# §1 Tutorial 2019-11-01. Normal Subgroups and Quotient Groups

## §1.1 Normal Subgroups

$N \subseteq G$ is normal if $gN = Ng, \quad \forall g \in G$. Equivalently $gNg^{-1} \subseteq N$ and $gNg^{-1} = N$.

$$gNg^{-1} = \{gng^{-1} : n \in N\}$$

Prooving that $gNg^{-1}$ is a subgroup.

Identity: $e \in N \Rightarrow$

$$geg^{-1} = gg^{-1} = e \in gNg^{-1}$$

Inverses: $gng^{-1} \in gNg^{-1}$. $(gng^{-1})^{-1} = gn^{-1}g^{-1}$. $n \in N$ so $gn^{-1}g^{-1} \in gNg^{-1}$

Closure: $gn_1g^{-1}, gn_2g^{-1} \in gNg^{-1}$

$$gn_1g^{-1}gn_2g^{-1} = gn_1n_2g^{-1}$$

$n_1n_2 \in N$ so $gn_1n_2g^{-1} \in gNg^{-1}$ ✓

## §1.2 Quotient Groups

Let $N \subseteq G$ be a normal subgroup. Then $G/N$ is a group with the operation $(aN)(bN) = (ab)N$

**Example 1.1**

Let $G = \mathbb{Z}$. Let $N = 24\mathbb{Z} = \{0, 24, 48, \dots\}$. $\mathbb{Z}/24\mathbb{Z} \cong \mathbb{Z}_{24}$

**Example 1.2**

$D_8 = \{id, r, \dots, r^7, s, sr, \dots, sr^7\}$
$N = \langle r^4 \rangle = \{id, r^4\}$ is a normal subgroup.
$|D_8/N| = 16/2 = 8$. Finding all the cosets of $N$ :

$$\begin{aligned} id \cdot N, &\quad sN \\ r \cdot N, &\quad srN \\ r^2 \cdot N, &\quad sr^2N \\ r^3 \cdot N, &\quad sr^3N \end{aligned}$$

**Exercise 1.3.** Let $G$ be a cyclic group where $G = \langle a \rangle$. Show that $G/N$ is cyclic.

Claim: $G/N = \langle aN \rangle$.

Let $bN \in G/N$. $b \in G$, so $b = a^k$ for some $k$. $bN = a^kN = (aN)^k \Rightarrow g/N = \langle aN \rangle$.

**Remark 1.4.** Let $G$ be a group, and $H, K \subseteq G$ be subgroups of $G$ such that $H \subseteq K \subseteq G$. Then $H$ being normal in $K$ and $K$ being normal in G does NOT imply that $H$ is normal in G.

**Example 1.5**

Consider the following:

$$\begin{aligned} D_4 &= \{id, r, r^2, r^3, \mu_1, \mu_2, \mu_3, \mu_4\} \\ K &= \{id, \mu_1, \mu_3, r^2\} \\ H &= \{id, \mu_1\} \end{aligned}$$

Show that $K$ is normal in $D_4$, and that $H$ is normal in $K$, but that $H$ is not normal in $D_4$.

**Note 1.6.** Tips for determining wether or not $H$ is normal in $G$:

1. If $G$ abelian, then all of its subgroups must be normal.
2. If $G$ is simple, then it has no normal non-trivial proper subgroups.
3. If $[G : H] = 2$, then $H$ is normal (we proved this in a previous assignment).
4. If all else fails, compute

$[D_4 : K] = 2$ so $K$ is normal in $D_4$. $[K : H] = 4/2 = 2$ so $H$ is normal in $K$.

Now to show that $H$ is not normal in $D_4$ with a counter example.

$$\begin{aligned} \mu_1 &= (24) \\ H &= \{(), (24)\} \\ r &= (1234) \in D_4 \\ rH &= \{(1234, (12)(34)\} \\ Hr &= \{(1234), (14)(23)\} \end{aligned}$$

Therefore $rH \neq Hr \Rightarrow H$ is not normal in $D_4$.

**Exercise.** Let $G$ be a group, and $N \subseteq G$ be a normal subgroup. Let $gN \in G/N$.

(a) Show that $|gN| = n$ in $G/N$ where $n$ is the smallest natural number such that $g^n \in N$.
Observe that $(gN)^n = g^n N = eN \Leftrightarrow g^n \in N$. Therefore the order of $(gN)$ is the smallest of $n \in \mathbb{N}$ such that $g^n \in N$.

(b) Give an example where $|gN|$ in $G/N$ is <u>strictly</u> smaller that $|g|$ in $G$.
Let $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$
Let $N = \langle 2 \rangle = \{0, 2\}$
Elements of $G/N$:

$$\begin{aligned} 0 + N &= \{0, 2\} = 2 + N \\ 1 + N &= \{1, 3\} = 3 + N \end{aligned}$$

$$\begin{aligned} &\mathbb{Z}/3\mathbb{Z} = \\ 0 + 3\mathbb{Z} &= \{0, 3\} \\ 1 + 3\mathbb{Z} &= \{1, 4\} \\ 2 + 3\mathbb{Z} &= \{2, 5\} \end{aligned}$$