§1 Lecture 01-13

Linear transformation is an additive group homomorphism that preserves
an Algebra is a vector space and ring at the same time.

$$\mathrm{End}_F(V) = \hom_F(V, V)$$

This is both a vector space over F and a ring where multiplication is the composition of functions.

Definition 1.1 (Dual Space)**.** $V^* = \hom_F(V, F)$

§1.1 Bases

∇ = vector space

Definition 1.2 (Collection Linear Independence)**.** A collection $\Sigma \subset V$ is linearly independent if, $\forall v_1, \ldots, v_n \in \Sigma$ (distinct) satisfies

$$\lambda v_1 + \cdots + \lambda_n v_n = 0 \Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$$

Can only talk about finite sums

Definition 1.3 (Spanning Set)**.** A collection Σ spans V if,

$$\forall v \in V, \quad \exists v_1, \ldots, v_n \in \Sigma, \quad \lambda_1, \ldots, \lambda_n \in F v = \lambda_1 + v_1 + \cdots + \lambda_n + v_n$$

Definition 1.4 (Basis)**.** A basis is a set $\Sigma \subset V$ that is both linearly independent and spans V.

Proposition 1.5

If Σ is a basis for V, then, for all $v \in V$, there is a unique

$$v_1, \ldots, v_n \in \Sigma, \quad \lambda_1, \ldots, \lambda_n \in F \text{ s.t. } v = \lambda_1 v_1 + \cdots + \lambda_n v_n$$

Proof. Existence of $(v_1, \ldots, v_n, \lambda_1, \ldots, \lambda_n) \Leftarrow \Sigma$ spans V.

Uniqueness $\Leftarrow$ linear independence of Σ. $\square$

Corollary 1.6
The vector space V is isomorphic to F^n or $F_0(\Sigma, F)$

Proof. We set up a linear transformation.

$$\phi : F_0(\Sigma, F) \to V$$
$$f \to \sum_{v \in \Sigma} f(v) \cdot v$$

Need to check

1. ϕ is linear
2. ϕ is injective
3. ϕ is surjective

$\square$

Theorem 1.7 (Every vector space over F has a basis)*Proof.* Let V be a vector space. Let L be a collection of all subsets of V that are linearly independent. Partial ordering on L is given by inclusion.

Completeness of ordering. If $\{A_{\alpha \alpha \in I}$ is a chain, $A = \sum_\alpha A_\alpha$.

Claim: $A \in L$. If $v_1 \dots v_n \in A$, $v_j \in A_\alpha$. $\exists n$ such that $v_1, \dots, v_n \in A_{\alpha_N}$ and $v_1, \dots, v_n$ are linearly independent.

Zorn's Lemma $\Rightarrow \exists$ a maxmial element $\Sigma \in L$. Claim: Σ spans V. Otherwise $\exists$ v which is not in span(Σ).

$$\Sigma \cup \{v\} \supsetneq \Sigma$$

and is linearly independent. $\Sigma \cup \{v\} \in L$. $\square$

Not as useful as you might think at first because basis is obtained in a non constructive way.

Definition 1.8. A set endowed with a partial ordering satisfies the maximal chain condition if, for all subsets of S, for which the ordering is a total ordering (chain condition), every totally ordered subset $A \subseteq S$ has an upper bound, $\exists B \in S$ such that $a \leq B$

A partially ordered set S is complete if, for all chains $A \subset S$, $\exists B \in S$ such that $a \leq B, \quad \forall a \in A$. Partial ordering means a relation less than or equal. Anti symmetric, transitive. Think of it as a directed graph with no back tracking. Some elements are not ordered.

Every complete partially ordered set has a maximal element. i.e. $\exists s \in S$ s.t. $s \leq a \Rightarrow a = s, \quad \forall a \in S$

Chain stands for a totally ordered subset.

Axiom of choice. If you have an infinite collection of sets $\{S_\alpha\}_{\alpha \in I}$. Then there exists

S' containing one $s_\alpha \in S_\alpha (\alpha \in I)$

Example 1.9

$F[x]$ is the ring of polynomials with coefficients in F. Basis: $\Sigma = \{1, x, x^2, x^3, \ldots, x^n, \ldots\}$

Example 1.10

$F[[x]] = \{a_0 + a_1 x + a_2 x^2 + \ldots, \quad a_i \in F$. The difference between $F[x]$ and $F[[x]$ is that elements in $F[[x]]$ are infinite.

Infinite sums don't make sense in algebra.