§1 Lecture 11-11

Definition 1.1 (Weierstrass)**.** The ϵ definition of the limit of a function.

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$, and $x_0 \in \mathbb{R}$. We say that L is a limit of f as x approaches x_0 if:

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x \in A : 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$$

This can be rewritten in several ways:

1. $$\forall \epsilon > 0,\ \exists \delta > 0 : x \in V_\delta^*(x_0) \cap A \Rightarrow f(x) \in V_\epsilon(L)$$

2. $$\forall \epsilon > 0,\ \exists \delta > 0 : f(V_\delta^*(x_0) \cap A) \subseteq V_\epsilon(L)$$

Theorem 1.2

Let $f : A \to \mathbb{R}$ be a function. Let $x_0 \in \mathbb{R}$ and $L \in \mathbb{R}$. Then:

$$\lim_{x \to x_0} f(x) = L$$

in the sequential sense if and only if this holds in the $\epsilon - \delta$ sense.

Proof.

1. "$\epsilon - \delta \Rightarrow$ Sequential":

   Let $\epsilon > 0$. Let $\delta > 0$ be such that $f(V_\delta^*(x_0) \cap A) \subseteq V_\epsilon(L)$.

   Let (x_n) be a sequence in $A \setminus \{x_0\}$ with $\lim(x_n) = x_0$. Then $\exists N \in \mathbb{N}$, $\forall n \geq N : x_n \in V_\delta(x_0)$.
   We also have that $x_n \neq x_0$ and $x_n \in A$ for all $n \in \mathbb{N}$. This implies that

   $$\begin{aligned} &\forall n \geq N : x_n \in V_\delta^*(x_n) \cap A \\ &\Rightarrow \forall n \geq N : f(x_n) \in V_\epsilon(L) \\ &\Rightarrow (f(x_n)) \text{ converges } toL \end{aligned}$$

2. "Sequential $\Rightarrow \epsilon - \delta$":

   Assume that the sequential definition holds but that there exists $\epsilon > 0$ for which ulno $\delta > 0$ exists that satisfies $\epsilon - \delta$.

   i.e. assume that $f(V_\delta^*(x_0) \cap A) \not\subseteq V_\epsilon(L)$ for all $\delta > 0$. Especially:

   $$\begin{aligned} \delta = 1 : \quad & f(V_1^*(x_0) \cap A) \not\subseteq V_\epsilon(L) \\ \Rightarrow \exists x_1 \in V_1^*(x_0) \cap A & \text{ such that } f(x_1) \notin V_\epsilon(L) \end{aligned}$$

   $$\begin{aligned} \delta = \frac{1}{2} : \quad & f(V_{\frac{1}{2}}^*(x_0) \cap A) \not\subseteq V_\epsilon(L) \\ \Rightarrow \exists x_2 \in V_{\frac{1}{2}}^*(x_0) \cap A & \text{ such that } f(x_2) \notin V_\epsilon(L) \end{aligned}$$

   $$\vdots$$

   We then obtain a sequence (x_n) such that $x_n \in V_{\frac{1}{n}}^*(x_0) \cap A$ but $f(x_n) \notin V_\epsilon(L)$.

   Thus $\lim(x_n) = x_0$ but $(f(x_n))$ does <u>not</u> converge to L. This contradicts the sequential definition of limit.

   Thus $\exists \delta > 0$ such that $f(V_\delta^*(x_0) \cap A) \subseteq V_\epsilon(L)$.

$\square$

Example 1.3

Show that:

$$\lim_{x \to x_0} x^2 = x_0^2$$

Solution.

1. Sequential:

   Let (x_n) be a sequence in $\mathbb{R} \setminus \{x_0\}$ with $\lim(x_n) = x_0$. Then $\lim(f(x_n)) = \lim(x_n^2) = [\lim(x_n)]^2 = x_0^2$

2. $\epsilon - \delta$:

   Let $\epsilon > 0$. Let $\delta > 0$ be arbitrary for now and assume that $|x - x_0| < \delta$. Then

   $$|f(x) - f(x_0)| = |x^2 - x_0^2| = \underbrace{|x - x_0|}_{<\delta} \cdot |x + x_0|$$

   $$\Rightarrow < |x + x_0|\delta = |x - x_0 + 2x_0|\delta \le (|x - x_0| + 2|x_0|)\delta$$
   $$< (\delta + 2|x_0|)\delta < (\delta + 2|x_0|) \cdot \delta < \epsilon$$

   Assume that $\delta < 1$. Then $|f(x) - f(x_0)| < (\delta + 2|x_0|)\delta < (1 + 2|x_0|)\delta < \epsilon$

   Now let:

   $$\delta < \min(1, \frac{\epsilon}{1 + 2|x_0|})$$

   Then if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon \Rightarrow$

   $$\lim_{x \to x_0} x^2 = x_0^2$$

□

Example 1.4

$$f : \mathbb{R}\{0\} \to \mathbb{R}, x \to \frac{1}{x}$$

Let $x_0 \in \mathbb{R} \setminus \{0\}$. Show that:

$$\lim_{x \to x_0} \frac{1}{x} = \frac{1}{x_0}$$

Solution.

