Math #254 Notes

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Contents

§1 Limit laws

Example 1.1

$$a_n = \frac{n}{4^n}$$

Show that $\lim(a_n) = 0$ Try using bernoulli but here it doesn't help much.

$$4^n = (1+3)^n \geq 1 + 3n$$

$\Rightarrow |a_n - 0| = \frac{n}{4^n} \leq \frac{n}{1+3n} \to \frac{1}{3} \neq 0$

Unfortunately $\frac{n}{1+3n}$ does not converge to 0 so this estimate is too weak to be useful. Note: This argument can be save (see next assignment).

Different approach: We'll show that $4^n \geq n^2$ for all $n \in \mathbb{N}$

Proof by Induction. .
n = 1: $4^1 = 4 \geq 1 = 1^2$
$n \to n+1$: Assume that $4^n \geq n^2$, then

$$\begin{aligned} 4^{n+1} = 4 \cdot 4^n \geq 4 \cdot n^2 = 2n^2 + n^2 + n^2 &= 2n^2 + (n+1)^2 + (n-1)^n - 2 \\ &= (2n^2 - 2) + (n-1)^2 + (n+1)^2 \geq (n+1)^2 \\ &\Rightarrow 4^n \geq n^2 \ \forall n \in \mathbb{N} \end{aligned}$$

Thus $|a_n - 0| = \frac{n}{4^n} \leq \frac{n}{n^2} \leq \frac{1}{n} \to 0$
Therefore $\lim(a_n) = 0$ ☐

Theorem 1.2
Every convergent sequence is bounded.

Proof. Let (a_n) be a sequence with $\lim(a_n) = L$, and let $\epsilon = 1$.
Then $\exists N \in \mathbb{N}\ \forall n \geq N : |a_n - L| < \epsilon = 1$

$$\Rightarrow |a_n| = |(a_n - L) + L| \leq |a_n - L| + |L| < 1 + |L| \quad \forall n \geq N$$

This proves that when $n \geq N$, a_n is bounded.

Now let $M = \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, 1 + |L|\}$
Then $|a_n| \leq M$ for <u>all</u> $n \in \mathbb{N}$. ☐

Remark 1.3. The convergence condition is essential. The sequence $(n) = (1, 2, 3, \dots)$ is unbounded.

Theorem 1.4

Let $(a_n), (b_n)$ be convergent sequences. Then $(a_n + b_n)$ is convergent with $\lim(a_n + b_n) = \lim(a_n) + \lim(b_n)$

Proof. Let $a = \lim(a_n), b = \lim(b_n)$. Let $\epsilon > 0$.

$$|a_n + b_n - (a+b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$$

Since $\lim(a_n) = a, \ \exists N_1 \in \mathbb{N} \ \forall n \geq N_1 : |a_n - a| < \epsilon/2$

Similarly, because $\lim(b_n) = b, \ \exists N_2 \in \mathbb{N} : \forall n \geq N_2 : |b_n - b| < \frac{\epsilon}{2}$.

Let $N = \max\{N_1, N_2\}$. Then

$$\forall n \geq N : |a_n - a| < \frac{\epsilon}{2} \wedge |b_n - b| < \frac{\epsilon}{2}$$

Therefore

$$|a_n + b_n - (a+b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall n \geq N$$

Thus $(a_n + b_n)$ converges and $\lim(a_n + b_n) = a + b = \lim(a_n) + \lim(b_n)$ $\square$

This is supposed to be relatively simple.

Example 1.5

$$\lim(\frac{n+1}{n}) = \lim(1 + \frac{1}{n}) = \lim(1) + \lim(\frac{1}{n}) = 1 + 0 = 1$$

Theorem 1.6

Let $(a_n), (b_n)$ be convergent. Then (a_nb_n) converges and $\lim(a_nb_n) = \lim(a_n) \cdot \lim(b_n)$

Proof. Let $a = \lim(a_n), b = \lim(b_n)$. Let $\epsilon > 0$.

$$\begin{aligned} |a_nb_n - ab| &= |a_nb_n - ab_n + ab_n - ab| \\ &= |(a_n - a)b_n + a(b_n - b)| \\ &\leq |a_n - a||b_n| + |a||b_n - b| \end{aligned}$$

Because (b_n) converges, (b_n) is bounded by a previous theorem. Thus $\exists M_1 > 0$ such that $|b_n| \leq M$ for all $n \in \mathbb{N}$.

$$|a_nb_n - ab| \leq M_1 \cdot |a_n - a| + |a| \cdot |b_n - b|$$

Let $M = \max\{M_1, |a|\}$

$$\leq M|a_n - a| + M|b_n - b| = M\left[|a_n - a| + |b_n - b|\right]$$

Since $\lim(a_n) = a,\ \exists N_1 \in \mathbb{N}\ \forall n \geq N_1 : |a_n - a| < \epsilon/2M$

Similarly, because $\lim(b_n) = b,\ \exists N_2 \in \mathbb{N} : \forall n \geq N_2 : |b_n - b| < \frac{\epsilon}{2M}$.

Let $N = \max\{N_1, N_2\}$. Then

$$\forall n \geq N : |a_n - a| < \frac{\epsilon}{2M} \wedge |b_n - b| < \frac{\epsilon}{2M}$$

Therefore

$$|a_nb_n - ab| \leq M\left[|a_n - a| + |b_n - b|\right] < M(\frac{\epsilon}{2M} + \frac{\epsilon}{2M}) = M \cdot \frac{\epsilon}{M} = \epsilon \quad \forall n \geq N$$

Thus (a_nb_n) converges and $\lim(a_nb_n) = ab = \lim(a_n) \cdot \lim(b_n)$ ☐

This can be applied to finitely many sequences.

Example 1.7

$\lim(\frac{1}{n^b}) = 0$ for all $k \in \mathbb{N}$

Proof. Because $(\frac{1}{n})$ converges to 0, $\lim(\frac{1}{n^k}) = \lim(\frac{1}{n}) \cdots \lim(\frac{1}{n}) = 0$ ☐

Note 1.8. Special case where (b_n) is constant. i.e. $b_n = c$ for all $n \in \mathbb{N}$. Let (a_n) be convergent with $\lim(a_n) = a$. Then $\lim(c \cdot a_n) = \lim(c) \cdot \lim(a_n) = c \cdot \lim(a_n)$

Example 1.9

$$\lim(\frac{n-1}{n}) = \lim(1-\frac{1}{n}) = \lim(1+(-\frac{1}{n})) = \lim(1)+\lim(-\frac{1}{n})$$
$$= 1+\lim(-1\cdot\frac{1}{n}) = 1+-1\cdot\lim(\frac{1}{n}) = 1+-1\cdot 0 = 1$$

Theorem 1.10
In general, if $(a_n),(b_n)$ converges, then (a_n-b_n) converges and $\lim(a_n-b_n) = \lim(a_n)-\lim(b_n)$

Proof.

$$\lim(a_n-b_n) = \lim(a_n+(-b_n)) = \lim(a_n)+\lim(-b_n) = \lim(a_n)+-1\lim(b_n) = \lim(a_n)-\lim(b_n)$$

□

Theorem 1.11
Let (a_n) be convergent with $\lim(a_n)\neq 0$ and $a_n \neq 0 \quad \forall n\in\mathbb{N}$. Then $(\frac{1}{a_n})$ converges and $\lim(\frac{1}{a_n}) = \frac{1}{\lim(a_n)}$

Proof. Let $\lim(a_n) = a, \quad a\neq 0$. Let $\epsilon > 0$. Then

$$|\frac{1}{a_n}-\frac{1}{a}| = |\frac{a-a_n}{a_n\cdot a}| = \frac{|a_n-a|}{|a_n|\cdot|a|} < \frac{|a_n-a|}{k|a|} = \frac{1}{k|a|}\cdot|a_n-a| = 0$$

By conv. criterion, $(\frac{1}{a_n})$ converges to $\frac{1}{a}$ □

Lemma 1.12
Let (a_n) be convergent with $a_n\neq 0 \quad \forall n\in\mathbb{N}$ and $\lim(a_n) = a\neq 0$. Then there exists $M>0$ such that $|\frac{1}{a_n}|\leq M \quad \forall n\in\mathbb{N}$.

Proof. Let $a = \lim(a_n)$ and $\epsilon = \frac{1}{2}|a|$. Then $\exists n\in\mathbb{N}$ such that $|a_n-a| < \epsilon = \frac{1}{2}|a|$ for all $n\geq N$, then $|a_n| = |a-(a-a_n)| \geq |a|-|a_n-a| > |a|-\frac{1}{2}|a| = \frac{1}{2}|a| > 0 \quad \forall n\geq N$
Let $k = \min\{|a_1|,|a_2|,\dots,|a_{n-1}|,\frac{1}{2}|a|\} > 0$, then $|a_n| > k > 0 \quad \forall n\in\mathbb{N}$

$$\Rightarrow |\frac{1}{a_n}| < \frac{1}{k} = M \quad \forall n\in\mathbb{N}$$

□

Theorem 1.13
Let $(a_n),(b_n)$ by convergent where $\forall n\in\mathbb{N}\ b_n\neq 0$ and $\lim(b_n)\neq 0$. Then $\frac{a_n}{b_n}$ converges and $\lim(\frac{a_n}{b_n}) = \frac{\lim(a_n)}{\lim(b_n)}$

§2 Monotone Sequences

Recall 2.1. Monotone means increasing or decreasing in the non strict sense.

Theorem 2.2

Let (x_n) be a monotone sequence. Then (x_n) is convergent if and only if it is bounded. This is useful because it is easier to check whether or not a sequence is bounded than to check whether or not it is convergent.

Proof. Assume that (x_n) is increasing. We will show that (x_n) converges ot the supremum.

What is the supremeum of a sequence. We take all the numbers and consider it a set in $\mathbb{R}$ and then find the supremeum. $x := \sup \underbrace{\{x_1, x_2, x_3, \dots\}}_{:=S}$.

Let $\epsilon > 0$, then $x - \epsilon$ is not an upper bound of S. Thus $\exists N \in \mathbb{N}$ such that $x - \epsilon < x_N \leq X$ but (x_n) is increasing. We also have $x - \epsilon < x_N \leq x_{N+1} \leq x_{N+2} \leq \cdots \leq x$. i.e. $\forall n \geq N : x - \epsilon < x_n \leq x$

$\Rightarrow x_n \in]x - \epsilon, x]$ for all $n \geq N \quad \subseteq]x - \epsilon, x + \epsilon[= V_\epsilon(x)$. i.e. $\forall n \geq N : x_n \in V_\epsilon(x)$. Thus (x_n) converges to $x := \sup\{x_1, x_2, \dots\}$. The case that (x_n) is decreasing is left as an exercise. $\square$

Example 2.3

$x_1 = 1, x_{n+1} = \frac{1}{2}x_n + 2$

Show that x_n convreges and determine its limit. We will show that (x_n) is increasing and bounded; by monotone convergence theorem, (x_n) converges. Lastly, we will show that $\lim(x_n) = 4$.

Proof. (x_n) is bounded from above by 4. We'll show this using induction.

$\underline{n = 1}$: $\quad 1 \leq 4$ ✓

$\underline{n \to n+1}$: Assume that $x_n \leq 4$. Then $x_{n+1} = \frac{1}{2}x_n + 2 \leq \frac{1}{2} \cdot 4 + 2 = 4$ ✓

Therefore (x_n) is bounded from above by 4. □

Proof. Proving that (x_n) is increasing. Consider $x_{n+1} - x_n = \frac{1}{2}x_n + 2 - x_n = 2 - \frac{1}{2}x_n \geq 0$.

$$\Rightarrow \forall n \in \mathbb{N} \quad x_{n+1} - x_n \geq 0$$
$$\Rightarrow \forall n \in \mathbb{N} \quad x_{n+1} \geq x_n$$

i.e. (x_n) is increasing. □

By showing that (x_n) is bounded from above and increasing, we know that (x_n) is convregent by the monotone convergence theorem. Now to find where it converges.

Let $x := \lim(x_n)$.

$$\forall n \in \mathbb{N} \quad x_{n+1} = \frac{1}{2}x_n + 2$$
$$\Rightarrow \lim(x_{n+1}) = \lim(\frac{1}{2}x_n + 2) = \frac{1}{2}\lim(x_n) + 2 = \frac{1}{2}x + 2$$
$$\Rightarrow x = \frac{1}{2}x + 2$$
$$\Rightarrow \frac{1}{2}x = 2 \Rightarrow x = 4$$

Note 2.4. It is essential for this argument that we knew in advance that (x_n) is convergent.

We've now shown that $\lim(x_n) = 4$.

Example 2.5

Exercise for the reader: $x_1 = 1$. $x_{n+1} = \sqrt{2 + x_n}$.

Prove that (x_n) converges to 2.

§2.1 Euler's constant

Consider the squence $x_n = (1+\frac{1}{n})^n$ and $y_n = (1+\frac{1}{n})^{n+1}$.

We will show that (x_n) increases and that (y_n) decreases.

Proof. (x_n) is increasing. We have to show that $\forall n \in \mathbb{N} : x_n \leq x_{n+1}$. i.e. that

$$\begin{aligned}
&(1+\frac{1}{n})^n \leq (1+\frac{1}{n+1})^n + 1 \\
&\Leftrightarrow (1+\frac{1}{n+1})^{n+1} \geq (1+\frac{1}{n})^n \\
&\Leftrightarrow 1+\frac{1}{n+1} \geq \sqrt[n+1]{(1+\frac{1}{n})^n}
\end{aligned}$$

Recall the inequality of the algebraic and geometric mean. If $a_1, a_2, \ldots, a_n \geq 0$, then

$$\frac{a_1+\cdots+a_n}{n} \geq \sqrt[n]{a_1 \times \cdots \times a_n}$$

Let $a_1 = \cdots = a_n = 1+\frac{1}{n}$ and $a_{n+1} = 1$. Then

$$\sqrt[n+1]{a_1 \times \cdots \times a_n \times a_{n+1}} = \sqrt[n+1]{(1+\frac{1}{n})^n}$$

$$\text{and} \quad \frac{a_1+\cdots+a_n+a_{n+1}}{n+1} = \frac{n(1+\frac{1}{n})+1}{n+1} = \frac{n+1+1}{n+1} = \frac{n+2}{n+1} = 1+\frac{1}{n+1}$$

Thus, by AGM-inequality, $1+\frac{1}{n+1} \geq \sqrt[n+1]{(1+\frac{1}{n})^n}$. □

Proof. Now to show that y_n is decreasing. Similar strategy, but take inverse to reverse inequality. □

It follows from the above proofs that, Claim:

$$\forall n, k \in \mathbb{N} : x_n < y_n$$

Definition 2.6.

$$e := \lim\left((1+\frac{1}{n})^n\right) = \lim\left((1+\frac{1}{n})^{n+1}\right)$$

In analysis 2, you'll see that

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

From which it can be shown that e is irrational.

