§1 Lecture 11-18

Definition of continuity: $\forall \epsilon > 0$, $\exists \delta > 0 : f(V_\delta(x_0) \cap A) \subseteq V_\epsilon(f(x_0))$

Remark 1.1. Let x_0 be an isolated point of A. Then any function $f : A \to \mathbb{R}$ is continuous at x_0.

Proof. Let $f : A \to \mathbb{R}$ and let $\epsilon > 0$. Since x_0 is an isolated point of A, $\exists \delta : V_\delta(x_0) \cap A = \{x_0\}$.

Then, $f(V_\delta(x_0) \cap A) = f(\{x_0\}) = \{f(x_0)\}$. Thus f is continuous at x_0. □

Theorem 1.2 (Algebraic Rules for Continuity)
Let $f, g : A \to \mathbb{R}$ and let $x_0 \in A$ be a cluster point of A. f, g is continuous at x_0, then:

(a) $f + g$ is continuous at x_0.

(b) $f \cdot g$ is continuous at x_0.

(c) $f - g$ is continuous at x_0.

(d) f/g is continuous at x_0 if $\forall x \in A$, $g(x) \neq 0$.

Proof.

(a) Let (x_n) be a sequence in A with $\lim(x_n) = x_0$.

Since f and g are continuous at x_0, we have that $\lim(f(x_n)) = f(x_0)$ and $\lim(g(x_n)) = g(x_0)$.

Thus,

$$\lim((f+g)(x_0)) = \lim(f(x_0) + g(x_0))$$
$$= \lim(f(x_n)) + \lim(g(x_n)) = f(x_0) + g(x_0) = (f+g)(x_0)$$
$$\Rightarrow f + g \text{ is continuous at } x_0$$

Alternatively, we can use the limits of functions. f, g are continuous at x_0 so

$$\lim_{x \to x_0} f(x) = f(x_0)$$
$$\lim_{x \to x_0} g(x) = g(x_0)$$

Thus

$$\lim_{x \to x_0} [(f+g)(x)] = \lim_{x \to x_0} [f(x) + g(x)]$$
$$= \lim_{x \to x_0} f_(x) + \lim_{x \to x_0} g(x) = f(x_0) + g(x_0) = (f+g)(x_0)$$
$$\Rightarrow f + g \text{ is continous at } x_0$$

(b) Left as an exercise

(c) Left as an exercise

(d) Left as an exercise

□

Theorem 1.3 (Compositions of continuous functions)
Let $f : A \to B$, and $g : B \to \mathbb{R}$ where $f(A) \subseteq B$. Let $x_0 \in A$, and let f be continuous at x_0, and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

Proof.

1. Proof with $\epsilon - \delta$

   Let $\epsilon > 0$. Because g is continuous at $f(x_0)$, we get that

   $$\exists \nu > 0 \text{ such that } g(V_\nu(f(x_0)) \cap B) \subseteq V_\epsilon(g(f(x_0))). \tag{1}$$

   And since f is continuous at x_0, we get that

   $$\exists \delta > 0 \text{ such that } f(V_\delta(x_0) \cap A) \subseteq V_\nu(f(x_0)) \tag{2}$$

   Combining (1) and (2) we get that

   $$(g \circ f)(V_\delta(x_0) \cap A) = g(f(V_\delta(x_0) \cap A) \subseteq g(V_\nu(f(x_0) \cap B) \subseteq V_\epsilon(g(f(x_0)))$$

   $$\Rightarrow (g \circ f)(V_\delta(x_0) \cap A) \subseteq V_\epsilon((g \circ f)(x_0))$$

   $$\Rightarrow g \circ f \text{ is continuous at } x_0$$

2. Proof with sequential method

   Let (x_n) be a sequence with $\lim(x_n) = x_0$. Since f is continuous at x_0, we have that $\lim(f(x_n)) = f(x_0)$.

   Because g is continuous at $f(x_0)$, we have that

   $$\lim(g(f(x_n))) = g(f(x_0))$$

   $$\Rightarrow \lim((g \circ f)(x_n)) = (g \circ f)(x_0)$$

   $$\Rightarrow g \circ f \text{ is continuous at } x_0$$

$\square$

Definition 1.4. A function $f : A \to \mathbb{R}$ is called continuous (on A) if f is continuous at all $x_0 \in A$.

Example 1.5

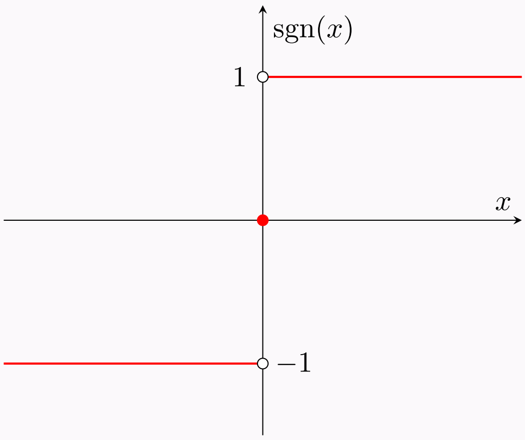
1. x is continuous on $\mathbb{R}$.
2. Because products of continuous functions are continuous, x^n is continuous on $\mathbb{R}$ for all $n \in \mathbb{N}$.

