§1 Monotone Sequences

Recall 1.1. Monotone means increasing or decreasing in the non strict sense.

Theorem 1.2

Let (x_n) be a monotone sequence. Then (x_n) is convergent if and only if it is bounded. This is useful because it is easier to check whether or not a sequence is bounded than to check whether or not it is convergent.

Proof. Assume that (x_n) is increasing. We will show that (x_n) converges ot the supremum.

What is the supremeum of a sequence. We take all the numbers and consider it a set in $\mathbb{R}$ and then find the supremeum. $x \coloneqq \sup \underbrace{\{x_1, x_2, x_3, \dots\}}_{\coloneqq S}$.

Let $\epsilon > 0$, then $x - \epsilon$ is not an upper bound of S. Thus $\exists N \in \mathbb{N}$ such that $x - \epsilon < x_N \leq X$ but (x_n) is increasing. We also have $x - \epsilon < x_N \leq x_{N+1} \leq x_{N+2} \leq \cdots \leq x$. i.e. $\forall n \geq N : x - \epsilon < x_n \leq x$

$\Rightarrow x_n \in]x - \epsilon, x]$ for all $n \geq N \quad \subseteq]x - \epsilon, x + \epsilon[= V_\epsilon(x)$. i.e. $\forall n \geq N : x_n \in V_\epsilon(x)$. Thus (x_n) converges to $x \coloneqq \sup\{x_1, x_2, \dots\}$. The case that (x_n) is decreasing is left as an exercise. □

Example 1.3

$x_1 = 1, x_{n+1} = \frac{1}{2}x_n + 2$

Show that x_n convreges and determine its limit. We will show that (x_n) is increasing and bounded; by monotone convergence theorem, (x_n) converges. Lastly, we will show that $\lim(x_n) = 4$.

Proof. (x_n) is bounded from above by 4. We'll show this using induction.

$\underline{n = 1}$: $\quad 1 \leq 4$ ✓

$\underline{n \to n+1}$: Assume that $x_n \leq 4$. Then $x_{n+1} = \frac{1}{2}x_n + 2 \leq \frac{1}{2} \cdot 4 + 2 = 4$ ✓

Therefore (x_n) is bounded from above by 4. □

Proof. Proving that (x_n) is increasing. Consider $x_{n+1} - x_n = \frac{1}{2}x_n + 2 - x_n = 2 - \frac{1}{2}x_n \geq 0$.

$$\begin{aligned} &\Rightarrow \forall n \in \mathbb{N} \quad x_{n+1} - x_n \geq 0 \\ &\Rightarrow \forall n \in \mathbb{N} \quad x_{n+1} \geq x_n \end{aligned}$$

i.e. (x_n) is increasing. □

By showing that (x_n) is bounded from above and increasing, we know that (x_n) is convregent by the monotone convergence theorem. Now to find where it converges.

Let $x := \lim(x_n)$.

$$\begin{aligned} \forall n \in \mathbb{N} \quad x_{n+1} &= \frac{1}{2}x_n + 2 \\ \Rightarrow \lim(x_{n+1}) = \lim(\frac{1}{2}x_n + 2) &= \frac{1}{2}\lim(x_n) + 2 = \frac{1}{2}x + 2 \\ \Rightarrow x &= \frac{1}{2}x + 2 \\ \Rightarrow \frac{1}{2}x &= 2 \Rightarrow x = 4 \end{aligned}$$

Note 1.4. It is essential for this argument that we knew in advance that (x_n) is convergent.

We've now shown that $\lim(x_n) = 4$.

Example 1.5

Exercise for the reader: $x_1 = 1$. $x_{n+1} = \sqrt{2 + x_n}$.

Prove that (x_n) converges to 2.

§1.1 Euler's constant

Consider the squence $x_n = (1 + \frac{1}{n})^n$ and $y_n = (1 + \frac{1}{n})^{n+1}$.

We will show that (x_n) increases and that (y_n) decreases.

