meth test wk2 notes

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Introduction

Hello! Here are some notes that I have written up based on the chapters for 3, 4, and 7. The size of this document may be intimidating, but these notes are intended to be understood and explanatory rather than something to be memorized. I'm not in any year 11 methods classes, so these notes aim to explain the Cambridge chapters that i believe are in the test in maybe a more accessible way.

1 Quadratics

1.1 Forms of a quadratic

There are 3 forms of a quadratic. (Please note that the following values of a, b and c are usually not equal between forms, and I just only want to use 3 letters.)

- Standard form: $ax^2 + bx + c = 0$
  Standard form is not particularly useful, but you will often get questions with quadratics in standard form and have to convert them to another form.
- Factored form: $a(x - b)(x - c) = 0$
  Factored form is especially useful for solving quadratics. Note that simply substituting b and c both give 0 on the left hand side, as it is multiplied by 0, so this quadratic has solutions $x = b$ and $x = c$
- Vertex form: $a(x - b)^2 + c = 0$
  This form is special as it helps us find the **maximum or minimum** (which is also the turning point or vertex). This is due to $(x - b)^2$ always being positive, and therefore $a(x - b)^2$ having the same sign as a. Therefore, the extreme cases, where the minimum or maximum is reached, occurs when $(x - b)^2$ is 0, or when $x = b$.

1.2 Factorising Quadratics

Why do we even want to factorise anyways? Well, suppose we want to solve the equation $x^2 + 11x + 24 = 0$. Lets expand a seemingly unrelated factored quadratic like $(x + 3)(x + 8)$.

$$\begin{aligned}(x+3)(x+8) &= x^2 + 3x + 8x + 3 \times 8 \\ &= x^2 + (3+8)x + 3 \times 8 \\ &= x^2 + 11x + 24\end{aligned}$$

Therefore, we have that $(x + 3)(x + 8) = x^2 + 11x + 24$. Shockingly, they are the same! So, we can solve this quadratic as follows:

$$\begin{aligned}0 &= x^2 + 11x + 24 \\ &= x^2 + 3x + 8x + 3 \times 8 \\ &= (x+3)(x+8)\end{aligned}$$

Notice we have $(x + 3)$ and $(x + 8)$ multiplying to 0. If neither are 0, their product wouldn't be 0, so one of the factors must be 0 (this is otherwise known as null factor law). Therefore, either $x + 3 = 0$ or $x + 8 = 0$, so

$x = 3$ or $x = 8$ and the equation is solved. Here, we saw that by rewriting the quadratic as two factors that multiplied to 0, the equation magically solved itself! However, how do we find these two factors?

How to Factorise: If we take a monic factorised quadratic in a general form, we can expand it to get $(x-a)(x-b) = x^2 - (a+b)x + ab$. Therefore, for **monic** quadratics $x^2 + px + q$ if we can find a and b such that $p = -(a+b)$ and $q = ab$, we have

$$x^2 + px + q = x^2 - (a+b)x + ab = (x-a)(x-b)$$

which factorises the quadratic. Unfortunately, we have to use intuition and trial and error to find these two numbers. What I like to do is list the pairs of factors in my head that multiply to the last number, and use trial and error adding them up until I find a pair that works. With practice it becomes much easier, so do some practice questions.

Example 1. Find the solutions of $x^2 + 2x - 15 = 0$.

Proof. We want to find $a + b = 2$ and $ab = -15$. Now we go through all the factors that multiply to -15 (be careful with negatives!)
We have $(1, -15)$ and $(-1, 15)$, $(3, -5)$, $(-3, 5)$ are all the pairs with two integers that multiply to -15. The corresponding sums of each pair are $-14, 14, -2$, and 2. Note that we are looking for $a + b = 2$ and $ab = -15$, and our search shows that the fourth pair $(-3, 5)$ fills this condition.

$$\therefore \text{ Since } -3+5 = 2, -3 \times 5 = -15, x^2 + 2x - 15 = (x-3)(x+5).$$
$$\therefore x - 3 = 0 \text{ or } x + 5 = 0$$
$$\therefore x = 3 \text{ or } x = -5$$

□

For non-monic quadratics, the process is quite similar. If u can, try and take a common factor. For example, $2x^2 + 4x - 6 = 2(x^2 + 2x - 3)$, which can then be factorised similar to before. If this is not possible to do without creating fractions, then we have to take a slightly different approach.