Estimates for e. Since (x_n) is increasing and (y_n) is decreasing, we have that $\forall n \in \mathbb{N} : x_n \leq e \leq y_n$.

$$\frac{5}{2} < e < 3 \Leftarrow \begin{cases} x_6 \geq \frac{5}{2} = 2.5 \\ y_5 < 3 \end{cases}$$

§2.2 Subsequences

Definition 2.7. Let $n_1 < n_2 < n_3 < \dots$ be natural numbers and let $(x_n) = (x_1, x_2, x_3, \dots)$ be a sequence. Then $(x_{n_k}) = (x_{n_1}, x_{n_2}, x_{n_3}, \dots)$ is called a subsequence of (x_n).

Example 2.8

Let $(x_1, x_2, x_3, \dots)$ be a sequence. Then $(x_1, x_3, x_5, x_7, \dots)$ is called the subsequence of odd indices; here $n_k = 2k - 1$.

Likewise, $(x_2, x_4, x_6, x_8, \dots)$ is called the subsequence of even indices; here $n_k = 2k$.

Theorem 2.9

Let (x_n) be convergent. Then every subsequence (x_{n_k}) of (x_n) also converges to the same limit.

Proof. Next class. □

Example 2.10

Let $0 < a < 1$; consider (a^n). We will show that $\lim(a^n) = 0$. Note that (a^n) is decreasing and is bounded from below. By monotone convergence theorem, (a^n) converges.

Let $x := \lim(a^n)$. Now consider the subsequence of even terms (a^{2n}). By the theorem above, this subsequence converges and has the same limit. i.e. $\lim(a^{2n}) = x$.

On the other hand, we can rewrite this as

$$\begin{aligned} \lim((a^n)^2) = [\lim(a^n)]^2 = x^2 = x \\ \Rightarrow x^2 - x = 0 \\ \Rightarrow x(x-1) = 0 \end{aligned}$$

This means that either $x = 0$ or $x = 1$. But $a^3 < a^2 < a^1 = a < 1 \Rightarrow x < 1 \Rightarrow x = 0$.

§3 10-28

Theorem 3.1

Let (x_n) be a convergent sequence, then every subsequence of (x_n) also converges to the same limit. i.e. $\lim(x_{n_k}) = \lim(x_n)$.

Lemma 3.2

If $n_1 < n_2 < n_3 < \ldots$ where $n_k \in \mathbb{N}$ for all k, then $n_k \geq k$ for all $k \in \mathbb{N}$.

Proof. By induction.

$k = 1$: Base case where $n_k \geq k$.

$k \to k+1$: Assume that $n_k \geq k$. Then

$$n_{k+1} > n_k \geq k \Rightarrow n_{k+1} > k \Rightarrow n_{k+1} \geq k+1$$

Thus $n_k \geq k$ for all $k \in \mathbb{N}$. □

Proof. Let $x := \lim(x_n)$. Let $\epsilon > 0$, then $\exists N \in \mathbb{N} \quad \forall n \geq N : |x_n - x| < \epsilon$.

Since $n_k \geq k$, by the lemma, we also have that $|x_{n_k} - x| < \epsilon$ for all $k \geq N$, since $n_k \geq k \geq N$.

Thus (x_{n_k}) converges to x. □

§3.1 Criterion for the divergence of sequences

Theorem 3.3 (1)

Let (x_n) be a sequence such that (x_n) has a subsequence (x_{n_k}) that diverges.

Proof. If (x_n) were convergent, (x_{n_k}) would converge, but it doesn't. Thus (x_n) diverges. □

Theorem 3.4

Let (x_n) be a sequence such that there exists two subsequences (x_{n_k}) and (x_{n_j}) that converge to different limits, then (x_n) diverges.

Proof. If (x_n) was convergent to x_1, then (x_{n_k}) and (x_{n_j}) would converge to x_1; but they don't. Thus (x_n) diverges. □

Example 3.5

$x_n = (-1)^n$. Consider the subsequences of the even and odd terms (x_{2n}) and (x_{2n-1}).

$x_{2n} = (-1)^{2n} = 1^{2n} = 1$. i.e. (x_{2n}) is a constant sequence and $\lim(x_{2n}) = 1$.

Similarly, $x_{2n-1} = (-1)(-1)^{2n} = -1$. i.e. (x_{2n-1}) is a constant sequence and $\lim(x_{2n-1}) = -1$.

According to one of the criterion for the divergence of sequences theorems, (x_n) diverges.

Example 3.6

$x_n : 1, 1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}$. Then $x_{2n-1} : 1, 2, 3, 4, \ldots$. Which diverges, thus (x_n) diverges.

Example 3.7

$x_n = \sqrt[n]{n}$; Prove that (x_n) converges to 1.

1st step: (x_n) is eventually decreasing.

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)^{\frac{1}{n+1}}}{n^{\frac{1}{n}}}$$
$$\Rightarrow (\frac{x_{n+1}}{x_n})^{n(n+1)} = \frac{1}{n} \cdot \frac{n+1}{n}^n = \frac{1}{n} \cdot (1 + \frac{1}{n})^n \leq \frac{1}{n} \cdot e < \frac{3}{n} \leq 1$$

As long as $n \geq 3$. Thus (x_n) is decreasing for all $n \geq 3$.

Furthermore, (x_n) is bounded from below by 1. Thus (x_n) is bounded and eventually decreasing $\Rightarrow$ $(x_n$ converges by monotone convergence theorem. Let $x := \lim(x_n)$.

Second step: Show that $x = 1$.
Consider the subsequence (x_{2n}) of even terms.

$$x_{2n} = \sqrt[2n]{2n} \Rightarrow x_{2n}^2 = \sqrt[n]{2n} = \sqrt[n]{2} \cdot \sqrt[n]{n} = \sqrt[n]{2} \cdot x_n$$

Thus

$$\lim(x_{2n}^2) = \lim(\sqrt[n]{2} \cdot x_n) = \underbrace{\lim(\sqrt[n]{2})}_{=1} \cdot \lim(x_n)$$
$$\lim(x_{2n}^2) = (\lim(x_{2n}))^2$$
$$\Rightarrow x^2 = x \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0$$
$$\Rightarrow x = 0 \vee x = 1. \text{ but } x_n \geq 1 \quad \forall n \in \mathbb{N}$$
$$\Rightarrow x = 1$$

Theorem 3.8 (Bolzano - Weirstrass)
Let (x_n) be a bounded sequence. Then (x_n) has a convergent subsequence.

Proof. Since (x_n) is bounded, $\exists \mu > 0$ such that $x_n \in \underbrace{[-M, M]}_{=I_1}$ for all $n \in \mathbb{N}$.

Divide I_1 into two subintervals of equal width. At least one of these subintervals contains infinitely many terms of (x_n). Choose this one of these intervals and call it I_2.

Divide I_2 into 2 subintervals of equal width. At least one of them, called I_3 contains infinitely many terms of (x_n). Etc...

We obtain an infinite sequence $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ of closed and bounded intervals. By the nested interval property of $\mathbb{R}$ we know that the intersection over all of these intervals is not empty. i.e. $\cap_{n\in\mathbb{N}} I_n \neq \varnothing$.

Let $x \in \cap_{n\in\mathbb{N}} I_n$. We will now show that there exists a subsequence (x_{n_k}) of (x_n) with $\lim(x_{n_k}) = x$.

Let $n_1 \in \mathbb{N}$ be arbitrary. We know that $x_{n_1} \in I_1$ because all elements are in I_1. I_2 contains infinitely many terms of (x_n). Thus there exists $n_2 > n_1$ such that $x_{n_2} \in I_2$. The same goes for I_3 ; etc...

We obtain $n_1 < n_2 < n_3 < \ldots$ such that $x_{n_k} \in I_k$ for all $k \in \mathbb{N}$.

We also have that $x \in I_k$ for all $k \in \mathbb{N}$. This gives that $|x_{n_k} - x| \leq |I_k|$ where $|I_1| = 2M, \quad |I_2| = M, \quad |I_3| = \frac{M}{2}, \quad \ldots.$

$$\Rightarrow |I_k| = \frac{2M}{2^{k-1}} = \frac{4M}{2^k} \Rightarrow |x_{n_k} - x| \leq 4M \cdot (\frac{1}{2})^k$$

for all $k \in \mathbb{N}$. By convergence criterion, $\lim(x_{n_k}) = x$; especially, (x_{n_k}) converges. Corner stone of the proof is the nested interval property of $\mathbb{R}$. □

Definition 3.9. Let (x_n) be a sequence and let (x_{n_k}) be a convergent subsequence. Let $x := \lim(x_{n_k})$. Then x is called an accumulation point or a subsequential limit (point) of (x_n).

Example 3.10
$x_n = (-1)^n$. The accumulation points of (x_n) are $+1$ and -1.

Example 3.11
Let x_n be an enumeration of Q. Every real number is an accumulation point because Q is dense in $\mathbb{R}$.

Theorem 3.12

Let (x_n) be a sequence. $x \in \mathbb{R}$ is an accumulation point of (x_n) iff $\forall \epsilon > 0 : V_\epsilon(x)$ contains infinitely many terms of (x_n).

Proof.

($\Rightarrow$) Let x be an accumulation point of (x_n). Thus there exists a subsequence (x_{n_k}) of (x_n) with $\lim(x_{n_k}) = x$. Then $\exists k \in \mathbb{N} : \forall k \geq N x_{n_k} \in V_\epsilon(x)$. Thus $V_\epsilon(x)$ contains infinitely many terms of (x_n).

($\Leftarrow$) Let $x \in \mathbb{R}$ be such that $\forall \epsilon > 0 : V_\epsilon(x)$ contains infinitely many terms of (x_n). Let $\epsilon := 1$. Then $V_1(x)$ contains infinitely many terms of (x_n) .Let $n_1 \in \mathbb{N}$ such that $x_{n_1} \in V_1(x)$.

Let $\epsilon := \frac{1}{2}$. Then $V_{\frac{1}{2}}(x)$ contains infinitely many terms of (x_n). Thus $\exists n_l > n_1$ such that $x_{n_2} \in V_{\frac{1}{2}}(x)$.

$\vdots$

$\epsilon = \frac{1}{k}$. Then $V_{\frac{1}{k}}(x)$ contains infinitely many terms of (x_n) thus $\exists n_k > n_{k-1}$ such that $x_{n_k} \in V_{\frac{1}{k}}(x)$

Since $n_1 < n_2 < n_3 < \dots$, we obtain a subsequence (x_{n_k}) of (x_n) with $x_{n_k} \in V_{\frac{1}{k}}(x)$. Now let $\epsilon > 0$ and let $k > \frac{1}{\epsilon} \Leftrightarrow \frac{1}{k} < \epsilon \Rightarrow x_{n_k}, x_{n_{k+1}}, x_{n_{k+2}}, \dots \in V_{\frac{1}{k}}(x) \subseteq V_\epsilon(x)$.

$$x_{n_k} \in V_\epsilon(x) \quad \forall k \geq K \Rightarrow x_{n_k} \text{ converges to } x$$

□

§4 Tutorial 10-30

§4.1 e

Example 4.1

1. $$\lim(1 - \frac{1}{n})^{-n} = e$$

2. $$(1 + \frac{1}{2n})^n = ((1 + \frac{1}{2n})^{2n})^{\frac{1}{2}} = (e)^{\frac{1}{2}}$$

   Because $(1 + \frac{1}{2n})$ is a subsequence of $(1 + \frac{1}{n})$ which converges to e.

3. $(1 + \frac{n}{2})^{\frac{n}{2}}$ is <u>not</u> a subsequence of $(1 + \frac{1}{n})^n$. It's the other way around.

Let $a > 0$. Pick $x_1 > 0$. Let $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n}) > 0$

Prove that $x_n \to \sqrt{a}$.

§5 10-30

Theorem 5.1
A bounded sequence converges if and only if it has exactly one accumulation point.

Proof.

($\Rightarrow$) Let (x_n) be convergent. $x := \lim(x_n)$. Then every subsequence (x_{n_k}) of (x_n) converges to x. Thus x is the only accumulation point of (x_n).

($\Leftarrow$) Let (x_n) be a bounded sequence which has only one accumulation point x. We will show that (x_n) converges to x. Assume that this is not the case.

Convergence: $\forall \epsilon > 0,\ \exists N \in \mathbb{N} : \forall n \geq N,\ |x_n - x| < \epsilon$
Negation: $\exists \epsilon > 0 : \forall N \in \mathbb{N},\ \exists n \geq N : |x_n - x| \geq \epsilon$

Thus $\exists$ infinitely many $n \in \mathbb{N}$ such that $|x_n - x| \geq \epsilon_0$.
Let $n_1 < n_2 < n_3 < \ldots$ such that $\forall k \in \mathbb{N} : |x_{n_k} - x| \geq \epsilon_0$.

Consider the subsequence (x_{n_k}) of $(x_n) \Rightarrow (x_{n_k})$ is bounded because (x_n) is bounded.

By Bolzano-weierstrass, (x_{n_k}) has a convergent subsequence $(x_{n_{k_j}})$. Let $\sim x := \lim(x_{n_{k_j}})$. Since it is a subsequence of (x_n) which has only one accumulation point. It follows that $\sim x = x$.

