§1 Lecture 11-27

§1.1 Application of Heine-Borel

Theorem 1.1
Let $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ be a nested sequence of compact sets. Then

$$\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$$

(This is by the nested interval property, but we are going to prove it using heine-borel)

Proof. $\forall n \in \mathbb{N}$, let $U_n := \mathbb{R} \setminus A_n \Rightarrow \forall n \in \mathbb{N} U_n$ is open and $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$

By de morgans law, we have that

$$\bigcup_{n\in\mathbb{N}} U_n = \bigcup_{n\in\mathbb{N}} \mathbb{R} \setminus A_n \underbrace{=}_{\text{De morgans}} \mathbb{R} \setminus \bigcap_{n\in\mathbb{N}} A_n$$

Now assume that $\cap_{n\in\mathbb{N}} A_n = \varnothing$. Then $\cup_{n\in\mathbb{N}} U_n = \mathbb{R} \setminus \varnothing = \mathbb{R}$.

i.e. The U_n cover all of $\mathbb{R}$ and thus especially A_1. By heine-borel, this open cover has a finite subcover.

$$\{U_{n_1}, \ldots, U_{n_k}\}, n_1 < \cdots < n_k$$

$$\Rightarrow A_1 \subseteq \bigcup_{i=1}^{k} U_{n_i} = U_{n_1} \cup \cdots \cup U_{n_k} = U_{n_k}$$

$$\Rightarrow A_1 \subseteq U_{n_k}$$

$$\Rightarrow A_{n_k} \subseteq A_1 \subseteq U_{n_k} = \mathbb{R} \setminus A_{n_k}$$

$$\Rightarrow A_{n_k} \subseteq \mathbb{R} \setminus A_{n_k} \quad ↯$$

$$\Rightarrow \bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$$

□

Definition 1.2 (Uniform Continuity)**.** Let's recall the definition of continuity of $f : A \to \mathbb{R}$:

$$(\forall x_0 \in A)(\forall \epsilon > 0)(\exists \delta = \delta(\epsilon, x_0)) : (\forall x \in A)(|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon)$$

Note 1.3. In general, δ will depend on both ϵ (unavoidable) and x_0.

It would be useful in many branches of analysis (e.g. Riemann integration) if δ would only depend on ϵ and <u>not</u> x_0.

i.e. we'd like to have this:

$$(\forall x_0 \in A)(\forall \epsilon > 0)(\exists \delta = \delta(\epsilon))(\forall x \in A) : (|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon)$$

$$\equiv$$

$$(\forall \epsilon > 0)(\exists \epsilon > 0)(\forall x_1, x_0 \in A) : (|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon)$$

Since x_0 is actually a variable, we'll use μ instead and obtain:

$f : A \subseteq \mathbb{R} \to \mathbb{R}$ is called uniformly continuous on A if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, \mu \in A) : (|x - \mu| < \delta \Rightarrow |f(x) - f(\mu)| < \epsilon)$$

Example 1.4

$f : \mathbb{R} \to \mathbb{R}$, $x \to x$. Claim: f is uniformaly continuous.

Proof. Let $\epsilon > 0$ and let $\delta := \epsilon$. Then $\forall x, \mu \in \mathbb{R}$, $|x - \mu| < \delta = \epsilon \Rightarrow |f(x) - f(\mu)| = |x - u| < \epsilon$ $\square$

Lemma 1.5

$\forall x, \mu > 0$ where $x \geq \mu$, we have that $\sqrt{x} - \sqrt{\mu} \leq \sqrt{x - \mu}$.

