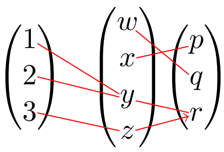
Definition - The Cartesian Product: $A \times B$ of A and $B = \{(a,b) : a \in A, b \in B\}$

i.e. $R \times R = R^2$ $[0,1] \times [0,2]$

Definition - Functions in Calculus: Let D and E be sets; a function $f : D \to E$ is a rule that takes an input from D and assigns to it an output in E.



In modern math we define a function $f : D \to E$ as a subset f of $D \times E$ s.t. $\forall x \in D$ there exists EXACTLY ONE $y \in E$ s.t. $(x, y) \in f$. Functions are thus just sets, there is thus just one fundamental concept (sets) we need to consider.

ex: $f : \{-1, 0, 1\} \to \{-1, 0, 1\}$
x "maps to" x^2
image vs. codomain
$\{(-1, 1), (0, 0), (1, 1)\} = f$

Definition - a function $f : D \to E$ is called injective or one-to-one if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ Everything gets mapped to its own unique point. Equivalently: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Definition - $f : D \to E$ is called surjective or "onto" if $\forall y \in E$ "there exists" $x \in D : f(x) = y$

Definition - $f : D \to E$ is called bijective if f is both injective and surjective

let $f : D \to E$, $A \subset D$ then $f(A) = \{f(x) : x \in A\} \subset E$ is called the image of A under f

Definition - let f: $D \to E, B \subset E$, then $f^{-1}(B) = \{x \in D : f(x) \in B\} \subset D$ is called the inverse image of B under f

CAUTION: The inverse image $f^{-1}(B)$ makes sense whether or not f is invertible!
ex: $f : \{-1, 0, 1\} \to \{-1, 0, 1\}$
x "maps to" x^2
image vs. codomain
$\{(-1, 1), (0, 0), (1, 1)\} = f$
note that f is NOT injective (because $(f(1) = f(-1)$ and is thus not invertible. none the less, inv. images make sense.
$f^1 = \{-1, 1\}$ $f^0 = \{0\}$ $f^{-1} = \{\} = \varnothing$
ex: let $f : D \to E$ be bijective.
then $f^{-1}(\{y_0\}) = \{x_0\}$ where $f(x_0) = y_0$
inv. function: $f^{-1}(y_0) = x_0$
inv. image: $f^{-1}(\{y_0\}) = \{x_0\}$

Theorem (i): let $f : D \to E$, $A, B \subset D$ then (a) $f(A \cup B) = f(A) \cup f(B)$ (b) $f(A \cap B) \subset f(A) \cap f(B)$ (ii) $let f : D \to E, \quad A, B, \subset E$ then (a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ (b) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

(ii)(a) will be shown in the tutorials (b) assign 1
we will prove (i):
(a) we have to show that the 2 sets $f(A \cup B)$ and $f(A) \cup f(B)$ are equal
Proof: let $y \in f(A \cup B) \Rightarrow$ "there exists" $x \in A \cup B : y = f(x) \Rightarrow$ "there exists" $x \in A : y = f(x) v$ "there exists" $x \in B : y = f(x) \Rightarrow y \in f(A) v y \in f(B) \Rightarrow y \in f(A) \cup f(B) \Rightarrow f(A \cup B) \subset f(A) \cup f(B)$

proof part 2:

let $y \in f(A) \cup f(B)$

$$y \in f(A) v y \in f(B)$$

$$\text{"there exists" } x \in A : y = f(x) v \text{ "There exists" } x \in B : y = f(x)$$

$$\text{"there exists" } x \in A \cup B : y = f(x)$$

$$\Rightarrow y \in f(A \cup B)$$

$$f(A) \cup f(B) \subset f(A \cup B)$$

$$\Rightarrow f(A \cup B) = f(A) \cup f(B)$$