Thus $\lim(it) = x$ and $\forall j \in \mathbb{N}, |it - x| \geq \epsilon_0 CONTRADICTION$
Thus our assumption was wrong which proves that (x_n) converges to x. $\square$

Theorem 5.2

Let (x_n) be a bounded sequence and let A be the set of all accumulation points of (x_n). Then $A \neq \varnothing$ and A is compact (i.e. A is closed and bounded).

Proof. By BOLZANO-WEIERSTRASS, (x_n) has at least one convergent subsequence. Its limit is an accumulation point of $(x_n) \Rightarrow A \neq \varnothing$.

A is bounded: (x_n) is bounded i.e. $\exists M > 0$ such that $\forall n \in \mathbb{N},\ -M \leq x_n \leq M$.

Let $x \in A$ be arbitrary. Then $\exists$ subsequence (x_{n_k}) of (x_n) with $x = \lim(x_{n_k})$.
We have that $\forall k \in \mathbb{N} : -M \leq x_{n_k} \leq M \Rightarrow -M \leq x \leq M$.
$\Rightarrow x \in [-M, M]$ for all accumulation points x of (x_n).
$\Rightarrow A \subseteq [-M, M] \Rightarrow A$ is bounded.

A is closed: Let $x \in \mathbb{R} \setminus A$ i.e. x is not an accumulation point. Thus $\exists \epsilon > 0 : V_\epsilon(x)$ contains at most finitely many terms of (x_n).
Let $t \in V_\epsilon(x)$. $V_\epsilon(x)$ is open. Thus $\exists \tilde{\epsilon} > 0 : V_{\tilde{\epsilon}(t)} \subseteq V_\epsilon(x)$.
Thus $V_{\tilde{\epsilon}(t)}$ contains at most finitely many terms of (x_n). Thus t is not an accumulation point $\Rightarrow$ no point in $V_\epsilon(x)$ is an accumulation point of (x_n) $\Rightarrow V_\epsilon(x) \subseteq \mathbb{R} \setminus A$. Thus $\mathbb{R} \setminus A$ is open $\Rightarrow A$ is closed.

□

We've just seen that the set of all accumulation points of a bounded sequence (x_n) is $\neq 0$, closed, and bounded.

Since A is bounded, it has a supremum and an infimum. Both sup and inf are boundary points. A is closed so it contains sup and inf. Therefore $\sup(A)$ is the Maximum of A and $\inf(A)$ is the minimum of A. i.e. $\sup(A)$ is an accumulation point of (x_n), the greatest accumulation point of (x_n). Similarly $\inf(A)$ is the least accumulation point of (x_n).

Definition 5.3.

1. Let (x_n) be a bounded sequence. Then the greatest accumulation point of (x_n) is called the LIMES SUPERIOR of (x_n). In symbols: $\limsup(x_n)$.
2. The least accumulation point of (x_n) is called the LIMES INFERIOR of (x_n). In symbols: $\liminf(x_n)$.

Theorem 5.4

Let (x_n) be a bounded sequence. Then (x_n) is convergent if and only if

$$\liminf(x_n) = \limsup(x_n)$$

Proof.

($\Rightarrow$) Let $x := \lim(x_n)$. Then every subsequence (x_{n_k}) of (x_n) converges to x.

$$\Rightarrow A = \{x\} \Rightarrow \liminf(A) = x = \limsup(A)$$

($\Leftarrow$) Assume that $\liminf(x_n) = \limsup(x_n) := x$.

$$A = \{x\}$$

i.e. (x_n) has only one accumulation point. By previous theorem, (x_n) converges.

$\square$

Example 5.5

1. $$x_n = (-1)^n$$

   Accumulation points are -1 and $1 \Rightarrow \liminf(x_n) = -1$ and $\limsup = 1$. Especially, $(-1)^n$ diverges because $\liminf \neq \limsup$.

2. Let (x_n) be an enumeration of $\mathbb{Q} \cap [a, b]$ where $a < b$. We'll show that $\liminf = a$ and that $\limsup = b$.

   Proof. Let $x > b$. Let $\epsilon := b - x > 0$. Then $\forall n \in \mathbb{N}$, $x_n \notin V_\epsilon(x) \Rightarrow$ x is <u>not</u> an accumulation point of (x_n).

   Let $x \in [a, b]$ and let $\epsilon > 0$; consider $V_\epsilon(x) =]x - \epsilon, x + \epsilon[$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, $V_\epsilon(x)$ contains infinitely many rational numbers, especially, $V_\epsilon(x_n)$ contains infinitely many terms of $(x_n) \Rightarrow x$ is an accumulation point of (x_n).

   <u>x = a</u>: By density of $\mathbb{Q}$ in $\mathbb{R}$, $]a, a + \epsilon[$ contains infinitely many terms of $(x_n) \Rightarrow a$ is an accumulation point of (x_n). Similarly for $x = b$.

   Therefore $A := [a, b] \Rightarrow \liminf(x_n) = a$ and $\limsup(x_n) = b$. $\square$

3. Find all accumulation points of the following sequence.

   $$x_n : 1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

   Claim: $A = \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

   Proof. For every $k \in \mathbb{N}$, the constant sequence $\frac{1}{n}, \frac{1}{n}, \frac{1}{n}$ is a subsequence of (x_n). Thus

   $$\frac{1}{n} = \lim(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots) \in A$$

   and $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ is a subsequence of (x_n). Thus

   $$0 = \lim(1, \frac{1}{2}, \frac{1}{3}, \dots) \in A$$

   Now let $x > 1$, $\epsilon := x - 1 > 0$. Then $\forall n \in \mathbb{N} : x_n \notin V_\epsilon(x) \Rightarrow x \notin A$.

   Similarly, $x \notin A$ for all $x < 0$. Let $0 < x < 1$; $x \notin A$. Then $\exists n \in \mathbb{N} : \frac{1}{n+1} < x < \frac{1}{n}$.

   Let $\epsilon := \min\{x - \frac{1}{n+1}, \frac{1}{n} - x\} > 0$. Then $\frac{1}{n+1} \notin V_\epsilon(x) \vee \frac{1}{n} \notin V_\epsilon(x)$

   $$\Rightarrow x_n \notin V_\epsilon(x) \ \forall n \in \mathbb{N}$$

   x is <u>not</u> an accumulation point of (x_n)

   Thus $A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ $\square$

§5.1 Properties of $\limsup, \liminf$

Theorem 5.6

Let (x_n) be a bounded sequence and let $\epsilon > 0$. Then $\exists N \in \mathbb{N}\ \forall n \geq N : x_n \in]\liminf(x_n), \limsup(x_n) + \epsilon[$. i.e. at most finitely many terms of (x_n) have the property that $x_n > \limsup(x_n) + \epsilon$ or $x_n < \liminf(x_n) - \epsilon$

Proof. assignment 8 □

Theorem 5.7

Let (x_n) be a bounded sequence. Then $\limsup(x_n) = \lim(\sup\{x_k : k \geq n\})$ and $\liminf(x_n) = \lim(\inf\{x_k : k \geq n\})$.

Remark 5.8. It is not clear initially whether this is well defined. We'll prove this.

Let $y_n := \sup\{x_k : k \geq n\}$. Then (y_n) is bounded because (x_n) is bounded.

Let A, B be bounded with $A \subseteq B$. Then $\sup(A) \leq \sup(B)$.

Note 5.9. $\{x_k : k \geq n+1\} \subseteq \{x_k : k \geq n\}$.

Therefore $\sup\{x_k : k \geq n+1\} \leq \sup\{x_k : k \geq n\}$.

Therefore (y_n) is bounded and decreasing and therefore converges.

Thus $\lim(\sup\{x_k : x \geq n\})$ exists. A similar argument applies to $\lim(\inf\{x_k : k \geq n\})$.

Proof. Examination material. This is the cutoff for the midterm exam. Next week coshy sequences. 3.4 in the textbook. Important: This doesn't mean that you don't have to remember the stuff from before. If you don't know stuff from before you will be closed. I used open and closed todayand left it to you to know what open and closed means. It did not contain interior and closure so that is midterm 2 material. And you need to know what boundary sets are in order to make sense of these things but I won't ask a separate question on these things. □

§6 11-06

§6.1 Divergence to infinity

Definition 6.1. Let (x_n) be a sequence. We say that (x_n) diverges to $+\infty$ if

$$\forall M > 0,\ \exists N \in \mathbb{N},\ \forall n \geq N : x_n > M$$

In symbols:

$$\lim(x_n) = +\infty$$

(x_n) diverges to $-\infty$ if

$$\forall M > 0 (\exists N \in \mathbb{N})(\forall \geq N) : x_n < -M$$

In symbols:

$$\lim(x_n) = -\infty$$

Remark 6.2. If $\lim(x_n) = +\infty$ or $\lim(x_n) = -\infty$, then the sequence diverges. The limit laws thus do NOT apply.

Example 6.3
$\lim(n^2) = +\infty$. Let $M > 0$. Then $n^2 > M \Leftrightarrow n > \sqrt{M}$.

Let $N > \sqrt{M}$. Then $\forall n \geq N : n^2 \geq N^2 > M \Rightarrow n^2 > M$ for all $n \geq M \Rightarrow (n^2)$ diverges to $+\infty$.

Example 6.4
Let $a > 1$. Show that $\lim(a^n) = +\infty$.

Since $a > 1$, $b := a - 1 > 0$. Then $a = 1 + b$ and $a^n = (1 + b)^n$. Applying bernoulli's:

$$(1 + b)^n \geq 1 + nb > nb > M \Leftrightarrow n > \frac{M}{b}$$

Let $N > \frac{M}{b}$. Then $\forall n \geq N$, we know that $a^n > nb \geq Nb > M$. Thus a^n diverges to $+\infty$.

§6.2 Chapter 4: Limits of functions

Preparatory definition:

Definition 6.5 (In A)**.** Let $A \subseteq \mathbb{R}$. A sequence (x_n) is said to be in A if $\forall n \in \mathbb{N} : x_n \in A$.

Definition 6.6 (Cluster point)**.** Let $A \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is called a cluster point of A if:

$$\forall \epsilon > 0 : \underbrace{V_\epsilon(x) \setminus \{x\}}_{\text{Punctured neighborhood}} \cap A \neq \varnothing$$

Note 6.7. Notation for punctered neighborhoods:

$$V_\epsilon^*(x) := V_\epsilon(x) \setminus \{x\}$$

i.e. x is a cluster point of A if $\forall \epsilon > 0 : V_\epsilon^*(x) \cap A \neq \varnothing$.

Remark 6.8. Cluster points of A are not necessarily elements of A.

Definition 6.9 (Isolated Point)**.** Let $A \subseteq \mathbb{R}$. $x \in A$ is called an isolated point of A if $\exists \epsilon > 0 : V_\epsilon^*(x) \cap A = \varnothing$.

i.e. x is the only element of A that is in $V_\epsilon(x)$.

Example 6.10
$S := \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$.

Claim: 0 is the only cluster point of S. All points $\frac{1}{n} : n \in \mathbb{N}$ are isolated points of S.

0 is a cluster point. Let $\epsilon > 0$. Then $V_\epsilon(0)$ contains infinitely many numbers of the form $\frac{1}{n}$ because $\lim(\frac{1}{n}) = 0$. Thus 0 is a cluster point of S.

Let $x \neq 0$. Then $\exists \epsilon > 0 : V_\epsilon^*(x) \cap S = \varnothing$ (left as exercise). Especially, such $\epsilon > 0$ exists for all $x = \frac{1}{n}$. Thus every $\frac{1}{n}$ is an isolated point of S. □

Example 6.11
Let $A := \mathbb{Q}$. Then every real number is a cluster point of A.

Proof. Let $x \in \mathbb{R}$ be arbitrary and let $\epsilon > 0$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $V_\epsilon(x)$ contains infinitely many rational numbers. Thus $V_\epsilon^*(x)$ contains at least one (in fact infinitely many) rational numbers. i.e.

$$V_\epsilon^*(x) \cap A \neq \varnothing \Rightarrow x \text{ is a cluster point of } A$$

□

Exercise 6.12. Let I be an interval. Then the set of all cluster points of I is $\overline{I}$

Theorem 6.13
Let $A \subseteq \mathbb{R}$. Then $x \in \mathbb{R}$ is a cluster point of A if and only if there exists a sequence (x_n) in $A \setminus \{x\}$ with $\lim(x_n) = x$.

Proof.

($\Rightarrow$) Let x be a cluster point of A.

Let $\epsilon := 1$. Then $V_\epsilon^*(x) \cap A \neq \varnothing$. Let $x_1 \in V_1^*(x) \cap A$.

Let $\epsilon := \frac{1}{2}$. Then $V_\epsilon^*(x) \cap A \neq \varnothing$. Let $x_2 \in V_{\frac{1}{2}}^*(x) \cap A$.

We obtain a sequence (x_n) in $A \setminus \{x\}$ with $\forall n \in \mathbb{N} : x_n \in V_{\frac{1}{n}}^*(x) \cap A$.

Let $\epsilon > 0$. Let $N > \frac{1}{\epsilon} \Leftrightarrow \frac{1}{N} < \epsilon$. Then

$$\forall n \geq N : x_n \in V_{\frac{1}{n}}^*(x) \cap A \subseteq V_{\frac{1}{n}}^*(x) \subseteq V_{\frac{1}{n}}(x) \subseteq V_{\frac{1}{N}}(x) \subseteq V_\epsilon(x).$$

i.e. $\forall n \geq N : x_n \in V_\epsilon(x) \Rightarrow (x_n)$ converges to x.

($\Leftarrow$) Let (x_n) be a sequence in $A \setminus \{x\}$ such that $\lim(x_n) = x$. Let $\epsilon > 0$. Then $\exists N \in \mathbb{N}, \forall n \geq N : x_n \in V_\epsilon(x)$. But since $x_n \in A \setminus \{x\}$, $x_n \neq x$. This means that $x_n \in V_\epsilon^*(x)$ and $x_n \in A$. Thus $\forall n \geq N : x_n \in V_\epsilon^*(x) \cap A$. Thus $v_\epsilon^*(x) \cap A \neq \varnothing \Rightarrow x$ is a cluster point.

□

Theorem 6.14

Let $A \subseteq \mathbb{R}$. Let x be a cluster point of A. Then $x \in \overline{A}$.