Proof.

$$\begin{aligned}
&\sqrt{x} - \sqrt{\mu} \leq \sqrt{x - \mu} \\
\Leftrightarrow\ &(\sqrt{x} - \sqrt{\mu})^2 \leq (\sqrt{x - \mu})^2 = x - \mu \\
\Leftrightarrow\ &x - 2\sqrt{x}\sqrt{\mu} + \mu \leq x - \mu \\
\Leftrightarrow\ &2\mu - 2\sqrt{x}\sqrt{\mu} \leq 0 \\
\Leftrightarrow\ &\underbrace{2\sqrt{\mu}}_{\geq 0}\underbrace{(\sqrt{\mu} - \sqrt{x})}_{\leq 0} \leq 0 \checkmark
\end{aligned}$$

Because we only used equivalence statements, this final true statement proves that

$$\sqrt{x} - \mu \leq \sqrt{x - \mu}$$

$\square$

Example 1.6

$f : \mathbb{R}_0^+ = [0, \infty[\to \mathbb{R}$, $x \to \sqrt{x}$. Claim: f is uniformally continuous.

Remark 1.7. We did prove in chapter 4 that $\sqrt{x}$ is continuous on $[0, \infty[$. Back then, the δ value we obtained <u>did</u> depend on both ϵ and x !

However, this does <u>not</u> necessarily mean that $\sqrt{}$ is not uniformally continuous! It might just mean that we need better estimates!

Proof. Let $\epsilon > 0$, let $\delta > 0$ be arbitrary for now. Let $x, \mu \in [0, \infty[$. We may assume without loss of generality that $x \geq \mu$. Let $|x - \mu| = x - \mu < \delta$. Then:

$$|f(x) - f(\mu)| = |\sqrt{x} - \sqrt{\mu}| = \sqrt{x} - \sqrt{\mu} \leq \sqrt{x - \mu} < \sqrt{\delta} < \epsilon$$
$$\Leftrightarrow \delta < \epsilon^2$$

Note that δ is independent of x and μ !

With this <u>uniform</u> δ, we have

$$|x - \mu| < \delta \Rightarrow |f(x) - f(\mu)| < \epsilon \Rightarrow \sqrt{x}$$

is uniform continuous on $[0, \infty[$. □

How can we see whether a funciton is <u>not</u> uniformally continuous?

$f : A \to \mathbb{R}$ <u>not</u> continuous:

$$\neg(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, \mu \in A) : (|x - \mu| < \delta \Rightarrow |f(x) - f(\mu)| < \epsilon)$$
$$\equiv \neg(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, \mu \in A) : (|x - \mu| \geq \delta \vee |f(x) - f(\mu)| < \epsilon)$$
$$\equiv (\exists \epsilon > 0)(\forall \delta > 0)(\exists x, \mu \in A) : (|x - \mu| < \delta \wedge |f(x) - f(\mu)| \geq \epsilon)$$

Recall 1.8. $P \Rightarrow Q \equiv \neg P \vee Q$

Theorem 1.9 (2 sequence criterion for non-uniform continuity)

Let $f : A \to \mathbb{R}$. Let $\epsilon_0 > 0$ and let $(x_n), (\mu_n)$ be sequences in A such that $\lim(x_n - \mu_n) = 0$ and $|f(x_n) - f(\mu_n)| \geq \epsilon_0$ for all $n \in \mathbb{N}$. Then f is not uniformally continuous on A.

Proof. Assume that f is uniform continuous. Then $\exists \delta > 0$ such that $\forall x, \mu \in A$: $|x - \mu| < \delta \Rightarrow |f(x) - f(\mu)| < \epsilon_0$. $(*)$

Now $\lim(x_n - \mu_n) = 0$. Then $(\exists N \in \mathbb{N})(\forall n \geq N) : |x_n - \mu_n| < \delta$. Especially, $|x_n - \mu_n| < \delta$.

In $(*) :\Rightarrow |f(x_N) - f(\mu_N)| < \epsilon_0$ ↯

Thus f is <u>not</u> uniformally continuous on A. □

Example 1.10
$f : \mathbb{R} \to \mathbb{R}$, $x \to x^2$.

Let $x_n := n$, $u_n := n + 1/n$

Then $|x_n - \mu_n| = 1/n \Rightarrow \lim(x_n - \mu_n) = 0$

But $|f(x_n) - f(\mu_n)| = |n^2 - (n + 1/n)^2| = |n^2 - n^2 - 2 - 1/n^2| = 2 + 1/n^2 > 2$.

