§1 Lecture 02-07

$spec(T)$ = set of eigenvalues of $T = \{\lambda \in F : \exists v \neq 0 : T(v) = \lambda v\}$.

$$\bigoplus_{\lambda \in (T)} V_\lambda \subseteq V$$
$$V_\lambda = \{v | T(v) = \lambda v\}$$
$$\Rightarrow \# spec(T) \leq \dim V$$

Two polynomials attached to T.

1. $p_T(x)$ is the minimal polynomial of T. $\deg p_T(x) \leq \dim(V)$
2. $f_T(x)$ = characteristic polynomial = $\det(xI - T)$

Theorem 1.1
If $\lambda \in F$, then

$$p_T(\lambda) = 0 \Leftrightarrow \lambda \in spec(T)$$

Proof.

($\Leftarrow$) $\exists v \neq 0$ such that $T(v) = \lambda v$. Then $T^2(v) = \lambda^2 v$. Then $T^j(v) = \lambda^j v$.

Let $g \in F[x]$. Then

$$g(T)(v) = g(\lambda)(v)$$
$$\lambda \in spec(T) \;\Rightarrow\; g(\lambda) \in spec(g(T))$$

$$p_T(T)(v) = p_T(\lambda)v$$
$$0(v) = p_T(\lambda)v$$
$$0 = p_T(\lambda)v$$
$$\Rightarrow p_T(\lambda) = 0$$

($\Rightarrow$)

$$p_T(\lambda) = 0$$
$$\Rightarrow p_T(x) = (x - \lambda)g(x)$$
$$\deg g(x) < \deg p_T(x) \Rightarrow g(T) \neq 0$$
$$0 = p_T(T) = (T - \lambda I) \circ g(T)$$
$$\Rightarrow \mathrm{Im}(g(T)) \subseteq \ker(T - \lambda I) = V_\lambda$$
$$V_\lambda \neq \{0\} \Rightarrow \lambda \in spec(T)$$

□

Theorem 1.2
If $\lambda \in F$, then

$$f_T(\lambda) = 0 \iff \lambda \in spec(T)$$

Proof.

$$\begin{aligned} f_T(\lambda) = 0 &\iff \det(\lambda I - T) = 0 \\ &\iff T - \lambda \text{ is non-invertible} \\ &\iff \ker(T - \lambda) \supsetneq \{0\} \\ &\iff V_\lambda \neq \{0\} \\ &\iff \lambda \in spec(T) \end{aligned}$$

□

§1.1 Voting with vectors

A, B, C candidates.

$$\begin{aligned} &A > B > C && (1, 1, -1) \\ &A > C > B && (1, -1, 1) \\ &B > A > C && (-1, 1, -1) \\ &B > C > A && (-1, 1, 1) \\ &C > A > B && (1, -1, 1) \\ &C > B > A && (-1, -1, 1) \end{aligned}$$

Where the vectors encode the following: $(A > B,\ B > C,\ C > A)$

If N_1 votes vote for $(-1, 1, 1)$, N_2 vote for $(1, -1, 1)$, and N_3 vote for $(1, 1, -1)$, then

$$N_1(-1, 1, 1) + N_2(1, -1, 1) + N_3(1, 1, -1) = (X, Y, Z)$$

where X represents the margin of voters who prefer A to B, Y represents the margin of voters who prefer B to C, and Z represents the margin of voters who prefer C to A.

Consider the following scenario. The population is $3N$. N people vote $(-1, 1, 1)$, N people vote $(1, -1, 1)$, and N people vote $(1, 1, -1)$. Then

$$\text{Vote } = (N, N, N)$$

So 66% prefer A to B, 66% prefer B to C, and 66% prefer C to A. So even though everyone voted rationally, a weird scenario arose.