Proof. Let x be a cluster point of A. By previous theorem, $\exists (x_n)$ is $A \setminus \{x\}$ such that $\lim(x_n) = 0$.

Since $\forall n \in \mathbb{N} : x_n \in A \setminus \{x\}$. We have that $\forall n \in \mathbb{N} : x_n \in \overline{A} \supseteq A \setminus \{x\}$.

Since $\overline{A}$ is closed, $\lim(x_n) \in \overline{A}$ (see assignment 6). □

Definition 6.15 (The limit of a function: Sequential Definition)**.**
Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$. Let $x_0 \in \mathbb{R}$, we say that L is a limit of f as $x \to x_0$. In symbols:

$$L = \lim_{x \to x_0} f(x)$$

if for all sequences (x_n) in $A \setminus \{x_0\}$ with $\lim(x_n) = x_0$, we have that $\lim(f(x_n)) = L$.

Example 6.16

Let

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, x \to \frac{x^2}{|x|}$$

Note that for $x \neq 0$ we have that

$$\frac{x^2}{|x|} = |x|$$

Claim: $\lim_{x \to 0} f(x) = 0$.

Let (x_n) be a sequence such that $x_n \neq 0$ for all $n \in \mathbb{N}$ and such that $\lim(x_n) = 0$. We need to show that $(f(x_n))$ converges to 0. Note that $f(x_n) = |x_n|$.

Let $\epsilon > 0$. Since $\lim(x_n) = 0$, there exists $(N \in \mathbb{N})(\forall n \geq N) : |x_n - 0| = |x_n| < \epsilon$.

Thus $\forall n \geq N : ||x_n| - 0| = ||x_n|| = |x_n| < \epsilon \Rightarrow \lim(f(x_n)) = 0$. Thus:

$$\lim_{x \to x_0} f(x) = 0$$

Example 6.17

Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ where $x \to \frac{1}{x}$. Let $x_0 \neq 0$. Show that

$$\lim_{x \to x_0} f(x) = \frac{1}{x_0}$$

Proof. Let (x_n) be a sequence in $\mathbb{R} \setminus \{0, x_0\}$ with $\lim(x_n) = x_0$. Then $\lim(f(x_n)) = \lim(\frac{1}{x_n}) = \frac{1}{\lim(x_n)} = \frac{1}{x_0}$. □

Example 6.18

Let $f : \mathbb{Z} \to \mathbb{R}$ where $x \to 0$. Let $L \in \mathbb{R}$ be arbitrary. Then

$$\lim_{x \to 0} f(x) = L$$

Since 0 is an isolated point in $\mathbb{Z}$, there doesn't exist any sequence in $\mathbb{Z} \setminus \{0\}$ that converges to 0. Thus *all* sequences (x_n) in $\mathbb{Z} \setminus \{0\}$ that converge to 0 hvae that property that

$$\lim_{x \to 0} f(x_0) = L$$

Thus $\lim_{x \to 0} f(x) = L$ for any $L \in \mathbb{R}$.

Remark 6.19. This example shows that we should avoid isolated points when considering limits.

Theorem 6.20

Let $f : A \to \mathbb{R}$ where x_0 is a cluster point of A.

Then: if f has a limit as x approaches x_0, then this limit is uniquely determined.

Proof. Let L_1, L_2 be limits of f as x approaches x_0. Then $\exists (x_n)$ is $A \setminus \{x_0\}$ with $\lim(x_n) = x_0$. Because f has a limit at x_0, $\lim(f(x_n))$ exists and $L_1 = \lim(f(x_n)) = L_2$. $\square$

§7 Lecture 11-11

Definition 7.1 (Weierstrass)**.** The ϵ definition of the limit of a function.

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$, and $x_0 \in \mathbb{R}$. We say that L is a limit of f as x approaches x_0 if:

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x \in A : 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$$

This can be rewritten in several ways:

1. $$\forall \epsilon > 0,\ \exists \delta > 0 : x \in V_\delta^*(x_0) \cap A \Rightarrow f(x) \in V_\epsilon(L)$$

2. $$\forall \epsilon > 0,\ \exists \delta > 0 : f(V_\delta^*(x_0) \cap A) \subseteq V_\epsilon(L)$$

Theorem 7.2

Let $f : A \to \mathbb{R}$ be a function. Let $x_0 \in \mathbb{R}$ and $L \in \mathbb{R}$. Then:

$$\lim_{x \to x_0} f(x) = L$$

in the sequential sense if and only if this holds in the $\epsilon - \delta$ sense.

Proof.

1. "$\epsilon - \delta \Rightarrow$ Sequential":

   Let $\epsilon > 0$. Let $\delta > 0$ be such that $f(V_\delta^*(x_0) \cap A) \subseteq V_\epsilon(L)$.

   Let (x_n) be a sequence in $A \setminus \{x_0\}$ with $\lim(x_n) = x_0$. Then $\exists N \in \mathbb{N}$, $\forall n \geq N : x_n \in V_\delta(x_0)$.
   We also have that $x_n \neq x_0$ and $x_n \in A$ for all $n \in \mathbb{N}$. This implies that

   $$\begin{aligned} &\forall n \geq N : x_n \in V_\delta^*(x_n) \cap A \\ \Rightarrow\ &\forall n \geq N : f(x_n) \in V_\epsilon(L) \\ \Rightarrow\ &(f(x_n)) \text{ converges } toL \end{aligned}$$

2. "Sequential $\Rightarrow \epsilon - \delta$":

   Assume that the sequential definition holds but that there exists $\epsilon > 0$ for which ulno $\delta > 0$ exists that satisfies $\epsilon - \delta$.

   i.e. assume that $f(V_\delta^*(x_0) \cap A) \not\subseteq V_\epsilon(L)$ for all $\delta > 0$. Especially:

   $$\begin{aligned} \delta = 1 : \quad & f(V_1^*(x_0) \cap A) \not\subseteq V_\epsilon(L) \\ \Rightarrow \exists x_1 \in V_1^*(x_0) \cap A & \text{ such that } f(x_1) \notin V_\epsilon(L) \end{aligned}$$

   $$\begin{aligned} \delta = \frac{1}{2} : \quad & f(V_{\frac{1}{2}}^*(x_0) \cap A) \not\subseteq V_\epsilon(L) \\ \Rightarrow \exists x_2 \in V_{\frac{1}{2}}^*(x_0) \cap A & \text{ such that } f(x_2) \notin V_\epsilon(L) \end{aligned}$$

   $$\vdots$$

   We then obtain a sequence (x_n) such that $x_n \in V_{\frac{1}{n}}^*(x_0) \cap A$ but $f(x_n) \notin V_\epsilon(L)$.

   Thus $\lim(x_n) = x_0$ but $(f(x_n))$ does <u>not</u> converge to L. This contradicts the sequential definition of limit.

   Thus $\exists \delta > 0$ such that $f(V_\delta^*(x_0) \cap A) \subseteq V_\epsilon(L)$.

□

Example 7.3

Show that:

$$\lim_{x \to x_0} x^2 = x_0^2$$

Solution.

1. Sequential:

   Let (x_n) be a sequence in $\mathbb{R} \setminus \{x_0\}$ with $\lim(x_n) = x_0$. Then $\lim(f(x_n)) = \lim(x_n^2) = [\lim(x_n)]^2 = x_0^2$

2. $\epsilon - \delta$:

   Let $\epsilon > 0$. Let $\delta > 0$ be arbitrary for now and assume that $|x - x_0| < \delta$. Then

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = \underbrace{|x - x_0|}_{<\delta} \cdot |x + x_0|$$

$$\begin{aligned} \Rightarrow < |x + x_0|\delta = |x - x_0 + 2x_0|\delta \leq (|x - x_0| + 2|x_0|)\delta \\ < (\delta + 2|x_0|)\delta < (\delta + 2|x_0|) \cdot \delta < \epsilon \end{aligned}$$

   Assume that $\delta < 1$. Then $|f(x) - f(x_0)| < (\delta + 2|x_0|)\delta < (1 + 2|x_0|)\delta < \epsilon$

   Now let:

$$\delta < \min(1, \frac{\epsilon}{1 + 2|x_0|})$$

   Then if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon \Rightarrow$

$$\lim_{x \to x_0} x^2 = x_0^2$$

□

Example 7.4

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, x \to \frac{1}{x}$$

Let $x_0 \in \mathbb{R} \setminus \{0\}$. Show that:

$$\lim_{x \to x_0} \frac{1}{x} = \frac{1}{x_0}$$

Solution

Solution.

1. Sequential:

   Let (x_n) be a sequence in $\mathbb{R} \setminus \{0, x_0\}$ with $\lim(x_n) = x_0$. Then:

$$\lim(f(x_n)) = \lim(\frac{1}{x_n}) = \frac{1}{\lim(x_n)} = \frac{1}{x_0}$$

2. With $\epsilon - \delta$:

   Let $\epsilon > 0$. Let $\delta > 0$ be arbitrary for now. Let $|x - x_0| < \delta$. Then:

$$\begin{aligned} |f(x) - f(x_0)| &= |\frac{1}{x} - \frac{1}{x_0}| = |\frac{x_0 - x}{x x_0}| \\ &= \frac{|x - x_0|}{|x||x_0|} < \frac{\delta}{|x||x_0|} \end{aligned}$$

   Let $\delta < \frac{1}{2}|x_0|$. Then for all x with $|x - x_0| < \delta$ we have:

$$|x| = |(x - x_0) + x_0| \geq |x| - |x - x_0| > |x_0| - \frac{1}{2}|x_0| = \frac{1}{2}|x_0|$$

   i.e. $|x| \geq \frac{1}{2}|x_0|$ Now:

$$\begin{aligned} |f(x) - f(x_0)| &< \frac{\delta}{|x||x_0|} \leq \frac{\delta}{\frac{1}{2}|x_0||x_0|} = \frac{2\delta}{x_0^2} < \epsilon \\ &\Leftrightarrow \delta < \frac{x_0^2}{2} \cdot \epsilon \end{aligned}$$

   Let $\delta < \min(\frac{1}{2}|x_0|, \frac{1}{2}x_0^2\epsilon)$. Then if $|x - x_0| < \delta$, we have that:

$$|f(x) - f(x_0)| < \epsilon \Rightarrow \lim_{x \to x_0} \frac{1}{x} = \frac{1}{x_0}$$

□

§7.1 Limit Laws

Theorem 7.5 (Limit of a Sum is the Sum of the Limits)
Let $f, g : A \to \mathbb{R}$, and x_0 be a cluster point of A. Assume that $\lim_{x \to x_0} f(x) = L_1$ and that $\lim_{x \to x_0} g(x) = L_2$.

Then

$$\begin{aligned}\lim_{x \to x_0} [(f+g)(x)] &= \lim_{x \to x_0} [f(x) + g(x)] = L_1 + L_2 \\ &= \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)\end{aligned}$$

i.e.

$$\lim_{x \to x_0} [(f+g)(x)] = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$$

Proof. We'll use the sequential criterion to prove this theorem. Let (x_n) be a sequence in $A \setminus \{x_0\}$ with $\lim(x_n) = x_0$. Then

$$\begin{aligned}\lim((f+g)(x_n)) &= \lim(f(x_n) + g(x_n)) \\ = \lim(f(x_n)) + \lim(g(x_n)) &= L_1 + L_2 = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)\end{aligned}$$

□

Theorem 7.6 (Limit of a Product is the Product of the Limits)
Let $f, g : A \to \mathbb{R}$ and x_0 be a cluster point of A. Assume that $\lim_{x \to x_0}$ and $\lim_{x \to x_0} g(x)$ exist. Then:

$$\lim_{x \to x_0} [(f \cdot g)(x)] = \lim_{x \to x_0} [f(x) \cdot g(x)] = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} g(x)$$

Proof. Let (x_n) be a sequence in $A \setminus \{x_0\}$ with $\lim(x_n) = x_0$. Then:

$$\lim_{x \to x_0} [(f \cdot g)(x)] = \lim(f(x_n) \cdot g(x_n)) = \lim(f(x_n)) \cdot \lim(g(x_n)) = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} g(x)$$

□

Especially, let $c \in \mathbb{R}$. Then

$$\lim_{x \to x_0} [c \cdot f(x)] = c \cdot \lim_{x \to x_0} f(x) \quad \text{Think of it as choosing } g = c$$

Therefore:

$$\begin{aligned}\lim_{x \to x_0} [f(x) - g(x)] &= \lim_{x \to x_0} [f(x) + (-1) \cdot g(x)] = \lim_{x \to x_0} f(x) + \lim[(-1)g(x)] \\ &= \lim_{x \to x_0} f(x) + (-1) \lim_{x \to x_0} g(x) = \lim_{x \to x_0} f(x) - \lim_{x \to x_0} g(x) \\ &\Rightarrow \lim_{x \to x_0} [f(x) - g(x)] = \lim_{x \to x_0} f(x) - \lim_{x \to x_0} g(x)\end{aligned}$$

Theorem 7.7
Let $f, g : A \to \mathbb{R}$ and x_0 be a cluster point of A. Furthermore, let $\forall x \in A,\ g(x) \neq 0$ and let $\lim_{x \to x_0} f(x), \lim_{x \to x_0} g(x)$ exist where $\lim_{x \to x_0} g(x) \neq 0$. Then:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$$

§8 Lecture 11-13

§8.1 Limits and Inequalities

Theorem 8.1 (Bounded Limit Theorem for Functions)
Let $f : A \to \mathbb{R}$, and x_0 bea cluster point of A. Assume that $\lim_{x \to x_0} f(x)$ exists.

Furthermore, assume that $\exists a, b \in \mathbb{R}$ such that $a \leq f(x) \leq b$ for all $x \in A \setminus \{x_0\}$. Then $a \leq \lim_{x \to x_0} f(x) \leq b$.

Proof. Let $\lim_{x \to x_0} f(x) = L$. Then $\forall (x_n)$ in $A \setminus \{x_0\}$ with $\lim(x_n) = x_0$, it holds that $\lim(f(x_n)) = L$.

