§1 Lecture 2019-09-09

Theorem 1.1

Suppose $f : A \to B$ and $g : B \to C$ are surjective then $g \circ f : A \to C$ is surjective

Proof: $c \in C$

since $g : B \to C$ there exists $b \in B$ s.t. $g(b) = c$ since $f : A \to B$ is surjective, exists $a \in A$ s.t. $f(a) = b$

thus $(g \circ f)(a) = g(f(a)) = g(b) = c$

Definition 1.2. A function $g : b \to A$ is inverse to function $f : A \to B$ if:

$$f \circ g = 1_B$$
$$g \circ f = 1_A$$

$$A \to^f B \to^g C$$
$$A \to^f B \to^g A$$
$$g \circ f = 1_A$$

$$B \to^g A \to^f B$$
$$f \circ g = 1_B$$

Note 1.3. They say f and g are inverible, use notation f^-1 for inverse of f

Theorem 1.4

Let $f : A \to B$ be a map: f is invertible if and only if f is a bijection

$$P \Leftrightarrow Q$$
$$P \Leftarrow Q$$
$$P \Rightarrow Q$$

Proof that f is invertible means f is a bijection:

$$\text{let } g = f^{-1} \text{ f is surjective since for all } b \in B$$
$$\text{we have } \quad f(g(b)) = f \circ g(b) = 1_B(b) = b$$

$$\text{f is injective since if } f(a_1) = f(a_2) \Rightarrow g(f(a_1)) = g(f(a_2))$$

injective: if $f(a_1) = f(a_2)$, then $a_1 = a_2$

Proof that f is a bijection means f is invertible define $f^{-1} : B \to A$ thus:

for each $b \in B$, there exists $a \in A$ s.t. $f(a) = b$ and a is unique with this property (by injectivity)

define $f^{-1}(b) = a$ then $f \circ f^{-1}(b) = f(f^{-1}(b)) = f(a) = b$ $f^{-1} \circ f(a) = f^{-1}(f(a)) = f^{-1}(b)$

so $f \circ f^{-1} = 1_B$

Definition 1.5 (Equivalence Relation)**.** Equivalence Relation on a set X is a relation $R \subset X \times X$

$$\begin{aligned} R \text{ is reflexive } (x, x) \in R \text{ for all } x \in X \\ \text{is symmetric } (x, y) \in R \to (y, x) \in R \\ \text{is transitive } (x, y) \in R \text{ and } (y, z) \in R \to (x, z) \in R \end{aligned}$$

Note 1.6. Usually denote equiv relations by $x \sim y$ instead of $(x, y) \in R$

$$\begin{aligned} \text{or } x = y \\ x \equiv y \end{aligned}$$

Definition 1.7. A partition of X is a collection of disjoint nonempty subsets of X whose union is X

Example 1.8

$\{X_k : k \in K\}$ $x_i \cap x_j = \varnothing$ for $i \neq j$
$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 0\} = \{1, 4, 5\} \cup \{6\} \cup \{9\} \cap \{2, 3, 7, 9, 0\}$
$X = X_1 \cup X_2 \cup X_3 \cup X_4$

§1.1 Creating a partition

let x be a set with equivalence relation $\sim$ for $y \in X$, let $[y] = \{x \in X : x \sim y\}$
$[y]$ is the equivalence class represented by y

Theorem 1.9

Theorem 1.25: The equivalence classes of an equivalence relation ($\sim$) form a partition of X.

Proof:

$$\begin{aligned} 1. \text{ each equiv class is nonempty since } \ y \in [y] \\ 2. \text{ equiv classes are either disjoint or equal since if } \ y \in [a] \text{ and } y \in [b] \\ \text{then } \ [a] \subset [b] \text{ since } \ c \in [a] \Rightarrow c \sim a \Rightarrow^{transitivity} c \sim y \Rightarrow^{transitivity} c \sim b \Rightarrow c \in [b] \\ \text{similarly } [b] \subset [a] \\ 3. X = \cup_{x \in X} [x] \end{aligned}$$

Conversely, given a partition of X you can define an equivalence relation by declaring $x \sim y \Rightarrow x, y$ lie in the same part of the partition

Note 1.10. An equivalence relation is a disguised version of a partition

Definition 1.11. Definition: congruence modulo n equivalence relation on Z

$a \equiv_n b$ if n divides $(b - a)$ i.e. $b - a = mn$ for some $m \in Z$ do NOT use $(a \equiv b(mod n)$
EX. $\equiv_2$ partition

$$\begin{aligned} \{, -4, -2, 0, 2, \cdot\} \\ \{, -3, -1, 1, 3, \cdot\} \end{aligned}$$

Proof: $\equiv_n$ is equiv relation

$$1.a \equiv_n a \text{ since } n|(a-a)$$
$$2.(a \equiv_n b) \Rightarrow (b \equiv_n a) \text{ since } n|(b-a) \text{ then } n|(a-b)$$
$$3.a \equiv_n b \text{ and } b \equiv_n c \text{ then } a \equiv_n c$$

$n|(b-a)$ and $n|(c-b)$ so $n|(b-a)+(c-b)$