§1 Lecture 03-11

§1.1 Inner product spaces

V over $F = \mathbb{R}, \mathbb{C}$.

1. Positivity:

$$\langle v, v \rangle \in \mathbb{R} \geq 0$$
$$\langle v, v \rangle = 0 \Leftrightarrow v = 0$$

2. Rather than imposing bilinearity, we were lead to impose hermition linearity.

3. Basic symmetry assumption requiring the $\langle v, w \rangle = \overline{\langle w, v \rangle}$.

4. We defined norm of v as

$$||v|| = \sqrt{\langle, v, v \rangle}$$

Theorem 1.1 (Cauchy-Schwartz Inequality)
For all $v, w \in V$,

$$|\langle v, w \rangle| \leq ||v|| \cdot ||w||$$
$$\text{With equal if} \quad \text{span}(v, w) \text{ is one dimensional}$$

Proof. We can assume without loss of generality that $v \neq 0$.

Positivity implies that for all $\lambda \in F$,

$$\langle \lambda v + w, \lambda v + w \rangle \in \mathbb{R} \geq 0$$
$$= |\lambda|^2 \langle v, v \rangle + \lambda \langle v, w \rangle + \overline{\lambda} \langle w, v \rangle + \langle w, w, \rangle \geq 0$$
$$|\lambda|^2 \langle v, v, \rangle + 2Re(\lambda \langle v, w \rangle) + \langle w, w, \rangle \geq 0, \quad \forall \lambda \in F$$

1. If $F = \mathbb{R}$.

$$f(\lambda) = \lambda^2 \langle v, v \rangle + 2 \langle v, w \rangle \lambda + \langle w, w \rangle \geq 0, \quad \forall \lambda \in \mathbb{R}$$
$$\text{This is a quadratic so either it has a root or it doesn't}$$
$$\Rightarrow (2 \langle v, w \rangle)^2 - 4 \langle v, v, \rangle \langle w, w, \rangle \leq 0$$
$$\text{with equal if there is a root}$$
$$\Rightarrow \langle v, w \rangle^2 \leq ||v||^2 ||w||^2$$
$$\Rightarrow |\langle v, w \rangle| \leq ||v|| ||w||$$
$$\text{with equal if } \exists \lambda_0 \text{ such that } f(\lambda) = 0$$
$$i.e. \ \langle \lambda_0 v + w, \lambda_0 v + w \rangle = 0 \Rightarrow \lambda_0 v + w = 0 \Rightarrow \text{span}(v, w) = \text{span}(v) \ \checkmark$$

2. If $F = \mathbb{C}$. Assume that $\lambda \langle v, w \rangle \in \mathbb{R}$.

$$|\lambda|^2 \langle v, v \rangle \pm 2|\lambda| |\langle v, w \rangle| + \langle w, w, \rangle \geq 0$$
$$\text{Doesn't matter on the sign because b term squared in discriminant}$$
$$4|\langle v, w \rangle|^2 - 4 \langle v, v \rangle \langle w, w \rangle \leq 0$$
$$|\langle v, w \rangle|^2 \leq ||v||^2 ||w||^2$$

The rest follows like the real case.

□

§1.2 Properties of $||v||$

1.

$$||v|| \in \mathbb{R} \geq 0$$

2.

$$||\lambda v|| = |\lambda| \cdot ||v||$$

3.

$$||v + w|| \leq ||v|| + ||w|| \text{ with equality if } (v, w) \text{ are colinear.}$$

Proof.

1. By definition
2.

$$||\lambda v|| = \sqrt{\langle\, \lambda v, \lambda v \,\rangle} = \sqrt{\lambda\bar{\lambda}}\sqrt{\langle\, v, v \,\rangle} = |\lambda| \cdot ||v||$$

3.

$$\begin{aligned} ||v+w||^2 = \langle\, v+w, v+w \,\rangle = ||v||^2 + 2Re\,\langle\, v, w \,\rangle + ||w||^2 \\ \leq ||v||^2 + 2|\,\langle\, v, w \,\rangle\,| + ||w||^2, \quad \text{because } Re(\lambda) \leq |\lambda| \\ \leq ||v||^2 + 2||v||||w|| + ||w||^2 = (||v|| + ||w||)^2 \\ \Rightarrow ||v+w|| \leq ||v|| + ||w|| \end{aligned}$$

□

Definition 1.2 (Orthogonality)**.** Two vectors are orthogonal if $\langle\, v, w \,\rangle = 0$.

Definition 1.3 (Orthonormal Basis)**.** An orthonormal basis of V is a basis Σ of V such that for all $v, w \in \Sigma$,

$$\langle\, v, w \,\rangle = \begin{cases} 0, & \text{if } v \neq w \\ 1, & \text{if } v = w \end{cases}$$

Example 1.4

1. $V = \mathbb{R}^n$ or $\mathbb{C}^n$ with dot product.

$$e_1 = (1, 0, \ldots, 0),\ e_2 = (0, 1, \ldots, 0),\ e_n = (0, 0, \ldots, 1)$$

   is an orthonormal basis.

2. $V = P_n([0,1])$, the space of polynomials of degree $\leq n$ with

$$\langle\, f, g \,\rangle = \int_0^1 f(t)\overline{g(t)}dt$$

   Then

$$\Sigma = (1, x, x^2, \ldots, x^n)$$

   is not an orthonormal basis because inner product is not zero pairwise between these elements

Theorem 1.5

If V is a finite-dimensional inner product space over $\mathbb{R}$ or $\mathbb{C}$, then it has an orthonormal basis.

Proof. We will prove something more precise. Let $(v_1, \ldots, v_n)$ be a basis for V, then there is $(e_1, \ldots, e_n)$ orthonormal with

$$\operatorname{span}(e_1, \ldots, e_j) = \operatorname{span}(v_1, \ldots, v_j), \quad j = 1, \ldots, n$$

We will prove the existence of $(e_1, \ldots, e_n)$ by induction on j.

1. Base case $j = 1$, let $e_1 = v_1/||v_1||$.

2. Inductive step $j \to j + 1$. Assume that we have an orthonormal collection $e_1, \ldots, e_j$ with $\operatorname{span}(e_1, \ldots, e_j)$, with $\operatorname{span}(e_1, \ldots, e_j) = \operatorname{span}(v_1, \ldots, v_j)$. We then must define e_{j+1}.

$$\widetilde{e_{j+1}} = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_i e_i + \lambda_{j+1} v_{j+1}$$

Want $\widetilde{e_{j+1}} \perp e_i, \quad i = 1, \ldots, j$.

$$0 = \langle \widetilde{e_{j+1}}, e_i \rangle = \lambda_i + \lambda_{j+1} \langle v_{j+1}, e_i \rangle$$
$$\lambda_i = -\lambda_{j+1} \langle v_{j+1}, e_i \rangle$$

Set $\lambda_{j+1} = 1$, then $\lambda_i = - \langle v_{j+1}, e_i \rangle$.

$$\widetilde{e_{j+1}} = v_{j+1} - \langle v_{j+1}, e_1 \rangle e_1 + \cdots + \langle v_{j+1}, e_j \rangle e_j$$

is orthogonal to $e_1, \ldots, e_j$ and hence $v_1, \ldots, v_j$.

$$e_{j+1} = \widetilde{e_{j+1}} \frac{1}{||\widetilde{e_{j+1}}}$$

□