§1 Normal Subgroups and Factor Groups

Definition 1.1. A subgroup $H \subset G$ is normal if $gH = Hg$ for all $g \in G$.

Example 1.2

1. Every subgroup of H is normal if G is abelian.
2. If $[G : H] = 2$, then H is normal. This is because $gH \cup H = G = H \cup Hg$.
3. Let $H \subset D_h$ be a subgroup of rotations. Then H is normal (because $[D_h : H] = 2$). However, let $R = \langle r \rangle$ where r is reflection, then R is not normal in D_n.
4. $\{e\} \subset G$ and $G \subset G$ are normal.

Theorem 1.3
Let $N \subset G$ be a subgroup. TFAE

1. N is normal in G.
2. $gNg^{-1} \subset N$ for all $g \in G$.
3. $gNg^{-1} = N$ for all $g \in G$.

Note 1.4. For $S \subset G$ and $x, y \in G$, $xSy = \{xsy : s \in S\}$

Proof.

$(1 \Rightarrow 2)$ We must show that $gng^{-1} \in N$ for all $n \in N$.

$$gN = Ng \Rightarrow \exists n' \in N \text{ such that } gn = n'g$$
$$\text{Hence:} \quad (gn)g^{-1} = (n'g)g^{-1} = n' \in N$$

$(2 \Rightarrow 3)$ Suffices to show that $N \subset gNg^{-1}$.

$$g^{-1}ng \in g^{-1}N(g^{-1})^{-1} \subset N \Rightarrow g^{-1}ng = n' \text{ for some } n' \in N$$
$$\text{So } n = gn'g^{-1}$$

$(3 \Rightarrow 1)$ Right multiply by g. $gNg^{-1} = N$ gives $gN = Ng$.

□

§1.1 Factor Group or Quotient Group

Definition 1.5. Let $N \subset G$ be a normal subgroup of G. The left cosets of N in G form a group whose operation is $(aN)(bN) = (abN)$. This is the quotient group of G and N, denoted by G/N.

Theorem 1.6
G/N is really a group!

Proof.

1. To show: Operation is well defined. If $aN = a'N$ and $bN = b'N$, then $abN = a'b'N$.

   We know that $a' = an_1$ and $b' = bn_2$ where $n_1, n_2 \in N$. Hence $a'b' = (an_1)(bn_2)$. Because $Nb = bN$, we hvae that $n_1 b = bn_3$ for some $n_3 \in N$. Therefore $a'b' = a(n_1 b)n_2 = a(bn_3)n_2 = abn_3n_2$.

   Thus $a'b'N = abN$ since $(ab)^{-1}(a'b') = b^{-1}a^{-1}abn_3n_2 = n_3n_2 \in N$.

2. To show: Associativity.

$$aN(bNcN) = aN(bcN) = a(bc)N = abcN$$
$$(aNbN)cN = (abN)cN = (ab)cN = abcN$$

To show: Identity. $eNxN = exN = xN = xeN = xNeN$

To show: Inverses. $(xN)(x^{-1}N) = xx^{-1}N = eN = x^{-1}xN = x^{-1}NxN$ $\square$

Recall 1.7. If G is finite, $|G/N| = [G : N] = |G|/|N|$

Example 1.8
$\mathbb{Z}_n$ is just notation for $\frac{\mathbb{Z}}{n\mathbb{Z}}$

Quotient Group $\mathbb{Z}/4\mathbb{Z}$:

$\circ$	$0+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$
$0+4\mathbb{Z}$	$0+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$
$1+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$	$4+4\mathbb{Z}$
$2+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$	$4+4\mathbb{Z}$	$5+4\mathbb{Z}$
$3+4\mathbb{Z}$	$3+4\mathbb{Z}$	$4+4\mathbb{Z}$	$5+4\mathbb{Z}$	$6+4\mathbb{Z}$

Example 1.9
$H \subset D_n$ be subgroup of rotations. $D_n/H \cong \mathbb{Z}_2$ since $[D_n : H] = 2$.

Example 1.10
$S_n/A_n \cong \mathbb{Z}_2$

Example 1.11

$N = \{\pm 1\}$ is normal in Q. It's cosets are:

$$\begin{aligned} 1N &= \{\pm 1\} = N1 \\ jN &= \{\pm j\} = Nj \\ kN &= \{\pm k\} = Nk \\ iN &= \{\pm i\} = Ni \end{aligned}$$

What is Q/N? Note: $|Q/N| = [Q : N] = 4$.

$$Q/N : \quad \begin{array}{c|cccc} \circ & 1N & iN & jN & kN \\ \hline 1N & 1N & iN & jN & kN \\ iN & iN & 1N & kN & jN \\ jN & jN & kN & 1N & iN \\ kN & kN & jN & iN & 1N \end{array}$$

Example 1.12

$$(\mathbb{Z}_4, +) : \quad \begin{array}{c|cccc} \circ & (0,0) & (1,0) & (0,1) & (1,1) \\ \hline (0,0) & (0,0) & (1,0) & (0,1) & (1,1) \\ (1,0) & (1,0) & (0,0) & (1,1) & (0,1) \\ (0,1) & (0,1) & (1,1) & (0,0) & (1,0) \\ (1,1) & (1,1) & (0,1) & (1,0) & (0,0) \end{array}$$