1.3 Completing the Square

For this chapter, I will first provide a little bit of motivation behind why we complete the square. First, lets solve a simple quadratic, $x^2 = 9$.

$$\begin{aligned} x^2 &= 9 \\ \sqrt{x^2} &= \pm\sqrt{9} \\ x &= 3 \text{ or } x = -3 \end{aligned}$$

Here the process was quite simple - we just square rooted both sides (and accounted for the negative solution as well!) Now, notice that if we have any square in place of x^2, we can do the same thing. For example, lets solve what seems to be a more complicated quadratic, $x^2 + 2x + 1 = 9$. However, this is not very difficult at all once we realise that $(x+1)^2 = x^2 + 2x + 1$.

$$\begin{aligned} x^2 + 2x + 1 &= 9 \\ (x+1)^2 &= 9 \\ x + 1 &= 3 \text{ or } x + 1 = -3 \\ x &= 2 \text{ or } x = -4 \end{aligned}$$

Furthermore, if we take another quadratic like $x^2 + 2x + 5 = 13$, we can break it up into $x^2 + 2x$

1.4 Graphs of Quadratics

Look at Section 2.1 for notes on transformations, as they apply here. To find the equation of a graph of a quadratic, you need 3 points or a 2 points, one of which is a vertex point. Generally, this means you need 3 pieces of information. So, if you are given or it is easy to read the x-intercepts, you can put it in factored form $y = a(x - b)(x - c)$ and use some other point to solve for a. If you have the vertex, put it into vertex form $a(x$

2 Graphs of relations

2.1 Transformations

Lets talk about transformations, as we can apply these ideas to any graph. There are 3 types of transformations that we care about:

- Translations - the graph moving around without the shape changing or stretching.
- Dilation - Stretching the graph around horizontally or vertically
- Reflection - the graph is flipped in some way horizontally or vertically

I'm going to use $f(x)$ as the base function that we are going to transform, and $g(x)$ as the transformed function.
Vertical translations: $g(x) = f(x) + c$
Here each point in the original graph $(x, f(x))$ corresponds with the point $(x, g(x)) = (x, f(x)+c)$, a translation of c units up.
Horizontal translations: $g(x) = f(x - c)$
Again each point $(x, f(x))$ corresponds to another point $(x + c, g(x + c)) = (x + c, f(x))$, which is a translation of c units to the right.
Vertical dilation: $g(x) = af(x)$
Each point $(x, f(x))$ has a corresponding point $(x, g(x)) = (x, af(x))$, which is a dilation of scale factor a from the x axis (vertically).
Horizontal dilation: $g(x) = f(ax)$
For each point $(x, f(x))$ there is a corresponding point $(\frac{x}{a}, g(\frac{x}{a}) = (\frac{x}{a}, f(x))$, which is a horizontal dilation of scale factor $\frac{1}{a}$, not a.

2.2 Hyperbolas

2.3 Parabolas... but different?

2.4 Circles

3 Powers and Polynomials

3.1 What is a Polynomial

This term sounds scary, but they are actually easy to understand. A polynomial is any expression of (positive) powers of x added up, or more precisely:

> A polynomial is anything that can be written in the form $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where $a_n, a_{n-1}, \ldots a_0$ are all real numbers.

For example, x, $x^2 + x + 1$ and $x^5 + 3x^2 + 100x$ are all polynomials. For convention, we will assume that a_n is not 0, since if it is we can just ignore it and start with a_{n-1}. As Cambridge says, we say a_nx^n is the leading term, a_0 is the constant term (unchanged by x) and if $a_n = 1$, then the polynomial is monic.
We say that a polynomial written out as such has a **degree** of n, or in other words the highest power of x in the polynomial. For example, polynomials we have seen before such as:

- $x + 5$ has a degree of 1

- $x^2 + 6x + 7$ has a degree of 2
- $x^{100} + 69x^{48} + 3x^2$ has a degree of 100.

Linear functions have a degree of 1, quadratics 2, cubics 3, quartics 4, and quintics 5.
We also say that two polynomials are equal if for every value of x you put in you get the same output from both. This happens if and only if their terms are equal, i.e.
If the two polynomials are $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ and $b_nx^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0$, they are equal polynomials if and only if $a_0 = b_0$, $a_1 = b_1$, e.t.c. up to $a_n = b_n$

3.2 Polynomial Arithmetic

Polynomials function remarkably similar to whole numbers. For example, with 2 polynomials I can add, subtract and multiply them just like numbers. For example, suppose we have two polynomials $P(x) = x^3 + 2x + 1$ and $Q(x) = x^2 + 7x + 3$. We can add them and multiply them by just doing so algebraically:
$P(x) + Q(x) = x^3 + x^2 + 9x + 4$
$P(x) - Q(x) = x^3 - x^2 - 5x - 2$
$P(x) \times Q(x) = x^5 + 7x^4 + 5x^3 + 15x^2 + 13x + 3$
So as you can see, we can add, subtract (basically adding a negative polynomial) and multiply to get another polynomial.
Now, we can probably guess that if we add two quadratics we are probably also going to get a quadratic, or maybe a linear function if the x^2 terms cancel out. We know that we can't get a cubic, as the x^3 term can't appear out of nowhere. This gives us some motivation for the following rule:

> **Degrees of adding polynomials:** $\deg(P + Q) \leq \max(\deg(P),\deg(Q))$

Here, I've written P instead of $P(x)$ for readability. Since the equation can be a bit confusing, all it says is that if you add 2 polynomials the degree of the resulting polynomial can't get larger.
Furthermore, we know that multiplying a polynomial by some linear function, like $(x + 2)$ increases the degree by 1, and multiplying by a quadratic increases the degree by 2. This gives some motivation for this rule:

> **Degrees of multiplying polynomials:** $\deg(P \times Q) = \deg(P)+\deg(Q)$

For the second rule, suppose that the leading terms of $P(x)$ and $Q(x)$ are a_nx^n and b_mx^m. This means that $\deg(P) = n$, $\deg(Q) = m$. Therefore, when we multiply these together, the leading term of the resulting polynomial will be $a_n \times b_m \times x^{n+m}$ so the resulting degree will be $m + n = \deg(P)+\deg(Q)$.

