# §1 Simple Groups

**Definition 1.1.** A group $G$ with no normal subgroups except $G$ and $\{1_G\} = \{e\}$ is called simple.

**Example 1.2**

1. $\mathbb{Z}_p$ with $p$ prime. The only subgroups are $G$ and $\{1_G\}$.
2. $A_n \quad \forall n \geq 5$.

In some sense, simple groups are like the primes. Every group can be built from simple groups.

# §2 Homomorphisms

**Definition 2.1.** A homomorphism from group $(G, \cdot)$ to $(H, \circ)$ is a map $\phi : G \to H$ such that is preserves multiplication. i.e. $\phi(g_1 \cdot g_2) = \phi(g_1) \circ \phi(g_2)$ for all $g_1, g_2 \in G$.

The range $\phi(G) \subset H$ is called the homomorphic image of $G$.

**Remark 2.2.** $\phi(G)$ is a subgroup of $H$.

**Note 2.3.** All isomorphisms are homomorphisms with the additional property that $\phi$ is a bijection.

**Example 2.4**

Let $g \in G$. There is a homomorphism $\phi : \mathbb{Z} \to G$ defined by $\phi(n) = g^n$.

Check: (review how binary operations apply below)

$$\phi(a+b) = g^{a+b} = g^a g^b = \phi(a)\phi(b)$$
$$\phi(\mathbb{Z}) = \langle g \rangle \subset G$$

**Example 2.5**

$$\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^*$$
$$\det(AB) = \det(A) \cdot \det(B)$$

**Example 2.6**

Let $G =$ the isometries of a tetrahedron.

$\phi : G \to \{\pm 1\}$. $\phi(g) = \pm 1$ if $g$ preserves orientation. $\phi(g) = -1$ if $g$ reverses orientation.

**Theorem 2.7**
Let $\phi : G_1 \to G_2$ be a homomorphism.

1. If $e_1$ is the identity element of $G_1$, the $\phi(e_1)$ is the identity element of $G_2$.
2. $\phi(g^{-1}) = [\phi(g)]^{-1}$
3. $H_1 \subset G_1$ is a subgroup $\Rightarrow \phi(H_1) \subset G_2$ is a subgroup
4. $H_2 \subset G_2$ is a subgroup $\Rightarrow \phi^{-1}(H_2) \subset G_1$ is a subgroup
5. $H_2 \subset G_2$ is a normal subgroup $\Rightarrow \phi^{-1}(H_2) \subset G_1$ is a normal subgroup

**Note 2.8.** Normal groups can be used to build factor and quotient groups.

*Proof.* Of the above statements.

1. $\phi(e_1) = \phi(e_1 e_1) = \phi(e_1)\phi(e_1)$. Therefore $e_2 = \phi(e_1)$.
2. $e_2 = \phi(e_1) = \phi(g \cdot g^{-1}) = \phi(g) \cdot \phi(g^{-1})$. Therefore $\phi(g)$ and $\phi(g^{-1})$ are inverse to one another.
3. Identity: $e_1 \in H_1 \Rightarrow e_2 = \phi(e_1) \in \phi(H_1)$, so image of $\phi$ contains identity element.

   Inverses: $g_2 \in \phi(H_1) \Rightarrow g_2 = \phi(g_1)$ for some $g_1 \in H_1 \Rightarrow g_1^{-1} \in H_1 \Rightarrow \phi(g^{-1}) = [\phi(g_1)]^{-1} = g_2^{-1} \in \phi(H_1)$. Therefore image contains inverses.

   Closure: Let $g_2, g_2' \in \phi(H_1)$. Therefore $\exists g_1, g_1' \in H_1$ such that $g_2 = \phi(g_1)$ and $g_2' = \phi(g_1')$. Therefore:

   $$g_1 g_1' \in H_1 \Rightarrow \phi(g_1 g_1') \in \phi(H_1) \Rightarrow g_2 g_2' = \phi(g_1)\phi(g_1') \in \phi(H_1)$$

4. Identity: $e_1 \in \phi^{-1}(H_2)$ because $\phi(e_1) = e_2 \in H_2$.

   Inverses: $g_1 \in \phi^{-1}(H_2) \Rightarrow g_1^{-1} \in \phi^{-1}(H_2)$ because $\phi(g_1^{-1}) = [\phi(g_1)]^{-1} \in H_2$.

   Closure: $g_1, g_1' \in \phi^{-1}(H_2) \Rightarrow g_1 g_1^{-1} \in \phi^{-1}(H_2)$ because $\phi(g_1 g_1') = \phi(g_1)\phi(g_1') \in H_2$

5. Show that for all $g_1 \in G_1$, $g_1 \phi^{-1}(H_2) g_1^{-1} \subset \phi^{-1}(H_2)$

   Let $k \in \phi^{-1}(H_2)$. Then $\phi(g_1 k g_1^{-1}) = \phi(g_1)\phi(k)\phi(g_1^{-1}) = \phi(g_1)\phi(k)[\phi(g_1)]^{-1} \in H_2$. Since we construct with $H_2 \subset G_2$ is normal. Remember that $\phi(k) \in H_2$.

□