§1 Lecture 01-27

Definition 1.1 (Grassmannian)**.** The Grassmannian of k-dimensional subspace of V is the collection of k-dimensional supspaces of V.

Note 1.2. (V, k). If $\dim(V) = n$, then (n, k)

If F is a finite field, then (n, k) is a finite set. $\#F = q$.

Question: What is $\#(n, k)$. Strategy is to fix $V \simeq F^n$.

Each subspace could have multiple basis so you might over count. Let WV be a subspace of $\dim(k)$. Orbit.

$$G = \mathrm{Aut}_F(V).Gactstransitivelyon(\text{V, k}).$$
$$\#(V, k) = \#G/stab_G(W)$$

Definition 1.3 (Action of a group G)**.** An action of a group G.

Combinatorics is usually concerned with counting the cardinality of finite sets.

Finite sets of cardinality n seem to resonate with a vector space of dimension n.

$$S \mapsto F(S, F) = functionsS \to F$$

"How many sets of size k are there in a set of size n?" resonates with "How many spaces of dimension k are there in a space of dim n where $\#F = q$.

$$\binom{n}{k}$$
$$\binom{n}{k}_q$$

§1.1 Determinants

Definition 1.4 (Linear functional)**.** A linear form (or linear functional) is a linear transfromation

$$l : V \to F$$

Definition 1.5 (Bilinear Form)**.** A bilinear form is a function $f : V \times V \to F$ such that $f(v, w)$ is linear in v when w is fixed, and linear in w when v is fixed.

$$f(v_1 \lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 f(v_1 w_1) + \lambda_2 f(v_1 w_2)$$
$$f(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 f(v_1, w) + \lambda_2 f(v_2, w)$$

An example of such a form is the dot product.

Definition 1.6 (k-linear form)**.** A k-linear form is a function

$$f : V \times V \times \cdots \times V \to F$$

which is linear in each argument, while others are fixed.

Definition 1.7. A k-multilinear form on V is symmetric, (resp alternating).

If $f(v_{\sigma 1}, v_{\sigma 2}, \ldots, v_{\sigma k}) = f(v_1, \ldots, v_k)$ where $\sigma \in S_k$.

Example 1.8

Dot product $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a symmetric bilinear form. $F^n \times F^n \to F$.

Example 1.9

Cross product $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$.

$$(x_1, y_1, 0) \times (x_2, y_2, 0) = (0, 0, x_1y_2 - y_1x_2)$$
$$(x_1, y_1) \times (x_2, y_2) = x_1y_2 - y_1x_2$$

The collection of all (symmetric or alternating) k-multiliear functions on V is an F vector space.

Lemma 1.10

Suppose V has basis $(e_1, \ldots, e_n)$. Then a bilinear form is completely determined by $f(e_i, e_j)$

$$M_f = (f(e_i, e_j))$$

A k-multilinear form is specified by

$$(f(e_{i_1}, e_{i_2}, \ldots, e_{i_k}))_{1 \leq i_1, \ldots, i_k \leq n}$$

§1.2 Alternating forms

Easy properties of alternating forms.

$$f(v_1, \ldots, v_k) = 0$$

if $v_i = v_j$ where $i \neq j$ because $f(\cdots) = -f(\cdots)$. We're using that $\lambda = -\lambda \Rightarrow \lambda = 0$.

$$\begin{aligned} &f(v_1, \ldots, v_{j-1}, v_j + \sum_{i \neq j} \lambda_i v_i, v_{j+1}, \ldots, v_k) \\ &= f(v_1, \ldots, v_j, \cdots, v_k) \end{aligned}$$

Proposition 1.11

A k-multilinear form is completely determined by its values

$$\{f(ei_1, e_{i_2}, \ldots, e_{i_j})\}_{1 \leq i_1 < i_2 < \cdots < i_k \leq n}$$