Math 254 Notes

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§1 Lecture 10-02

§1.1 Open and Closed Sets

Definition 1.1. Open interval does not contain any of its boundary points. Closed interval contains all of its boundary points.

Theorem 1.2

Every open interval is open. This is not self evident because definition of open is very specific.

Proof. Let I be an open interval. We need to show that I is always open.

1. Case: $I =]a, \infty]$

   Let $x \in I$ be arbitrary. Let $\epsilon = x - a$. Then $V_\epsilon(x) =]x - \epsilon, x + \epsilon] =]a, 2x - a] \subset]a, \infty]$. i.e. $V_\epsilon(x) \subset]a, \infty] \Rightarrow I$ is open.

2. Case: $I =]-\infty, b]$. Do yourself. Let $x \in I$ be arbitrary. Let $\epsilon = b - x$. Then $V_\epsilon(x) =]x - \epsilon, x + \epsilon[=]2x - b, b[\subseteq]-\infty, b[$. Therefore I is open.

3. Case: $I =]a, b[$.

   Let $\epsilon = \min \{x - a, b - x\} > 0$.

   Then $V_\epsilon(x) =]x - \epsilon, x + \epsilon[$.

   Note that $x + \epsilon \leq x + (b - x) = b$ and $x - \epsilon \geq x - (x - a) = a \Rightarrow]x - \epsilon, x + \epsilon] \subset]a, b[$. Therefore I is open and therefore any open interval is open.

□

Theorem 1.3

Every closed interval is closed.

Proof. Let I be a closed interval. We need to show that $\mathbb{R} \setminus I$ is open.

1. Case: $I = [a, \infty] \Rightarrow \mathbb{R} \setminus I =]-\infty, a[$ which as an open interval is open $\Rightarrow$ I is closed.

2. Case: I = $[-\infty, b]$. do yourself. $\mathbb{R} \setminus I =]b, \infty[$ which is an open interval $\Rightarrow$ I is closed.

3. Case: $I = [a, b] \Rightarrow \mathbb{R} \setminus I =]-\infty, a] \cup]b, \infty]$. Union of open with open is open so I is closed. Therefore any closed interval is closed.

□

Theorem 1.4

a. Let J be an index set and let u_j be open for all $j \in J$. Then

$$\bigcup_{j \in J} u_j$$

is open. "Arbitrary unions of open sets are open.

Proof. Let $u = \bigcup_{j \in J} u_j$. Let $x \in u$ be arbitrary $\Rightarrow \exists j \in J$ such that $x \in u_j$ open $\Rightarrow \exists \epsilon > 0 : V_\epsilon(x) \subset u_j \subset U$. Can't follow. Basically uses definition of union and definition of openness. □

b. Let $u_1, \ldots, u_n$ be open. Then

$$\bigcap_{j=1}^{n} u_j$$

is open. "Finite intersections of open sets are open.

c. Finite unions of closed sets are closed. Let $v_1, \ldots, v_n$ be closed, then $\bigcup_{j=1}^{n} v_j$ is closed.

Proof. Let $v_1, \ldots, v_n$ be closed, then $\mathbb{R} \setminus v_1, \ldots, \mathbb{R} \setminus v_n$ are all open. $\Rightarrow \mathbb{R} \setminus v_1 \cap \cdots \cap \mathbb{R} \setminus v_n$ is open. By demorgans law this equals $\mathbb{R} \setminus (v_1 \cup \cdots \cup v_n)$ is closed. $\Rightarrow v_1 \cup \cdots \cup v_n$ is closed. (because closed is comp lement of open) □

d. Arbitrary intersections of closed sets are closed. Let J be an arbitraryIndex set and let $\forall j \in J v_j$ be closed, then $\cap_{j \in J} v_j$ is closed.

Proof. Let v_j be closed for all $j \in J$. $\Rightarrow \mathbb{R} \setminus v_j$ is open. $\Rightarrow \cup_{j \in J}(\mathbb{R} \setminus v_j)$ is open. Demorgans law gives us that $\mathbb{R} \setminus \cap_{j \in J} v_j$ is open. Therefore $\cap_{j=1} v_j$ is closed. □

Example 1.5

Every finite subset of $\mathbb{R}$ is closed.

Proof. Let's first consider $\{x\}$. For some $x \in \mathbb{R}$, $\{x\} =]x, x]$ is closed. Finite unions of singleton sets are thus closed $\Rightarrow$ all finite subsets of $\mathbb{R}$ are closed. ☐

Example 1.6

$S_1 = \{\frac{1}{n} : n \in \mathbb{N}\}$ is NOT closed.

Proof. Assume it is closed. Then the comp lement u is open. we have $0 \in u$, but every ϵ neighborhood V_ϵ intersects S_1 and is thus not contained in $u \Rightarrow u$ is not open. ☐

Example 1.7

$S_2 = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is closed.

Definition 1.8. Boundary.

Let $S \subset \mathbb{R}$. A point $x \in \mathbb{R}$ is called a boundary point of S if every epsilon neighborhood cenetered around x intersects both S and the comp lement of S.

A point $x \in \mathbb{R}$ is not a boundary point of S if $\exists \epsilon > 0 : V_\epsilon(x) \cap S = \varnothing \vee V_\epsilon(x) \cap (\mathbb{R} \setminus S) = \varnothing$.

Example 1.9

$S =]a, \infty]$ Claim:

§2 Lecture 10-07

Example 2.1

Definition 2.2. Let $S \subseteq \mathbb{R}$. The interior $\mathring{S}$ "S with dot on top" or int(S) is defined as:

$$\mathring{S} = \bigcup_{U \subset S, \quad U open} U$$

Note that $\mathring{S}$ is open as a union of open sets. It is the largest open subset of S.

Definition 2.3. Let $S \subset \mathbb{R}$. The closure $\tilde{S}$ of S is defined as:

$$\tilde{S} = \bigcap_{V \supset S, \quad V closed} V$$

Note that $\tilde{S}$ is closed as an intersection of closed sets.

Theorem 2.4

Let $S \subset \mathbb{R}$. Then $\mathring{S} = S \setminus \delta S$

Proof. 1. " $\subseteq$ ". Let $x \in \mathring{S} \Rightarrow \exists U$ open $, U \subset S$ with $x \in U$.

Thus $\exists \epsilon > 0 s.t. V_\epsilon(x) \subset U \subset S$.

□

Theorem 2.5

Let $S \subset \mathbb{R}$. Then $\mathbb{R} \setminus \tilde{S} = int(\mathbb{R} \setminus S)$.