   Note also that if $c_n \in \mathbb{R}$, $c_n x^n$ is continuous on $\mathbb{R}$.
3. Since sums of continuous functions are continuous, every polynomial $p(x) := a_0 + a_1 x + \cdots + a_n x^n$ is continuous on $\mathbb{R}$.
4. Since quotients of continuous functions are continuous, wherever the denominator is non-zero, we have that all rational functions $R(x) := \dfrac{P(x)}{Q(x)}$, P, Q polynomials are continuous on $\mathbb{R}/N$ where $N := \{x \in \mathbb{R} : Q(x) = 0\}$.
5. We've seen that $\lim\limits_{x \to x_0} \sqrt{x} = \sqrt{x_0}$ for all $x_0 \in \mathbb{R}_0^+$. Thus $\sqrt{}$ is continuous on $\mathbb{R}_0^+$.
6. sin and cos are continuous on $\mathbb{R}$. See assignment 11.

Example 1.6 (Examples of discontinuous functions. sgn, Dirichlet, Thomae)

1.

$$\operatorname{sgn}(x) := \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$



Let (x_n) be a sequence with $x_n > 0$ for all $n \in \mathbb{N}$ and $\lim(x_n) = 0$ (e.g. $x_n = 1/n$. Then $\operatorname{sgn}(x_n) = 1$ for all $n \in \mathbb{N}$. Thus $(\operatorname{sgn}(x_n))$ converges to 1.

But! $\operatorname{sgn}(0) = 0 \neq 1 = \lim(\operatorname{sgn}(x_n))$. Thus sgn is discontinuous at 0.

2. Dirichlet's Function. $f : [0, 1] \to \mathbb{R}$ where f is defined as follows:

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Claim: f is discontinuous at all $x_0 \in [0, 1]$.

Proof. Proof by cases where $x_0 \in \mathbb{Q}$ and $x_0 \in \mathbb{R} \setminus \mathbb{Q}$:

a) Let x_0 be rational. Because $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$, we know that $\exists (x_n) \in [0, 1]$ such that $\lim(x_n) = x_0$ and that $\forall n \in \mathbb{N} : x_n \in \mathbb{R} \setminus \mathbb{Q}$.

Then $\forall n \in \mathbb{N}$ we have that $f(x_n) = 0 \Rightarrow \lim(f(x_n)) = 0 \neq 1 = f(x_0)$.

b) Let $x_0 \in \mathbb{R} \setminus \mathbb{Q}$. Because $\mathbb{Q}$ is dense in $\mathbb{R}$, we know that $\exists (x_n) \in [0, 1]$ with $\lim(x_n) = x_0$ and $\forall n \in \mathbb{N} : x_n \in \mathbb{Q}$.

Then $\forall n \in \mathbb{N} : f(x_n) = 1 \Rightarrow \lim(f(x_n)) = 1 \neq 0 = f(x_0)$.

□

3. Thomae's Function Consider $f : [0, 1] \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 1/q, & x = n/q, \ \gcd(n, q) = 1 \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Claim: f is <u>continuous</u> at all irrational numbers, but <u>discontinuous</u> at all rational numbers.

§1.1 Topological consequences of continuity

Exercise.

1. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be continuous. Is $f(I)$ an interval? (Yes, we will see later)
2. If $U \subseteq \mathbb{R}$ is open and $f : U \to \mathbb{R}$ is continuous, is $f(U)$ open? (No. Find a counterexample).
3. If $V \subseteq \mathbb{R}$ is closed, is $f(V)$ closed? (No)
4. If $S \subseteq \mathbb{R}$ is bounded, is $f(S)$ bounded (No)
5. If $C\mathbb{R}$ is compact (recall that this means closed and bounded), is $f(C)$ compact?

Solution.

1. We will see later.
2. Let $f :]-1, 1[\to \mathbb{R}$ where $x \to x^2$. Then $]-1, 1[$ is open, but $f(]-1, 1[) = [0, 1[$ which is <u>not</u> open.
3. $f : [1, \infty[\to \mathbb{R}$ where $x \to 1/x$. Then $f([1, \infty[) =]0, 1]$ which is <u>not</u> closed.
4. $f :]0, 1] \to \mathbb{R}$ where $x \to 1/x$. The domain of f is bounded. But $(]0, 1]) = [1, \infty[$ is unbounded.
5.

$\square$