1. Sequential:

   Let (x_n) be a sequence in $\mathbb{R} \setminus \{0, x_0\}$ with $\lim(x_n) = x_0$. Then:

$$\lim(f(x_n)) = \lim(\frac{1}{x_n}) = \frac{1}{\lim(x_n)} = \frac{1}{x_0}$$

2. With $\epsilon - \delta$:

   Let $\epsilon > 0$. Let $\delta > 0$ be arbitrary for now. Let $|x - x_0| < \delta$. Then:

$$\begin{aligned} |f(x) - f(x_0)| &= |\frac{1}{x} - \frac{1}{x_0}| = |\frac{x_0 - x}{x x_0}| \\ &= \frac{|x - x_0|}{|x||x_0|} < \frac{\delta}{|x||x_0|} \end{aligned}$$

   Let $\delta < \frac{1}{2}|x_0|$. Then for all x with $|x - x_0| < \delta$ we have:

$$|x| = |(x - x_0) + x_0| \geq |x| - |x - x_0| > |x_0| - \frac{1}{2}|x_0| = \frac{1}{2}|x_0|$$

   i.e. $|x| \geq \frac{1}{2}|x_0|$ Now:

$$\begin{aligned} |f(x) - f(x_0)| < \frac{\delta}{|x||x_0|} \leq \frac{\delta}{\frac{1}{2}|x_0||x_0|} = \frac{2\delta}{x_0^2} < \epsilon \\ \Leftrightarrow \delta < \frac{x_0^2}{2} \cdot \epsilon \end{aligned}$$

   Let $\delta < \min(\frac{1}{2}|x_0|, \frac{1}{2}x_0^2\epsilon)$. Then if $|x - x_0| < \delta$, we have that:

$$|f(x) - f(x_0)| < \epsilon \Rightarrow \lim_{x \to x_0} \frac{1}{x} = \frac{1}{x_0}$$

$\square$

§1.1 Limit Laws

Theorem 1.5 (Limit of a Sum is the Sum of the Limits)
Let $f, g : A \to \mathbb{R}$, and x_0 be a cluster point of A. Assume that $\lim_{x \to x_0} f(x) = L_1$ and that $\lim_{x \to x_0} g(x) = L_2$.

Then

$$\begin{aligned} \lim_{x \to x_0} [(f+g)(x)] &= \lim_{x \to x_0} [f(x) + g(x)] = L_1 + L_2 \\ &= \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x) \end{aligned}$$

i.e.

$$\lim_{x \to x_0} [(f+g)(x)] = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$$

Proof. We'll use the sequential criterion to prove this theorem. Let (x_n) be a sequence in $A \setminus \{x_0\}$ with $\lim(x_n) = x_0$. Then

$$\lim((f+g)(x_n)) = \lim(f(x_n) + g(x_n))$$
$$= \lim(f(x_n)) + \lim(g(x_n)) = L_1 + L_2 = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$$

□

Theorem 1.6 (Limit of a Product is the Product of the Limits)
Let $f, g : A \to \mathbb{R}$ and x_0 be a cluster point of A. Assume that $\lim_{x \to x_0}$ and $\lim_{x \to x_0} g(x)$ exist. Then:

$$\lim_{x \to x_0} [(f \cdot g)(x)] = \lim_{x \to x_0} [f(x) \cdot g(x)] = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} g(x)$$

Proof. Let (x_n) be a sequence in $A \setminus \{x_0\}$ with $\lim(x_n) = x_0$. Then:

$$\lim_{x \to x_0} [(f \cdot g)(x)] = \lim(f(x_n) \cdot g(x_n)) = \lim(f(x_n)) \cdot \lim(g(x_n)) = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} g(x)$$

□

Especially, let $c \in \mathbb{R}$. Then

$$\lim_{x \to x_0} [c \cdot f(x)] = c \cdot \lim_{x \to x_0} f(x) \quad \text{Think of it as choosing } g = c$$

Therefore:

$$\begin{aligned} \lim_{x \to x_0} [f(x) - g(x)] &= \lim_{x \to x_0} [f(x) + (-1) \cdot g(x)] = \lim_{x \to x_0} f(x) + \lim[(-1)g(x)] \\ &= \lim_{x \to x_0} f(x) + (-1) \lim_{x \to x_0} g(x) = \lim_{x \to x_0} f(x) - \lim_{x \to x_0} g(x) \\ &\Rightarrow \lim_{x \to x_0} [f(x) - g(x)] = \lim_{x \to x_0} f(x) - \lim_{x \to x_0} g(x) \end{aligned}$$

Theorem 1.7

Let $f, g : A \to \mathbb{R}$ and x_0 be a cluster point of A. Furthermore, let $\forall x \in A,\ g(x) \neq 0$ and let $\lim_{x \to x_0} f(x), \lim_{x \to x_0} g(x)$ exist where $\lim_{x \to x_0} g(x) \neq 0$. Then:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$$