Since $\forall n \in \mathbb{N} : x_n \in A \setminus \{x_0\}$, we have that

$$a \leq f(x_n) \leq b \qquad \underbrace{\Rightarrow}_{\text{Theorem from Chapter 3}} \qquad a \leq L = \lim(f(x_n)) \leq b$$
$$\Rightarrow a \leq \lim_{x \to x_0} f(x) \leq b$$

□

Theorem 8.2 (Squeeze Theorem for Functions)
Let $f, g, h : A \to \mathbb{R}$, and let x_0 be a cluster point of A. Assume that

$$g(x) \leq f(x) \leq h(x)$$

For all $x \in A \setminus \{x_0\}$.

Furthermore, assume that

$$L \coloneqq \lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x)$$

Then the limit of $f(x)$ as $x \to x_0$ exists and equals L.

Proof. Let (x_n) be a sequence in $A \setminus \{x_0\}$ such that $\lim(x_n) = x_0$. Then $\lim(g(x_n)) = L$ and $\lim(h(x_n)) = L$.

And since $\forall n \in \mathbb{N} : x_n \in A \setminus \{x_0\}$, we know that

$$g(x_n) \leq f(x_n) \leq h(x_n)$$

By the squeeze theorem for sequences it now follows that $(f(x_n)$ converges to L. Since this holds for any (x_n) in $A \setminus \{x_0\}$ with $\lim(x_n) = x_0$, it follows from sequence criterion that

$$\lim_{x \to x_0} f(x) = L$$

□

Example 8.3

Consider the following function :

$$f(x) : \mathbb{R} \setminus \{0\} \text{ where } x \to x \cdot \sin(\frac{1}{x})$$

Solution.

$$|x \cdot \sin(\frac{1}{x})| = |x| \cdot |\sin(\frac{1}{x})| \leq |x|$$
$$\Rightarrow -|x| \leq x \sin(\frac{1}{x}) \leq |x|$$

for all $x \in \mathbb{R} \setminus \{0\}$.

Note that

$$\lim_{x \to x_0} |x| = 0$$
$$\lim_{x \to x_0} (-|x|) = - \lim_{x \to x_0} |x| = 0$$

Therefore, by squeeze theorem we have that

$$-|x| \leq x \sin(\frac{1}{x}) \leq |x| \quad \underbrace{\Rightarrow}_{\text{Squeeze Theorem}} \quad \lim_{x \to x_0} (x \sin(\frac{1}{x})) = 0$$

□

Example 8.4

Let $f : \mathbb{R}^+ \to \mathbb{R}$ and $x \to x^{3/2}$. We want to find $\lim_{x \to 0} x^{3/2}$.

Restrict f to the interval $[0, 1]$. On this interval we have that

$$0 \leq x \leq x^{1/2}$$
$$\Rightarrow 0 \leq x^{3/2} \leq x$$

and $\lim_{x \to 0} x = 0$.

Therefore, by squeeze theorem,

$$\underbrace{0}_{=0} \leq x^{3/2} \leq \underbrace{x}_{=0} \Rightarrow \lim_{x \to 0} x^{3/2} = 0$$

§8.2 Criteria for non-existence of limits of functions

Theorem 8.5 (Non-existence criteria where $(f(x_n))$ diverges.)
Let $f : A \to \mathbb{R}$ and x_0 be a cluster point of A. If $\exists(x_n)$ in $A \setminus \{0\}$ such that $\lim(x_n) = x_0$ but such that $\lim(f(x_n))$ diverges, then $\lim_{x\to x_0} f(x)$ DNE.

Proof. If $\lim_{x\to x_0} f(x)$ would exist, then $\lim(f(x_n) = \lim_{x\to x_0} f(x)$ but $f(x_n))$ diverges $\Rightarrow \lim_{x\to x_0} f(x)$ DNE. □

Theorem 8.6 (Non-existence criteria where $(f(x_n))$ and $(f(t_n))$ converge to different limits)
Let $f : A \to \mathbb{R}$ and x_0 be a cluster point of A. Assume that $\exists(x_n), (t_n)$ in $A \setminus \{x_n\}$ such that $\lim(x_n) = x_0 = \lim(t_n)$ and such that both $(f(x_n))$ and $(f(t_n))$ converge but to different limits. Then $\lim_{x\to x_0} f(x)$ does not exist.

Proof. Assume that $\lim_{x\to x_0} f(x) = L$. Then $\lim(f(x_n)) = L = \lim(f(t_n))$. Contradiction because $\lim(f(x_n)) \neq \lim(f(t_n))$. Thus $\lim_{x\to x_0} f(x)$ diverges. □

Example 8.7
Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ and $x \to \sin(1/x)$. Show that $\lim_{x\to 0} f(x)$ DNE.

1. Solution using the 2-sequence criterion.

   Choose (x_n) where $x_n := \frac{1}{\pi n}$ for all $n \in \mathbb{N}$. Then $f(x_n) = \sin(\pi n) = 0$ for all $n \in \mathbb{N}$. i.e. $\lim(f(x_n)) = 0$.

   Now choose (t_n) where $t_n := \frac{1}{\pi/2+2\pi n}$. Then $f(t_n) = \sin(\pi/2 + 2\pi n) = \sin(\pi/2) = 1$ for all $n \in \mathbb{N}$.

$$\Rightarrow \lim(f(t_n)) = 1 \neq 0 = \lim(f(x_n))$$
$$\Rightarrow \lim_{x\to 0} f(x) \text{ DNE}$$

2. Solution using the 1-sequence criterion.

   Let $x_n := \frac{1}{(2n-1)\pi/2}$. Then $\lim(x_n) = 0$ and $f(x_n) = \sin((2n-1)\pi/2) = (-1)^n$ for all $n \in \mathbb{N}$. i.e. $(f(x_n)) = ((-1)^n)$ which diverges!

$$\Rightarrow \lim_{x\to 0} f(x) \text{ DNE}$$

§8.3 One-sided limits (Brief)

In calculus you've seen

$$\lim_{x\to x_0+} f(x) \text{ and } \lim_{x\to x_0^-} f(x)$$

How do we define these properly?

Definition 8.8 (Definition of limit from left and right)**.** Let $f : A \to \mathbb{R}$ and $x_0 \in \mathbb{R}$.

$$\lim_{x \to x_0^+} f(x) \coloneqq f\big|_{A \cap]x_0, \infty[}(x)$$
$$\lim_{x \to x_0^-} f(x) \coloneqq f\big|_{A \cap]-\infty, x_0[}(x)$$

Example 8.9

$f : \mathbb{R} \to \mathbb{R}$ where $x \to |x|$. Determine $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$.

$$\lim_{x \to 0} x = 0 \Rightarrow \lim_{x \to x^+} |x| = 0$$
$$\lim_{x \to 0} x = 0 \Rightarrow \lim_{x \to x^-} |x| = 0$$

Theorem 8.10 (Limit of function exists iff limits from left and right exists and are equal)

Let $f : A \to \mathbb{R}$ and x_0 be a cluster point of A. Then $\lim_{x \to x_0} f(x)$ exists if and only if $\lim_{x \to x_0^+} f(x)$ and $\lim_{x \to x_0^-} f(x)$ exist and are equal.

Proof. Assignment 11. □

§8.4 Chapter 5: Continuity

Definition 8.11 (Defining a continuous function)**.** Let $f : A \to \mathbb{R}$ and $x_0 \in A$. We say that f is continuous at x_0 if

$$\lim x \to x_0 f(x)$$

exists and is equal to $f(x_0)$. i.e $\lim_{x \to x_0} f(x) = f(x_0)$.

Remark 8.12. In the case that x_0 is an isolated point, this definition should be read as follows: f is continuous at x_0 if it has a limit at x_0 which equals $f(x_0)$. In other words, all functions are continuous at all isolated points. Continuous is thus only interesting at cluster points.

§9 Lecture 11-18

Definition of continuity: $\forall \epsilon > 0$, $\exists \delta > 0 : f(V_\delta(x_0) \cap A) \subseteq V_\epsilon(f(x_0))$

Remark 9.1. Let x_0 be an isolated point of A. Then any function $f : A \to \mathbb{R}$ is continuous at x_0.

Proof. Let $f : A \to \mathbb{R}$ and let $\epsilon > 0$. Since x_0 is an isolated point of A, $\exists \delta : V_\delta(x_0) \cap A = \{x_0\}$.

Then, $f(V_\delta(x_0) \cap A) = f(\{x_0\}) = \{f(x_0)\}$. Thus f is continuous at x_0. □

Theorem 9.2 (Algebraic Rules for Continuity)
Let $f, g : A \to \mathbb{R}$ and let $x_0 \in A$ be a cluster point of A. f, g is continuous at x_0, then:

(a) $f + g$ is continuous at x_0.

(b) $f \cdot g$ is continuous at x_0.

(c) $f - g$ is continuous at x_0.

(d) f/g is continuous at x_0 if $\forall x \in A$, $g(x) \neq 0$.

Proof.

(a) Let (x_n) be a sequence in A with $\lim(x_n) = x_0$.

Since f and g are continuous at x_0, we have that $\lim(f(x_n)) = f(x_0)$ and $\lim(g(x_n)) = g(x_0)$.

Thus,

$$\lim((f+g)(x_0)) = \lim(f(x_0) + g(x_0))$$
$$= \lim(f(x_n)) + \lim(g(x_n)) = f(x_0) + g(x_0) = (f+g)(x_0)$$
$$\Rightarrow f + g \text{ is continuous at } x_0$$

Alternatively, we can use the limits of functions. f, g are continuous at x_0 so

$$\lim_{x \to x_0} f(x) = f(x_0)$$
$$\lim_{x \to x_0} g(x) = g(x_0)$$

Thus

$$\lim_{x \to x_0} [(f+g)(x)] = \lim_{x \to x_0} [f(x) + g(x)]$$
$$= \lim_{x \to x_0} f_(x) + \lim_{x \to x_0} g(x) = f(x_0) + g(x_0) = (f+g)(x_0)$$
$$\Rightarrow f + g \text{ is continous at } x_0$$

(b) Left as an exercise

(c) Left as an exercise

(d) Left as an exercise

□

Theorem 9.3 (Compositions of continuous functions)
Let $f : A \to B$, and $g : B \to \mathbb{R}$ where $f(A) \subseteq B$. Let $x_0 \in A$, and let f be continuous at x_0, and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

Proof.

1. Proof with $\epsilon - \delta$

   Let $\epsilon > 0$. Because g is continuous at $f(x_0)$, we get that

   $$\exists \nu > 0 \text{ such that } g(V_\nu(f(x_0)) \cap B) \subseteq V_\epsilon(g(f(x_0))). \tag{1}$$

   And since f is continuous at x_0, we get that

   $$\exists \delta > 0 \text{ such that } f(V_\delta(x_0) \cap A) \subseteq V_\nu(f(x_0)) \tag{2}$$

   Combining (1) and (2) we get that

   $$(g \circ f)(V_\delta(x_0) \cap A) = g(f(V_\delta(x_0) \cap A) \subseteq g(V_\nu(f(x_0) \cap B) \subseteq V_\epsilon(g(f(x_0)))$$

   $$\Rightarrow (g \circ f)(V_\delta(x_0) \cap A) \subseteq V_\epsilon((g \circ f)(x_0))$$

   $$\Rightarrow g \circ f \text{ is continuous at } x_0$$

2. Proof with sequential method

   Let (x_n) be a sequence with $\lim(x_n) = x_0$. Since f is continuous at x_0, we have that $\lim(f(x_n)) = f(x_0)$.

   Because g is continuous at $f(x_0)$, we have that

   $$\lim(g(f(x_n))) = g(f(x_0))$$

   $$\Rightarrow \lim((g \circ f)(x_n)) = (g \circ f)(x_0)$$

   $$\Rightarrow g \circ f \text{ is continuous at } x_0$$

□

Definition 9.4. A function $f : A \to \mathbb{R}$ is called <u>continuous</u> (on A) if f is continuous at all $x_0 \in A$.

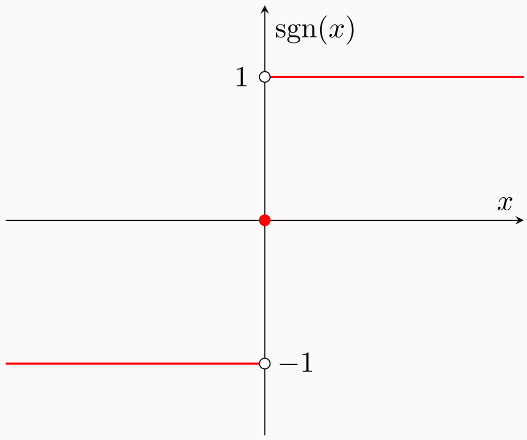
Example 9.5

1. x is continuous on $\mathbb{R}$.
2. Because products of continuous functions are continuous, x^n is continuous on $\mathbb{R}$ for all $n \in \mathbb{N}$.

   Note also that if $c_n \in \mathbb{R}$, $c_n x^n$ is continuous on $\mathbb{R}$.
3. Since sums of continuous functions are continuous, every polynomial $p(x) := a_0 + a_1 x + \cdots + a_n x^n$ is continuous on $\mathbb{R}$.
4. Since quotients of continuous functions are continuous, wherever the denominator is non-zero, we have that all rational functions $R(x) := \dfrac{P(x)}{Q(x)}$, P, Q polynomials are continuous on $\mathbb{R}/N$ where $N := \{x \in \mathbb{R} : Q(x) = 0\}$.
5. We've seen that $\lim_{x \to x_0} \sqrt{x} = \sqrt{x_0}$ for all $x_0 \in \mathbb{R}_0^+$. Thus $\sqrt{\ }$ is continuous on $\mathbb{R}_0^+$.
6. sin and cos are continuous on $\mathbb{R}$. See assignment 11.

Example 9.6 (Examples of discontinuous functions. sgn, Dirichlet, Thomae)

1.
$$\operatorname{sgn}(x) := \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$



Let (x_n) be a sequence with $x_n > 0$ for all $n \in \mathbb{N}$ and $\lim(x_n) = 0$ (e.g. $x_n = 1/n$. Then $\operatorname{sgn}(x_n) = 1$ for all $n \in \mathbb{N}$. Thus $(\operatorname{sgn}(x_n))$ converges to 1.

