§1 Lecture 01-24

Recommend Colmez. Drawback is that it is in french.

Let V be a finite dimensional vector space. Let B be a basis for V. $B = (v_1, \ldots, v_n) \in V^n$. $v \in V$ has coordinates

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in F^n$$

if $v = Bx$

If T is a linear transformation,

$$T : V \to V$$
$$V \simeq_B F_1^n$$
$$V \simeq_{B'} F_2^n$$
$$F_1^n \to_{M_{T,B}} F_2^n$$

Fact 1.1. If B and B' are different bases for V, then the matrices $T_{T,B}$ and $T_{T,B'}$ are conjugate. i.e. $\exists P \in \mathrm{GL}_n(F)$ such that $M_{T,B'} = PM_{T,B}P^{-1}$

§1.1 Determinant

Proposition 1.2

There is a unique function $\det : M_n(F) \to F$ satisfying:

1. det is multilinear. i.e. it is a linear function in each row with all other rows being fixed.
2. det is alternating, namely, the determinant changes sign after interchanging two rows.

$$\det(M^\sigma) = \mathrm{sign}(\sigma)\det(M), \sigma \in S_n$$

§1.2 Proof of existence and uniqueness

$$\det(AB) = \det(A)\det(B)$$
$$\det(A+B) = ???$$
$$\det(PAP^{-1}) = \det(P)\det(A)\det(P)^{-1} = \det(A)$$

Definition 1.3. The determinant of $T : V \to V$ is the determinant of any matrix representing T.

Definition 1.4 (Trace)**.** Trace $(A) = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$ where $A = (a_{ij})$.

$$\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$$
$$\mathrm{Tr}(AB) = ??$$

Lemma 1.5

$$A \cdot B = \sum_{i,j} a_{ij} b_{ij}$$
$$\text{Tr}(AB) = A \cdot B^T = \sum_{i,j} a_{ij} b_{ji}$$
$$\text{Tr}(BA) = B \cdot A^T = \sum_{i,j} b_{ij} a_{ji}$$
$$\text{Tr}(AB) = \text{Tr}(BA)$$
$$\text{Tr}(PAP^{-1}) = \text{Tr}(AP^{-1}P) = \text{Tr}(A)$$

So trace is also invariant over conjugation.

Definition 1.6. The trace of $T : V \to V$ is the trace of any matrix representing T.

Exercise 1.7. Show that $\text{End}_F(V) = \text{End}_F(V)$

1. First show that $M_n(F) \simeq M_n(F)^*$ where $A \mapsto (x \mapsto \text{Tr}(AX))$.
2. Then show that $\text{End}_F(V) \simeq \text{End}_F(V)^*$. Solution the mapping $T \mapsto (U \mapsto \text{Tr}(TU))$.

If $T : V \to V$ is a linear transformation, study the structure of T acting on V (nullspace, eigenspaces, eigenvalues, characteristic polynomial, minimal polynomial.)

$$F[T] = \{a_0 + a_1 T + a_n T^n + \cdot\} \in \text{End}_F(V) \subseteq \text{End}_F(V)$$

$F[T]$ is a sub F-algebra of $\text{End}_F(V)$.

Remark 1.8. If $\dim(V) > 1$, then $F[T] \neq \text{End}_F(V)$.

$F[T]$ is a quotient ring of $F[x]$, the ring of polynomials. Ther eis a natural ring homomoprhism

$$\varphi_T : F[x] \to F[T] \subseteq \text{End}(V)$$
$$p(x) \mapsto P(T)$$

But $F[X]$ is infinite dimensional. So this means that there is a nontrivial kernal because $F[T]$ is not infinite dimensional.

Definition 1.9 (Defining Ideal)**.** The kernel of φ_T is called the defining ideal of T.

$I_T = \ker(\varphi_T)$.

I_T is generated by a unique polynomial in $F[x]$ which is monic. $I_T = (P_T(X))$.

$$P_T(X) = X^+ a_{m-1} x^{m-1} + \cdots + a_0, \quad a_j \in F$$

What is $P_T(x)$?

$$\varphi_T(p_t) = 0$$
$$\varphi_T(T) = 0$$

P_T is called the minimal polynomial. $f \in F[x]$. $f(T) = 0$. $p_T(x) | f(x)$.