Analyzing Fixed Point Arithmetic Rounding Error

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Abstract

We examine the magnitude of rounding error resulting from different combinations of fixed point arithmetic operations used to achieve the same goal, in an environment with truncating integer semantics.

Contents

1 Definitions

1.0.1 Epsilon Notation

By definition of the floor operation, the following is true:

$$\frac{x}{y} - 1 < \left\lfloor \frac{x}{y} \right\rfloor \leq \frac{x}{y}$$

To simplify the inequality above for use in larger expressions, we will define epsilon (ϵ) as a non-deterministic number in the interval $[0, 1)$. Using this new notation, we can rewrite the inequality above like so:

$$\frac{x}{y} - \epsilon = \left\lfloor \frac{x}{y} \right\rfloor$$

2 Analysis

A common use case for fixed point arithmetic is multiplying a quantity of one asset by a conversion rate to another asset, in a programming language with no floating point arithmetic, only integer operations. A notable example of a language with these properties is the Solidity smart contract language, which only has basic truncating integer operations.

2.1 Introduction

Let σ_n be an arbitrary quantity of asset A that we wish to exchange for a quantity of asset B, μ_n. The rate of exchange between asset A and asset B is defined as the ratio between the separate quantities of μ and σ.

Below we will define and analyze multiple implementations of the desired function, $\text{to}_\mu(\sigma_n)$, which takes a quantity of asset A (σ_n) and returns the amount of asset B that the quantity of asset A is worth (μ_n) using the rate of exchange formula described briefly above ($\frac{\mu}{\sigma}$).

2.2 Fixed Loss Implementation

In an environment without truncating division, we can observe that these two implementations are equivalent:

$$\mu_n = \text{to}_\mu(\sigma_n) = \sigma_n \cdot \frac{\mu}{\sigma} = \frac{\sigma_n \mu}{\sigma}$$

However, in an environment with truncating division, these implementations differ:

$$\mu_n = \text{to}_\mu(\sigma_n) = \sigma_n \cdot \left\lfloor \frac{\mu}{\sigma} \right\rfloor \leq \left\lfloor \frac{\sigma_n \mu}{\sigma} \right\rfloor$$

Rewriting using the epsilon notation introduced earlier:

$$\mu_n = \text{to}_\mu(\sigma_n) = \sigma_n \cdot \left(\frac{\mu}{\sigma} - \epsilon \right) \leq \frac{\sigma_n \mu}{\sigma} - \epsilon$$

Distributing multiplication:

$$\mu_n = \text{to}_\mu(\sigma_n) = \frac{\mu}{\sigma}\sigma_n - \epsilon\sigma_n \leq \frac{\sigma_n \mu}{\sigma} - \epsilon$$

Now, with the expressions fully expanded, we can see why this simple advice is so effective. Compared to the scaled rounding loss in the left equation ($\epsilon\sigma_n$), where rounding error is unbounded, rounding error in the equation on the right is bounded between $[0, 1)$.

2.3 Fixed Point Arithmetic Implementations

2.3.1 $\sigma_n.fmul(\mu.fdiv(\sigma))$

$$\mu_n = \mathrm{to}_\mu(\sigma_n) = \left\lfloor \frac{\left\lfloor \frac{\mu \cdot 10^{18}}{\sigma} \right\rfloor \cdot \sigma_n}{10^{18}} \right\rfloor$$

Using epsilon notation:

$$\mu_n = \mathrm{to}_\mu(\sigma_n) = \frac{\left(\frac{\mu \cdot 10^{18}}{\sigma} - \epsilon\right) \cdot \sigma_n}{10^{18}} - \epsilon$$

Simplified:

$$\mu_n = \mathrm{to}_\mu(\sigma_n) = \frac{\mu\sigma_n}{\sigma} - \frac{\epsilon\sigma_n}{10^{18}} - \epsilon$$

Error scales proportionally with σ_n.

2.3.2 $\sigma_n.fmul(\mu).fdiv(\sigma)$

$$\mu_n = \mathrm{to}_\mu(\sigma_n) = \left\lfloor \frac{\left\lfloor \frac{\sigma_n \cdot \mu}{10^{18}} \right\rfloor \cdot 10^{18}}{\sigma} \right\rfloor$$

Epsilon notation:

$$\mu_n = \mathrm{to}_\mu(\sigma_n) = \frac{\left(\frac{\sigma_n \cdot \mu}{10^{18}} - \epsilon\right) \cdot 10^{18}}{\sigma} - \epsilon$$

Simplified:

$$\mu_n = \mathrm{to}_\mu(\sigma_n) = \frac{\mu\sigma_n}{\sigma} - \frac{\epsilon 10^{18}}{\sigma} - \epsilon$$

Error is inversely proportional with σ.

2.3.3 $\sigma_n.fdiv(\sigma).fmul(\mu)$

$$\mu_n = \mathrm{to}_\mu(\sigma_n) = \left\lfloor \frac{\left\lfloor \frac{\sigma_n \cdot 10^{18}}{\sigma} \right\rfloor \cdot \mu}{10^{18}} \right\rfloor$$

Epsilon notation:

$$\mu_n = \mathrm{to}_\mu(\sigma_n) = \frac{\left(\frac{\sigma_n \cdot 10^{18}}{\sigma} - \epsilon\right) \cdot \mu}{10^{18}} - \epsilon$$

Simplified:

$$\mu_n = \mathrm{to}_\mu(\sigma_n) = \frac{\mu\sigma_n}{\sigma} - \frac{\epsilon\mu}{10^{18}} - \epsilon$$

Error scales proportionally with μ.