§1 Lecture 02-21

Theorem 1.1
If M is a symmetric $n \times n$ matrix with real entries, then M is diagonalizable.

If M is symmetric, then $M = (a_{ij})_{i,j=1,\ldots,n}, \quad a_{ij} = a_{ji}$.
Language for approaching this result: self adjoint operators on inner product spaces.

§1.1 Duality

V vector space. V^* is the space of linear functionals $V \to F$.

If $T : V_1 \to V_2$ is a linear transformation, then it induces

$$T^* : V_2^* \to V_1^*$$
$$T^*(l) = l \circ T$$

$$V_1 \to_{T_1} V_2 \to_{T_2} V_3$$
$$(T_2 \circ T_1)^* : V_3^* \to V_1^*$$
$$(T_2 \circ T_1)^* = T_1^* \circ T_2^*$$

$$l \in V_3^*$$
$$(T_2 \circ T_1)^*(l) = l \circ (T_2 \circ T_1) = (l \circ T_2) \circ T_1$$
$$= [T_2^*(l)] \circ T_1 = T_1^*(T_2^*(l)) = T_1^* \circ T_2^*(l)$$

Lemma 1.2

1. If $T : W \to V$ is injective, then $T^* : V^* \to W^*$ is surjective.
2. If $T : V \to W$ is surjective, then T^* is injective.

Proof.

1. If T is injective, then it realises co inclusion of W into V and
$$T^*(l) = l|_{\text{Im}(T)=W}$$
Surjectivity of T^* means that given $l_0 : \text{Im}(T) \to F$, $\exists$ an extension $l : V \to F$ such that $l|_W = l_0$. After choosing a complementary W' such that $W \oplus W' = V$, we let $l(w + w') = l_0(w)$.
2. If $T : V \to W$ is surjective, then $\ker(T^*) = \{l : W \to F \text{ such that } l \circ T = 0\}$
$$\begin{aligned} l \circ T = 0 &\Leftrightarrow l \circ T(v) = 0 && \forall v \in V \\ &\Leftrightarrow l(T(v)) = 0 && \forall v \in V \\ &\Leftrightarrow l(w) = 0 && \forall w \in \text{Im}(T) \\ &\Leftrightarrow l(w) = 0 && \forall w \in W \end{aligned}$$
So $\ker(T^*) = 0 \Rightarrow T^*$ is injective.

If W is a subspace of V, then W^* is a quotient of V^*. If W is a quotient of V, then W^* is a subspace of V^*.

$$W^* = \{l : V \to F \text{ such that } l|_{\ker(V \to W)=0}\}$$

□

GIven a $W \subseteq V$, there is a canonical subspace of V^* attached to W,

$$W^\perp = \ker(V^* \to W^*)$$
$$W^\perp = \{l : V \to F \text{ such that } l(W) = 0\}$$

The assignment $W \mapsto W^\perp$ sets up an inclusion reversion bijection between subspaces of V and subspaces of V^*.

$$WW^\perp$$
$$0V^*$$
$$V0$$

Claim: $\dim(W) + \dim(W^\perp) = \dim(V) = \dim(V^*)$.
Caveat: $W \oplus W^\perp$ does not make sense.

Proof.

$$i : VV$$
$$i^* : V^* \to W^*$$
$$W^\perp = \ker(i^* : V^* \to W^*)$$

□

§1.2 Rank-nullity Theorem

$$\dim(W^{\perp}) + \dim(W^*) = \dim(V^*)$$
$$\dim(W^{\perp}) + \dim(W) = \dim(V)$$

If $W \subseteq V^*$, then $W^{\perp} \subseteq V$. $W^{\perp} = \{v \in V \text{ such that } l(v) = 0 \quad \forall l \in W\}$.