§1 Lecture 02-17

§1.1 Dunford Decomposition

$$T : V \to V$$

If $p_T(x) = (x - \lambda_1)^{e_1}(\cdots)(x - \lambda_r)^{e_r}$, then $\exists\ D, N$ where D and N commute, D is diagonalizable, N is nilpotent, and $T = D + N$.

Definition 1.1 (Nilpotent)**.** Nilpotent if $N^d = 0$ for some $d \in \mathbb{N}$.

Application. Given $g(x) \in F[x]$, evaluate $g(T)$.

$$g(D+N) = g(D) + g'(D)N + \frac{g''(D)}{2!}N^2 + \cdots + \frac{g^j(D)}{j!}N^j + \frac{g^{e-1}(D)}{(e-1)!}N^{e-1}$$

where $N^e = 0$.

Relative to an eigenbasis, we have

$$D \sim \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{pmatrix}$$
$$g(D) \sim \begin{pmatrix} g(\lambda_1) & 0 \\ 0 & g(\lambda_n) \end{pmatrix}$$

More generally if g is defined by a convergent power series, and $\lambda_1, \ldots, \lambda_n$ belong to the domain of convergence, we have

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$
$$g(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_n T^n + \cdots$$

If $g(x)$ is $(e-1)$ times differentiable, and $\lambda_1, \ldots, \lambda_n$ belong to the domain of convergence for $g(x)$, then

$$g(T) = g(D+N) = \sum_{j=0}^{e-1} \frac{g^j(D)}{j!} N^j$$

Example 1.2

$$g(x) = e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$$
$$e^D \sim \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_n} \end{pmatrix}$$

Focusing on a single generalized eigenbasis, what is the "nicest" basis for V_λ.

$T = \lambda + N$. We can choose a basis for V in such a way that

$$\text{Upper Triangular } M_{T,B} = \begin{pmatrix} \lambda & x \\ 0 & \lambda \end{pmatrix}$$

A jordon subspace W for N is a subspace of V that admins a cyclic vector. i.e. a vector $v \in W$ such that $v, Nv, \dots, N^{e-1}v$ spans W.

Relative ot the basis $N^{e-1}v, N^{e-2}v, \dots, Nv, v$, N is represented by

$$J_{0,e} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

T is represented by

$$J_{\lambda,e} = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

Definition 1.3 (Jordan Matrix)**.** The vector $J_{\lambda,e}$ is called the Jordan matrix, or Jordan block of size e and eigenvalue λ.

Theorem 1.4 (Jordon Decomposition)

If $N : V \to V$ is a nilpotent endomorphism, then V can be decomposed into a direct sum of Jordon subspaces

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_j$$

Proof. Pianful. This decomposition is not unique.

Remark 1.5. Let $V_0 \subseteq V$. V_0 need not admit an N-stable complement.

□

Theorem 1.6 (Concrete Form)

If M is a matrix with char polynomial $(x-\lambda_1)^{e_1} \cdots (x-\lambda_r)^{e_r}$, then M is similar to a matrix of the form: Wait wait start over.

If $p_T(x) = (x-\lambda)^e$ and $f_T(x) = (x-\lambda)^d$, $e \leq d$, then $\exists$ basis B for V such that

$$M_{T,B} = \begin{pmatrix} J_{\lambda,e_1} & 0 & 0 \\ 0 & J_{\lambda,e_2} & 0 \\ 0 & 0 & J_{\lambda,e_r} \end{pmatrix}$$

$$e_1 + e_2 + \cdots + e_r = d$$
$$\max(e_1, \dots, e_r) = e$$
$$T(J_{\lambda,e} - \lambda I)^e = 0$$