Let $\epsilon_0 := 2$. Then $\lim(x_n - \mu_n) = 0$, but $\forall n \in \mathbb{N} : |f(x_n) - f(\mu_n)| \geq \epsilon_0$.

$\Rightarrow x^2$ is <u>not</u> uniformally continuous on $\mathbb{R}$.

Example 1.11
$f :]0, \infty[\to \mathbb{R}$, $x \to 1/x$

Let $x_n := 1/n$, $\mu_n := 1/(n + 1)$.

Then, $|x_n - \mu_n| = |1/n - 1/(n + 1)| = |(x + 1 - x)/(n(n + 1))| = 1/(n(n + 1)) \leq 1/n^2 \to 0$.

By convergence criterion, $\lim(x_n - \mu_n) = 0$.

But, $|f(x_n) - f(\mu_n)| = |n - (n + 1)| = 1$. Let $\epsilon_0 := 1$.

Then $\lim(x_n - \mu_n) = 0$. But $|f(x_n) - f(\mu_n)| \geq \epsilon_0$.

Therefore $1/x$ is <u>not</u> uniformally continuous on $]0, \infty[$.

Theorem 1.12
Every continuous function on a compact domain is uniformally continuous.

Proof. Let $f : A \to \mathbb{R}$, A be compact, and f continuous on A.

Let $\epsilon > 0$, then $(\forall x \in A)(\exists \delta_x > 0) : (|x - \mu| < \delta_x \Rightarrow |(f(x) - f(\mu)| < \epsilon/2)$

Now consider the neighborhoods $V_{(1/2)\delta_x}(x)$ for all $x \in A$.

Then $\varphi := \{V_{(1/2)\delta_x}(x) : x \in A\}$ is an open cover of A. (Even just the centres of these neighborhoods already cover A)

By Heine-Borel, φ has a finite subcover $\{V_{(1/2)\delta_{x_1}}, \ldots, V_{(1/2)\delta_{x_n}}\}$ where $x_1, \ldots, x_n \in A$.

Let $\delta := \min\{\frac{1}{2}\delta_{x_1}, \ldots, \frac{1}{2}\delta_{x_n}\} > 0$.

We'll prove that with this δ, we have that $|x - \mu| < \delta \Rightarrow |f(x) - f(\mu)| < \epsilon$.

Let $x, \mu \in A$ such that $|x - \mu| < \delta$. Since $x \in A$, $\exists 1 \leq k \leq n$ such that $x \in V_{(1/2)\delta_{x_k}}(x_k)$

$\Rightarrow |x - x_k| < \frac{1}{2}\delta_{x_k} < \delta_{x_k}$

and

$$\begin{aligned}
|\mu - x_k| = |(\mu - x) + (x - x_k)| \leq |x - \mu| + |x - x_k| < \delta + \frac{1}{2}\delta_{x_k} = \delta_{x_k} \\
\Rightarrow x, \mu \in V_{\delta_{x_k}}(x_k) \\
\Rightarrow |f(x) - f(\mu)| = |(f(x) - f(x_k)) + f(x_k) - f(\mu))| \\
\leq \underbrace{|f(x) - f(x_k)|}_{\leq \epsilon/2} + \underbrace{|f(\mu) - f(x_k)|}_{\leq \epsilon/2} < \epsilon
\end{aligned}$$

Because $|x - x_k| < \delta_{x_k}$ and $|\mu - x_k| < \delta_{x_k}$.

i.e. if $|x - \mu| < \delta \Rightarrow |f(x) - f(\mu)| < \epsilon \Rightarrow f$ is uniform continuous on A ◻

Example 1.13
x^2 is uniform continuous on all intervals $[-a, a]$ where $a > 0$.

Example 1.14
$1/x$ is uniform continuous on all intervals $[a, 1]$ where $0 < a < 1$.