But! $\operatorname{sgn}(0) = 0 \neq 1 = \lim(\operatorname{sgn}(x_n))$. Thus sgn is discontinuous at 0.

2. Dirichlet's Function. $f : [0,1] \to \mathbb{R}$ where f is defined as follows:
$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Claim: f is discontinuous at all $x_0 \in [0,1]$.

Proof. Proof by cases where $x_0 \in \mathbb{Q}$ and $x_0 \in \mathbb{R} \setminus \mathbb{Q}$:

a) Let x_0 be rational. Because $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$, we know that $\exists (x_n) \in [0,1]$ such that $\lim(x_n) = x_0$ and that $\forall n \in \mathbb{N} : x_n \in \mathbb{R} \setminus \mathbb{Q}$.

Then $\forall n \in \mathbb{N}$ we have that $f(x_n) = 0 \Rightarrow \lim(f(x_n)) = 0 \neq 1 = f(x_0)$.

b) Let $x_0 \in \mathbb{R} \setminus \mathbb{Q}$. Because $\mathbb{Q}$ is dense in $\mathbb{R}$, we know that $\exists (x_n) \in [0,1]$ with $\lim(x_n) = x_0$ and $\forall n \in \mathbb{N} : x_n \in \mathbb{Q}$.

Then $\forall n \in \mathbb{N} : f(x_n) = 1 \Rightarrow \lim(f(x_n)) = 1 \neq 0 = f(x_0)$.

$\square$

3. Thomae's Function Consider $f : [0,1] \to \mathbb{R}$ such that
$$f(x) = \begin{cases} 1/q, & x = n/q, \ \gcd(n,q) = 1 \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Claim: f is <u>continuous</u> at all irrational numbers, but <u>discontinuous</u> at all rational numbers.

§9.1 Topological consequences of continuity

Exercise.

1. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be continuous. Is $f(I)$ an interval? (Yes, we will see later)
2. If $U \subseteq \mathbb{R}$ is open and $f : U \to \mathbb{R}$ is continuous, is $f(U)$ open? (No. Find a counterexample).
3. If $V \subseteq \mathbb{R}$ is closed, is $f(V)$ closed? (No)
4. If $S \subseteq \mathbb{R}$ is bounded, is $f(S)$ bounded (No)
5. If $C \subseteq \mathbb{R}$ is compact (recall that this means closed and bounded), is $f(C)$ compact?

Solution.

1. We will see later.
2. Let $f :]-1,1[\to \mathbb{R}$ where $x \to x^2$. Then $]-1,1[$ is open, but $f(]-1,1[) = [0,1[$ which is not open.
3. $f : [1,\infty[\to \mathbb{R}$ where $x \to 1/x$. Then $f([1,\infty[) =]0,1]$ which is not closed.
4. $f :]0,1] \to \mathbb{R}$ where $x \to 1/x$. The domain of f is bounded. But $(]0,1]) = [1,\infty[$ is unbounded.
5.

□

§10 Lecture 11-20

§10.1 Preservation of compactness

We'll need the following theorem:
$A \subseteq \mathbb{R}$ is closed iff every cauchy sequence in A has its limit in A.

Proof. Let A be closed and let (x_n) be a cauchy sequence in A. Assume that $x_0 :=$ □

§11 Lecture 11-25

Definition 11.1. Let $A \subseteq \mathbb{R}$ and let $c := \{U_i : i \in I\}$, where I is an index set, U_i is open for all $i \in I$.

Then c i scalled an open cover of A if $A \subseteq U_{i \in I} U_i$. i.e. every $x \in A$ is contained.

If $y \subseteq I$ such that $\{U_j : j \in J\} coloneqq \varphi$ is still a cover of A, we say that φ' is a finite subcover of φ.

Example 11.2
Let $A = [0,1]$ and let $\varphi := \{V_{1/2}(x) : x \in [0,1]\}$.

Then φ is an open cover of $[0,1]$ because

$$[0,1] \subseteq \cup_{x \in [0,1]} V_{1/2}(x) : x \in [0,1] \subseteq \,]-1/2, 3/2[$$

Theorem 11.3 (Heine-Borel)
$A \subseteq \mathbb{R}$ is compact (closed and bounded) if and only if every open cover of A has a finite subcover.

Proof.

- $\Rightarrow$ Special Case: A is a closed and bounded interval $[a,b] := I_0$. Assume that c is an open cover of I_0 that doesn't have a finite subcover. Divide I_0 into two closed subintervals of equal width $[a,c]$ and $[c,b]$ where $c = \frac{a+b}{2}$.

  For at least one of these subintervals, φ does not have a finite subcover. Otherwise, φ would have a finite subcover φ' of $[a,\varphi]$ and φ'' of $[\varphi,b]$. Then $\varphi' \cup \varphi''$ would be a finite open cover of I_0, which doesn't exist.

  Let I_1 be (one of) the subinterval(s) without finite subcover. Divide I_1 into 2 closed subintervals of equal width. At least one of them doesn't have A.

  We obtain a nested sequence $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ of closed and bounded intervals. Then
  $$\cap_{n\in\mathbb{N}_0} I_n \neq \varnothing$$
  by the nested interval property.

  Let $x_0 \in \cap_{n\in\mathbb{N}_0} I_n$. Then $x_0 \in I_0$, thus $\exists i \in I$ such that $x_0 \in U_i$ which is open. Thus, $\exists \epsilon > 0 : V_\epsilon(x_0) \subseteq U_i$.

  Claim: $\exists n \in \mathbb{N}_0 : I_n \subseteq V_\epsilon(x_0)$.

  Proof. $|I_n| = 1/2^n |I_0|$. Let $n \in \mathbb{N}_0$ such that $1/2^n|I_0| < \epsilon$.

  Let $x \in I_n$ be arbitrary. Then $|\underbrace{x}_{\in I_n} - \underbrace{x_0}_{\in I_n}| \leq 1/2^n|I_0| < \epsilon \Rightarrow x \in V_\epsilon(x_0)$.

  $\Rightarrow I_n \subseteq V_\epsilon(x_0)$. Now we have:
  $$I_n \subseteq V_\epsilon(x_0) \subseteq U_i$$
  i.e. $\{U_i\}$ covers I_n

  φ has a finite (of length 1) subcover for I_n. CONTRADICTION.

  $\Rightarrow \varphi$ does have a finite subcover. $\square$

  General Case; $A \subseteq \mathbb{R}$ compact. φ open cover. Since A is bounded, $\exists M > 0$ such that $A \subseteq [-M, M]$. Let $U := \mathbb{R}/A$ which is open.
  Consider $\varphi' := \varphi \cap \{U\}$. Then φ' covers $\mathbb{R}$. Thus φ' covers $[-M, M]$ which is closed and bounded interval by special case.
  By special case, φ' has a finite subcover φ''. φ'' may not be a subcover of φ because φ'' may contain U. However, if φ'' should contain U, we can simply remove it.
  i.e. if $U \in \varphi''$, let $\varphi''' = \varphi''/\{U\}$. If $U \notin \varphi''$, let $\varphi''' := \varphi''$.
  Since $U = \mathbb{R}/A$, φ''' will still cover A. Thus we've obtained a finite subcover of A.

$\square$

Theorem 11.4

$A \subseteq \mathbb{R}$ is compact (closed and bounded) if and only if every open cover of A has a finite subcover.

Proof.

$\Leftarrow$ Let A not be compact. We need to find an open cover of A without a finite subcover. A not closed: assignment 12.

A **unbounded**
Let $\varphi := \{U_n : n \in \mathbb{N}\}$ where $U_n :=]-n, n[$. Then φ covers $\mathbb{R}$ and thus A. Consider any finite subset $m\{U_{n_1}, \cdots, U_{n_k}$.

□

Remark 11.5. THe "classical" definition of compacness is closed and bounded, however this definition doesn't generalize will beyond $\mathbb{R}^n$ since there isn't even a notion of boundedness on general "topological spaces" However, open covers still make perfect sense on topological spaces. Thus, the def of compactness was revised to

Definition 11.6 (Modern definition of compactness)**.** A is called compact if every open cover of A has a finite subcover.

"Modern" heine borel becomes:

Definition 11.7. $A \subseteq \mathbb{R}$ is compact if and only if A is closed and bounded.

Applications of heine borel: It can often be useful to generalize "local" properties of functions to "global" properties if the domain is compact.

Definition 11.8. $f : A \to \mathbb{R}$ is called locally bounded if $\forall x_0 \in A$, $\exists \epsilon > 0 : f$ is bounded on the domain $V_\epsilon(x_0)$.

Example 11.9

$f :]0, \infty[\to \mathbb{R}$, $x \to 1/x$.

f is bounded on any neighborhood about x_0 that does not contain 0 is in its boundary. Thus f is locally bounded, but not (globally) bounded!

However, this can't happen if the domain is compact

Theorem 11.10

Let $A \subseteq \mathbb{R}$ be compact. $f : A \to \mathbb{R}$ be locall bounded. Then f is bounded (on A).

Proof. Let $x \in A$ be arbitrary. f locally bounded $\Rightarrow \exists \epsilon_x > 0$ such that f is bounded on interval $V_{\epsilon_x}(x)$.

Then $\varphi := \{V_{\epsilon_x} : x \in A$ is an open cover of A. Since A is compact, φ has a finite subcover $\{V_{\epsilon_{x_1}}, \cdots, V_{\epsilon_{x_n}}(x_n)\}$.

On each of these n neighborhoods, f is bounded.

$$\Rightarrow \exists M_1, \cdots, M_n \geq 0$$

such that $|f|(x) \leq M_1, \cdots, |f|(x) \leq M_n$ bounded on $V_{\epsilon_n}(x_n)$.

Let $M := \max\{M_1, \cdots, M_n\}$. Then $|f|(x) \leq M, \cdots, |f| \leq M$. $\square$

§12 Lecture 11-27

§12.1 Application of Heine-Borel

Theorem 12.1

Let $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ be a nested sequence of compact sets. Then

$$\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$$

(This is by the nested interval property, but we are going to prove it using heine-borel)

Proof. $\forall n \in \mathbb{N}$, let $U_n := \mathbb{R} \setminus A_n \Rightarrow \forall n \in \mathbb{N} U_n$ is open and $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$

By de morgans law, we have that

$$\bigcup_{n\in\mathbb{N}} U_n = \bigcup_{n\in\mathbb{N}} \mathbb{R} \setminus A_n \underbrace{=}_{\text{De morgans}} \mathbb{R} \setminus \bigcap_{n\in\mathbb{N}} A_n$$

Now assume that $\cap_{n\in\mathbb{N}} A_n = \varnothing$. Then $\cup_{n\in\mathbb{N}} U_n = \mathbb{R} \setminus \varnothing = \mathbb{R}$.

i.e. The U_n cover all of $\mathbb{R}$ and thus especially A_1. By heine-borel, this open cover has a finite subcover.

$$\begin{aligned}
&\{U_{n_1}, \ldots, U_{n_k}\}, n_1 < \cdots < n_k \\
\Rightarrow A_1 \subseteq \bigcup_{i=1}^{k} U_{n_i} &= U_{n_1} \cup \cdots \cup U_{n_k} = U_{n_k} \\
&\Rightarrow A_1 \subseteq U_{n_k} \\
&\Rightarrow A_{n_k} \subseteq A_1 \subseteq U_{n_k} = \mathbb{R} \setminus A_{n_k} \\
&\Rightarrow A_{n_k} \subseteq \mathbb{R} \setminus A_{n_k} \quad \lightning \\
&\Rightarrow \bigcap_{n\in\mathbb{N}} A_n \neq \varnothing
\end{aligned}$$

□

Definition 12.2 (Uniform Continuity)**.** Let's recall the definition of continuity of $f : A \to \mathbb{R}$:

$$(\forall x_0 \in A)(\forall \epsilon > 0)(\exists \delta = \delta(\epsilon, x_0)) : (\forall x \in A)(|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon)$$

Note 12.3. In general, δ will depend on both ϵ (unavoidable) and x_0.

It would be useful in many branches of analysis (e.g. Riemann integration) if δ would only depend on ϵ and <u>not</u> x_0.

i.e. we'd like to have this:

$$(\forall x_0 \in A)(\forall \epsilon > 0)(\exists \delta = \delta(\epsilon))(\forall x \in A) : (|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon)$$
$$\equiv$$
$$(\forall \epsilon > 0)(\exists \epsilon > 0)(\forall x_1, x_0 \in A) : (|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon)$$

Since x_0 is actually a variable, we'll use μ instead and obtain:

$f : A \subseteq \mathbb{R} \to \mathbb{R}$ is called uniformly continuous on A if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, \mu \in A) : (|x - \mu| < \delta \Rightarrow |f(x) - f(\mu)| < \epsilon)$$

Example 12.4

$f : \mathbb{R} \to \mathbb{R}$, $x \to x$. Claim: f is uniformaly continuous.

Proof. Let $\epsilon > 0$ and let $\delta := \epsilon$. Then $\forall x, \mu \in \mathbb{R}$, $|x - \mu| < \delta = \epsilon \Rightarrow |f(x) - f(\mu)| = |x - u| < \epsilon$ □

Lemma 12.5

$\forall x, \mu > 0$ where $x \geq \mu$, we have that $\sqrt{x} - \sqrt{\mu} \leq \sqrt{x - \mu}$.

Proof.

$$\begin{aligned}
&\sqrt{x} - \sqrt{\mu} \leq \sqrt{x - \mu} \\
\Leftrightarrow\ &(\sqrt{x} - \sqrt{\mu})^2 \leq (\sqrt{x - \mu})^2 = x - \mu \\
\Leftrightarrow\ &x - 2\sqrt{x}\sqrt{\mu} + \mu \leq x - \mu \\
\Leftrightarrow\ &2\mu - 2\sqrt{x}\sqrt{\mu} \leq 0 \\
\Leftrightarrow\ &\underbrace{2\sqrt{\mu}}_{\geq 0}\underbrace{(\sqrt{\mu} - \sqrt{x})}_{\leq 0} \leq 0 \ \checkmark
\end{aligned}$$

Because we only used equivalence statements, this final true statement proves that

$$\sqrt{x} - \sqrt{\mu} \leq \sqrt{x - \mu}$$

□

Example 12.6

$f : \mathbb{R}_0^+ = [0, \infty[\to \mathbb{R}$, $x \to \sqrt{x}$. Claim: f is uniformally continuous.

