \frac{1}{\lim(a_n)}$

Proof. Let $\lim(a_n) = a, \quad a \neq 0$. Let $\epsilon > 0$. Then

$$|\frac{1}{a_n} - \frac{1}{a}| = |\frac{a-a_n}{a_n \cdot a}| = \frac{|a_n - a|}{|a_n|\cdot|a|} < \frac{|a_n - a|}{k|a|} = \frac{1}{k|a|}\cdot|a_n - a| = 0$$

By conv. criterion, $(\frac{1}{a_n})$ converges to $\frac{1}{a}$ □

Lemma 1.12
Let (a_n) be convergent with $a_n \neq 0 \quad \forall n \in \mathbb{N}$ and $\lim(a_n) = a \neq 0$. Then there exists $M > 0$ such that $|\frac{1}{a_n}| \leq M \quad \forall n \in \mathbb{N}$.

Proof. Let $a = \lim(a_n)$ and $\epsilon = \frac{1}{2}|a|$. Then $\exists n \in \mathbb{N}$ such that $|a_n - a| < \epsilon = \frac{1}{2}|a|$ for all $n \geq N$, then $|a_n| = |a - (a - a_n)| \geq |a| - |a_n - a| > |a| - \frac{1}{2}|a| = \frac{1}{2}|a| > 0 \quad \forall n \geq N$

Let $k = \min\{|a_1|, |a_2|, \ldots, |a_{n-1}|, \frac{1}{2}|a|\} > 0$, then $|a_n| > k > 0 \quad \forall n \in \mathbb{N}$

$$\Rightarrow |\frac{1}{a_n}| < \frac{1}{k} = M \quad \forall n \in \mathbb{N}$$

□

Theorem 1.13
Let $(a_n), (b_n)$ by convergent where $\forall n \in \mathbb{N}$ $b_n \neq 0$ and $\lim(b_n) \neq 0$. Then $\frac{a_n}{b_n}$ converges and $\lim(\frac{a_n}{b_n}) = \frac{\lim(a_n)}{\lim(b_n)}$