§1 Lecture 02-24

Definition 1.1 (Bilinear Forms)**.** A bilinear form $B : V \times V \to F$ is said to be left non degenerate if $B(v, w) = 0, \ \forall w \in V \Rightarrow v = 0$.

Note 1.2.

$$B(v, w) = \langle v, w\rangle$$

Key remark: A non-degenerate bilinear form induces a linear injection

$$\begin{aligned} l : V &\to V^* \\ v &\mapsto l_v \\ l_v(w) &= \langle v, w\rangle \end{aligned}$$

Now to show the following:

1. l_v is indeed a linear transformation (follows from the linearity of $\langle , \rangle$ in the second variable)
2. The assignment $v \mapsto lv$ is linear (follows from the linearity of $\langle , \rangle$ in the first variable.

Lemma 1.3
If $\dim V < \infty$, then l is an isomorphism between V and V^*.

Proof. $\langle , \rangle$ is left-nondegenerate $\Rightarrow l : V \hookrightarrow V^*$ is injective.

The rank-nullity theorem implies that since $\dim V = \dim V^*$, l is also surjective. ☐

Is it possible to classify all possible bilinear forms on V, up to isomorphism?

If V is finite dimensional, we can choose a basis $\sum = (e_1, \ldots, e_n)$ for V such that

$$\begin{aligned} v &= \sum_{i=1}^{n} x_i e_i \\ w &= \sum_{j=1}^{n} y_j e_j \\ \langle v, w\rangle &= \langle \sum x_i e_i, \sum y_j e_j\rangle \\ &= \sum_{i,j=1}^{n} x_i y_j \langle e_i, e_j\rangle \end{aligned}$$

Definition 1.4. The pairing matrix associated to $B(v, w) = \langle v, w\rangle$, and the basis $\sum$.

$$M_{B,\sum} = (\langle e_i, e_j\rangle)_{i,j=1,\ldots,n}$$

$$\langle v, w\rangle = (x_1, \ldots, x_n) M_{B,\sum} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

The most general bilinear form on F^n is given by a matrix M, by

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = (x_1, \ldots, x_n) M \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Lemma 1.5

$\langle , \rangle$ is left-nondegenerate $\Leftrightarrow M_{B,\sum}$ is invertible.

Proof. Left as exercise. □

§1.1 Change of Basis

Let $\sum = (e_1, \ldots, e_n)$ and $\sum' = (e_1', \ldots, e_n')$ be two bases for V. How are $M_{B,\sum}$ and $M_{B,\Sigma'}$ related?

$$\begin{pmatrix} e_1' \\ \vdots \\ v_n' \end{pmatrix} = P_i \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

$$p \in M_n(F), \text{ invertible}$$

$$m_{B,\Sigma} = \langle \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}, (e_1, \ldots, e_n) \rangle$$

$$M_{B,\Sigma'} = \langle \begin{pmatrix} e_1' \\ \vdots \\ e_n' \end{pmatrix}, (e_1', \ldots, e_n') \rangle$$

$$(e_1', \ldots, e_n') = (e_1, \ldots, e_n) P^t$$

$$P = (a_{ij}) \quad p^t = (a_{ji})$$

$$M_{B,\Sigma'} =$$

$$M_{B,\Sigma'} = P M_{B,\Sigma} P^t$$

Corollary 1.6

Two matrices M_1 and M_2 represent the same bilinear form $\Leftrightarrow$ there exists an invertible linear transformation P such that $M_1 = PM_2P^t$

Isomorphism classes of linear transformations on $F^n = M_n(F)/\operatorname{GL}_n(F)$ where the group $\operatorname{GL}_n(F)$ acts on the set $M_n(F)$ by conjugation $M^g = gMg^{-1}$.

Isomorphism classes of bilinear forms we likewise identified with

$$M_n(F)/\operatorname{GL}_n(F)$$

, but where the action of $_n(F)$ on $M_n(F)$ is very different

$$g * M = gMg^t$$

Example 1.7

1. Orbit of I_n for the conjugation action $= \{I_n\}$.
2. Orbit of I_n for the second action is the set of $\{pp^t, p \in \operatorname{GL}_n(F)\}$

Exercise 1.8. There are no orbits of size 1 for the action $M \mapsto gMg^t$.

Definition 1.9. A vector space equipped with a non-degenerate bilinear form B is called a quadratic space (V, B).

An isomorphism $T : (V_1, B_2) \to (V_2, B_2)$ is the natural notion. A linear isomorphism $T : V_1 \to V_2$, $\forall v, w \in V_1$,

$$\langle v, w\rangle_{B_1} = \langle Tv, Tw\rangle_{B_2}$$

The adjoint of a linear transformation $T : V \to V$ when V is a quadratic space, endowed with a nondegerenate form.

$$\begin{gathered} T : V \to V \\ T^* : V^* \to V^* \\ T^*(l) = l \circ T \end{gathered}$$

The adjoint of T on the quadratic space V is the linear transformation defined by

$$\begin{gathered} T^*(lv) = l_{T^*(v)} \\ T^*(lv)(w) = l'_{T^*v}(w) \\ lv \circ T(w) = \langle T^*v, w\rangle \\ \langle v, T(w)\rangle \end{gathered}$$

$$\langle v, Tw\rangle = \langle T^*v, w\rangle$$