3.3 Dividing polynomials

Similar to whole numbers, we have a division algorithm for polynomials. Lets look at the division algorithm for whole numbers again:

> **Division algorithm for whole numbers:** For 2 positive whole numbers a, b, there are 2 other non-negative numbers q, r such that $a = bq + r$ and $0 \leq r < b$.

This just describes long division, where q is a divided by b, and r is the remainder. For example, we can divide 7 into 25 3 times, with a remainder of 4. Therefore, $25 = 7 \times 3 + 4$. Note that here, we want the remainder to be less than 7. If we do end up dividing and the remainder is more than 7, we can just divide Furthermore, note that we can do this for any 2 numbers, which describes division with whole numbers. We can do something similar with polynomials.

> **Division algorithm for polynomials:** For any two polynomials $P(x)$ and $Q(x)$, there are two more polynomials $A(x)$ and $R(x)$ such that $P(x) = A(x)Q(x) + R(x)$ and $\deg(R) < \deg(Q)$.

Notice how this looks very similar to division with normal numbers. In fact, the process for long division on polynomials is very similar.

3.4 How to do polynomial division

Long division for numbers: I'm going to start this section with dividing some numbers, as polynomial division is almost exactly the same process. Lets do long division of $23|\overline{1094}$.

$$\begin{array}{r} 47 \text{ r}13 \\ 23|\overline{1094} \\ -92\downarrow \\ 174 \\ -161 \\ 13 \end{array}$$

- 23 divides into 109 4 times, so we subtract $4 \times 23 = 92$ from 109 to get a remainder of 17.
- We bring the 4 down.
- 23 goes into 174 7 times so we subtract $161 = 23 \times 7$ from 174 to get a remainder of 13
- No more remaining digits, so we are done with $47 \times 23 + 13 = 1094$

Long division for polynomials:
We can do a similar thing with polynomials. There is a small difference however. Notice how at each step we always aim to have a remainder that is less than our divisor, in our case 23. The two remainders that we generate are 17 and 13, both less than 23. We aim to do the same thing with polynomials, except we aim for the remainder at each step to have a lesser degree than before. Lets divide $x+1$ into x^3+4x^2+2x+3.

$$\begin{array}{r} x^2+3x-1 \text{ r}4 \\ x+1|\overline{x^3+4x^2+2x+3} \\ -x^3-x^2 \quad\downarrow \quad \downarrow \\ 3x^2+2x \quad \downarrow \\ -3x^2-3x \quad \downarrow \\ -x+3 \\ -(-x-1) \\ 4 \end{array}$$

- $x+1$ divides x^2 times into x^3+4x^2 with remainder $x^3+4x^2-x^3-x^2=3x^2$
- $x+1$ divides $3x$ times into $3x^2+2x$ with remainder $3x^2+2x-3x^2-3x=-x$ (the remainder can be negative)
- $x+1$ divides -1 times into $-x+3$ with remainder $-x+3+x+1=4$

Notice that at every step, we try to eliminate the highest power of x. For example, we eliminate x^3 by subtracting x^3+x^2.

3.5 Remainder Theorem and Factor Theorem

> **Remainder Theorem:** The remainder of $P(x)$ when divided by $(x-a)$ is $P(a)$.

Here we can use our division algorithm - We can express $P(x)$ as $(x-a)A(x)+R(x)$. Furthermore, we know that $R(x)$ has a degree less than 1 and must therefore be a constant, which we can denote as r. In other words, dividing by a linear gives a constant remainder.

$$\begin{aligned} \therefore P(x) &= (x-a)A(x)+r \\ P(a) &= (a-a)A(a)+r \\ &= 0 \times A(a)+r=r \end{aligned}$$

So the remainder is $P(a)$ as desired. In our previous example, we divided $x+1$ into x^3+4x^2+2x+3 to get a remainder of 4. We could also get the same remainder by substituting $x=-1$ into x^3+4x^2+2x+3 to get $(-1)^+4(-1)^2+2(-1)+3=-1+4-2+3=4$. This also leads to an extremely useful **corollary** (a theorem that easily logically follows or a side case).

Factor Theorem: If $P(a)=0$, then $(x-a)$ divides $P(x)$.

This follows from the remainder theorem. If $P(a)=0$, then the remainder when we divide $P(x)$ by $x-a$ will be 0, so $x-a$ divides into $P(x)$.