Remark 12.7. We did prove in chapter 4 that $\sqrt{x}$ is continuous on $[0, \infty[$. Back then, the δ value we obtained did depend on both ϵ and x !

However, this does not necessarily mean that $\sqrt{\ }$ is not uniformally continuous! It might just mean that we need better estimates!

Proof. Let $\epsilon > 0$, let $\delta > 0$ be arbitrary for now. Let $x, \mu \in [0, \infty[$. We may assume without loss of generality that $x \geq \mu$. Let $|x - \mu| = x - \mu < \delta$. Then:

$$|f(x) - f(\mu)| = |\sqrt{x} - \sqrt{\mu}| = \sqrt{x} - \sqrt{\mu} \leq \sqrt{x - \mu} < \sqrt{\delta} < \epsilon$$
$$\Leftrightarrow \delta < \epsilon^2$$

Note that δ is independent of x and μ !

With this uniform δ, we have

$$|x - \mu| < \delta \Rightarrow |f(x) - f(\mu)| < \epsilon \Rightarrow \sqrt{x}$$

is uniform continuous on $[0, \infty[$. □

How can we see whether a funciton is not uniformally continuous?

$f : A \to \mathbb{R}$ not continuous:

$$\begin{aligned}
&\neg(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, \mu \in A) : (|x - \mu| < \delta \Rightarrow |f(x) - f(\mu)| < \epsilon) \\
\equiv\ &\neg(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, \mu \in A) : (|x - \mu| \geq \delta \vee |f(x) - f(\mu)| < \epsilon) \\
\equiv\ &(\exists \epsilon > 0)(\forall \delta > 0)(\exists x, \mu \in A) : (|x - \mu| < \delta \wedge |f(x) - f(\mu)| \geq \epsilon)
\end{aligned}$$

Recall 12.8. $P \Rightarrow Q \equiv \neg P \vee Q$

Theorem 12.9 (2 sequence criterion for non-uniform continuity)

Let $f : A \to \mathbb{R}$. Let $\epsilon_0 > 0$ and let $(x_n), (\mu_n)$ be sequences in A such that $\lim(x_n - \mu_n) = 0$ and $|f(x_n) - f(\mu_n)| \geq \epsilon_0$ for all $n \in \mathbb{N}$. Then f is not uniformally continuous on A.

Proof. Assume that f is uniform continuous. Then $\exists \delta > 0$ such that $\forall x, \mu \in A$: $|x - \mu| < \delta \Rightarrow |f(x) - f(\mu)| < \epsilon_0$. $(*)$

Now $\lim(x_n - \mu_n) = 0$. Then $(\exists N \in \mathbb{N})(\forall n \geq N) : |x_n - \mu_n| < \delta$. Especially, $|x_n - \mu_n| < \delta$.

In $(*) :\Rightarrow |f(x_N) - f(\mu_N)| < \epsilon_0$ ↯

Thus f is not uniformally continuous on A. □

Example 12.10
$f : \mathbb{R} \to \mathbb{R}$, $x \to x^2$.

Let $x_n := n$, $u_n := n + 1/n$

Then $|x_n - \mu_n| = 1/n \Rightarrow \lim(x_n - \mu_n) = 0$

But $|f(x_n) - f(\mu_n)| = |n^2 - (n + 1/n)^2| = |n^2 - n^2 - 2 - 1/n^2| = 2 + 1/n^2 > 2$.

Let $\epsilon_0 := 2$. Then $\lim(x_n - \mu_n) = 0$, but $\forall n \in \mathbb{N} : |f(x_n) - f(\mu_n)| \geq \epsilon_0$.

$\Rightarrow x^2$ is <u>not</u> uniformally continuous on $\mathbb{R}$.

Example 12.11
$f :]0, \infty[\to \mathbb{R}$, $x \to 1/x$

Let $x_n := 1/n$, $\mu_n := 1/(n + 1)$.

Then, $|x_n - \mu_n| = |1/n - 1/(n + 1)| = |(x + 1 - x)/(n(n + 1))| = 1/(n(n + 1)) \leq 1/n^2 \to 0$.

By convergence criterion, $\lim(x_n - \mu_n) = 0$.

But, $|f(x_n) - f(\mu_n)| = |n - (n + 1)| = 1$. Let $\epsilon_0 := 1$.

Then $\lim(x_n - \mu_n) = 0$. But $|f(x_n) - f(\mu_n)| \geq \epsilon_0$.

Therefore $1/x$ is <u>not</u> uniformally continuous on $]0, \infty[$.

Theorem 12.12
Every continuous function on a compact domain is uniformally continuous.

Proof. Let $f : A \to \mathbb{R}$, A be compact, and f continuous on A.

Let $\epsilon > 0$, then $(\forall x \in A)(\exists \delta_x > 0) : (|x - \mu| < \delta_x \Rightarrow |(f(x) - f(\mu)| < \epsilon/2)$

Now consider the neighborhoods $V_{(1/2)\delta_x}(x)$ for all $x \in A$.

Then $\varphi := \{V_{(1/2)\delta_x}(x) : x \in A\}$ is an open cover of A. (Even just the centres of these neighborhoods already cover A)

By Heine-Borel, φ has a finite subcover $\{V_{(1/2)\delta_{x_1}}, \dots, V_{(1/2)\delta_{x_n}}\}$ where $x_1, \dots, x_n \in A$.

Let $\delta := \min\{\frac{1}{2}\delta_{x_1}, \dots, \frac{1}{2}\delta_{x_n}\} > 0$.

We'll prove that with this δ, we have that $|x - \mu| < \delta \Rightarrow |f(x) - f(\mu)| < \epsilon$.

Let $x, \mu \in A$ such that $|x - \mu| < \delta$. Since $x \in A$, $\exists 1 \leq k \leq n$ such that $x \in V_{(1/2)\delta_{x_k}}(x_k)$

$\Rightarrow |x - x_k| < \frac{1}{2}\delta_{x_k} < \delta_{x_k}$

and

$$\begin{aligned}
|\mu - x_k| = |(\mu - x) + (x - x_k)| \leq |x - \mu| + |x - x_k| < \delta + \frac{1}{2}\delta_{x_k} = \delta_{x_k} \\
\Rightarrow x, \mu \in V_{\delta_{x_k}}(x_k) \\
\Rightarrow |f(x) - f(\mu)| = |(f(x) - f(x_k)) + f(x_k) - f(\mu))| \\
\leq \underbrace{|f(x) - f(x_k)|}_{\leq \epsilon/2} + \underbrace{|f(\mu) - f(x_k)|}_{\leq \epsilon/2} < \epsilon
\end{aligned}$$

Because $|x - x_k| < \delta_{x_k}$ and $|\mu - x_k| < \delta_{x_k}$.

i.e. if $|x - \mu| < \delta \Rightarrow |f(x) - f(\mu)| < \epsilon \Rightarrow f$ is uniform continuous on A ☐

Example 12.13
x^2 is uniform continuous on all intervals $[-a, a]$ where $a > 0$.

Example 12.14
$1/x$ is uniform continuous on all intervals $[a, 1]$ where $0 < a < 1$.

§13 Lecture 12-02

Theorem 13.1
Let $f : A \to \mathbb{R}$ be uniformly continuous on A.

Let (x_n) be a cauchy sequence in A. Then $(f(x_n))$ is also a cauchy sequence.

Proof. Let $\epsilon > 0$. Then $\exists \delta > 0$ such that $|x - \mu| < \delta \Rightarrow |f(x) - f(\mu)| < \epsilon$

(x_n) cauchy then $\exists N \in \mathbb{N}$ such that $\forall n, m \geq N : |x_n - x_m| < \delta \Rightarrow |f(x_n) - f(x_m)| < \epsilon$.

i.e. $(\exists N \in \mathbb{N})(\forall n, m \geq N : |f(x_n) - f(x_m)| < \epsilon \Rightarrow (f(x_n)$ is a cauchy sequence. $\square$

Remark 13.2. This result is, in general, false, if f is just continuous on A.

Example 13.3
$f :]0, \infty[\to \mathbb{R}, x \to 1/x$.

f is continuous but <u>not</u> uniformally continuous on $]0, \infty[$.

Consider $x_n := 1/n$. Then (x_n) is a cauchy sequence but $(f(x_n)) = (n)$ which diverges.

$$\Rightarrow (f(x_n)) \text{ is } \underline{\text{not}} \text{ a cauchy sequence}$$

However: if $f : A \to \mathbb{R}$ is continuous, (x_n) is a convergent sequence in A such that $\lim(x_n) \in A$. Then:

$\lim(x_n) := x \in A$. Then f is continuous at x. Thus let $\lim(f(x_n)) = f(x)$ be the sequence of continuity. Especially, $(f(x_n))$ is cauchy sequence in this case.

This can be turned into another criterion for non-uniform continuous functions.

Theorem 13.4 (One sequence criterion for a non-uniform continuous function)
Let $f : A \to \mathbb{R}$. If (x_n) is cauchy sequence in A such that $(f(x_n))$ is not cauchy, then f is not uniformally continuous on A.

Example 13.5
$f :]0, \infty[\to \mathbb{R}, x \to 1/x$.

$$x_n := \frac{1}{n}$$

cauchy but $(f(x_n)) = (n)$ is not cauchy.

$$\Rightarrow f \text{ is not uniformly continuous on }]0, \infty[$$

Theorem 13.6

Let $f : A \to \mathbb{R}$, A bounded, f a uniformly continuous on A, then f is bounded (i.e. $f(A)$ is bounded.

Proof. Assume that f is unbounded. Then $\forall n \in \mathbb{N}$, $\exists x_n \in A : |f(x_n)| \geq n$.

Consider (x_n). Since A is bounded, (x_n) is bounded and thus has a convergent subsequence (x_{n_k}). Thus (x_{n_k}) is cauchy $\Rightarrow (f(x_{n_k}))$ is cauchy and thus especially bounded. But $|f(x_{n_k})| \geq n_k \geq k$ for all $k \in \mathbb{N}$.

This implies that $f(x_{n_k})$ is unbounded. Contradiction!

Thus f is bounded. □

Example 13.7

$f :]0,1[\to \mathbb{R}, x \to 1/x$. Then f is unbounded on the bounded domain $]0,1[\Rightarrow f$ is not continuous on $]0,1[$.

§14 Lecture 12-03

Lipschitz Continuous.

Example 14.1

Last class: $\sqrt{x}$ is not lipschitz on $[0,\infty[$, however $\sqrt{x}$ is lipschitz on $[a,\infty[$ for any $a > 0$.

Proof. Let $x, \mu \in [a,\infty[$. Then

$$\begin{aligned}|\sqrt{x} - \sqrt{\mu}| &= |\frac{(\sqrt{x} - \sqrt{\mu})(\sqrt{x} + \sqrt{\mu})}{\sqrt{x} + \sqrt{u}}| \\ &\leq \frac{1}{2\sqrt{a}}|x - \mu|\end{aligned}$$

i.e. $\sqrt{x}$ is lipschitz continuous on $[a,\infty[$ with lipschitz constant $k = \frac{1}{2\sqrt{a}}$ □

Example 14.2

Last class: x^2 is lipschitz on $]-a,a[,\ a > 0$.

However, x^2 is not lipschitz on $\mathbb{R}$.

Proof. x^2 isn't even uniformly continuous on $\mathbb{R}$ and thus cannot be lipschitz. □

Definition 14.3 (Geometric interpretation of lipschitz continuous)**.** Geometric interpretation of lipschitz continuous:

$f : A \to \mathbb{R}$ is lipschitz if

$$\exists k > 0 \ : \ \forall x, \mu \in A \ : \ |f(x) - f(\mu)| \leq k \cdot |x - \mu|$$
$$\text{if } x \neq \mu \Leftrightarrow \underbrace{|\frac{f(x) - f(\mu)}{x - \mu}|}_{\text{Difference Quotient}} \leq k$$

i.e. f is lipschitz if and only if the average slope of f is bounded on A.

§14.1 Another method for proving that $\sqrt{x}$ is uniformly continuous on $[0, \infty[$.

Idea: If $x \geq 1$, $\sqrt{x}$ is lipschitz on $[1, \infty[$ and thus uniformly continuous. And: if $0 \leq x \leq 1$: $\sqrt{x}$ is uniformly continuous since it is continuous and $[0, 1]$ is compact.

Q: $if\sqrt{x}$ is uniformly continuous on $[0, 1]$ and $[1, \infty[$, does it follow that f is uniformly continuous on $[0, \infty[$.

A: Yes; this requries proof!

Theorem 14.4

Let f be uniformly continuous on intervals I_1, I_2 where I_1 is closed on the right with $\sup I_1 = \max I_1 = b$. And I_2 is closed on the left with $\inf I_2 = \min I_2 = b$, then f is uniformly continuous on $I = I_1 \cup I_2$.

Proof. Let $\epsilon > 0$, f uniformly continuous on I_1, thus $\exists \delta_1 > 0$ such that $|x - \mu| < \delta_1 \Rightarrow |f(x) - f(\mu)| < \epsilon/2$.

f is uniformly continuous on I_2. Thus $\exists \delta_2 > 0$ such that $|x - \mu| < \delta_2 \Rightarrow |f(x) - f(\mu)| < \epsilon/2$.

Let $\delta := \min\{\delta_1, \delta_2\}$.