Proof. (x_n) is increasing. We have to show that $\forall n \in \mathbb{N} : x_n \leq x_{n+1}$. i.e. that

$$\begin{aligned} &(1 + \frac{1}{n})^n \leq (1 + \frac{1}{n+1})^n + 1 \\ \Leftrightarrow\ &(1 + \frac{1}{n+1})^{n+1} \geq (1 + \frac{1}{n})^n \\ \Leftrightarrow\ &1 + \frac{1}{n+1} \geq \sqrt[n+1]{(1 + \frac{1}{n})^n} \end{aligned}$$

Recall the inequality of the algebraic and geometric mean. If $a_1, a_2, \ldots, a_n \geq 0$, then

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \times \cdots \times a_n}$$

Let $a_1 = \cdots = a_n = 1 + \frac{1}{n}$ and $a_{n+1} = 1$. Then

$$\sqrt[n+1]{a_1 \times \cdots \times a_n \times a_{n+1}} = \sqrt[n+1]{(1 + \frac{1}{n})^n}$$

$$\text{and} \quad \frac{a_1 + \cdots + a_n + a_{n+1}}{n+1} = \frac{n(1 + \frac{1}{n}) + 1}{n+1} = \frac{n+1+1}{n+1} = \frac{n+2}{n+1} = 1 + \frac{1}{n+1}$$

Thus, by AGM-inequality, $1 + \frac{1}{n+1} \geq \sqrt[n+1]{(1 + \frac{1}{n})^n}$. □

Proof. Now to show that y_n is decreasing. Similar strategy, but take inverse to reverse inequality. □

It follows from the above proofs that, Claim:

$$\forall n, k \in \mathbb{N} : x_n < y_n$$

Definition 1.6.

$$e := \lim \left((1 + \frac{1}{n})^n \right) = \lim \left((1 + \frac{1}{n})^{n+1} \right)$$

In analysis 2, you'll see that

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

From which it can be shown that e is irrational.

Estimates for e. Since (x_n) is increasing and (y_n) is decreasing, we have that $\forall n \in \mathbb{N} : x_n \leq e \leq y_n$.

$$\frac{5}{2} < e < 3 \Leftarrow \begin{cases} x_6 \geq \frac{5}{2} = 2.5 \\ y_5 < 3 \end{cases}$$

§1.2 Subsequences

Definition 1.7. Let $n_1 < n_2 < n_3 < \dots$ be natural numbers and let $(x_n) = (x_1, x_2, x_3, \dots)$ be a sequence. Then $(x_{n_k}) = (x_{n_1}, x_{n_2}, x_{n_3}, \dots)$ is called a subsequence of (x_n).

Example 1.8

Let $(x_1, x_2, x_3, \dots)$ be a sequence. Then $(x_1, x_3, x_5, x_7, \dots)$ is called the subsequence of odd indices; here $n_k = 2k - 1$.

Likewise, $(x_2, x_4, x_6, x_8, \dots)$ is called the subsequence of even indices; here $n_k = 2k$.

Theorem 1.9

Let (x_n) be convergent. Then every subsequence (x_{n_k}) of (x_n) also converges to the same limit.

Proof. Next class. $\square$

Example 1.10

Let $0 < a < 1$; consider (a^n). We will show that $\lim(a^n) = 0$. Note that (a^n) is decreasing and is bounded from below. By monotone convergence theorem, (a^n) converges.

Let $x := \lim(a^n)$. Now consider the subsequence of even terms (a^{2n}). By the theorem above, this subsequence converges and has the same limit. i.e. $\lim(a^{2n}) = x$.

On the other hand, we can rewrite this as

$$\begin{aligned} \lim((a^n)^2) = [\lim(a^n)]^2 = x^2 = x \\ \Rightarrow x^2 - x = 0 \\ \Rightarrow x(x-1) = 0 \end{aligned}$$

This means that either $x = 0$ or $x = 1$. But $a^3 < a^2 < a^1 = a < 1 \Rightarrow x < 1 \Rightarrow x = 0$.