§1 Lecture 01-29

§1.1 Multilinear functions or forms

Note 1.1. A form is just another way of saying function.

$$f : \underbrace{V \times \cdots \times V}_{k} \to F$$

Given a basis $e_1, \ldots, e_n$ of V, the k-multilinear form f is determined by

$$(f(e_{i1}, \ldots, e_{ik}))_{1 \le i_1, \ldots, i_k \le n}$$

Definition 1.2. f is <u>symmetric</u> if

$$f(v_{\sigma 1}, \ldots, v_{\sigma k}) = f(v_1, \ldots, v_k) \quad \forall \sigma \in S_k$$

Definition 1.3. f is <u>alternating</u> if

$$f(v_{\sigma 1}, \ldots, v_{\sigma k}) = \mathrm{sign}(\sigma) f(v_1, \ldots, v_k) \quad \forall \sigma \in S_k$$

Sign is defined as follows:

$$S_k \to \{1, -1\}$$
$$\sigma \mapsto (-1)^{\text{number of transposition needed to write } \sigma}$$

Remark 1.4. If f is symmetric, then f is determined by

$$(f(v_{i1}, \ldots, v_{ik}))_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n}$$

If f is alternating, then f is determined by

$$(f(v_{i1}, \ldots, v_{ik}))_{1 < i_1 < i_2 < \cdots < i_k < n}$$

Theorem 1.5

The set of alternating n-multilinear functions on a vector space of dimension n is a one-dimensional vector space.

Example 1.6

$n = 2$. $V = Fe_1 \oplus Fe_2$.

Two different approaches.

1.

$$\begin{aligned} f(ae_1 + be_2, ce_1 + de_2) &= acf(e_1, e_1) + adf(e_1, e_2) + bcf(e_2, e_1) + bdf(e_2, e_2) \\ \text{If alternating:} \quad &= (ad - bc)f(e_1, e_2) \end{aligned}$$

2.

$$\begin{aligned} f(ae_1 + be_2, ce_1 + de_2) &= f(ae_1 + be_2, (-\frac{c}{a}b + d)e_2) \\ &= (-\frac{bc}{a} + d)f(ae_1 + be_2, e_2) \\ &= (-\frac{bc}{a} + d)f(ae_1, e_2) \\ &= (-bc + ad)f(e_1, e_2) \end{aligned}$$

Definition 1.7. The unique n-multilinear alternating function f satisfying $f(e_1, \dots, e_n) = 1$ is called the determinant relative to $(e_1, \dots, e_n)$.

$$\det : V^n \to F$$

Note 1.8. $\det_B(v_1, \dots, v_n)$ is the value of the determinant relative to B, at $(v_1, \dots, v_n)$.

Properties: $\det_B(v_1, \dots, v_n) = 0 \Leftrightarrow (v_1, \dots, v_n)$ are linearly dependent.

Proof. $\Leftarrow$ If $(v_1, \dots, v_n)$ are linearly dependent, then WLOF, $v_1 = \lambda_2 v_2 + \cdots + \lambda_n v_n$.

$$\begin{aligned} \det(v_1, \dots, v_n) &= \det(\lambda_2 v_2 + \cdots + \lambda_n v_n, v_2 \dots, v_n) \\ &= \lambda_2 \det(v_2, v_2, v_3, \dots, v_n) + \lambda_3 \det(v_3, v_2, v_3, \dots, v_n) + \cdots + \lambda_n \det(v_n, v_2, v_3, \dots, v_n) \\ &= \lambda_2 0 + \cdots + \lambda_n 0 = 0 \end{aligned}$$

$\Rightarrow$ Left as an exercise

□

Proposition 1.9
For $(v_1, \dots, v_n)$ in a vector space of dim n, the following are equivalent:

1. $\det_B(v_1, \dots, v_n) \neq 0$
2. $(v_1, \dots, v_n)$ are linearly independent
3. $(v_1, \dots, v_n)$ span V
4. $(v_1, \dots, v_n)$ form a basis.

§1.2 Determinent of $T : V \to V$

Proposition 1.10
There is a unique scalar d_T such that $\det_B(T(v_1), \dots, T(v_n)) = d_T \det_B(v_1, \dots, v_n)$.

Proof. The function

$$(v_1, \dots, v_n) \mapsto \det_B(T(1), \dots, T(v_n))$$

is a function $V^n \to F$ which is also n-multilinear and alternating.

$$\det{}'(v_1, \dots, v_n) = \det(T(v_1), \dots, T(v_n))$$

$$\begin{aligned}
\det{}'(\lambda_1 v_1 + \lambda_1' v_1', v_2, \dots, v_n) &= \det(T(\lambda_1 v_1 + \lambda_1' v_1'), T(v_2), \dots, T(v_n)) \\
&= \det(\lambda_1 T(v_1) + \lambda_1' T(v_1'), T(v_2), \dots, T(v_n)) \\
&= \lambda_1 \det(T(v_1), T(v_2), \dots, T(v_n)) + \lambda_1' \det(T(v_1'), T(v_2), \dots, T(v_n)) \\
&= \lambda_1 \det{}'(v_1, \dots, v_n) + \lambda_1' \det{}'(v_1', v_2, \dots, v_n)
\end{aligned}$$

This proves that this function is still multi-linear. We know that it's a multiple because we showed that the set of alternating functions is one-dimensional.

Therefore $\det_B(T(-), \dots, T(-))$ is a scalar multiple of $\det_B$. □

Definition 1.11. The determinent of T is the unique scalar $\det(T)$ such that

$$\det_B(T(v_1), \dots, T(v_n)) = \det(T) \cdot \det_B(v_1, \dots, v_n)$$

Note that this defining property is independent of B.

§1.3 Next week

Let $T : V \to V$. Then T generates a subring of $\text{End}_F(V)$.

$$F[T] = \{a_0 I + a_1 T + a_2 T^2 + \cdots + a_k T^k\} \quad a_0, \dots, a_k \in F$$

$F[T]$ is a quotient of $F[x]$. $F[x] \to F[T]$, $p(x) \mapsto p(T)$.

$$I_T = \{p(x) \in F[x] \text{ such that } p(T) = 0_v\}$$

I_T is an ideal in $F[x]$.

$\exists! P_T(x)$ monic such that $I_T = (p_T(x))$. $P_T(x)$ is the min poly.

Characteristic Poly: $\det(xI - T)$.