1. Case $x, \mu \in I_1$

$$|x - \mu| < \delta \leq \delta_1 \Rightarrow |f(x) - f(\mu)| < \epsilon/2 < \epsilon$$

2. Case $x, \mu \in I_2$

$$|x - \mu| < \delta \leq \delta_2 \Rightarrow |f(x) - f(\mu)| < \epsilon/2 < \epsilon$$

3. Case $x \in I_1, \mu \in I_2$

$$\begin{gathered}
|x - \mu| < \delta \Rightarrow |x - b|\delta \wedge |u - b| < \delta \\
\text{Thus } |f(x) - f(b)| < \frac{\epsilon}{2} \text{ and } |f(\mu) - f(b)| < \frac{\epsilon}{2} \\
\text{Now: } |f(x) - f(\mu)| = |[f(x) - f(b)] - [f(\mu) - f(b)]| \\
\leq |f(x) - f(b)| + f(\mu) - f(b)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\
\text{i.e. } |x - \mu| < \delta \Rightarrow |f(x) - f(\mu)| < \epsilon \\
\Rightarrow f \text{ is uniformly continous on } I = I_1 \cup I_2
\end{gathered}$$

□

Application: $\sqrt{x}$ is uniformly continuous on $[0, 1]$ and $[1, \infty[\Rightarrow \sqrt{x}$ is uniformly continuous on $[0, \infty[$.

§14.2 Differentiation

Definition 14.5 (Differentiable Definition)**.** Let $f : I \to \mathbb{R}$, I be an interval, $x_0 \in I$.

We say that f is differentiable at x_0, if

$$\lim_{x \to x_0} \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\text{Difference Quotient}} \quad \text{exists.}$$

If the limit exists, we call its value the derivative of f at x_0, denoted by

$$f'(x_0) = \frac{df}{dx}(x_0)$$

If f is differentiable at all $x_0 \in I$, we say that f is differentiable on I.

Theorem 14.6 (Caratheodory Alternative Description of Differentiability)
Let $f : I \to \mathbb{R}$, $x_0 \in I$, then f is differentiable at x_0 if and only if there exists a function $\phi : I \to \mathbb{R}$ continuous at x_0 such that

$$\forall x \in I \quad f(x) = f(x_0) + \phi(x)(x - x_0)$$

If ϕ exists, it holds that $\phi(x_0) = f'(x_0)$.

Proof. "$\Rightarrow$" Let f be differentiable at x_0. Let

$$\phi(x) := \begin{cases} \frac{f(x)-f(x_0)}{x-x_0}, & x \neq x_0 \\ f'(x_0), & x = x_0 \end{cases}$$

Then

$$\lim_{x \to x_0} \phi(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) = \phi(x_0)$$
$$\Rightarrow \phi \text{ is continuous at } \ x_0$$

"$\Leftarrow$" Let $\phi : I \to \mathbb{R}$, continuous at x_0 such that

$$f(x) = f(x_0) + \phi(x)(x - x_0)$$

Let $x \neq x_0$. $\Rightarrow \phi(x) = \frac{f(x)-f(x_0)}{x-x_0}$

ϕ continuous at x_0 $\Rightarrow$ $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists and equals $\phi(x_0)$ $\Rightarrow$ f is differentiable at x_0 and $f'(x_0) = \phi(x_0)$

$\square$

Applications: Differentiable implies continuous. i.e. if $f : I \to \mathbb{R}$ is differentiable at $x_0 \in I$, then f is continuous at x_0.

Proof. f differentiable at x_0 $\Rightarrow$ $\exists \phi : I \to \mathbb{R}$, continuous at x_0 such that $\forall x \in I$, $f(x) = \underbrace{f(x_0) + \phi(x) \cdot (x - x_0)}_{\text{continuous at } x_0}$ $\square$

Theorem 14.7 (Product Rule)
Let $f, g : I \to \mathbb{R}$ be differentiable at x_0. Then $f \cdot g$ is differentiable at x_0 and $(f \cdot g)'(x_0) = f'(x_0)g(x_0) - f(x_0) \cdot g'(x_0)$.

Proof. f, g differentiable at $x_0 \Rightarrow \exists \phi, \psi : I \to \mathbb{R}$ continuous at x_0 such that

$$f(x) = f(x_0) + \phi(x)(x - x_0)$$
$$g(x) = g(x_0) + \psi(x)(x - x_0)$$
$$\Rightarrow (f \cdot g)(x) = f(x) \cdot g(x)$$
$$= f(x_0)g(x_0) + f(x_0)\psi(x)(x - x_0) + g(x_0)\psi(x)(x - x_0) + \phi(x)\psi(x)(x - x_0)^2$$
$$\Rightarrow (f \cdot g)(x) = f(x_0)g(x_0) + [f(x)g(x_0) + f(x_0)\psi(x) + \phi(x)\psi(x)(x - x_0)] \cdot (x - x_0)$$

$\square$

Theorem 14.8 (Chain Rule)
Let $f : I \to \mathbb{R}$, $f : J \to \mathbb{R}$, $f(I) \subseteq J$, $x_0 \in I$, f differentiable at x_0, g differentiable at $y_0 := f(x_0)$, then $g \circ f$ is differentiable at x_0, and $(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x)$
f differentiable at $x_0 \Rightarrow \exists \phi : I \to \mathbb{R}$, continuous at x_0 such that $f(x) = f(x_0) + \phi(x)(x - x_0)$.

g differentiable at $y_0 \Rightarrow \exists \psi : J \to \mathbb{R}$ continuous at y_0 such that $g(y) = g(y_0) + \psi(y) \cdot (y - y_0)$. Therefore

$$\begin{aligned} g(f(x)) &= g(f(x_0)) + \psi(f(x_0) + \phi(x)(x - x_0)) \cdot [f(x_0) + \phi(x)(x - x_0) - f(x_0)] \\ &= g(f(x_0)) + \psi(f(x_0) + \phi(x)(x - x_0)) \cdot \phi(x) \cdot (x - x_0) := \Theta(x) \end{aligned}$$

Then Θ is continuous at x_0 as a composition of 2 continuous functions.
$\Rightarrow g \circ f$ is differentiable at x_0

$$\begin{aligned} (g \circ f)'(x_0) &= \Theta(x_0) \\ &= \psi(f(x_0) + \phi(x_0) \cdot 0) \cdot \phi(x_0) \\ &= \psi(f(x_0)) \cdot \phi(x_0) \\ &= \psi(y_0) \cdot \phi(x_0) \\ &= g'(y_0) \cdot f'(x_0) \\ &= g'(f(x_0)) \cdot f'(x_0) \end{aligned}$$

§14.3 Relationship Between Lipschitz Continuity and Differentiability

Recall 14.9 (Mean Value Theorem)**.** The mean value theorem. Let $I = [a, b]$, $f : I \to \mathbb{R}$ differentiable on $]a, b[$ and continuous on the entire interval. Then there exists $c \in]a, b[$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Theorem 14.10

Let $f : I \to \mathbb{R}$ be differentiable. Then f is lipschitz on I if and only if f' is bounded on I.

Proof. "$\Rightarrow$" Let f be lipschitz with lipchitz constant k.

$$-k \leq \frac{f(x) - f(\mu)}{x - \mu} \leq k$$

$$\begin{aligned} &\Rightarrow -k \leq \lim_{x \to \mu} \frac{f(x) - f(\mu)}{x - \mu} \leq k \\ &\Rightarrow -k \leq f'(\mu) \leq k \\ &\Rightarrow |f'(\mu)| \leq k \; \forall \; \mu \in I \\ &\Rightarrow f' \text{ is bounded on } I \end{aligned}$$

"$\Leftarrow$" Assume that f' is bounded on I.

Let $k > 0$ such that $|f'(x)| \leq k$ for all $x \in I$.

Let $x < \mu$, $x, \mu \in I$. Apply mean value theorem to f on $[x, \mu]$ then $\exists c \in]x, \mu[$ such that

$$\begin{aligned} \frac{f(x) - f(\mu)}{x - \mu} = f'(c) \Rightarrow \frac{|f(x) - f(\mu)|}{|x - \mu|} &= |f'(c)| \leq k \\ \Rightarrow |f(x) - f(\mu)| &\leq k|x - \mu| \\ \Rightarrow f \text{ is lipschitz on } I& \end{aligned}$$

□

§15 Sequences

Definition 15.1. Limit. $x_n \to x$ if $\forall \epsilon > 0$, $\exists k \in \mathbb{N}$ such that $|x_n - x| < \epsilon$. $\forall n \geq K$.

Example 15.2

$$\lim(\frac{2n}{n+1}) = 2$$

Let $\epsilon > 0$. Compute (for any $n \in \mathbb{N}$)

$$|\frac{2n}{n+1} - 2| = |\frac{2n - 2n - 2}{n+1}| = \frac{2}{n+1} < \frac{2}{n}$$

By A.P, $\exists k \in \mathbb{N}$ such that $K > \frac{2}{\epsilon}$. Then $\forall n \geq K$:

$$|\frac{2n}{n+1} - 2| < \frac{2}{n} \leq \frac{2}{k} < \epsilon$$

Example 15.3

$$\lim \frac{3n+1}{2n+5} = \frac{3}{2}$$

First, for any $n \in \mathbb{N}$, we have that

$$|\frac{3n+1}{2n+5} - \frac{3}{2}| = |\frac{6n+2-6N-15}{2(2n+5)}| = \frac{13}{4n+10} \leq \frac{10^6}{n}$$

Note: If unsure, use number much bigger i.e. $10^6 > 13$.

Now, for any $\epsilon > 0$, by A.P, $\exists k \in \mathbb{N}$ such that $k > \frac{10^6}{\epsilon}$. Then, $\forall n \geq K$:

$$|\frac{3n+1}{2n+5} - \frac{3}{2|} \leq \frac{10^6}{n} \leq \frac{10^6}{k} < \epsilon$$

Example 15.4

$$\lim \frac{n^2-1}{2n^2+3} = \frac{1}{2}$$

First, $\forall n \in \mathbb{N}$,

$$|\frac{n^2-1}{2n^2+3} - \frac{1}{2}| = |\frac{2n^2-2-2n^2-3}{2(2n^2+3)}| = \frac{5}{4n^2+6} \leq \frac{5}{n^2}$$

$\forall \epsilon > 0$, $\exists k \in \mathbb{N}$ such that $k > \sqrt{\frac{5}{\epsilon}}$

Then, for any $n \geq k$

$$|\frac{n^2-1}{2n^2+3} - \frac{1}{2}| \leq \frac{5}{n^2} \leq \frac{5}{k^2} < \epsilon$$

Example 15.5

$$\lim \frac{\sqrt{n}}{n+1} = 0$$

For any $n \in \mathbb{N}$:

$$|\frac{\sqrt{n}}{n+1} - 0| = \frac{\sqrt{n}}{n+1} \leq \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$$

So, $\forall \epsilon > 0$, let $k \in \mathbb{N}$ be such that $k > \frac{1}{\epsilon^2} \Rightarrow \epsilon^2 > \frac{1}{k} \Rightarrow \epsilon > \frac{1}{\sqrt{k}}$ Then for any $n \geq k$,

$$|\frac{\sqrt{n}}{n+1} - 0| \leq \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{k}} < \epsilon$$

Note: $\epsilon > \frac{1}{\sqrt{k}} \Leftrightarrow \epsilon^2 > \frac{1}{k} \Leftrightarrow k > \frac{1}{\epsilon^2}$

Proposition 15.6
If $x_n \to x$, then $|x_n| \to |x|$.

Proof. Let $\epsilon > 0$ be arbitrary. We know that $\exists k \in \mathbb{N}$ such that $|x_n - x| < \epsilon \quad \forall n \geq K$.

$$||x_n| - |x|| \leq |x_n - x| < \epsilon \quad \forall n \geq k$$

□

Side proof

Proof.

$$|x_n| = |x_n - x + x| \leq |x_n - x| + |x|$$
$$\Rightarrow |x_n| - |x| \leq |x_n - x|$$
$$\dots$$

□

Proposition 15.7
If $|x_n| \to 0$, then $x_n \to 0$.

Proof. Let $\epsilon > 0$. Then $\exists k \in \mathbb{N}$ such that

$$|x_n - 0| = |x_n| = ||x_n| - 0| < \epsilon \quad \forall n \geq k$$

□

Exercise 15.8. Show that if $a > 1$, then $\frac{1}{a^n} \to 0$.

Proof. If $a > 1$, then $a = 1 + r$ where $r > 0$.

$$a^n = (1+r)^n \geq 1 + rn \text{ Bernoulli}$$
$$\Rightarrow |\frac{1}{a^n} - 0| = \frac{1}{a^n} \leq \frac{1}{1+rn} \leq \frac{1}{rn}$$

For any $\epsilon > 0$, we can pick $K \in \mathbb{N}$ such that $K > \frac{1}{r\epsilon}$. Then $\forall n \geq k$

$$|\frac{1}{a^n} - 0| \leq \frac{1}{rn} \leq \frac{1}{rK} < \epsilon$$

□

Exercise 15.9. Show that if $a \in (-1, 1)$, then $a^n \to 0$.

Proof. First, if $a = 0$, we are done.

If $a > 0$, pick $b = \frac{1}{a}$. $a^n = \frac{1}{b^n} \to 0$.

If $a < 0$, then $0 < |a| < 1 \Rightarrow |a|^n \to 0 \Rightarrow |a^n| \to 0 \Rightarrow a^n \to 0$ □

Note 15.10.

$$\lim_{m\to\infty}\lim_{n\to\infty} a_{n,m} \neq \lim_{n\to\infty}\lim_{m\to\infty} a_{n,m}$$

Definition 15.11. Another definition of limit: We have $x_n \to x$ if and only if for any open set $x \in U$, $\forall \epsilon > 0$, $\exists K \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq K$.

($\Rightarrow$) First, suppose $x_n \to x$. Let $U \ni x$ where U is open. We know that $\exists \epsilon > 0$ such that $V_\epsilon(x) \subseteq U$. This means that $y \in \mathbb{R}$ such that $|x - y| < \epsilon \Rightarrow y \in U$.

$\exists K \in \mathbb{N}$ such that $|x_n - x| < \epsilon \quad \forall n \geq K$. So, if $n \geq K$, then $|x_n - x| < \epsilon \Rightarrow x_n \in V_\epsilon(x) \subseteq U$

($\Leftarrow$) Fix $\epsilon > 0$. We know that $V_\epsilon(x)$ is open. So, $\exists K \in \mathbb{N}$ such that $x_n \in V_\epsilon(x) \forall n \geq K \Rightarrow |x_n - x| < \epsilon \quad \forall n \geq K$

Proposition 15.12

Let x_n be a positive sequence. If lim...