§1 Lecture 02-03

Question:
Calculate

$$\#\{(V_1, V_2), \dim(V_1) = k_1, \dim(V_2) = k_2, \dim(V_1 \cap V_2) = d, V_1, V_2 \subseteq V\}$$

Given $k_1, k_2, d, \dim V = n, \#F = q$.

New approach to solution. Let $d = 0$. Understand the set of linearly disjoint pairs (V_1, V_2) with $\dim(V_1) = k_1, \dim(V_2) = k_2$.

Number of possibilities for V_1 is

$$\binom{n}{k_1}_q = \frac{(q^n - 1)(q^{n-1} - 1)(\cdots)(q^{n-k_1+1} - 1)}{(q^{k_1} - 1)(\cdots)(q - 1)}$$

Next, the number of possibilities for V_2 once V_1 is chosen:

$$(q^n - q^{k_1})(q^n - q^{k_1+1})(\cdots)(q^n - q^{k_1+k_2-1})$$

Dividing by the possible bases for a single subspace of dim k_2.

$$\begin{aligned}
&\frac{(q^n - q^{k_1})(q^n - q^{k_1+1})(\cdots)(q^n - q^{k_1+k_2-1})}{(q^{k_2} - 1)(q^{k_2} - q)(\cdots)(q^{k_2} - q^{k_2-1})} \\
&= \frac{q^{k_1+(k_1+1)+(k_1+2)+\cdots+(k_1+k_2-1)}}{q^{0+1+2+\cdots+(k_2-1)}}\binom{n-k_1}{k_2}_q
\end{aligned}$$

$$= q^{k_1k_2}\binom{n-k_1}{k_2}_q$$

So

$$\begin{aligned}
&\#\{(V_1, V_2), \dim(V_1) = k_1, \dim(V_2) = k_2, \dim(V_1 \cap V_2) = 0, V_1, V_2 \subseteq V\} \\
&= \binom{n}{k_1}_q \binom{n-k_1}{k_2}_q q^{k_1k_2}
\end{aligned}$$

Remark 1.1.

$$\binom{n}{k_1}\binom{n-k_1}{k_2}$$

is the number of disjoint subsets of cardinality k_1 and k_2 in a set of cardinality n.

Now to solve for general d.

Lemma 1.2

The set $\{(V_1, V_2)$ of dim (k_1, k_2) with $\dim(V_1 \cap V_2) = d$ is a natural bijection with the set of triples $\{(W, \overline{V_1}, \overline{V_2})$ where $W \subseteq V$, $\dim W = d$,
$\overline{V_1} \subseteq V/W$, $\dim \overline{V_1} = k_1 - d$
$\overline{V_2} \subseteq V/W$, $\dim \overline{V_2} = k_2 - d$
$\overline{V_3} \subseteq V/W$, $\dim \overline{V_3} = k_3 - d$
$\overline{V_1}, \overline{V_2}$ are linearly disjoint.

Proof.

$$(V_1, V_2) \mapsto (V_1 \cap V_2, V_1 \setminus W, V_2 \setminus W)$$
$$(\pi^{-1}(\overline{V_1}), \pi^{-1}(\overline{V_2})) \leftarrow (W, \overline{V_1}, \overline{V_2})$$

$\square$

$$\#\Sigma = q^{(k_1-d)(k_2-d)} \binom{n}{d}_q \binom{n-d}{k_1-d}_q \binom{n-k_1}{k_2-d}_q$$

Number of choices for W =

$$\binom{n}{d}_q$$

Number of choices for $(\overline{V_1}, \overline{V_2})$ given W

$$\binom{n-d}{k_1-d}_q \binom{n-k_1}{k_2-d}_q q^{(k_1-d)(k_2-d)}$$

Number of linearly disjoint spaces of dims k_1, k_2 in $\mathbb{F}^n$ =

$$\binom{n}{k_1}_q \binom{n-k_1}{k_2}_q q^{k_1 k_2}$$

Question 3 from homework.

Show that if $T : V \to V$, $\dim V = n$, then T satisfies a polynomial of degree $\leq n$.

$$p(x) = x^m + a_{m-1}x^{n-1} + \cdots + a_1 x + a_0$$
$$p(T) = T^n + a_{m-1}T^{m-1} + \cdots + a_1 T + a_0 I$$

This shows that the space generated by

$$\underbrace{(1, T, T^2, T^3, \dots)}_{\leq n} \subseteq \underbrace{\operatorname{End}(V)}_{\leq n^2}$$

We show by induction of n that if W is any vector space of dimension n, $T : W \to W$ any endomorphism, then $\exists p(x), \deg(p(x) \leq n$, such that $p(T) = 0$.

$n = 1$. $T : V \to V, T(v) = \lambda v, \lambda \in F$.

Case 1. $\exists v \in V$ such that $v_1, Tv, T^2v, \dots, T^{n-1}v$ span V.

$-T^n v = a_0 v + a_1 Tv + a_2 T^2 v + \cdots + a_{n-1} T^{n-1} v$.

$p(x) = x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$.

$p(T)(v) = 0$. $T(p(T)(v)) = 0$.